Topological Structures of Large Scale Interacting Particle Systems

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May 11-15, 2023

1 The Topological Parts of a Particle System

A *locale* $X = (X, E)$ is a locally finite, infinite, simple, symmetric directed, connected graph. You should think of this as the nearest neighbour lattice $(\mathbb{Z}^d, \mathbb{E}^d)$, also called Euclidean lattice. Other examples include the triangular or the hexagonal lattices.

A *state space* S is a finite set S with a distinguished *base state* $* \in S$. (It is a pointed space.) You should think of this as the $\{0, 1\}$ set, where 0 is the distinguished base state (no particle).

A *group* G ***acting on*** X via a group action $G \circlearrowleft X$. In the Euclidean lattice, $G = \mathbb{Z}^d$ is acting via translation: $g.x := x + g$.

An *interaction* ϕ is a map $\phi : S \times S \to S \times S$ satisfying the property that

$$\hat{\iota} \circ \phi \circ \hat{\iota} \circ \phi = id_{S \times S},$$

where $\hat{\iota}(s_1, s_2) := (s_2, s_1)$ is the exchange operator. You should think of ϕ as being exactly the exchange operator $\hat{\iota}$. In this context, the above is trivial. In a more general context, the above ensures that particles can switch back.

2 The Abstract Configuration Space

The *space of configurations* is given as S^X. In the setting of the SSEP on $\mathbb{Z}^d$, this is $\{0, 1\}^{\mathbb{Z}^d}$.

The *transition structure* (S^X, Φ) is given by the configuration space together with *transitions* (=edges) φ of the form

$$\varphi = (\eta, \eta^e)$$

for any configuration $\eta \in S^X$ and any edge $e \in E$ in the underlying locale, where η^e is the configuration obtained from η by applying the interaction ϕ to the sites connected by e. More precisely, if $e = (x, y)$, then $\eta^e_z := \eta_z$ if $z \notin \{x, y\}$ and

$$(\eta^e_x, \eta^e_y) := \phi(\eta_x, \eta_y).$$

Note that Φ can contain edges of the form (η, η). Furthermore, using the property of the interaction ϕ, one shows that the graph (S^X, Φ) is symmetric! Note that this is what we really care about. In this sense, we may also look at interactions ϕ which do not satisfy this reversability property as long as the transition structure is symmetric. An example for this are the Glauber dynamics.

In analogy with the ideas from Differential Geometry, one can view the configuration space $M = S^X$ as a manifold. In this setting, all transitions of the form $(\eta, \eta^e) \in \Phi$ represent the tangent space at the configuration $\eta \in S^X$. In particular, we may view Φ as the *tangent bundle* of the manifold (= the union of all tangent spaces).

Define the modified transition structure (S^X_*, Φ_*) by restricting S^X to configurations

$$S^X_* := \{\eta \in S^X \ : \ \eta_x = * \text{ except for a finite number of sites}\}$$

that are mostly "void". Consequently, $\Phi_* := \Phi \cap (S^X_* \times S^X_*)$ is the restriction to transitions between mostly void configurations.

3 Function Spaces

For a set A, define $C(A) := \mathbb{R}^A$ as the set of all functions $A \to \mathbb{R}$.

Note that for $\Lambda \subset \Lambda' \subset\subset X$ (finite subset)

$$\begin{array}{ccc} S^X_* & & C(S^X_*) \\ \downarrow\!\!\!\!\downarrow & & \uparrow \\ S^{\Lambda'} & \rightsquigarrow & C(S^{\Lambda'}) \\ \downarrow\!\!\!\!\downarrow & & \uparrow \\ S^{\Lambda} & & C(S^{\Lambda}) \end{array}$$

where the surjection is given by the restriction of a configuration and the inclusion $i : C(S^\Lambda) \hookrightarrow C(S^X_*)$, then, is defined via duality by

$$i(f)(\eta) = f(\eta|_\Lambda).$$

In the following, we will identify $C(S^\Lambda)$ as a subset of $C(S^{\Lambda'})$ as of $C(S^X_*)$. This sort of identification will happen all the time and will not be stated explicitly any more. Every time, there is an inclusion arrow, you should interpret it as an identification.

The space of *local functions* is given as

$$C_{loc}(S^X) := \bigcup_{\Lambda \subset\subset X} C(S^\Lambda) \subseteq C(S^X) \cap C(S^X_*).$$

Note that $C_{loc}(S^X)$ can also be viewed as the direct limit of the family $C(S^\Lambda)$ with respect to the inclusion defined before. It consists of functions that depend only on the states at a finite number of sites.

Note that $C_{loc}(S_*^X) \subsetneq C(S_*^X)$ as, for example,

$$\left(\eta \mapsto \sum_{x\in X} x\eta_x\right) \in C(S_*^X) \setminus C_{loc}(S^X).$$

The space of *0-forms* is simply

$$C^0(S_*^X) := C(S_*^X).$$

The space of *1-forms* is given by

$$C^1(S_*^X) := C(\Phi_*)^{alt} := \{\omega \in C(\Phi_*) \;:\; \omega(\overline{\varphi}) = -\omega(\varphi)\},$$

where $\overline{\varphi}$ denotes the opposite edge: if $\varphi = (\eta, \eta^e)$, then $\overline{\varphi} = (\eta^e, \eta)$. Note that $\overline{\varphi} \in \Phi_*$ as we discussed before. This trick of only looking at alternating forms also gets rid of edges of the form (η, η), as all 1-forms are necessarily 0 on them.

Note that

$$C^1(S_*^X) \hookrightarrow C(\Phi_*) \hookrightarrow C(E \times S_*^X) = \prod_{e\in E} C(S_*^X),$$

where we used the fact that a transition $\varphi = (\eta, \eta^e) \in \Phi_*$ is uniquely determined by the source configuration $\eta \in S_*^X$ and the edge $e \in E$. This allows us to view a 1-form ω as a family $(\omega_e)_{e\in E}$ of functions $\omega_e : S_*^X \to \mathbb{R}$. More precisely, $C^1(S_*^X)$ corresponds to the subspace of families $\omega \in \prod_{e\in E} C(S_*^X)$ satisfying

$$\begin{cases} \omega_{\overline{e}}(\eta^e) = -\omega_e(\eta) & \\ \omega_e(\eta) = 0 & \text{if } \eta^e = \eta \\ \omega_e(\eta) = \omega_{e'}(\eta) & \text{if } \eta^e = \eta^{e'} \end{cases}.$$

Again, we can make a link with Differential Geometry. If we consider a 1-form in coordinate form

$$\alpha = f \ \mathrm{d}x + g \ \mathrm{d}y + h \ \mathrm{d}z,$$

then $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ represent the possible different directions. Another representation would simply to represent α as the vector (f, g, h) with $f, g, h : M \to \mathbb{R}$. Here, we do the same thing: every edge e can be viewed as a direction, so that it becomes natural to simply consider the vector $(\omega_e)_{e\in E}$ with entries for every direction.

The space Z^1 of *closed forms* contains all 1-forms $\omega \in C^1$ satisfying

$$\int_\gamma \omega = 0$$

for all closed paths, see Section 4.2 for a rigorous definition of the path integral.

We conclude by defining *conserved quantities*.

A *conserved quantitiy* is a function $\xi : S \to \mathbb{R}$ satisfying $\xi(*) = 0$ as well as the balance equation

$$\xi(s_1) + \xi(s_2) = \xi(s_1') + \xi(s_2')$$

for any $s_1, s_2 \in S$ and $(s_1', s_2') := \phi(s_1, s_2)$.

The space of conserved quantities $\text{Consv}^{\phi}(S)$ is the $\mathbb{R}$-vector space of all conserved quantities w.r.t. the interaction ϕ.

Note that $\text{Consv}^{\phi}(S) \hookrightarrow C(S_*^X)$ via the identification $\xi \mapsto \xi_X$ with the total conserved quantity

$$\xi_X(\eta) := \sum_{x\in X} \xi(\eta_x).$$

The interaction ϕ is *irreducibly quantified* if, for any locale X, any conserved quantity ξ and for any η, η',

$$\xi_X(\eta) = \xi_X(\eta') \qquad \Rightarrow \qquad \eta \leftrightarrow \eta'.$$

Note that one easily shows that ξ_X is constant on connected components. The property of being irreducibly quantified ensures that the reciprocal holds true: if ξ_X is constant on a set, then this set must be connected.

In the language of probability theory, this means that the associated Markov Chain is irreducible on the connected components.

We will always assume that the interaction ϕ is irreducibly quantified.

4 The Naïve Cohomology

In this section, we will consider the "natural" cohomology associated to the complex

$$0 \longrightarrow C^0 \xrightarrow{\partial^0} C^1 \longrightarrow 0,$$

where the differential ∂^0 is very naturally defined as

$$\partial^0 f = (\nabla_e f)_{e\in E} \qquad \text{where} \qquad \nabla_e f(\eta) := f(\eta^e) - f(\eta)$$

is the discrete gradient along the edge e. More precisely, the cohomology is defined as usual through

$$H^0_{naive} = \ker(\partial^0) \qquad \text{and} \qquad H^1_{naive} := {C^1}\big/{\partial^0(C^0)}.$$

4.1 The Algebraic Approach

You can skip this part as it is only of peripheral interest. Here, I will only discuss how we obtain the above cohomology from the associated homology. It doesn't add to the discussion (in my opinion).

It appears that the complex we should look at is given by

$$0 \longleftarrow C_0 \underset{\partial_1}{\longleftarrow} C_1 \longleftarrow 0,$$

where

$$C_0 := \mathbb{Z}[S_*^X] \qquad \text{and} \qquad C_1 := \mathbb{Z}[\Phi_*]^{alt} := {\mathbb{Z}[\Phi_*]}\big/{\varphi = \overline{\varphi}}$$

are the spaces consisting of (finite) formal sums with coefficients in $\mathbb{Z}$. For C_1, we quotient out formal sums which are equal to their reversed sum in which all transitions are reversed. For example, the formal sum

$$\varphi - \overline{\varphi} = 0.$$

Similarly, loops of the form (η, η) will also be identified with 0. The differential ∂_1, then, is defined through

$$\partial_1 \varphi = \eta^e - \eta$$

for any transition $\varphi = (\eta, \eta^e) \in \Phi_*$, and continued to all of C_1 by linearity.

Then, the associated cocomplex is given by

$$0 \longrightarrow C^0 \xrightarrow{\partial^0} C^1 \longrightarrow 0,$$

where

$$C^0 := Hom_{\mathbb{Z}}(C_0; \mathbb{R}) \cong C(S_*^X) \qquad \text{and} \qquad C^1 := Hom_{\mathbb{Z}}(C_1; \mathbb{R}) \cong C(\Phi_*)^{alt}$$

which corresponds to the complex that we defined at the very beginning of the section. Here, the first isomorphism is given by

$$C(S_*^X) \ni f \mapsto F \in C^0, \qquad F\left(\sum_{i=1}^n n_i \eta_i\right) := \sum_{i=1}^n n_i f(\eta_i)$$

and similarly for the second.

The differential ∂^0 is defined through the duality relationship $\partial^0 f := f \circ \partial_1$ which gives indeed

$$\partial^0 f\big((\eta, \eta^e)\big) := f\big(\partial_1(\eta, \eta^e)\big) = f(\eta^e - \eta) = f(\eta^e) - f(\eta) = \nabla_e f(\eta)$$

as we expected.

4.2 Studying the Naïve Cohomology

From now on, we will simply write ∂ instead of ∂^0.

The 0-th cohomology H^0_{naive} is easily computable. Recall that

$$H^0_{naive} = \ker \partial.$$

Clearly, this is the space of functions $f : S_*^X \to \mathbb{R}$ that are constant on every connected component of (S_*^X, Φ_*). In particular,

$$\dim H^0_{naive} = \#\text{connected components}.$$

Already in the simplest case of exclusion processes on $(\mathbb{Z}^d, \mathbb{E}^d)$, this number is infinite as the connected components are in bijection with the number of particles in the configuration.

This can be generalized in the following way. First, recall that we identify conserved quantities ξ with the associated functions

$$\xi_X(\eta) := \sum_{x \in X} \xi(\eta_x).$$

One easily verifies that ξ_X is constant on every connected component of (S_*^X, Φ_*), yielding

$$\mathrm{Consv}^\phi(S) \hookrightarrow H^0_{naive}.$$

However, the right-hand side must be much larger, as it contains also functions like $(\xi_X)^2$ if ξ_X is a conserved quantity. To get an explicit description of H^0_{naive} in terms of the conserved quantities, suppose that $\dim \mathrm{Consv}^\phi(S) = c^\phi$ and choose a basis $\xi^1, \dots, \xi^{c^\phi}$. Define the function

$$\boldsymbol{\xi}_X : S_*^X \to \mathbb{R}^{c^\phi}, \quad \eta \mapsto \big(\xi^1_X(\eta), \dots, \xi^{c^\phi}_X(\eta)\big).$$

This map associated to a configuration η the values of the conserved quantities for the configuration η. In particular, if $\eta \leftrightarrow \eta'$, then

$$\boldsymbol{\xi}_X(\eta) = \boldsymbol{\xi}_X(\eta').$$

Since we assumed that the interaction is irreducibly quantified, the converse holds true: if $\boldsymbol{\xi}_X(\eta) = \boldsymbol{\xi}_X(\eta')$, then η and η' are in the same connected component. Hence, the connected components corresponds bijectively with the elements of the set

$$\mathcal{M} := \boldsymbol{\xi}_X(S_*^X).$$

Note that $\mathcal{M}$ has the structure of a monoid w.r.t. the usual addition on $\mathbb{R}^{c^\phi}$. Indeed, for two configurations $\eta, \eta' \in S_*^X$, we can easily construct a configuration η'' such that

$$\boldsymbol{\xi}_X(\eta'') = \boldsymbol{\xi}_X(\eta) + \boldsymbol{\xi}_X(\eta').$$

To do so, let Λ and Λ' be two finite sets containing all nonempty sites of η and η' respectively. Choose any injection $i : \Lambda' \hookrightarrow X \setminus \Lambda$ and define $\eta''_x := \eta_x$ if $x \in \Lambda$, $\eta''_{i(x)} := \eta'_x$ for all $x \in \Lambda'$ and $\eta''_x = *$ otherwise.

In particular, if there is at least one conserved quantity ($c^\phi \geq 1$), the monoid $\mathcal{M}$ contains at least a copy of $\mathbb{N}$, implying that

$$\dim H^0_{naive} = +\infty.$$

If no quantity is conserved, then the transition structure is connected and $\dim H^0_{naive} = 1$.

The 1st cohomology H^1_{naive}, then, encodes information on the "holes" in the transition structure. Recall that

$$H^1_{naive} := {C^1}\big/{\partial(C^0)}.$$

To analyse H^1_{naive}, we first give an explicit description of the exact forms $\partial(C^0)$. Along a path $\gamma = (\varphi_1, \dots, \varphi_n)$, define the path integral

$$\int_\gamma \omega := \sum_{i=1}^n \omega(\varphi_i).$$

We say that a form ω is *closed* if

$$\int_\gamma \omega = 0$$

for all closed paths γ. Write $Z^1 \subseteq C^1$ for the space of all closed forms. Similarly to the continuous case, one can show

Proposition 4.1. *One has*

$$\partial(C^0) = Z^1,$$

i.e. a form is exact iff it is closed.

Proof. Clearly, if $\omega = \partial f$ and γ is a path from η to η', then

$$\int_\gamma \omega = f(\eta') - f(\eta).$$

In particular, if γ is closed, the integral vanishes and we may conclude that ω is closed.

Conversely, if ω is closed, choose some η_0 for every connected component. Then, for all $\eta \leftrightarrow \eta_0$, set

$$f(\eta) := \int_{\eta_0 \to \eta} \omega$$

for any path $\eta_0 \to \eta$. Since ω is closed, this integral does not depend on the choice of the path. For any transition $\varphi = (\eta, \eta')$, we know that η and η' are in the same connected component. So $f(\eta)$ and $f(\eta')$ are defined w.r.t. the same η_0. Consider some path $\eta_0 \to \eta$, then

$$\partial f(\varphi) = f(\eta') - f(\eta) = \int_{(\eta_0 \to \eta, \varphi)} \omega - \int_{\eta_0 \to \eta} \omega = \omega(\varphi).$$

This proves that ω is exact. □

In particular, to find elements in H^1_{naive}, it suffices to find forms that are not closed. Consider the example of SSEP. For every $x \in X$, define the configuration $\delta_x \in S_*^X$ that places a particle at site x and nothing at all the other sites. Denote by $\delta_e = (\delta_x, \delta_{x+e_1})$ the transition moving the particle along the edge $e = (x, x + e_1)$. We may define a differential form ω by

$$\omega(\delta_e) = 1 = -\omega(\overline{\delta_e})$$

and 0 otherwise. As soon as $X = \mathbb{Z}^d$ has dimension $d \geq 2$, we can go around in circles. More precisely, there is a path $\gamma : (x, x+e_1), (x+e_1, x+e_1+e_2), (x+e_1+e_2, x+e_2), (x+e_2, x)$. However, integrating the above form along this path gives

$$\int_\gamma \omega = \omega(\eta, \eta^e) = 1 \neq 0.$$

In particular, ω is not exact. Now, note that we can define such a differential form for every edge $e \in E$ and these forms define different classes in H^1_{naive}, yielding

$$\dim H^1_{naive} = +\infty$$

as soon as $d \geq 2$. Somehow, the first cohomolgy reflects on the number of holes in the transition structure. And we have shown that every hole in the locale X leads to a hole in the transition structure.

5 The Uniform Cohomology

5.1 Some More Definitions

The problem with the naïve cohomology is that there are too many badly behaved functions. For example, although $(\xi_X)^2$ is, strictly speaking, a conserved quantity, we know from experience that it doesn't matter to us. One of its "bad" properties is the long range correlation it exhibits, as

$$(\xi_X(\eta))^2 = \sum_{x,y\in X} \eta_x \eta_y.$$

The idea, then, is to discard all functions that have long range correlations. To make this rigorous, we first need to define what we want to understand under "correlations". Note that, classically, correlations is of the form $\prod_{x\in\Lambda} \eta_x$ for a finite subset $\Lambda \subset\subset X$. This leads to the definition of Λ-correlations as the set

$$C_\Lambda(S^X) := \{f \in C(S^\Lambda) \ : \ f(\eta) = 0 \text{ as soon as there is } x \in \Lambda \text{ with } \eta_x = *\}$$

of functions that behave like a product if the base state was 0. To isolate short range correlations, we will use the map

$$\iota^\Lambda : C(S_*^X) \longrightarrow C(S_*^X), \quad \iota^\Lambda f(\eta) := f(\eta|_\Lambda \cup \star|_{\Lambda^c}),$$

where $\star \in S_*^X$ denotes the base configuration $(*)_{x\in X}$. If f is a Λ'-correlation with $\Lambda' \not\subseteq \Lambda$, this implies

$$\iota^\Lambda f \equiv 0.$$

Furthermore, note that $\iota^\Lambda \iota^{\Lambda'} = \iota^{\Lambda\cap\Lambda'}$.

Proposition 5.1 (Proposition 3.3). *For any $f \in C(S_*^X)$, there exists a unique decomposition*

$$f = \sum_{\Lambda\subset\subset X} f_\Lambda$$

such that $f_\Lambda \in C_\Lambda(S^X)$.

Hence, to discard long range correlations, we define

the space $C_{unif}(S^X)$ of *uniform functions* as those functions $f \in C(S_*^X)$ for which the correlation decomposition is given by

$$f = \sum_{\substack{\Lambda\subset\subset X \\ \operatorname{diam}\Lambda \le R}} f_\Lambda$$

for some finite R.

the space $C^0_{unif}(S^X)$ of *normalized uniform functions* as the subset of functions $f \in C_{unif}(S^X)$ such that

$$f(\star) = 0.$$

These should be thought of as *mean zero* functions.

To describe the cohomology, we need a suitable analogue for 1-forms. Recall that

$$C^1(S_*^X) \hookrightarrow \prod_{e \in E} C(S_*^X).$$

This leads to the definition of

the space $C^1_{unif}(S^X)$ of *uniform 1-forms* as the union

$$C^1_{unif}(S^X) := \bigcup_{R>0} C^1_R(S^X),$$

where

$$C^1_R(S^X) := C^1(S_*^X) \cap \prod_{e \in E} C(S^{B(e,R)})$$

and $B(e,R)$ is the R-ball around the two vertices connected by e,

the space $Z^1_{unif}(S^X)$ of *closed uniform forms* as

$$Z^1_{unif}(S^X) := Z^1(S_*^X) \cap C^1_{unif}(S^X).$$

5.2 The Uniform Cohomology

The new complex, then, is given by

$$0 \longrightarrow C^0_{unif} \xrightarrow{\partial} Z^1_{unif} \longrightarrow 0,$$

where we use that ∂ does indeed map uniform functions to uniform forms. Furthermore, we have restricted ourselves directly to closed forms as non-closed forms cannot be exact. The fact that we only consider normalized uniform functions stems from the fact that we want to give special attention to the base state for which conserved quantities should vanish. In this sense, we are considering a cohomology of the pointed space $(S^X, \star)$.

As usual, the cohomology is defined as

$$H^0_{unif} := \ker \partial \qquad \text{and} \qquad H^1_{unif} := Z^1_{unif} \big/ \partial(C^0_{unif}).$$

Theorem 5.2 (Theorem 5.8)**.** *Suppose that the interaction ϕ is irreducibly quantified and that the locale is infinite and transferable (i.e. if I cut out a ball, the complement stays connected). Then,*

$$H^0_{unif} \cong \mathrm{Consv}^\phi(S) \qquad \textit{and} \qquad H^1_{unif} = \{0\}.$$

In particular, a uniform form is exact iff it is closed.

The Theorem is equivalent to showing that the short sequence

$$0 \longrightarrow \mathrm{Consv}^\phi(S) \hookrightarrow C^0_{unif} \xrightarrow{\partial} Z^1_{unif} \longrightarrow 0$$

is exact. The injection $\mathrm{Consv}^{\phi}(S) \hookrightarrow C^0_{unif} \cap \ker \partial$ follows from the fact that conserved quantities do not have cross-correlations as

$$\xi_X(\eta) = \sum_{x \in X} \xi(\eta_x)$$

and $\xi_X(\star) = 0$ by definition. It is harder to see why every element in the kernel necessarily is a conserved quantity. This is uses that X is infinite and that the interaction is irreducibly quantified.

The next step is to show that ∂ is indeed surjective. First, notice that every uniform form ω can be written as

$$\omega = \partial f$$

for some $f \in C(S_*^X)$ as we have seen before. It remains to show that f is uniform. This proof is the most technical one of the paper and uses quite some algebra. At this point, we need that the locale is transferable.

5.3 Towards Invariant Forms

At the moment, we have considered all (uniform) functions, which gave us the short exact sequence

$$0 \longrightarrow \mathrm{Consv}^{\phi}(S) \hookrightarrow C^0_{unif} \overset{\partial}{\twoheadrightarrow} Z^1_{unif} \longrightarrow 0$$

However, we are interested in the invariant functions under the group action $G \circlearrowleft X$. Note that this action induces group actions on the configuration space and the different function spaces. For example,

$$(g.\eta)_x := \eta_{g.x} \qquad \text{and} \qquad (g.f)(\eta) := f(g.\eta).$$

For a space V, write $V^G := \{v \in V \ : \ g.v = v \text{ for all } g \in G\}$ for the invariant subspace. As

$$g.\xi_X(\eta) = \sum_{x \in X} \xi(\eta_{g.x}) = \sum_{x \in X} \xi(\eta_x) = \xi_X(\eta),$$

one has

$$\left(\mathrm{Consv}^{\phi}(S)\right)^G = \mathrm{Consv}^{\phi}(S).$$

Proposition 5.3 (Lemma 5.15)**.** *The space* $\left(C^0_{unif}\right)^G$ *of shift-invariant normalized uniform functions consists of functions*

$$F = \sum_{g \in G} g.f$$

for some $f \in C_{loc}(S^X)$*.*

If we write $H^k(G, V)$ for the k-th group cohomology of V (to be defined later), then the short exact sequence from before induces the long exact sequence

$$0 \longrightarrow \mathrm{Consv}^{\phi}(S) \hookrightarrow \left(C^0_{unif}\right)^G \xrightarrow{\partial} \left(Z^1_{unif}\right)^G \xrightarrow{\delta} H^1(G, \mathrm{Consv}^{\phi}(S)) \longrightarrow H^1(G, C^0_{unif}) \longrightarrow \cdots$$

Now, write $\mathcal{C} := \left(Z^1_{unif}\right)^G$ and $\mathcal{E} := \partial\left(\left(C^0_{unif}\right)^G\right)$ for the closed and exact uniform forms respectively. Since the above implies that $\mathcal{E} = \ker \delta$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{E} \hookrightarrow \mathcal{C} \xrightarrow{\delta} H^1(G, \mathrm{Consv}^{\phi}(S)).$$

If we can show that δ is surjective, this would yield the short exact sequence

$$0 \longrightarrow \mathcal{E} \hookrightarrow \mathcal{C} \xrightarrow{\delta} H^1(G, \mathrm{Consv}^{\phi}(S)) \longrightarrow 0$$

and thus

$$H^1(G, \mathrm{Consv}^{\phi}(S)) \cong \mathcal{C}/\mathcal{E}.$$

Under the (technical) assumption that δ has a section, we then obtain the decomposition

$$\mathcal{C} \cong \mathcal{E} \oplus H^1(G, \mathrm{Consv}^{\phi}(S)).$$

Finally, if G is abelian and free of rank d, then

$$H^1(G, \mathrm{Consv}^{\phi}(S)) \cong \bigoplus_{i=1}^{d} \mathrm{Consv}^{\phi}(S)$$

and thus

$$\mathcal{C} \cong \mathcal{E} \oplus \bigoplus_{i=1}^{d} \mathrm{Consv}^{\phi}(S).$$