ON CRITICAL POINTS OF RANDOM SPHERICAL HARMONICS AND ISOTROPIC STATIONARY GAUSSIAN FIELDS

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Number and positions of critical points of a function are important qualitative descriptor.

The geometry of nodal lines, nodal domains, level curves and excursion sets are closely related to the set of critical points.

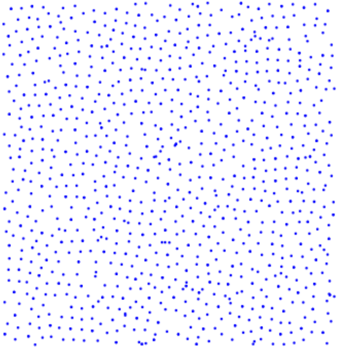
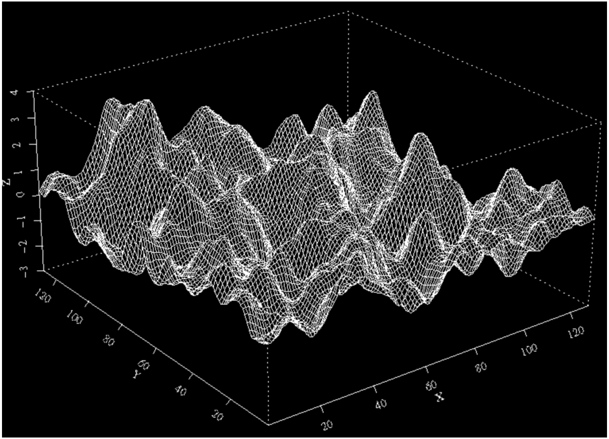


Figure: Critical points of a Berry's Random Plane Wave Model. [Beliaev, C., Wigman, 2019]

Random Plane Wave



Random Plane Wave is a stationary Gaussian field F in $\mathbb{R}^2$ with covariance kernel $K(x, y) = \mathbb{E}[F(x)F(y)] = J_0(|x - y|)$.

Random Plane Wave is a stationary Gaussian field F in $\mathbb{R}^2$ with covariance kernel

$$\begin{aligned} K(x,y) &= \mathbb{E}[F(x)F(y)] \\ &= J_0(|x-y|) \\ &= \sqrt{\frac{2}{\pi|x-y|}} \cos\left(|x-y| - \frac{\pi}{4}\right) + O(|x-y|^{-3/2}), \quad x, y \in \mathbb{R}^2. \end{aligned}$$

- point-wise covariances go to zero as points move away from each other
- but the rate is quite slow
- the covariance kernel is as oscillating function
- RPW is a stationary field i.e. $F(\cdot)$ and $F(\cdot + z)$ have the same distribution for every z or $K(x,y) = K(x-y)$

The corresponding spectral measure ρ is the normalized arc-length on the unit circle. Since the support of ρ is on the unit circle, the field is an eigenfunction of the Laplacian.

$$K(x) = \int_{\mathbb{R}^2} e^{-2\pi i x \cdot t} \rho(dt),$$

One can think RPW as (the limiting ensemble of) a random linear combination of plane waves with the same energy $E = k^2$ travelling in all possible directions

$$u_k(x) = \lim_{J\to\infty} \frac{1}{\sqrt{J}} \Re \left(\sum_{j=1}^{J} e^{k\langle \theta_j , x\rangle + \phi_j} \right)$$

θ_j random directions drawn uniformly on the unit circle, $\phi_j \in [0, 2\pi)$ random phases.

In this sense, one can think RPW is a natural notion of a ‘typical’ eigenfunction of the Laplacian in $\mathbb{R}^2$.

[Berry 1977] proposed to compare eigenfunctions with eigenvalue λ to a ‘typical’ instance of an isotropic, monochromatic random wave with wavenumber $k = \sqrt{\lambda}$ (now called RPW).

Berry conjectured that *high energy behaviour of deterministic eigenfunctions* on generic chaotic surfaces is universal and has statistically the same behaviour as Random Plane Waves.

This vague relation is subject to many numerical tests with very positive outcomes.

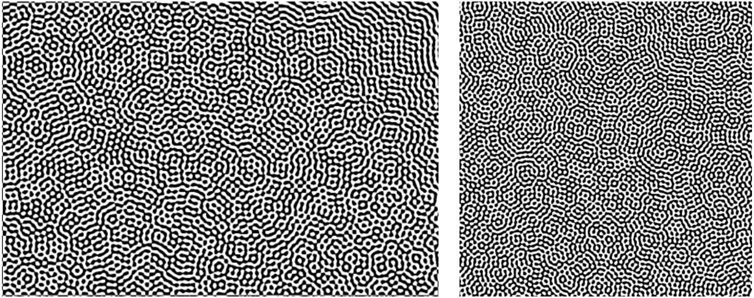


Figure: Nodal domains. Left: eigenfunction of a quarter of the stadium. Right: RPW [Bogomolny and Schmit 2007]

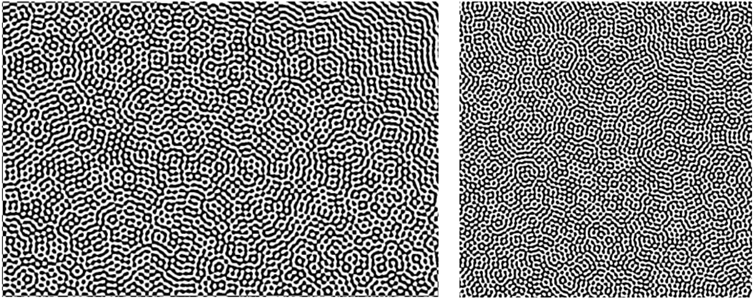


Figure: Nodal domains. Left: eigenfunction of a quarter of the stadium. Right: RPW [Bogomolny and Schmit 2007]

Nodal lines (nodal length, boundary intersections, intersections with a test curve...) of RPW should also model nodal lines of honest eigenfunctions.

Example: we are interested in the nodal length on the torus. We choose a representative planar domain $U \subseteq \mathbb{R}^2$ (e.g. a rectangle with the same aspect ratio and area as the torus), and study the distribution of the nodal length of RPW with wavenumber $k = \sqrt{\lambda}$ inside U.

Nodal line of F: $F^{-1}(0) = \{x \in U : F(x) = 0\}$.

Nodal length of RPW with wavenumber $k = \sqrt{\lambda}$ inside U (random variable): $\mathcal{L}_{U;\sqrt{\lambda}}$

Via Kac-Rice formula: $\mathbb{E}[\mathcal{L}_{U;\sqrt{\lambda}}] \sim \mathrm{const} \cdot \sqrt{\lambda}\,|U|$.

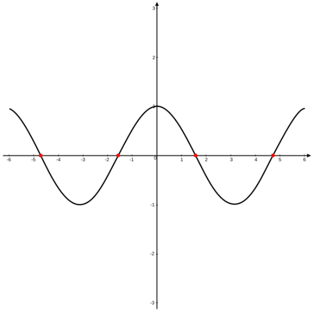
In accordance with Yau conjecture [Yau 1982]: for any smooth $\mathcal{M}$, the nodal length of the eigenfunctions ϕ_j is commensurable to $\sqrt{\lambda_j}$

$$c_{\mathcal{M}}\sqrt{\lambda_j} \le \mathcal{L}_{\mathcal{M};\sqrt{\lambda_j}} \le C_{\mathcal{M}}\sqrt{\lambda_j}$$

where the constants $c_{\mathcal{M}}$ and $C_{\mathcal{M}}$ depend on $\mathcal{M}$. Yau conjecture was proved under the assumption that the metric is real analytic (e.g. flat torus) [Donnelly and Fefferman 1988]. More recently the optimal lower bound [Logunov 2018] and polynomial in λ upper bound [Logunov 2018] were established for the more general, smooth, case.

Kac-Rice Formula

One-dimensional case: level sets are isolated points and excursion sets are intervals between them.



The number of level crossings is a local observable: the number of level crossings in a disjoint union of two sets is equal to the sum of level crossings inside these sets.

This allows writing an integral formula for moments of the number of level crossings. This formula is now known as Kac-Rice formula.

f Gaussian process on an interval I, f has C^1-paths ($f \in C^1$ with probability one). For pairwise distinct points $t_1, \ldots, t_k \in I$ the joint distribution of $(f(t_1), ..., f(t_k))$ is non-degenerate. Let N_u be the number of points where $f(t) = u$

$$\mathbb{E}[N_u(N_u - 1) \cdots (N_u - k + 1)]$$
$$= \int_{I^k} \mathbb{E}\Big[\prod_{i=1}^{k} |f'(t_i)| \Big| f(t_1) = \cdots = f(t_k) = u \Big] p_{t_1,...,t_k}(u, \ldots, u) dt_1 \cdots dt_k$$

$p_{t_1,...,t_k}$ joint density of $(f(t_1), ..., f(t_k))$.

Example: f is a **centred stationary process** with covariance kernel K

$$\mathbb{E}[N_u] = \int_I \mathbb{E}\Big[|f'(t)| \Big| f(t) = u \Big] p_t(u) dt.$$

The law of $(f'(t), f(t))$

- does not depend on t
- is centred
- its covariance matrix is

$$\begin{pmatrix} -K''(0) & 0 \\ 0 & K(0) \end{pmatrix}.$$

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So $f'(0)$ and $f(0)$ are independent, and

$$\mathbb{E}[N_u] = |I|\ \mathbb{E}\Big[|f'(0)|\Big] p_{f(0)}(u) = |I|\ \sqrt{-\frac{2K''(0)}{\pi}} \frac{1}{\sqrt{2\pi K(0)}} e^{-\frac{u^2}{2K(0)}}$$

$$= |I|\ \frac{1}{\pi}\sqrt{-\frac{K''(0)}{K(0)}} e^{-\frac{u^2}{2K(0)}}.$$

Critical points of RPW

Expected number of critical point

Critical points lying in a ball or radius R

$$\mathcal{C}_F(B(R)) = \{x \in B(R) : \nabla F(x) = 0\}.$$

F *smooth Gaussian* $\implies$ set of critical points is a point process on $\mathbb{R}^2$.

F *stationary* $\implies$ it is possible to employ Kac-Rice method to count the zeros of the map $x \to \nabla F(x)$: if $\nabla F(x)$ is nonsingular for all $x \in B(R)$

$$\mathbb{E}[\#\mathcal{C}_F(B(R))] = \int_{B(R)} \phi_{\nabla F(x)}(0) \cdot \mathbb{E}[|\det H_F(x)| \,\big|\nabla F(x) = 0]dx.$$

F *isotropic* $\implies$ $\mathbb{E}[\#\mathcal{C}_F(B(R))] = \mathrm{Vol}(B(R)) \frac{1}{2\pi\sqrt{3}}$.

Compare RPW with two very well known translation invariant processes.

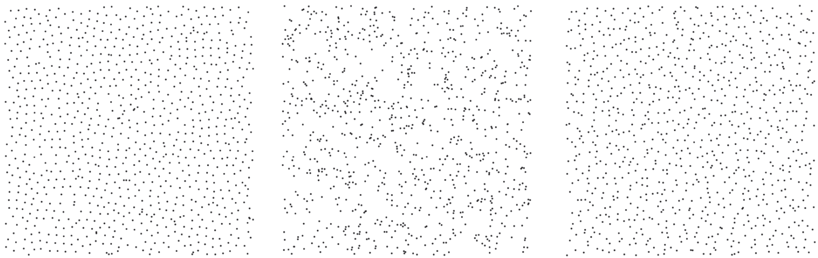


Figure: Left: Critical points of a Random Plane Wave. Centre: Poisson point process. Right: Ginibre ensemble.

The number of critical points in a square of side-length n is $c \cdot n^2$ where $c = \frac{1}{2\sqrt{3}\pi}$ is the natural intensity of critical points.

The other two point processes are rescaled to have the same intensity.

Fluctuations of the number of critical points

The second factorial moment of the number of critical points is

$$\mathbb{E}[\#\mathcal{C}_F(B(R)) \cdot (\#\mathcal{C}_F(B(R)) - 1)] = \int\limits_{B(R)\times B(R)} K_2(x,y)dxdy,$$

where the 2-point correlation function $K_2(x,y) = K_2(x-y)$ of the critical point process can be derived via the Kac-Rice formula

$$\begin{aligned} &K_2(x,y) \\ &= \lim_{\epsilon_1,\epsilon_2\to 0} \frac{1}{\mathrm{Vol}(B(\epsilon_1))\cdot \mathrm{Vol}(B(\epsilon_2))} \mathbb{E}[\#\mathcal{C}_F(B_x(\epsilon_1)) \cdot \#\mathcal{C}_F(B_y(\epsilon_2))] \\ &= \phi_{(\nabla F(x),\nabla F(y))}(0,0) \cdot \mathbb{E}\left[|\det H_F(x) \cdot \det H_F(y)| \,\middle|\, \nabla F(x) = \nabla F(y) = 0 \right]. \end{aligned}$$

F *isotropic* $\implies$ $K_2(x,y)$ is a function of the Euclidean distance $r = \|x-y\|$.

Fluctuations of the number of points in a square depend a lot on the point process.

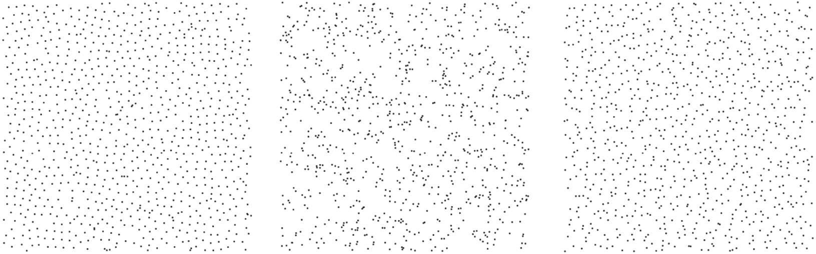


Figure: Left: Critical points of a Random Plane Wave. Centre: Poisson point process. Right: Ginibre ensemble.

- When $C_F(r)$ decays sufficiently rapidly, the *long range* asymptotics of $K_2(r)$, $r \to \infty$ yields the asymptotic variance of the number of critical points in large balls. From random spherical harmonics (equivalent to RPW in the scaling limit), variance for the number of critical points scales like $n^2 \log n$ [CMW 2016] and [CW 2017].
- Poisson: variance is asymptotic to $c \cdot n^2$.
- Ginibre ensemble: variance is of order n.

Attraction and repulsion

The *short range* asymptotics of $K_2(r)$, $r \to 0$ yields the asymptotic second factorial moment of the number of critical points in a *small* balls.

Informally, for R small,

$$\mathbb{P}(\text{one critical points in } B(R)) \approx R^2$$
$$\mathbb{P}(\text{two critical points in } B(R)) \approx \iint\limits_{B(R)\times B(R)} K_2(x,y)dxdy.$$

- We say that the critical points *attract* each other if $K_2(r) \to \infty$ as $r \to 0$, i.e.

$$\{\mathbb{P}(\text{one critical points in } B(R))\}^2 \ll \mathbb{P}(\text{two critical points in } B(R)).$$

- We say that the critical points *repel* each other if $K_2(r) \to 0$ as $r \to 0$, i.e.

$$\mathbb{P}(\text{two critical points in } B(R)) \ll \{\mathbb{P}(\text{one critical points in } B(R))\}^2.$$

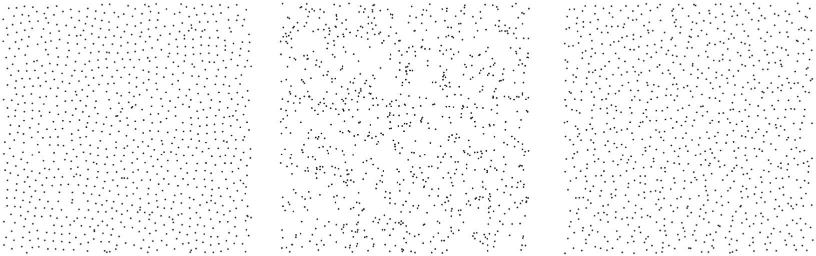


Figure: Left: Critical points of a Random Plane Wave. Centre: Poisson point process. Right: Ginibre ensemble.

- Ginibre ensemble: $\mathbb{P}$(two points in $B(R)) \approx R^6$, points repel each other.
- For Poisson process (or a finite collection of independent points) on the plane

$$\mathbb{P}(\text{two points in } B(R)) = \frac{1}{2}(c\pi R^2)^2 e^{-c\pi R^2} \approx R^4$$

- Critical points of RPW: $\mathbb{P}$(two points in $B(R)) \approx R^4$! Critical points exhibit no repulsion nor attraction [Beliaev, C., Wigman 2019].

The minor difference (by a constant) does not explain the big difference in the appearance of Poisson and critical points (highly regular).

Theorem (Beliaev, C., Wigman 2019 and 2020)

As $\rho \to 0$

$$\mathbb{E}[\mathcal{N}_\rho^c(\mathcal{N}_\rho^c - 1)] = \frac{1}{2^5 3\sqrt{3}}\rho^4 + O(\rho^6).$$

Proof:

- 2-point correlation function $K_2 : \mathcal{B}(\rho) \times \mathcal{B}(\rho) \to \mathbb{R}$

$$K_2(z,w) = \phi_{(\nabla\Psi(z), \nabla\Psi(w))}(0,0)\mathbb{E}[|\det H_\Psi(z)| \cdot |\det H_\Psi(w)| \big| \nabla\Psi(z) = \nabla\Psi(w) = 0].$$

- $(\nabla\Psi(z), \nabla\Psi(w))$ non degenerate for all $z \neq w$,

$$\mathbb{E}[\mathcal{N}_\rho^c(\mathcal{N}_\rho^c - 1)] = \iint_{\mathcal{B}(\rho)\times\mathcal{B}(\rho)} K_2(z,w)dzdw.$$

- 2-point correlation function K_2 depends on $r = |z - w|$

$$K_2(r) = \frac{1}{(2\pi)^5\sqrt{\mathrm{det}A(r)}} \int_{\mathbb{R}^6} |x_1x_3 - x_2^2| \cdot |x_4x_6 - x_5^2|$$
$$\times \frac{1}{\sqrt{\mathrm{det}\Delta(r)}} \exp\left\{-\frac{1}{2}\mathbf{x}^t\Delta(r)^{-1}\mathbf{x}\right\} d\mathbf{x}.$$

- For every $r > 0$, $\Delta(r)$ is symmetric, we diagonalise $\Delta(r)$

$$\Delta(r) = P(r)^t\Lambda(r)P(r)$$

i.e. compute the eigenvalues and eigenvectors of $\Delta(r)$.

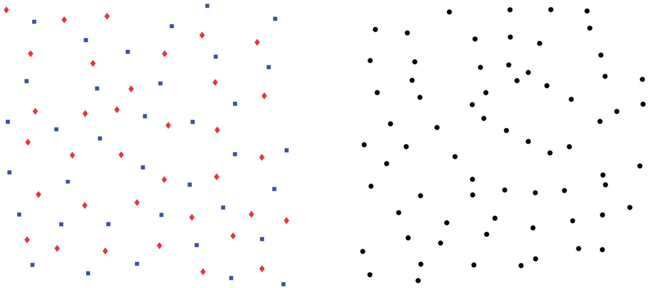
- Note that
$$\frac{1}{\sqrt{\mathrm{det}\Delta(r)}} \exp\{-\frac{1}{2}\mathbf{x}^t\Delta(r)^{-1}\mathbf{x}\} = \frac{1}{\sqrt{\prod \lambda_i(r)}} \exp\{-\frac{1}{2}(P(r)\mathbf{x})^t\Lambda(r)^{-1}P(r)\mathbf{x}\}.$$

- Change variable: $\mathbf{z} = P(r)\mathbf{x}$ i.e. $\mathbf{x} = P(r)^{-1}\mathbf{z}$.

- Finally

$$
\begin{aligned}
&K_2(r)\\
&= \frac{1}{(2\pi)^5\sqrt{\mathrm{det}A(r)}}\int_{\mathbb{R}^6} \left|f(r,\mathbf{z})\right| \exp\left\{-\frac{1}{2}\sum_{i=1}^{6} z_i^2\right\} d\mathbf{z}\\
&= \frac{1}{\pi^5 2\sqrt{3}r^2 + O(r^4)}\left[\frac{r^2}{384}\int_{\mathbb{R}^6} z_4^2 z_6^2 \exp\left\{-\frac{1}{2}\sum_{i=1}^{6} z_i^2\right\} d\mathbf{z} + O(r^4)\right]\\
&= \frac{1}{96\sqrt{3}\,\pi^2} + O(r^2).
\end{aligned}
$$

We believe that the isotropic assumption is not essential, used to reduce the number of variables and to make explicit computations. In the isotropic case there are no mysterious cancellations, suggesting the same for the asymptotic behaviour in the generic case.



Left: Extrema only. Right: Saddles only.

Both processes exhibit strong repulsion:

$$\mathbb{P}(2 \text{ points in } \mathcal{B}(\rho)) \approx \rho^7 \log 1/\rho.$$

The apparent ‘rigid’ structure that is observed for the critical points comes from the regularity of both these point processes.

$d > 2$: [Azaïs, Delmas 2022] attraction due to critical points with adjacent indexes and strong repulsion, growing with d, between maxima and minima.
$d = 1$: repulsion.

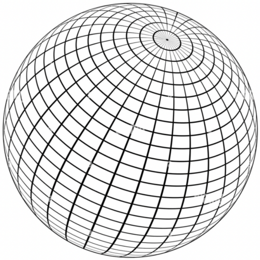
Random spherical harmonics

$\mathbb{S}^2$ two-dimensional unit sphere with the round metric

$\Delta_{\mathbb{S}^2}$ spherical Laplacian

$\Delta_{\mathbb{S}^2} f + \lambda_\ell f = 0$ Helmholtz equation

$-\lambda_\ell = -\ell(\ell+1)$, $\ell \in \mathbb{N}$ eigenvalues

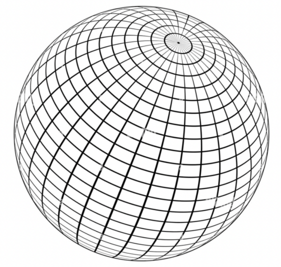


for any eigenvalue choose an arbitrary L^2-orthonormal basis $\{Y_{\ell m}(\cdot)\}_{m=-\ell,\dots,\ell}$ and consider *random Gaussian eigenfunctions*

$$f_\ell(x) = \frac{1}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \qquad x \in \mathbb{S}^2,$$

$\{a_{\ell m}\}$ are zero-mean, independent Gaussian.

$$f_\ell(x) = \frac{1}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x)$$



Standardised s.t. $\mathrm{Var}(f_\ell(x)) = 1$.

From the addition formula for the Legendre polynomials the covariance kernel is

$$K_\ell(x,y) = \mathbb{E}[f_\ell(x) f_\ell(y)] = P_\ell(\cos d(x,y)),$$

P_ℓ Legendre polynomial, $d(x,y) = \arccos\langle x, y\rangle$ geodesic distance on the sphere.

From this formula one can immediately see that this is a (spherically) stationary field. Alternatively, this could be seen from the invariance of the eigenspace and L^2 norm under rotations.

RPW is the scaling limit of random spherical harmonics

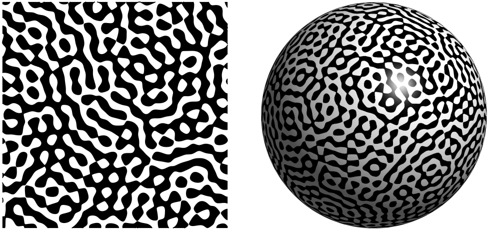


Figure: RPW (left) and random spherical harmonic of degree 100 (right). [Beliaev 2022]

$x_0 \in \mathbb{S}^2$, exponential map from the tangent plane to the sphere: $\exp_{x_0} : T_{x_0}\mathbb{S}^2 \to \mathbb{S}^2$

$$g_\ell(x) := f_\ell(\exp_{x_0}(x/\ell))$$

rescale by $\ell \sim \sqrt{\lambda}$ so the wavelength becomes of order 1.

$$\mathbb{E}[g_\ell(x)g_\ell(y)] = P_\ell(\cos\theta(\exp_{x_0}(x/\ell), \exp_{x_0}(y/\ell)))$$

Since the exponential map is almost isometry near x_0, the spherical distance between images is almost the distance between the points, hence uniformly in x and y the covariance behaves as

$$P_\ell\left(\cos\frac{|x-y|}{\ell}\right) \to J_0(|x-y|)$$

where the limit follows from Hilb's asymptotics for Legendre polynomials.

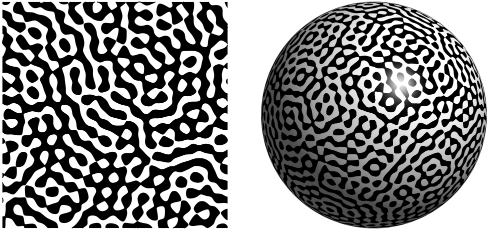


Figure: RPW (left) and random spherical harmonic of degree 100 (right). [Beliaev 2022]

RPW also appears as the scaling limit of 'narrow-band' functions.

It is possible to define a similar field on any compact manifold (with no large eigenspaces we take a linear combination of order n eigenfunctions around n^2-th eigenfunction).

$$f_n(x) = \sum_{k=n^2}^{n^2+n} a_k \phi_k(x)$$

a_k i.i.d. Gaussian. [Zelditch 2009] shows that under some mild assumptions of the manifold, a similarly rescaled field converges to RPW.

Critical points

Number of **critical points**:

$$\mathcal{N}_\ell^c = \#\{x \in \mathbb{S}^2 : \nabla f_\ell(x) = 0\}.$$

Let $I \subseteq \mathbb{R}$ be any interval, number of critical points of f_ℓ with value in I, number of **critical values**:

$$\mathcal{N}_\ell^c(I) = \#\{x \in \mathbb{S}^2 : \nabla f_\ell(x) = 0, f_\ell(x) \in I\}.$$

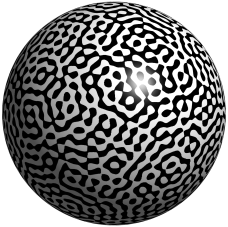
We investigate how much the **number of critical points and critical values characterizes the geometry of the random spherical eigenfunctions in the high frequency limit**, i.e. the excursion sets

$$A_u(f_\ell) = \{x \in \mathbb{S}^2 : f_\ell(x) \geq u\},$$

for arbitrary levels $u \in \mathbb{R}$.

Geometric functionals

$$A_0(f_\ell) = \{x \in \mathbb{S}^2 : f_\ell(x) \geq 0\}$$



Functionals which describe the geometry of the excursion sets $A_u(f_\ell)$:

- **area** $\mathcal{L}_2(u, \ell)$ of the excursion sets
- (half of the) **boundary length** $\mathcal{L}_1(u, \ell)$ of the excursion sets
- **Euler characteristic** $\mathcal{L}_0(u, \ell)$ of the excursion sets

Relationship between geometric functionals of excursion sets of f_ℓ at different levels u: [Wigman, 2011], [Marinucci-Wigman, 2014], [Marinucci-Rossi, 2015], [Rossi, 2019], [C.-Marinucci, 2019 and 2020], [C.-Todino, 2022].

Critical points: asymptotic variance

[C.-Marinucci-Wigman, 2016] and [C.-Wigman, 2017] show that as $\ell \to \infty$

The expected number of critical values behaves like

$$\mathbb{E}[\mathcal{N}_\ell^c(I)] = \frac{2}{\sqrt{3}}\ell^2 \int_I \frac{\sqrt{3}}{\sqrt{8\pi}}(2e^{-t^2} + t^2 - 1)e^{-\frac{t^2}{2}}\,dt + O(1),$$

the constant in the $O(\cdot)$ term is universal, i.e. the integral of the error term on any interval I is uniformly bounded by its value when $I = \mathbb{R}$.

The investigation of the **asymptotic variance** is more challenging

$$\text{Var}(\mathcal{N}_\ell^c(I)) = \ell^3[\nu^c(I)]^2 + O(\ell^{5/2}),$$

$$\nu^c(I) = \int_I \frac{1}{\sqrt{8\pi}}[2 - 6t^2 - e^{t^2}(1 - 4t^2 + t^4)]e^{-\frac{3}{2}t^2}\,dt, \qquad \nu^c(\mathbb{R}) = 0,$$

for $I = \mathbb{R}$ the leading term vanishes and

$$\text{Var}(\mathcal{N}_\ell^c) = \frac{1}{3^3\pi^2}\ell^2 \log \ell + O(\ell^2).$$

Similar results hold for extrema and saddles. **Proof: via (approximate) Kac-Rice formula for moments**.

Interpretation in terms of Wiener chaoses

$L^2(\Omega)$ (unique) Wiener-Itô decomposition of the number of critical points into Wiener chaoses

$$\mathcal{N}_\ell^c(I) = \sum_{q=0}^{\infty} \mathcal{N}_\ell^c(I)[q],$$

$\mathcal{N}_\ell^c(I)[q]$ projection on the q-order chaos component i.e. closed linear subspace of $L^2(\mathbb{P})$ generated by all real, finite, linear combinations of random variables of the form

$$H_{q_1}(\xi_1) \cdot H_{q_2}(\xi_2) \cdots H_{q_k}(\xi_k), \qquad k \geq 1,$$

H_{q_i} are Hermite polynomials, $q_i \in \mathbb{N}$ s.t. $q_1 + \cdots + q_k = q$ and $(\xi_1, \ldots, \xi_k)$ standard real Gaussian vector. Wiener chaoses are **orthogonal**.

A **single term dominates** the $L^2(\Omega)$ chaos expansion of $\mathcal{N}_\ell^c(I)$ and $\mathcal{N}_\ell^c$.

$\mathcal{N}_\ell^c(I)$ is dominated by the projection into the second chaotic component [C.-Marinucci, 2020].

The asymptotic behaviour of $\mathcal{N}_\ell^c$ is dominated by the projection into the fourth chaotic component [C.-Marinucci, 2021].

Correlation between $\mathcal{N}_\ell^c(I)$ and $\mathcal{N}_\ell^c$ is asymptotically zero while the partial correlation, after controlling the random L^2-norm on the sphere of the eigenfunctions, is asymptotically one [C.-Todino, 2022] .

- $\mathcal{N}_\ell^c$ and $\mathcal{N}_\ell^c(I)$ are asymptotically independent, but, when the effect of random fluctuations of the norm of f_ℓ is properly subtracted, their joint distribution is completely degenerate and the behaviour of the fluctuations of $\mathcal{N}_\ell^c(I)$ is **fully explained** by $\mathcal{N}_\ell^c$, in the high energy limit.

- As a simple corollary, a **quantitative Central Limit Theorem** holds for $\mathcal{N}_\ell^c(I)$ [Nourdin-Peccati 2005] and $\mathcal{N}_\ell^c$ [Marinucci-Wigman, 2014].

- While the computation of $\mathcal{N}_\ell^c$ and $\mathcal{N}_\ell^c(I)$ via Kac-Rice formula requires the evaluation of **gradient and Hessian fields**, the dominant terms depend, in the high frequency limit, only on the second-order and fourth-order Hermite polynomials evaluated at the eigenfunctions f_ℓ (Green's formula).

Critical values and Lipschitz-Killing curvatures at $u \neq 0$

[Wigman 2011], [Marinucci-Wigman, 2014], [Marinucci-Rossi, 2015], [C.-Marinucci, 2018] show that the three Lipschitz-Killing curvatures are asymptotically fully correlated: for all $u_1, u_2 \neq 0$ (and $u \neq 1, -1$ for the Euler characteristic)

$$\lim_{\ell \to \infty} \text{Corr}(\mathcal{L}_j(u_1, \ell), \mathcal{L}_k(u_2, \ell)) = 1, \qquad j, k = 0, 1, 2.$$

The number of critical values is then perfectly correlated, as $\ell \to \infty$, with the area, the Euler characteristic and the boundary length at any nonzero level u

$$\lim_{\ell \to \infty} \text{Corr}(\mathcal{L}_k(u, \ell), \mathcal{N}_\ell(u, \infty)) = 1, \qquad k = 0, 1, 2.$$

Thank you!

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