Scaling Limits of Random Growth Models

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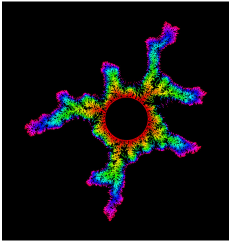


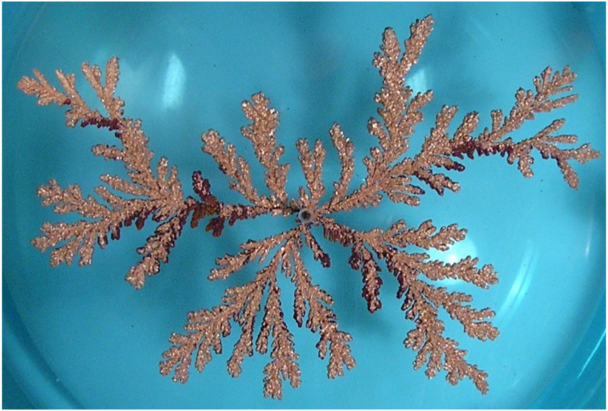
Biological growth

Photo by James Wearn

Biological growth

Gift by Sir Alexander Fleming to Edinburgh University Library, Scotland

Mineral deposition



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Mineral deposition

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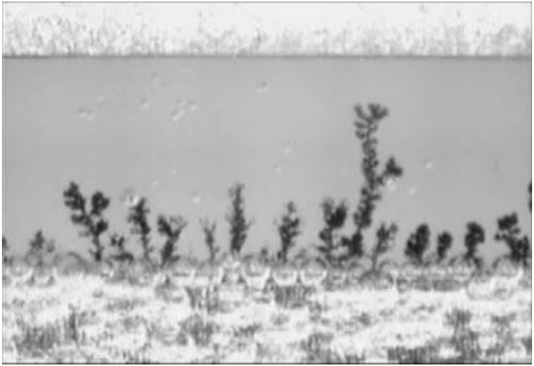
Dendrite Growth in a Lithium Battery



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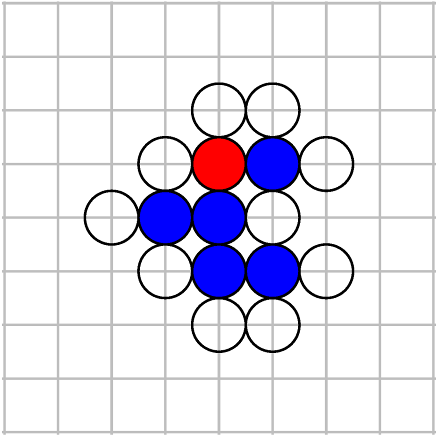
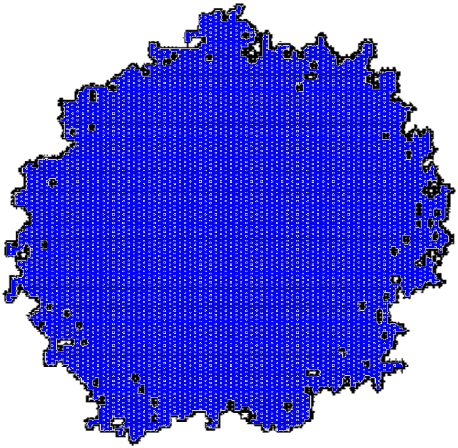
Lightning strikes

From “Lichtenberg Figures Due to a Lightning Strike” by Yves Domart, MD, and Emmanuel Garet, MD

Lattice models for random growth

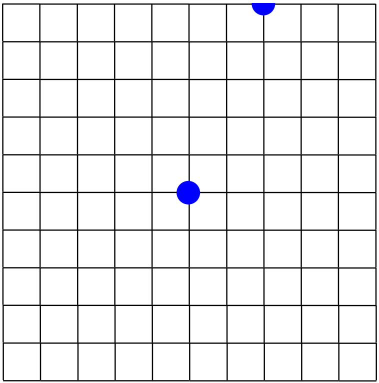


Eden cluster with 1,500 particles



Simulation by H.J. Herrmann

Diffusion-limited aggregation (DLA) model for mineral deposition



Simulation by Eviatar Procaccia

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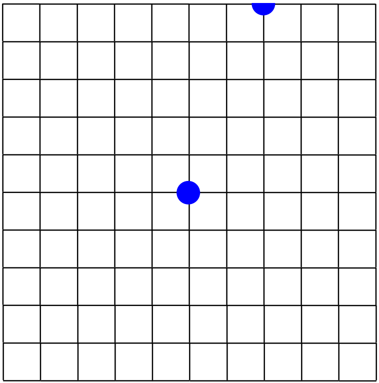

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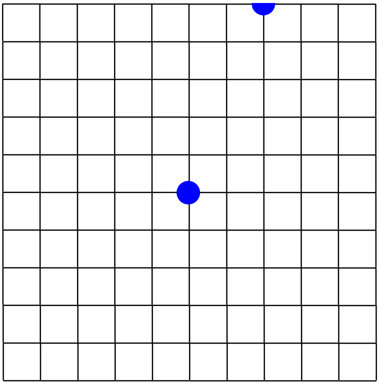

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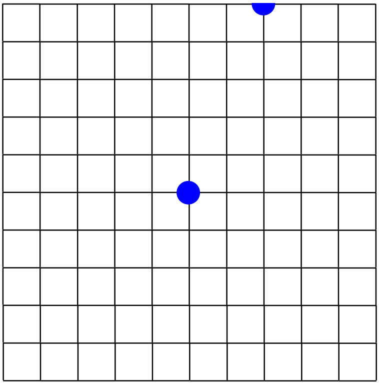

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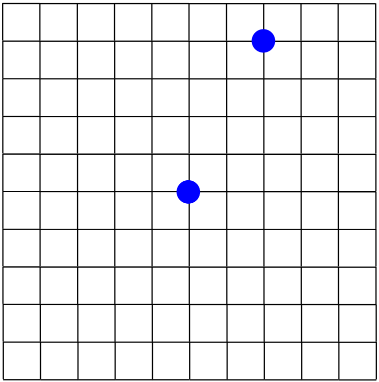

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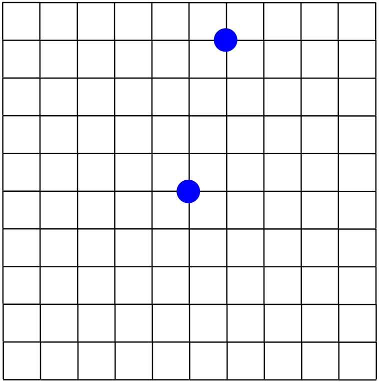

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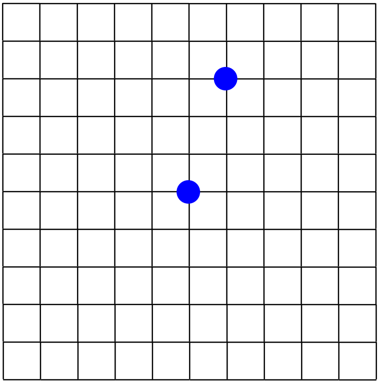

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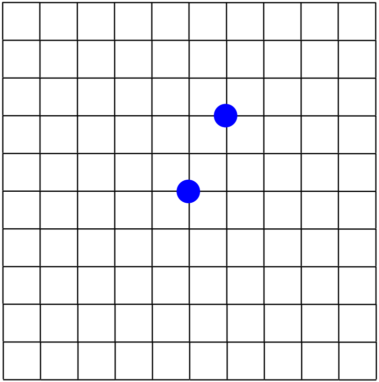

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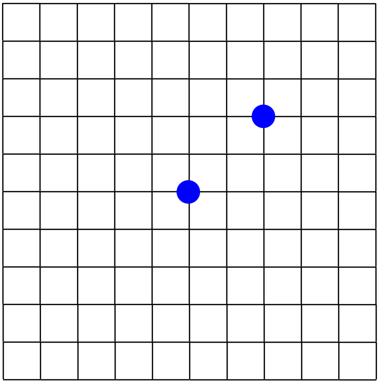

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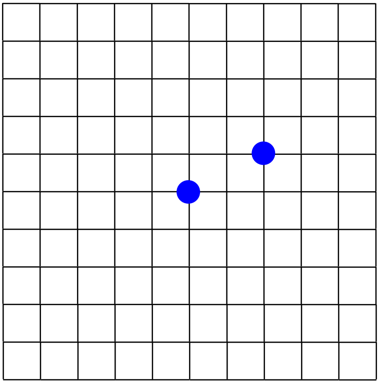

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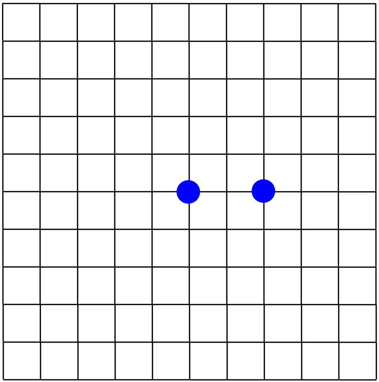

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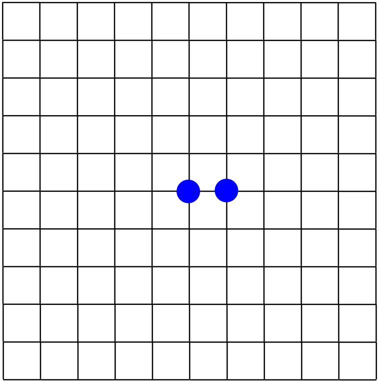

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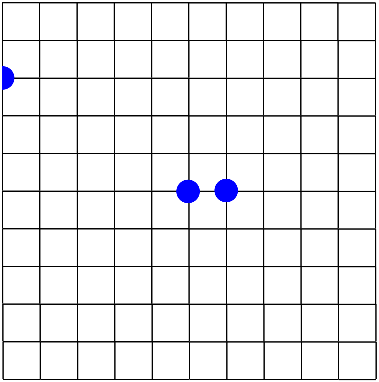

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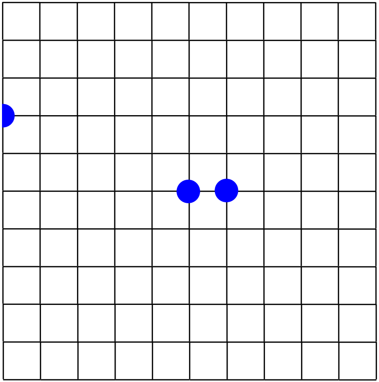

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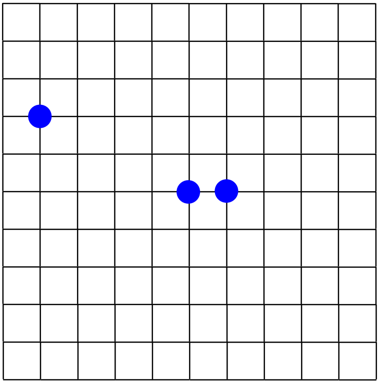

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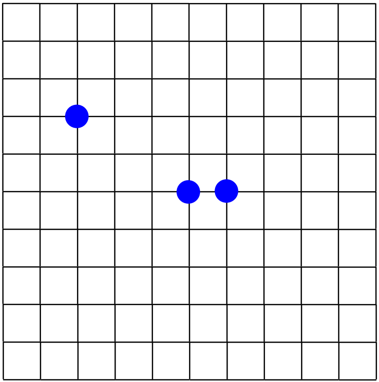

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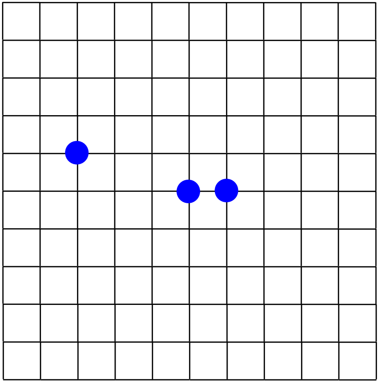

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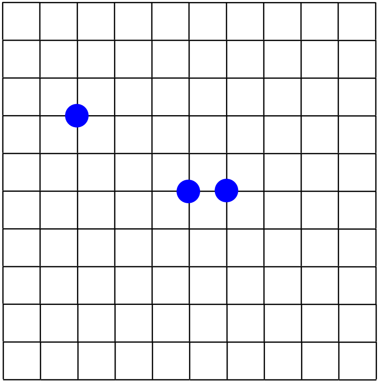

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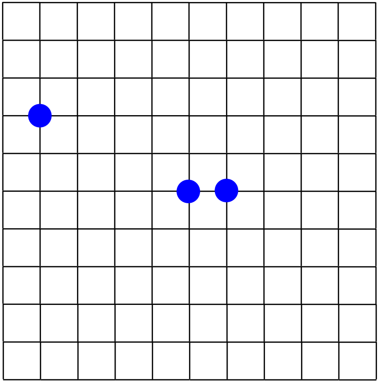

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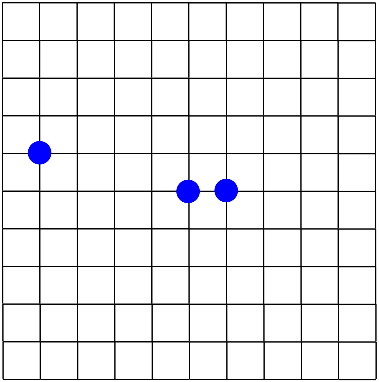

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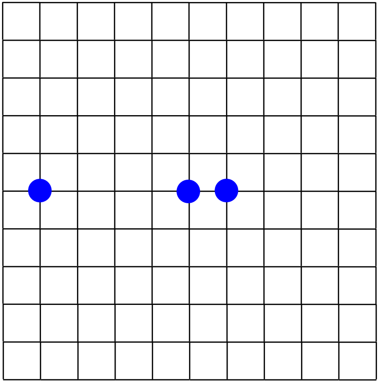

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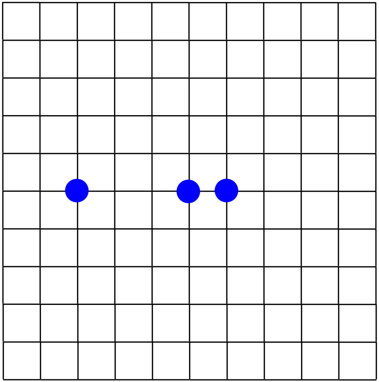

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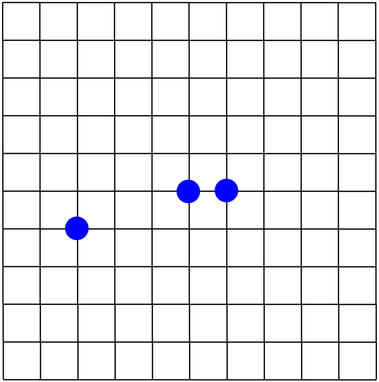

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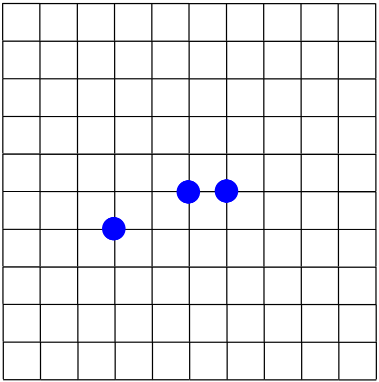

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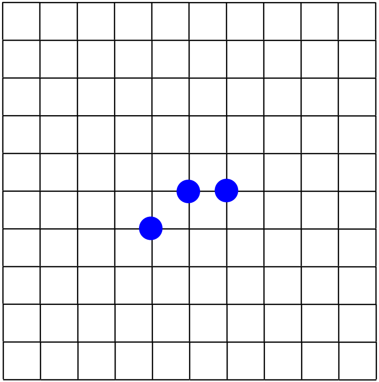

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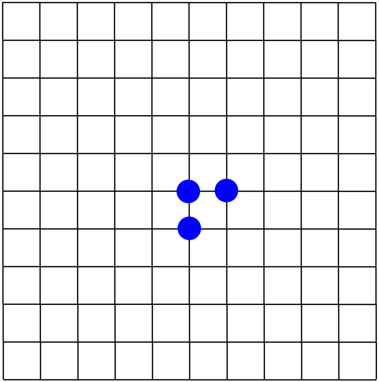
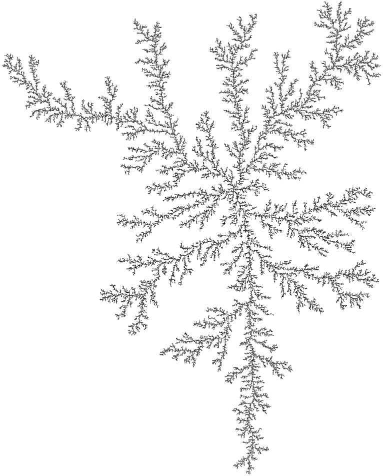


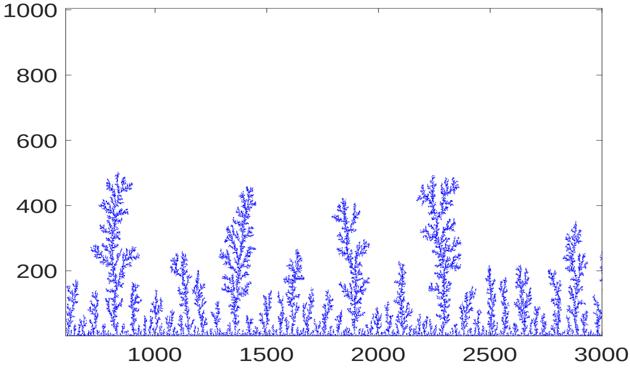
Simulation by Eviatar Procaccia

DLA cluster with 2,000 particles



Simulation by Vincent Beffara

Stationary DLA



Simulation by Eviatar Procaccia

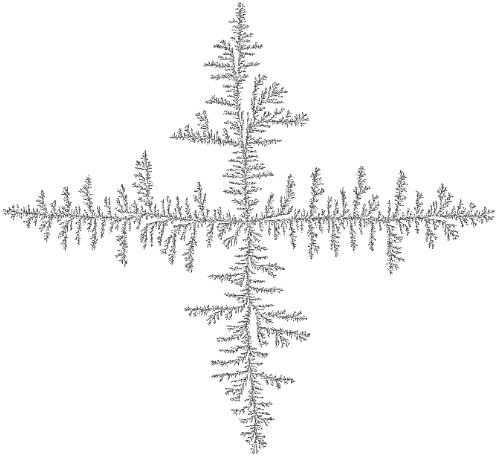
Other lattice models for random growth

- Dielectric breakdown models (DBM)
- Internal diffusion-limited aggregation (IDLA)
- First passage percolation (FPP)
- Interface models: ballistic deposition, corner growth model, etc.

What do we know about lattice models for random growth?

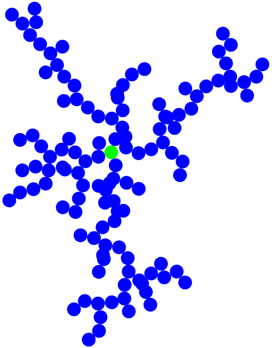
- Not much!
- DLA first formulated in 1981 by Witten and Sander.
- Original paper has almost 7,000 citations.
- Only one rigorous result in over 40 years: At time t DLA is contained in a ball of radius $t^{2/3}$ (Kesten, 1987).
- No proof (or even convincing explanation) that DLA does not converge to a ball.
- Main open problems:
  - Existence of universal limit.
  - Growth rate of the cluster.
  - Structure of the limiting set (e.g. fractal dimension).
  - Number of arms.

DLA cluster with 145,199,976 particles



Simulation by B.Grebenkov and D.Beliaev

Off-lattice DLA



Ball shaped particles perform BM (from infinity) until they attach to the aggregate.

Harmonic measure

- The attachment point is distributed according to harmonic measure on the cluster boundary (from infinity).
- By conformal invariance of BM, harmonic measure is conformally invariant.
- An algorithm for sampling a boundary point of a set A:
  Let D_0 denote the exterior unit disk in the complex plane $\mathbb{C}$ and let $\Phi : D_0 \to A^c$ be conformal. Choose a point $y \in \partial D_0$ uniformly. Then take $\Phi(y)$.
- But how do you find Φ?

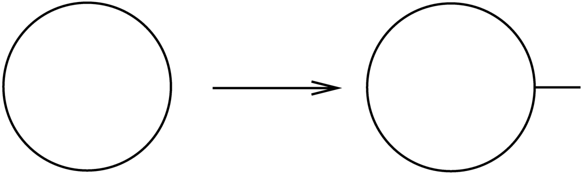
Conformal mapping representation of a slit-shaped particle

Let P denote the slit $[1, 1+\delta]$ in the complex plane.

There exists a unique conformal mapping $F : D_0 \to D_0 \setminus P$ that fixes ∞ in the sense that

$$F(z) = e^c z + O(1) \quad \text{as} \quad |z| \to \infty,$$

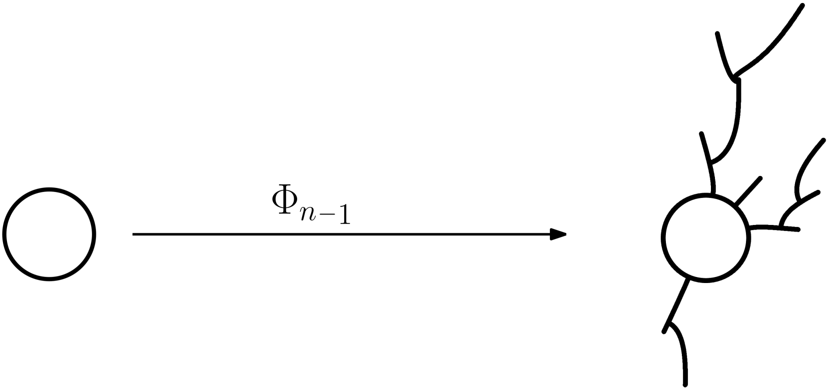
for some $c > 0$, the (log of the) capacity, which satisfies $e^c = 1 + \frac{\delta^2}{4(1+\delta)}$.



Conformal mapping representation of a cluster

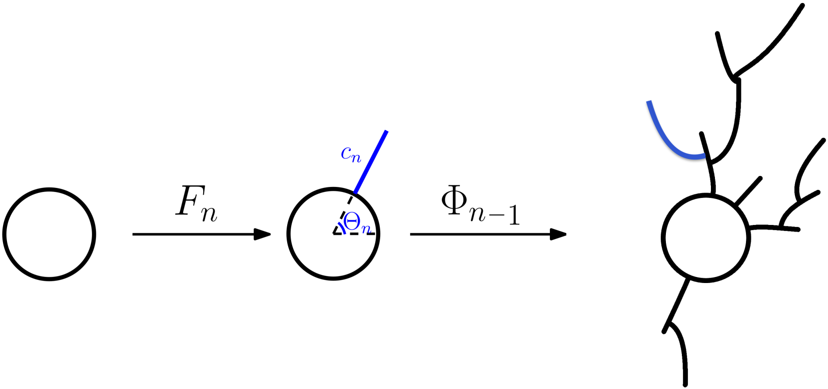
- Suppose $P_1, P_2, \ldots$ is a sequence of particles, where P_n has capacity c_n (or length δ_n) and attachment angle Θ_n, $n = 1, 2, \ldots$. Let F_n be the particle map corresponding to P_n.
  - Set $\Phi_0(z) = z$.
  - Recursively define $\Phi_n(z) = \Phi_{n-1} \circ F_n(z)$, for $n = 1, 2, \ldots$.
- This generates a sequence of conformal maps $\Phi_n : D_0 \to K_n^c$, where $K_{n-1} \subset K_n$ are growing compact sets, which we call clusters.

Cluster formed by iteratively composing mappings



Picture by Vittoria Silvestri

Cluster formed by iteratively composing mappings



$$\Phi_n = \Phi_{n-1} \circ F_n = F_1 \circ F_2 \circ \cdots \circ F_n$$

Parameter choices for physical models

- By varying the sequences $\{\Theta_n\}$ and $\{c_n\}$, it is possible to describe a wide class of growth models.
- For biological growth (Eden model)

$$\mathbb{P}(\Theta_n \in (a,b)) \propto \int_a^b |\Phi'_{n-1}(e^{i\theta})| d\theta$$

and

$$c_n \approx c|\Phi'_{n-1}(e^{i\Theta_n})|^{-2}$$

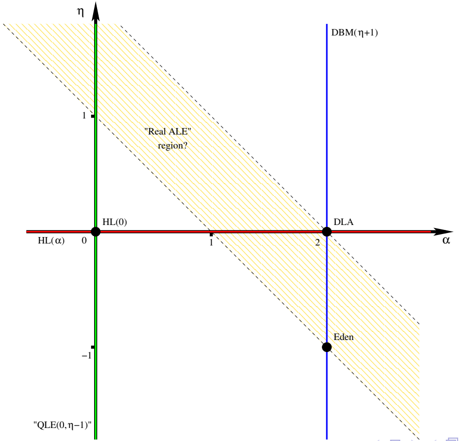
- For DLA, c_n is as above and

$$\mathbb{P}(\Theta_n \in (a,b)) = \mathbb{P}(\Phi_{n-1}^{-1}(B_\tau) \in (a,b)) \propto (b-a)$$

where B_t is Brownian motion started from ∞ and τ is the hitting time of the cluster K_{n-1}.

Aggregate Loewner Evolution, ALE(α, η, σ)

- Θ_n distributed $\propto |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta} d\theta$; $c_n = c|\Phi'_{n-1}(e^{\sigma+i\Theta_n})|^{-\alpha}$.



Regularization for ALE

- Even after the arrival of a single slit particle, the map $\theta \mapsto |\Phi_n'(e^{i\theta})|$ is badly behaved and takes the values 0 and ∞.
- For some values of η,

$$\int_{-\pi}^{\pi} |\Phi_{n-1}'(e^{i\theta})|^{-\eta} d\theta = \infty,$$

  so regularization is necessary to even define the measure.
- A solution is to let Θ_n have distribution

$$\propto |\Phi_{n-1}'(e^{\sigma+i\theta})|^{-\eta} d\theta$$

  for $\sigma > 0$ and take the limit $\sigma \to 0$.
- Models are very sensitive to the rate at which $\sigma \to 0$. Can be argued that $\sigma \sim c^{1/2}$ is natural from a physical point of view.

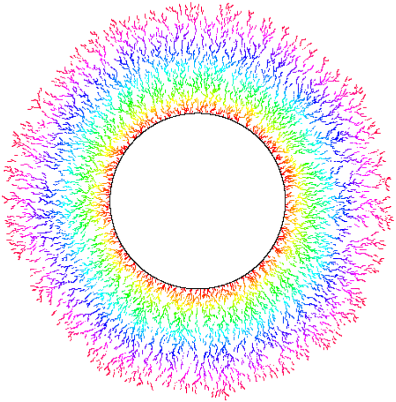
Phase transition

Open Problem:

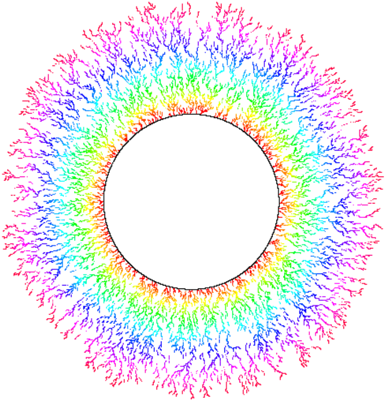
Does ALE(α, η, σ) exhibit a phase transition from disks to non-disks along the line $\alpha + \eta = 1$ (for 'broad' choices of the regularization parameter σ)?

- Longstanding conjectures:
    - HL(α) has a phase transition at $\alpha = 1$.
    - DBM(η) has a phase transition at $\eta = 0$.

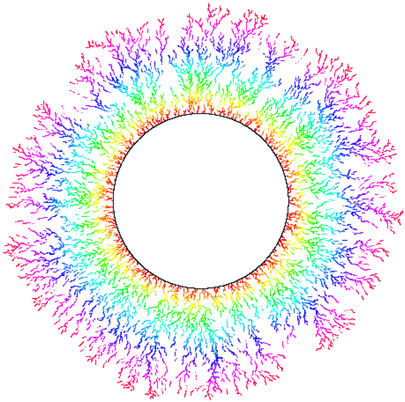
ALE(0,0) cluster with 8,000 particles for $c = 10^{-4}$



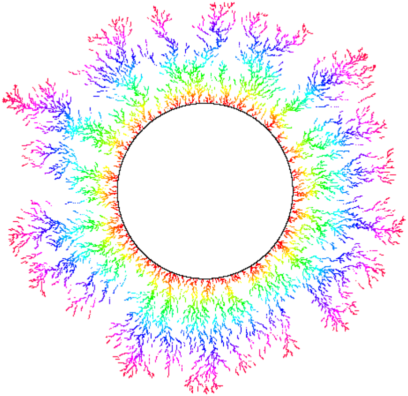
ALE(0,0.5,0.02) cluster with 8,000 particles for $c = 10^{-4}$



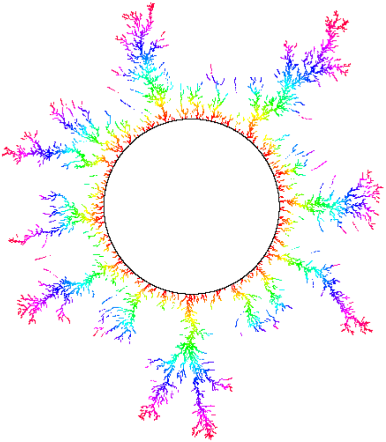
ALE(0,1,0.02) cluster with 8,000 particles for $c = 10^{-4}$



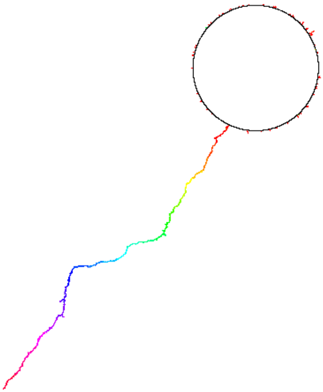
ALE(0,1.5,0.02) cluster with 8,000 particles for $c = 10^{-4}$



ALE(0,2,0.02) cluster with 8,000 particles for $c = 10^{-4}$



ALE(0,5,0.02) cluster with 8,000 particles for $c = 10^{-4}$



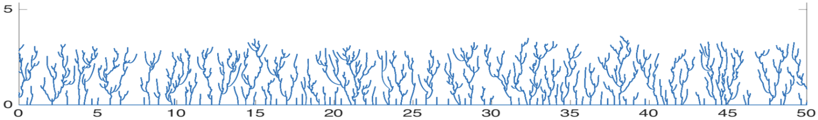
Previous results: HL(0)

Much of the previous work relates to HL(0) as particle maps are i.i.d. so the model is mathematically the most tractable.

- Norris and Turner (2012):
  - small-particle scaling limit of HL(0) is a growing disk: $\Phi_n(z) \approx e^{cn} z$
  - branching structure is related to the Brownian web
  - expected size of the n^{th} particle is roughly $\delta \exp cn$, so HL(0) is "unphysical".
- Silvestri (2017): fluctuations converge to a log-correlated Fractional Gaussian Field.

HL(0)-like model variants

- Sola, Turner, Viklund (2012): Anisotropic HL(0)
- Berestycki and Silvestri (2021): Constrained HL(0)
- Berger, Turner, Procaccia (2021): Stationary HL(0)



HL(α) for $\alpha \neq 0$

All results for HL(α) with $\alpha \neq 0$ require some kind of regularization.

- Rohde and Zinsmeister (2005) obtained estimates on the dimension of scaling limits for a regularized version of HL(α).
- Sola, T., Viklund (2015) showed small-particle scaling limit of regularized HL(α) is a growing disk for all α provided regularization parameter σ is large enough.
- Liddle and T. (2021) obtained disk-limits and fluctuation results for HL(α) under capacity rescaling, regularized at ∞, when $\alpha \in (0, 2)$.

ALE family in the singular regime

- Sola, T., Viklund (2019) showed scaling limit of ALE(α, η, σ) is a single slit if $\alpha \geq 0$ and $\eta > 1$ when using slit particles, provided σ is very small.
- Higgs (2021) showed scaling limit of ALE$(0, \eta, \sigma)$ converges to a SLE$_4$ for $\eta < -2$ when using slit particles, provided σ is very small. Other SLE$_\kappa$'s with $\kappa > 4$ can be obtained by using different particle shapes.

ALE family in the disk-regime

Norris, Turner, Silvestri (2022 and 2024)

- **Scaling limit:** For all $\alpha + \eta \leq 1$, provided $\sigma \gg c^{1/2}$ as $c \to 0$ ($\sigma \gg c^{1/3}$ when $\alpha + \eta = 1$),

$$\Phi_n(z) \approx (1 + \alpha c n)^{1/\alpha} z.$$

- **Fluctuations:** Set $\mathcal{F}_n^{(c)}(z) = c^{-1/2}\left((1 + \alpha c n)^{-1/\alpha}\Phi_n(z) - z\right)$. Then $\mathcal{F}^{(c)}_{\lfloor t/c \rfloor}(z) \to \mathcal{F}_t(z)$ where

$$\dot{\mathcal{F}}_t(z) = \frac{1}{1+\alpha t}\left((1-(\alpha+\eta))\, z\mathcal{F}_t'(z) - \mathcal{F}_t(z) + \sqrt{2}\xi_t(z)\right).$$

Here $\xi_t(z)$ is complex space-time white noise on the circle, analytically continued to the exterior unit disk.

Fluctuations for ALE(α, η, σ) when $\alpha + \eta \leq 1$

Specifically

$$\mathcal{F}_t(z) = \sum_{m=0}^{\infty} (A_t^m + iB_t^m) z^{-m}$$

where

$$dA_t^m = -\frac{(m(1-\alpha-\eta)+1)\, A_t^m}{1+\alpha t} dt + \frac{\sqrt{2}}{1+\alpha t} d\beta_t^m$$
$$dB_t^m = -\frac{(m(1-\alpha-\eta)+1)\, B_t^m}{1+\alpha t} dt + \frac{\sqrt{2}}{1+\alpha t} d\beta'^m_t.$$

Here $\beta_t^m, \beta_t'^m$ are i.i.d. Brownian motions for $m = 0, 1, \ldots,$ so

$$A_t^m, B_t^m \sim \mathcal{N}\left(0, \frac{1 - e^{-2(m(1-\alpha-\eta)+1)\tau_t}}{m(1-\alpha-\eta)+1}\right).$$

Remarks

- The map $z \mapsto \mathcal{F}_t(z)$ is determined (by analytic extension) by the boundary process $\theta \mapsto \mathcal{F}_t(e^{i\theta})$.
- When $\alpha = \eta = 0$, these boundary fluctuations are the same as for internal diffusion limited aggregation (IDLA).
- As $t \to \infty$, $\mathcal{F}_t(e^{i\theta})$ converges to a Gaussian field.
  - When $\alpha + \eta = 0$, $\mathcal{F}_\infty(e^{i\theta})$ is known as the augmented Gaussian Free Field.
  - When $\alpha + \eta < 1$, $\mathrm{Cov}\left(\mathcal{F}_\infty(e^{ix}), \mathcal{F}_\infty(e^{iy})\right) \asymp \log|x - y|$.
  - When $\alpha + \eta = 1$, $\mathcal{F}_\infty(e^{i\theta})$ is complex white noise.

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