An introduction to Schrödinger and Sinkhorn bridges

P. Del Moral, INRIA Bordeaux Sud Ouest

Berlin Summer School on Applied Stoch. Analysis, July 2026.

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$\rightsquigarrow$ `New trends/Sinkhorn [Review article] (+Jasra) (Arxiv-Jan.26):`

- *Gaussian models (+ Akyldiz/Miguez) (FoDS-26)*/(Arxiv-24)
- *Lyapunov (+ Akyldiz/Miguez) (FoDS-26)*/(Arxiv-25)
- *Contraction Theory via Log-Sob (SAA-26)*/(Arxiv-25)
- *Conditional mean/covariance (CIS-Kailath-25)*/(Arxiv-25)

Part I: Foundations

Part I

Foundations

Terminology, Entropic bridges, Sinkhorn bridges

Some terminology

Meas., transition, function $= \mu(dx)$, $K(x, dy)$, $f(y)$

Integral/matrix operations $\left[(\mu, K, \mathbf{f}) \overset{finite-state}{=} (\textbf{row}, \textbf{matrix}, \textbf{column})\right]$

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$$K(\mathbf{f})(x) := \int K(x,dy)\ \mathbf{f}(y) \qquad (\mu K)(dy) := \int \mu(dx)\ K(x,dy)$$

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Composition $(K_1K_2) \overset{finite-state}{=}$ **(matrix product)**

$$(K_1K_2)(x,dy) = \int K_1(x,du)K_2(u,dy)$$

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Gaussian distributions

- $\nu_{m,\sigma}$ = **multivariate Gaussian with mean** m **and cov.** σ
- G = Centered unit covariance Gaussian $\sim \nu_{0,I}$

Product space

$$\mathbb{Z} := (\mathbb{X} \times \mathbb{Y}) \ni (x, y) = z$$

Disintegration proba on $\mathbb{Z}$

$$(\mu \times K)(dz) := \mu(dx)\ K(x, dy)$$

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Set of couplings/disintegration/transport plans $\mu \rightsquigarrow \eta$

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Note measures on $\mathbb{Z}^{\flat} := (\mathbb{Y} \times \mathbb{X}) \ni z^{\flat} = (y, x)$

$$Q = \eta \times L$$

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Bayes' rule/Dual op. - meas. on $(\mathbb{X} \times \mathbb{Y}) \ni z := (x, y)$

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Posterior/conjugate/dual/backward transition:

$$K^{\sharp}_{\mu}(y, dx) := \frac{\mu(dx)\ K(x, dy)}{(\mu K)(dy)} \quad \Longrightarrow \quad \mu K K^{\sharp}_{\mu} = \mu$$

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Continuous/discrete: *Forward Markov*$\rightsquigarrow$*Backward Markov (end times)*

Example $\mu(dx) = e^{-U(x)}dx$ and K symmetric-Gaussian:

$$\mu(dx)K(x, dy)$$

$$= (\mu K)(dy) \underbrace{K^{\sharp}_{\mu}(y, dx)}_{\propto \ \exp\left(-U(x) - \frac{1}{2t} \ \|x-y\|_2^2\right) \ dx}$$

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"Auxiliary/Proximal sampler" with a "restricted **oracle** *sampling"* $K^{\sharp}_{\mu}\ldots$

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Oxford Language/Dictionary: **Priest/Medium/ AI?**

"prophecy/response/message (from the gods) **especially an ambiguous one.***"*

Bayes' rule for linear-Gauss models: $X \sim \mu = \nu_{m,\sigma}$

$$x \overset{K_\theta}{\rightsquigarrow} Z_\theta(x) = \alpha + \beta x + \tau^{1/2} G \quad \text{with} \quad \theta = (\alpha, \beta, \tau)$$

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$$\Longrightarrow \quad \mathrm{Cov}_{X, Z_\theta(X)} = \sigma\beta' \quad \text{and} \quad \mathrm{Cov}_{Z_\theta(X), Z_\theta(X)} = \beta\sigma\beta' + \tau$$

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$\rightsquigarrow$ **Regression/Dual/Backward/Conjugate/Update formula**

$$\left\{ \begin{array}{l} \mu \times K_\theta = ((\mu K_\theta) \times K_{\mathbb{B}_{m,\sigma}(\theta)})^\flat \\ \\ \end{array} \right.$$

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with the gain/regression matrix

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$\Longleftrightarrow$ **Bayes' map**

$$\theta \mapsto \mathbb{B}_{m,\sigma}(\theta) = (m - \kappa_\theta(\alpha + \beta m), \kappa_\theta, \varsigma_\theta)$$

Example: Linear-diffusion flow $(X_{s,s}(x) = x)$

$$t \in [s, \infty[\rightsquigarrow d_t X_{s,t}(x) = (A_t X_{s,t}(x) + b_t)\ dt + \Sigma_t^{1/2} dW_t$$

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Semigroup & Marginals = Internal states distributions :

$$P_{s,t}(x, dy) := \mathbb{P}(X_{s,t}(x) \in dy) \quad \text{and} \quad \nu_{\mathbf{t}} = \nu_{\mathbf{0}} \mathbf{P}_{\mathbf{0},\mathbf{t}} =: \mathrm{Law}(\mathbf{X}_{\mathbf{t}})$$

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On any given final time t:

$$P_{0,t} = K_{\theta[t]} \quad \text{with} \quad \theta[t] = (\alpha[t], \beta[t], \tau[t])$$

and the parameters

$$\alpha[t] \;:=\; \int_0^t \mathcal{E}_{s,t}(A)\ b_s\ ds \quad \text{with} \quad \partial_t\, \mathcal{E}_{s,t}(A) := A_t\, \mathcal{E}_{s,t}(A)$$

$$\beta[t] \;:=\; \mathcal{E}_{0,t}(A) \quad \text{and} \quad \tau[t] := \int_0^t \mathcal{E}_{s,t}(A)\ \Sigma_s\ \mathcal{E}_{s,t}(A)'\ ds.$$

Backward/Dual semigroup on $[0,t] \ni s$

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Example $\nu_s(dx) = p_s(x)dx$ and $X_{t,t}(x) = x$:

$$-d_s X_{t,s}(x) = (-(A_s X_{t,s}(x) + b_s) + \Sigma_s(\nabla \log p_s)(X_{t,s}(x)))\, ds + \Sigma_s^{1/2} dW_s$$

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On any given final/terminal time $t \rightsquigarrow P_{0,t} = K_{\theta[t]}$:

$$\mu = \nu_{m,\sigma}$$

$$\Longrightarrow \mu \times K_{\theta[t]} = \left((\mu K_{\theta[t]}) \times P_{t,0}\right)^\flat \quad \text{and} \quad P_{t,0} = K_{\mathbb{B}_{m,\sigma}(\theta[t])}$$

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Notes:

- **Pb with continuous time $\rightsquigarrow$ Sol. discretize SDE**

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Notes:

- **Pb with continuous time $\rightsquigarrow$ Sol. discretize SDE**
- **Pb with unknown $p_s \rightsquigarrow$ Sol. with scores/neural networks**

Entropic bridges

Schrödinger bridges

Relative entropy

$$\mathcal{H}(Q \mid P) = Q\left(\log\left(\frac{dQ}{dP}\right)\right)$$

Schrödinger bridges

Relative entropy

$$\mathcal{H}\left(Q \mid P\right) = Q\left(\log\left(\frac{dQ}{dP}\right)\right)$$

Entropic bridge map/reference P = probability on $(\mathbb{X} \times \mathbb{Y})$:

$$P \in \Pi(\mu, .) \mapsto P_{\mu,\eta} := \underset{Q \in \Pi(\mu,\eta)}{\arg\min}\ \mathcal{H}(Q|P) \in \Pi(\mu, \eta)$$

Static vs **Dyn. bridges Q** $\ll$ **P** on $\Omega := \mathcal{C}([0,T],\mathbb{R}^d)$

$$\mathbf{P}^z(d\omega) \overset{z=(x,y)}{:=} \mathbf{P}(d\omega \mid (\omega_0, \omega_T) = z)$$

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Entropy factorization

$$\mathcal{H}(\mathbf{Q} \mid \mathbf{P}) = \mathcal{H}(Q \mid P) + \int \underbrace{\mathcal{H}(\ \mathbf{Q}^z \mid \mathbf{P}^z)}_{0\ \Leftarrow\ \mathbf{Q}^z=\mathbf{P}^z}\ Q(dz).$$

Static vs **Dyn. bridges Q** $\ll$ **P** on $\Omega := \mathcal{C}([0, T], \mathbb{R}^d)$

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Solution = Static bridge & Pinned process

$$\mathbf{P}_{\mu,\eta}(d\omega) := \int P_{\mu,\eta}(dz)\ \mathbf{P}^z(d\omega)$$

Lin-Gauss: $dX_t(x) = (A_t X_t(x) + b_t)\ dt + \Sigma_t^{1/2}\ dW_t$

Pinned process | $(X_0(x), X_T(x)) = z = (x, y)$ **:**

$$
\begin{aligned}
d\mathbf{X}_t^z &= (A_t\ \mathbf{X}_t^z + b_t)dt + \Sigma_t^{1/2} dW_t \\
&\quad + \Sigma_t\ \left(\mathcal{E}_{t,T}(A)'\ \Sigma_{t,T}^{-1}\ \left(y - \left(\mathcal{E}_{t,T}(A)\ \mathbf{X}_t^z + \int_t^T \mathcal{E}_{s,T}(A)\ b_s\ ds\right)\right)\right) dt
\end{aligned}
$$

$$
\partial_t \mathcal{E}_{s,t}(A) := A_t \mathcal{E}_{s,t}(A) \quad \text{and} \quad \Sigma_{t,T} := \int_t^T \mathcal{E}_{s,T}(A)\ \Sigma_s\ \mathcal{E}_{s,T}(A)'\ ds
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$$

In terms of the stoch. flow $X_t(x)$:

$$
\mathbf{X}_t^z \stackrel{\text{law}}{=} X_t(x) + C_{t,T}\ C_{T,T}^{-1}\ (y - X_T(x))
$$

with the covariance matrices

$$
C_{t,T} := \mathbb{E}((X_t(x) - \mathbb{E}(X_t(x)))(X_T(x) - \mathbb{E}(X_T(x)))')
$$

Bayes/Dual Op. = Entropic half-bridge/projection

Note: Bridge disintegration formula

$$\Pi(\mu, \boldsymbol{.}) \ni P$$

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Cost functions $W(x,y) \overset{ex}{=} (x-y)^2/t + cte$

$$P = \mu \times K \quad \text{with} \quad K(x,dy) = e^{-W(x,y)}\ \nu(dy)$$

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$$\Downarrow$$

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Quadratic cost function:

$$W(x,y) \overset{ex}{=} \|x-y\|^2/t + cte$$

***t*-regularized entropic optimal transport:**

$$\begin{aligned} P_{\mu,\eta} \quad &= \quad \underset{Q \in \Pi(\mu,\eta)}{\arg\min} \left[\frac{1}{2} \int \|x-y\|^2 Q(d(x,y)) + t\ \mathcal{H}\left(Q \mid \mu \otimes \eta\right) \right] \\ &\simeq_{t \to 0} \quad \underset{Q \in \Pi(\mu,\eta)}{\arg\min} \quad \int \|x-y\|^2 Q(d(x,y)) \end{aligned}$$

Sinkhorn bridges

Sinkhorn eq. $(n = 0)$ ref. $\mathcal{K}_0(x, dy) = e^{-W(x,y)}\ \nu(dy)$

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$\Longleftrightarrow$ Half-bridges/Synthetic form, start $\mathcal{K}_0 = K$

Entropic projection on $\Pi(\mu, \centerdot)$

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"Correct" $\pi_{2n} \Longleftrightarrow$ **Entropic projection on $\Pi(\centerdot, \eta)$**

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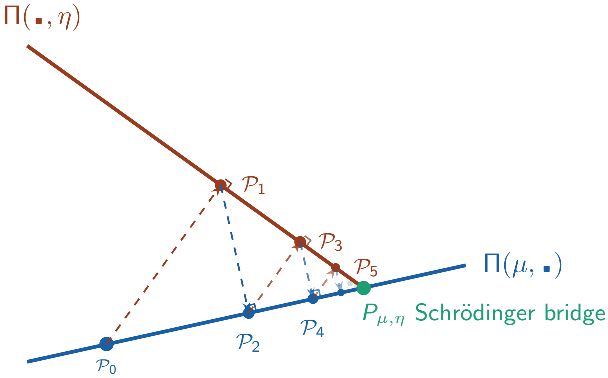
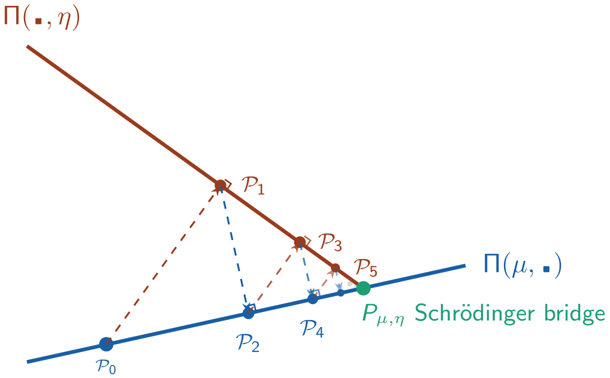
Equivalent form

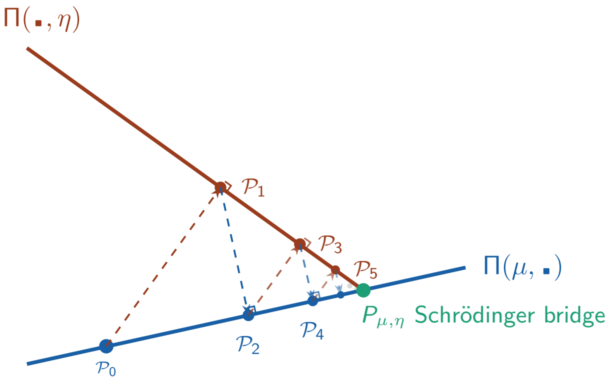
$$\begin{aligned} \mathcal{P}_{2n+1} &:= \operatorname*{arg\,min}_{\mathcal{Q}\in\Pi(\pi_{2n+1},\eta)} \mathcal{H}(\mathcal{Q} \mid \mathcal{P}_{2n}) \\ \mathcal{P}_{2(n+1)} &:= \operatorname*{arg\,min}_{\mathcal{Q}\in\Pi(\mu,\pi_{2(n+1)})} \mathcal{H}(\mathcal{Q} \mid \mathcal{P}_{2n+1}) \end{aligned}$$



$\Pi(\,\cdot\,,\eta)$
$\mathcal{P}_1$
$\mathcal{P}_3$
$\mathcal{P}_5$
$\Pi(\mu,\,\cdot\,)$
$P_{\mu,\eta}$ Schrödinger bridge
$\mathcal{P}_4$
$\mathcal{P}_2$
$\mathcal{P}_0$

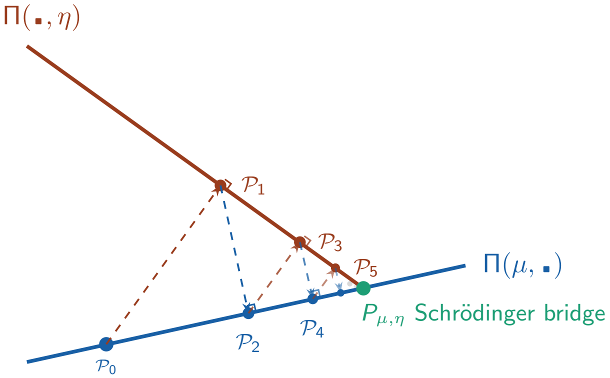


// **Filtering** // **Bayes stats $\rightsquigarrow$ sequential Bayes updates**



// Filtering // Bayes stats $\rightsquigarrow$ sequential Bayes updates

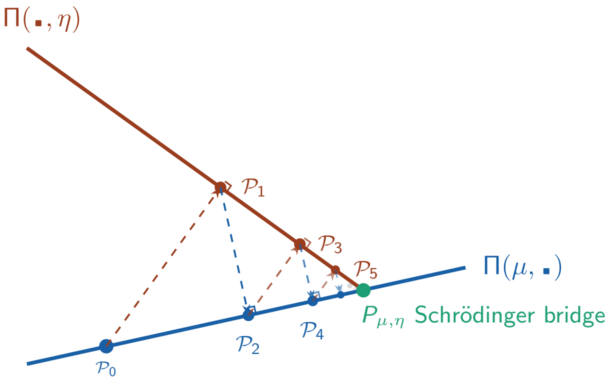
Finite state $\rightsquigarrow$ matrix operations (= Iterative proportional fitting)



$/\!\!/$ Filtering $/\!\!/$ Bayes stats $\rightsquigarrow$ sequential Bayes updates

Finite state $\rightsquigarrow$ matrix operations (= Iterative proportional fitting)

Non finite/Non gaussian: Sinkhorn eq. <u>UNSOLVABLE</u> $+$ Algo.



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$/\!\!/$ Nonlinear filtering equation + filtering algorithm

Part II: Stability toolbox and Gaussian models

Part II

Stability toolbox and Gaussian models

Sinkhorn bridges (continued), Elementary stability results, Linear-Gaussian models

$\Longleftrightarrow$ Time varying Gibbs loop/Markov chain formulation

Fixed point equations

$$\mu[\mathcal{K}_{2n}\mathcal{K}_{2n+1}] = \mu \overset{(1)}{=} \pi_{2n}\mathcal{K}_{2n+1} \quad \text{and} \quad \eta\,[\mathcal{K}_{2n-1}\mathcal{K}_{2n}] = \eta \overset{(2)}{=} \pi_{2n-1}\mathcal{K}_{2n}$$

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Time varying forward/backward Markov chain

$$\pi_{2(n+1)} \quad := \quad \mu\mathcal{K}_{2(n+1)} \stackrel{(1)}{=} \pi_{2n}\left[\mathcal{K}_{2n+1}\ \mathcal{K}_{2(n+1)}\right]$$

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Note:

$$\frac{d\mathcal{P}_{2n+1}}{d\mathcal{P}_{2n}}(x, y) = \frac{d\eta}{d\pi_{2n}}(y) \quad \text{and} \quad \frac{d\mathcal{P}_{2n}}{d\mathcal{P}_{2n-1}}(x, y) = \frac{d\mu}{d\pi_{2n-1}}(x)$$

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$$\Downarrow$$

Prop: [Commutation formulae]

$$\mathcal{K}_{2n-1}\left(\frac{d\mu}{d\pi_{2n-1}}\right) = \frac{d\pi_{2n}}{d\eta} \quad \text{and} \quad \mathcal{K}_{2n+1}\left(\frac{d\pi_{2n-1}}{d\mu}\right) = \frac{d\eta}{d\pi_{2n}}.$$

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Same with $(\mathcal{K}_n, \pi_n, \mu) \rightsquigarrow (\mathcal{K}_{n+1}, \pi_{n+1}, \eta)$.

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In terms of Schrödinger potentials $(\mu, \eta) = (e^{-U}, e^{-V})$

Reference $\mathcal{P}_0 = \mu \times \mathcal{K}$ **:**

$$\mathcal{K}(x, dy) = Q(x, dy) := q(x, y)dy \quad \text{and set} \quad R(y, dx) := q(x, y)dx$$

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$$\mathcal{P}_{2n}(d(x, y)) \overset{n=0}{=} e^{-U_{2n}(x)} \; q(x, y) \; e^{-V_{2n}(y)} \; dxdy \quad \text{with} \quad (U_0, V_0) = (U, 0)$$

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Schrödinger system $(V_{2(n+1)} = V_{2n+1})$**:**

$$U_{2n+1} = U + \log Q(e^{-V_{2n}}) \quad \text{and} \quad V_{2(n+1)} := V + \log R(e^{-U_{2n+1}}).$$

Some elementary stability results

An entropy formula $(n > p)$

$$
\begin{aligned}
\frac{d\mathcal{P}_{2n}}{d\mathcal{P}_{2p}}(x,y) &= \left[\prod_{p\leqslant l<n} \frac{d\mathcal{P}_{2l+1}}{d\mathcal{P}_{2l}}(x,y)\right] \left[\prod_{p\leqslant l<n} \frac{d\mathcal{P}_{2(l+1)}}{d\mathcal{P}_{2l+1}}(x,y)\right] \\
&= \left[\prod_{p\leqslant l<n} \frac{d\eta}{d\pi_{2l}}(y)\right] \left[\prod_{p\leqslant l<n} \frac{d\mu}{d\pi_{2l+1}}(x)\right].
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\end{aligned}$$

$\Longrightarrow \forall Q \in \Pi(\mu, \eta)$

$$\begin{aligned}
Q\left(\log \frac{d\mathcal{P}_{2n}}{d\mathcal{P}_{2p}}\right) &= \mathcal{H}(Q|\mathcal{P}_{2p}) - \mathcal{H}(Q|\mathcal{P}_{2n}) \\
&= \sum_{p\leqslant l<n} \left[\mathcal{H}(\eta|\pi_{2l}) + \mathcal{H}(\mu|\pi_{2l+1})\right].
\end{aligned}$$

$Q = P_{\mu,\eta} \rightsquigarrow$

$$\mathcal{H}(P_{\mu,\eta}|\mathcal{P}_{2p}) = \mathcal{H}(P_{\mu,\eta}|\mathcal{P}_{2n}) + \sum_{p\leqslant l<n} \left[\mathcal{H}(\eta|\pi_{2l}) + \mathcal{H}(\mu|\pi_{2l+1})\right]$$

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Some direct consequences

- $p = 0$ & *Markov transport* $\downarrow$ *entropy* $\Rightarrow$ **linear decays**

$$\mathcal{H}(P_{\mu,\eta}|\mathcal{P}_0) \quad \geqslant \quad \sum_{0\leqslant l\leqslant n} [\mathcal{H}(\eta|\pi_{2l}) + \mathcal{H}(\mu|\pi_{2l+1})]$$

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Some direct consequences

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$$\begin{aligned}\mathcal{H}(P_{\mu,\eta}|\mathcal{P}_0) &\geqslant \sum_{0\leqslant l\leqslant n} \left[\mathcal{H}(\eta|\pi_{2l}) + \mathcal{H}(\mu|\pi_{2l+1})\right] \\ &\geqslant (n+1)\ \left[\mathcal{H}(\eta|\pi_{2n}) + \mathcal{H}(\mu|\pi_{2n+1})\right]\end{aligned}$$

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- **Bridge convergence** $\mathcal{H}(P_{\mu,\eta}|\mathcal{P}_{2n}) \longrightarrow_{n\uparrow\infty} 0$ **iff**

$$\mathcal{H}(P_{\mu,\eta}|\mathcal{P}_{2p}) = \sum_{p\leqslant l} \left[\mathcal{H}(\eta|\pi_{2l}) + \mathcal{H}(\mu|\pi_{2l+1})\right]$$

Bounded costs W, $K(x,dy) = e^{-W(x,y)}\ \nu(dy)$

$$\mathcal{K}_{2n}(x,dy) = \frac{\eta(dy)\mathcal{K}_{2n-1}(y,dx)}{(\eta\mathcal{K}_{2n-1})(dx)} \quad \& \quad \mathcal{K}_{2n+1}(y,dx) = \frac{\mu(dx)\mathcal{K}_{2n}(x,dy)}{(\mu\mathcal{K}_{2n})(dy)}$$

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$\Longrightarrow$ **Homogeneous cross-products (a.k.a. structure conservation)**

$$\frac{d\delta_{x_1}\mathcal{K}_{2n}}{d\delta_{x_2}\mathcal{K}_{2n}}(y)\ \frac{d\delta_{x_2}\mathcal{K}_{2n}}{d\delta_{x_1}\mathcal{K}_{2n}}(\overline{y})$$

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$$= \exp\left(\left(W(x_2,y) - W(x_1,y)\right) + \left(W(x_1,\overline{y}) - W(x_2,\overline{y})\right)\right)$$

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$$\geqslant \epsilon_W := \exp\left(-2\ \mathrm{osc}(W)\right)$$

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$\rightsquigarrow$ Overdeveloped Hilbert projective metrics approaches for Sinkhorn:

Franklin-Lorenz (1989), Bornwein-Lewis-Nussbaum (1994), Ruschendorf (1995), Chen-Georgiou-Pavon (2016), Deligiannidis-de Bortoli-Doucet (2021), Eckstein (2025),... $\rightsquigarrow$ **next bypass Hilbert metrics entirely**

Bounded costs $\Longrightarrow$ Global Minorization prop.

$$\frac{d\delta_{x_1}\mathcal{K}_{2n}}{d\delta_{x_2}\mathcal{K}_{2n}}(y)\ \frac{d\delta_{x_2}\mathcal{K}_{2n}}{d\delta_{x_1}\mathcal{K}_{2n}}(z) \geqslant \epsilon_W := \exp\left(-2\ \mathrm{osc}(W)\right)$$

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$\Longrightarrow$ **Uniform minorisation condition**

$$\delta_{x_1}\mathcal{K}_{2n} \geqslant \epsilon_W\ \delta_{x_2}\mathcal{K}_{2n}$$

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Dobrushin coef. contracts any Φ-entropy [*(+Miclo/*`PTRF-03`*)*]

$$\mathrm{dob}(\mathcal{K}_{2n}) := \sup \frac{\|(\mu_1 - \mu_2)\mathcal{K}_{2n}\|_{tv}}{\|(\mu_1 - \mu_2)\|_{tv}}$$

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$\supset$ **Φ-entropies/Divergence criteria:** $\mathbb{L}_p$*-norms, Havrda-Charvat entropies,* α*-divergences, Jensen-Shannon divergence, squared Hellinger integrals, Kakutani-Hellinger, Jeffreys, Rényi divergences,...*

Linear-Gaussian models

Gaussian models $(\mu,\eta) = (\nu_{m,\sigma},\nu_{\overline{m},\overline{\sigma}})$ and ref. transition:

$$\mathcal{K}_0(x,dy) = K_\theta(x,dy) = \mathbb{P}(Z_\theta(x) \in dy)$$

with the random map

$$\theta = (\alpha,\beta,\tau) \ \mapsto \ Z_\theta(x) := \alpha + \beta x + \tau^{1/2} G$$

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Some advantages: *Applies to any linear-Gaussian transition model, closed-form Schrödinger & dual potential bridges,*

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$$\upsilon_{n+1} = \mathrm{Ricc}_{\varpi_\theta}(\upsilon_n) := (I + (\varpi_\theta + \upsilon_n)^{-1})^{-1}$$

$\rightsquigarrow$ **same tools for log-concave marginals** *(vs Cramer-Rao & Brascamp-Lieb), links with Bayesian inference, MCMC/Gibbs/. . . ,*

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Some advantages: *Applies to any linear-Gaussian transition model, closed-form Schrödinger & dual potential bridges, finite-dimensional (sequential) solutions of Sinkhorn bridges ("true" Sinkhorn Algorithm), Riccati matrix diff. equations/steady states,*

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$\rightsquigarrow$ **same tools for log-concave marginals** *(vs Cramer-Rao & Brascamp-Lieb), links with Bayesian inference, MCMC/Gibbs/..., complete convergence analysis with sharp rates,*

Gaussian models $(\mu, \eta) = (\nu_{m,\sigma}, \nu_{\overline{m},\overline{\sigma}})$ and ref. transition:

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⊕ *Brief review ⊂ (Arxiv-24)/(FoDS-26)*

Brief review of $v_{n+1} = \mathrm{Ricc}_{\varpi}(v_n) := (I + (\varpi + v_n)^{-1})^{-1}$

Uniform estimates $1 \leqslant p \leqslant n$

$$\mathrm{Ricc}_{\varpi}^{p}(0) \leqslant \mathrm{Ricc}_{\varpi}^{n}(0) \leqslant \mathrm{Ricc}_{\varpi}^{n}(v) \leqslant \mathrm{Ricc}_{\varpi}^{n-1}(I) \leqslant \mathrm{Ricc}_{\varpi}^{p-1}(I) \leqslant I$$

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Unique (closed-form) positive fixed point

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$$\|v_n - r_{\varpi}\| \leqslant c_{1,\varpi}\ \delta^{n}_{\varpi}\ \|v_0 - r_{\varpi}\| \quad \text{with} \quad \delta_{\varpi} := (1 + \ell_{min}(\varpi + r_{\varpi}))^{-2}$$

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Matrix products

$$\|(I - v_n)\ldots(I - v_1)(I - v_0)\| \leqslant c_{2,\varpi}\ \delta_{\varpi}^{n/2}$$

Note on the (positive definite) Riccati fixed point

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Gaussian Schrödinger map = Riccati steady-state

$$Z_{\mathbb{S}(\theta)}(x) := \overline{m} + \varsigma_\theta\ \chi_\theta\ (x - m) + \varsigma_\theta^{1/2}\ G \quad \text{with} \quad \chi_\theta := \tau^{-1}\beta$$

$\Longleftrightarrow$ **fixed point of Riccati matrix equation:**

$$(\varsigma_\theta\ \chi_\theta)\ \sigma\ (\varsigma_\theta\ \chi_\theta)' + \varsigma_\theta = \overline{\sigma}$$

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Closed form Discrete Algebraic Riccati equation (DARE)

$$r_\theta = -\frac{\varpi_\theta}{2} + \left(\varpi_\theta + \left(\frac{\varpi_\theta}{2}\right)^2\right)^{1/2} = \mathrm{Ricc}_{\varpi_\theta}(r_\theta)$$

Ex.: Gauss targets + Lin-Gauss ref (= Quadratic cost)

Riccati ref. matrix:

$$\tau = t\ I \Longrightarrow \frac{\varpi_\theta}{t^2} = \overline{\sigma}^{-1/2}\ \sigma_\beta^{-1}\ \overline{\sigma}^{-1/2} \quad \text{with} \quad \sigma_\beta := \beta\sigma\beta'$$

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Riccati fixed point $\simeq$ Geometric mean covariance:

$$\frac{r_\theta}{t} = -\frac{\varpi_\theta}{2t} + \left(\frac{\varpi_\theta}{t^2} + \left(\frac{\varpi_\theta}{2t}\right)^2\right)^{1/2} \simeq \left(\overline{\sigma}^{-1/2}\ \sigma_\beta^{-1}\ \overline{\sigma}^{-1/2}\right)^{1/2}$$

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Limiting optimal transport map from $\nu_{m,\sigma}$ to $\nu_{\overline{m\sigma}}$
$\rightsquigarrow$ Monge map/2-Wasserstein optimal (deterministic) transport

$$Z_{\mathbb{S}(\theta)}(x) = \overline{m} + \frac{\varsigma_\theta}{t}\ \beta\ (x-m) + \varsigma_\theta^{1/2}\ G \simeq_{t\downarrow 0} \overline{m} + (\sigma_\beta^{-1}\ \sharp\ \overline{\sigma})\ \beta\ (x-m)$$

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$\underline{\textit{Same Riccati theory}}$ $\subset$ *Log-concave marginals*
(vs Cramer-Rao & Brascamp-Lieb) $\rightsquigarrow$ *(SAA-26)/(Arxiv-25)*

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Same Riccati theory $\subset$ Log-concave marginals
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$\rightsquigarrow$ $\ddagger$ Approaches to Gaussian Schrödinger bridges :

via continuous time/Stochastic control/solving integral fixed point equations/Schröd. potentials/ mainly for $(\alpha, \beta, \tau) = (0, I, tI)$,...

Chen-Georgiou-Pavon (14/21); Bolijov-Galichon (14), del Barrio et Loubes (20), Janati-Muzellec-Peyré-Cuturi (20),Mallasto-Gerolin-Minh (20),Bunne-Hsieh-Cuturi-Krause (23), ...

Gaussian Sinkhorn $(\theta_0 \overset{ref}{=} \theta)$ & $\theta_{2n} \rightsquigarrow \theta_{2n+1} \rightsquigarrow \theta_{2(n+1)}$

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with the Bayes'/Regression/Dual/Conjugate maps

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$\rightsquigarrow \theta_n := (\alpha_n, \beta_n, \tau_n)$ **& Sinkhorn Gauss marginals**

$$\mu K_{\theta_{2n}} = \nu_{m_{2n},\sigma_{2n}} \quad \text{and} \quad \eta K_{\theta_{2n+1}} = \nu_{m_{2n+1},\sigma_{2n+1}}$$

Gaussian Schrödinger map = Riccati steady state

$$Z_{\mathbb{S}(\theta)}(x) = \overline{m} + \varsigma_\theta \; \chi_\theta \; (x - m) + {\varsigma_\theta}^{1/2} \; G \quad \text{with} \quad \chi_\theta := \tau^{-1}\beta$$

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and the expected values

$$m_{2n} - \overline{m} = \overline{\sigma}^{1/2}\ (I - \upsilon_{2n})\ \overline{\sigma}^{-1/2}\ (m_{2(n-1)} - \overline{m})$$

Some convergence theorems (reference $\theta = \theta_0$)

$$\mathcal{P}_{2n} = \mathbb{P}_{\theta_{2n}} := \mu \times K_{\theta_{2n}} \longrightarrow_{n\to\infty} P_{\mu,\eta} = \mathbb{P}_{\mathbb{S}(\theta)} := \mu \times K_{\mathbb{S}(\theta)}$$

In terms of the parameter

$$\rho_\theta := \left(1 + \ell_{min}(\varpi_\theta + r_{\varpi_\theta})\right)^{-2}$$

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Entropy estimates *(sharper than any log-concave marg. studies)*

$$\mathrm{Ent}\left(P_{\theta_{2n}} \mid \mathbb{P}_{\mathbb{S}(\theta)}\right) \leqslant c_\theta\ \rho_\theta^{\mathbf{n}}\ \left(\|\tau - \varsigma_\theta\| + \|m_0 - \overline{m}\|^2\right).$$

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***p*-Wasserstein distance**

$$\mathbb{W}_p\left(\mathbb{P}_{\theta_{2n}}, \mathbb{P}_{\mathbb{S}(\theta)}\right) \leqslant c_{1,\theta}(p) \; \rho_\theta^{\mathbf{n}} \; \|\tau - \varsigma_\theta\| + c_{2,\theta} \; \rho_\theta^{\mathbf{n}/2} \; \|m_0 - \overline{m}\|$$

Part III: Contraction and Lyapunov techniques

Part III

Contraction and Lyapunov techniques

Contraction theory via Log-Sobolev, Lyapunov techniques, Weighted norms contractions

Contraction theory via Log-Sobolev

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Theo [SAA-26/Arxiv-25]: For "good ref. $P = (\mu \times K)$" $\exists \varepsilon > 0$ s.t.

$$\mathcal{H}(\eta K \mid \mu K) \leqslant \mathcal{H}(P_{\mu,\eta K} \mid P_{\mu,\mu K}) \leqslant \varepsilon\, \mathcal{H}(\eta \mid \mu)$$

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$\sim$ *"Good" P example: μ and K satisfy Log-Sob ($\mathbb{X} = \mathbb{R}^d = \mathbb{Y}$)*
(example: K lin-Gauss and targets/marginals (μ, η) log-concave at ∞)

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Applying to $(P, K, \eta) = (\mathcal{P}_{2n}, \mathcal{K}_{2n}, \pi_{2n-1})$ **& recalling** $\pi_{2n-1}\mathcal{K}_{2n} = \eta$

$$\mathcal{H}\left(P_{\mu,\eta} \mid \mathcal{P}_{2n}\right) \leqslant \epsilon\ \mathcal{H}(\pi_{2n-1}|\mu)$$

Contraction inequalities

$$\epsilon^{-1}\ \mathcal{H}\left(P_{\mu,\eta} \mid \mathcal{P}_{2n}\right) \leqslant\ \mathcal{H}(\pi_{2n-1}|\mu)$$

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$$\mathcal{H}(P_{\mu,\eta}\mid\mathcal{P}_{2n})\;=\;\mathcal{H}(P_{\mu,\eta}|\,\mathcal{P}_{2(n-1)})-\left(\mathcal{H}(\eta\mid\pi_{2(n-1)})+\mathcal{H}(\mu\mid\pi_{2n-1})\right)$$

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$\Longrightarrow$ **Theo [SAA-26/Arxiv-25]**

$$\mathcal{H}(P_{\mu,\eta}\mid\mathcal{P}_{2n})\leqslant(1+\varepsilon^{-1})^{-n}\ \mathcal{H}(P_{\mu,\eta}\mid\mathcal{P}_0)$$

$\rightsquigarrow$ *Contraction Theory via Log-Sob (SAA-26)*/(Arxiv-25)

$\supset$ *more refined results, direct application to log-concave (at infinity) marginals, Riccati theory/estimates for log-concave/sub-Gaussian marginals (vs Cramer-Rao & Brascamp-Lieb), general costs/reference transitions, improved/better/more sharp expo rates & 2-Wasserstein stability,. . . ,*

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$\rightsquigarrow$ Next = Weighted norms and Wasserstein distances

Lyapunov techniques

Kantorovich semi-distance, φ semi-distance on $\mathbb{X}$

$$D_\varphi(\mu_1, \mu_2) := \inf_{Q \in \Pi(\mu_1, \mu_2)} Q(\varphi) \tag{1}$$

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Weighted g-norms

$$\varphi_g(x_1, x_2) := 1_{x_1 \neq x_2} \, (g(x_1) + g(x_2)) \Longrightarrow D_{\varphi_g}(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_g$$

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Wasserstein metrics

$$\varphi(x_1, x_2) := \|x_1 - x_2\|_2^p \Longrightarrow D_{\varphi}(\mu_1, \mu_2) = \mathbb{W}_p(\mu_1, \mu_2)^p$$

Comparisons - Weighted & ψ-semi-distances

$$g(x_1) + g(x_2) \geqslant \psi(x_1, x_2) \Longrightarrow \|\mu_1 - \mu_2\|_g \geqslant D_\psi(\mu_1, \mu_2)$$

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- $e^{c\|x_1\|^p} + e^{c\|x_2\|^p} \geqslant \psi(x_1, x_2) := 2\left(e^{\left(\frac{c}{2} \ (\|x_1\|^p + \|x_2\|^p)\right)} - 1\right)$

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Example $\mathbb{X} =]0,1[$ not complete for Euclidian but for the metric

$$\varrho_c\,(x_1, x_2) := \left|\frac{1}{x_1} - \frac{1}{x_2}\right| + \left|\frac{1}{1-x_1} - \frac{1}{1-x_2}\right|$$

Note $(a, b > 0)$**:**

$$g(x) = \left(\frac{1}{x}\right)^a\ \left(\frac{1}{1-x}\right)^b$$

$$\Downarrow$$

$$g(x_1) + g(x_2) \geqslant c\ \psi(x_1, x_2) := c\ \left(\left|\frac{1}{x_1} - \frac{1}{x_2}\right|^a + \left|\frac{1}{1-x_1} - \frac{1}{1-x_2}\right|^b\right)$$

Lipschitz/contraction constant

$$\mathrm{lip}_{g,h}(K) := \sup \frac{D_{\varphi_h}(\mu_1 K, \mu_2 K)}{D_{\varphi_g}(\mu_1, \mu_2)} = \sup \frac{\|\mu_1 K - \mu_2 K\|_h}{\|\mu_1 - \mu_2\|_g}$$

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Note: $g = 1/2 = h \Longrightarrow \|\centerdot\|_{1/2} = \|\centerdot\|_{tv} \Longrightarrow \mathrm{lip}_{g,h}(K) = \mathrm{dob}(K)$

Lyapunov conditions ((g, h) compact sub-levels)

$$K(h) \leqslant \epsilon\ g + c \quad \text{and} \quad L(g) \leqslant \epsilon\ h + c$$

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Theo: $\exists a > 0$ and $\varrho \in]0,1[\ \sim (\epsilon, c, \iota(l))$ s.t.

$$\mathrm{lip}_{g_a,h_a}(K) \vee \mathrm{lip}_{h_a,g_a}(L) \leqslant \varrho$$

$$\text{with} \quad g_a := \frac{1}{2} + a\ g \quad \text{and} \quad h_a := \frac{1}{2} + a\ h$$

Lyapunov + local minorization

Composition/Product rule

$$\mathrm{lip}_{g_a,g_a}(KL) \vee \mathrm{lip}_{h_a,h_a}(LK) \leqslant \mathrm{lip}_{g_a,h_a}(K)\ \mathrm{lip}_{h_a,g_a}(L) \leqslant \varrho^2.$$

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$\rightsquigarrow$ **Contraction inequalities/Example**

$$\|\mu_1(KL)^n - \mu_2(KL)^n\|_{g_a} \leqslant \varrho^{2n}\ \|\mu_1 - \mu_2\|_{g_a}$$

Sinkhorn/Gibbs-loop process:
Loc compact Polish space & $K(x, dy) = e^{-W(x,y)}\ \nu(dy)$

- Marginal potentials U, V tends to ∞ at ∞ (i.e. compact sub-levels)

$$\mu(dx) = \lambda_U(dx) := e^{-U(x)}\ \lambda(dx) \quad \text{and} \quad \eta(dy) = \nu_V(dy) := e^{-V(y)}\nu(dy)$$

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$$\begin{aligned} \mathcal{U}_\delta(x) &:= \delta U(x) - \int \nu_V(dy)\ W(x,y), \\ \mathcal{V}_\delta(y) &:= \delta V(y) - \int \lambda_U(dx)\ W(x,y) \quad \geqslant \quad \text{tends to } \infty \text{ at } \infty \end{aligned}$$

$$\mathcal{U}_\delta \text{ and } \mathcal{V}_\delta \quad \geqslant \quad \text{tends to } \infty \text{ at } \infty \text{ (i.e. compact sub-levels)}$$

Example on $\mathbb{X} = \mathbb{Y} = \mathbb{R}^d$ $(q > p)$ or $\left(q = p \text{ and } b_2 > 2a_2\right)$

$$W(x,y) \leqslant a_1 + a_2 \left(\|x\|^p + \|y\|^p\right) \quad \& \quad U(x) \wedge V(x) \geqslant b_1 + b_2 \|x\|^q.$$

$\mathcal{U}_\delta$ and $\mathcal{V}_\delta$ $\geqslant$ tends to ∞ at ∞ (i.e. compact sub-levels)

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Other examples (*(FoDS-26)*/(Arxiv-25)): *bi-Laplace, heavy-tailed transitions, exponential, states with boundaries, Beta, Weibull, semi-compact/semi-discrete models, linear Gaussian, Gaussian mixtures, ...*

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Example on $\mathbb{X} =]0, 1[$, $a \wedge b > 0 \rightsquigarrow$ Beta distribution

$$\lambda_U(dx) \propto x^a (1-x)^b\ dx$$

$$\Longrightarrow g(x) := e^{\delta U(x)} \geqslant c\ \left(\left(\frac{1}{x}\right)^{\delta a} + \left(\frac{1}{1-x}\right)^{\delta b} \right)$$

$\rightsquigarrow$ Lyapunov functions

$$(g,h) := (\exp(\delta U), \exp(\delta V)) \quad \text{tends to } \infty \text{ at } \infty$$

$$\stackrel{\forall n\geqslant 1}{\Longrightarrow} \quad \mathcal{K}_{2n}(h)/g \leqslant c\ e^{-\mathcal{U}_\delta} \quad \text{and} \quad \mathcal{K}_{2n+1}(g)/h \leqslant c\ e^{-\mathcal{V}_\delta}$$

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$\rightsquigarrow$ Minorisation for some $\imath > 0$

$$\begin{aligned} \imath\ \mathcal{K}_{2n}(x,dy) &\geqslant K_V(x,dy) := e^{-W(x,y)}\ \nu_V(dy) \\ \imath\ \mathcal{K}_{2n+1}(y,dx) &\geqslant K_U^\flat(y,dx) := e^{-W(x,y)}\ \lambda_U(dx) \end{aligned}$$

Note: $\Rightarrow$ For any $g(x_i) \vee h(y_i) \leqslant l$ and $n \geqslant 1$ we have

$$\|\delta_{x_1}\mathcal{K}_{2n} - \delta_{x_2}\mathcal{K}_{2n}\|_{tv} \vee \|\delta_{y_1}\mathcal{K}_{2n+1} - \delta_{y_2}\mathcal{K}_{2n+1}\|_{tv} \leqslant 1 - \iota(l)$$

Theo: $\exists a > 0$ and $\varrho \in]0,1[$ s.t. for all $n \geqslant 1$ we have

$$\mathrm{lip}_{g_a,h_a}(\mathcal{K}_{2n}) \vee \mathrm{lip}_{h_a,g_a}(\mathcal{K}_{2n+1}) \leqslant \varrho$$

$$\text{with} \quad g_a := \frac{1}{2} + a\ g \quad \text{and} \quad h_a := \frac{1}{2} + a\ h,$$

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Cor: Weighted norms contraction

$$\|\pi_{2(n+1)} - \eta\|_{h_a} \leqslant \varrho^2\, \|\pi_{2n} - \eta\|_{h_a} \quad \text{and} \quad \|\pi_{2n+1} - \mu\|_{g_a} \leqslant \varrho^2\, \|\pi_{2n-1} - \mu\|_{g_a}.$$

Prop: RN-ratio decays

$$\left\| \frac{d\pi_{2n}}{d\eta} - 1 \right\|_h \vee \left\| \frac{d\eta}{d\pi_{2n}} - 1 \right\|_h \vee \left\| \log \frac{d\eta}{d\pi_{2n}} \right\|_h \leqslant c\ \varrho^{2n}$$

Same with $(\pi_n, \eta, h) \rightsquigarrow (\pi_{n+1}, \mu, g)$.

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Cor: Entropy estimates

$$\mathcal{H}(P_{\mu,\eta} \mid \mathcal{P}_{2n+1}) \leqslant \mathcal{H}(P_{\mu,\eta} \mid \mathcal{P}_{2n}) \leqslant c\ \varrho^{2n}$$

Some references

Sinkhorn & Schrödinger bridges

- `New trends/Sinkhorn (+Jasra) (Arxiv-Jan.26)`
- *Gaussian models (+ Akyldiz/Miguez) (FoDS-26)*/(Arxiv-24)
- *Lyapunov (+ Akyldiz/Miguez) (FoDS-26)*/(Arxiv-25)
- *Contraction Theory via Log-Sob (SAA-26)*/(Arxiv-25)
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Tools : Contraction Markov & Riccati matrix equations:

- *Kantorovich contraction (+Gerber), (Arxiv-25)*
- *Riccati eq./Floquet form (+Horton),* $\rightsquigarrow$ *(Arxiv-21)/(SIAM-22)*
- *Lyap. sg approaches (+Arnaudon/Ouhabaz) (SAA-24)*
- *Dobrushin contraction coefficient (+Miclo)*`(PTRF-03)`