Discontinuous Galerkin methods

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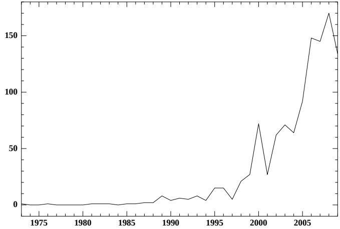
Introduction

- Discontinuous Galerkin (dG) methods can be viewed as
  - finite element methods allowing for **discontinuous discrete functions**
  - finite volume methods with **more than one dof per mesh cell**
- Advantages of such methods include
  - a **high level of flexibility** (choice of basis functions, nonmatching meshes, variable approximation order, local time stepping)
  - the possibility to **enforce locally basic conservation principles**
- The main drawback is higher computational costs w.r.t. stabilized FE or FV **on a fixed mesh**

A brief historical perspective I

- DG methods were introduced almost 40 years ago
  - moderate impact at that time
- Vigorous development over the last decade
  - numerical analysis
  - range of applications
- DG-related publications/year (Source: Mathscinet)



A brief historical perspective II

First-order PDEs

- DG methods first coined for neutronics simulations [Reed & Hill '73]
- **Convergence analysis for steady advection-reaction**
  - $O(h^k)$ L^2-error estimate if polynomials of degree k are used and exact solution is smooth enough [Lesaint & Raviart '74]
  - sharper $O(h^{k+1/2})$ estimate [Johnson & Pitkäranta '86]
- **Time-dependent conservation laws**
  - Runge–Kutta DG (RKDG) with slope limiter [Cockburn & Shu '89-91]: formal accuracy in smooth regions, sharp shock resolution
  - extension to multidimensional systems [Cockburn & Shu '98] and numerous applications

A brief historical perspective III

Elliptic PDEs

- Boundary penalty methods [Nitsche '71]
- **Interior penalty methods** [Babuška '73, Douglas & Dupont '75, Baker '77, Wheeler '78, Arnold '82]
- **Further developments**
  - liftings and application to NS [Bassi, Rebay et al '97]
  - analysis for Poisson problem [Brezzi et al '99]
  - mixed dG approximation [Cockburn & Shu '98]
  - variations on symmetry [Oden, Babuška & Baumann '98, Rivière, Wheeler & Girault '99]
  - weighted averages for heterogeneous diffusion [ESZ '09, DEG '08]
  - locally conservative diffusive flux reconstruction [NEV '07]
- **Unified analysis** for Poisson problem [Arnold, Brezzi, Cockburn & Marini '01]
- **Discrete functional analysis**, convergence with minimal regularity [Di Pietro & AE '10]

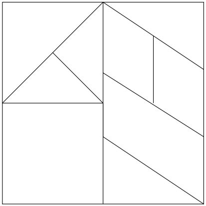
A brief historical perspective IV

Friedrichs systems

- Introduced by Friedrichs in '58
- Linear systems of first-order PDE's endowed by **symmetry and positivity (L^2-coercivity) properties**
- Encompass many important examples of elliptic and hyperbolic PDE's
  - advection–reaction, diffusion(–AR), elasticity, Stokes, Maxwell in diffusive regime, . . .
- **Unified analysis of dG methods based on Friedrichs systems** [AE & Guermond, '06–'08]

Some basic notation I

- Mesh family $\{\mathcal{T}_h\}_h$ of computational domain $\Omega \subset \mathbb{R}^d$
  - shape-regularity in the usual sense
  - the meshes can be nonmatching (hanging nodes); some contact-regularity is then enforced
  - for simplicity, the meshes are affine and cover Ω exactly
  - h: maximum mesh size
- Example of admissible mesh



Some basic notation II

- Broken polynomial space ($k \geq 0$)

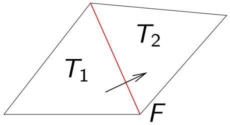
$$\mathbb{P}_d^k(\mathcal{T}_h) = \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_d^k(T)\}$$

  - $\mathbb{P}_d^k$: polynomials in d variables of total degree $\leq k$
  - $\mathbb{P}_d^0(\mathcal{T}_h)$ spanned by piecewise constant functions as in FV
- **No matching condition at interfaces** $\Longrightarrow$ dof's can be taken elementwise
- Other broken polynomial spaces can be considered, and also discrete spaces not spanned by piecewise polynomials
- Broken Sobolev spaces $H^s(\mathcal{T}_h)$ ($s \geq 0$)
- Broken gradient (defined elementwise) $\nabla_h : H^1(\mathcal{T}_h) \to [L^2(\Omega)]^d$

Some basic notation III

- Mesh faces collected into $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$ (split into interfaces and boundary faces)
- Discrete functions can be two-valued at interfaces
- Interface $\mathcal{F}_h^i \ni F = T_1 \cap T_2$, normal n_F from T_1 to T_2
- Mean values and jumps at interfaces

$$\{\varphi\} = \frac{1}{2}(\varphi_1 + \varphi_2) \qquad [\![\varphi]\!] = \varphi_1 - \varphi_2$$



- On the boundary, $\{\varphi\} = [\![\varphi]\!] = \varphi$

Outline

- Advection–reaction (Monday 11)
- The Laplacian (Wednesday 13)
- PDEs with diffusion (Friday 15)
- Incompressible Navier–Stokes (Wednesday 20)
- Most of the material (and much more!) can be found in a forthcoming book:
  Di Pietro & AE, Mathematical aspects of DG methods, Springer Mathématiques et Applications, 2011

Some topics not covered in these lectures

- **Time-dependent problems**
  - abundant numerical techniques/recipes
  - theoretical aspects are much less covered
  - see [Zhang & Shu '04, Burman, AE & Fernandez '10]
- **Implementation issues**
  - see, e.g., [Karniadakis & Spencer '99, Hesthaven & Warburton '08]
- **A posteriori error analysis**
  - Laplacian [Becker, Hansbo & Larson '03, Karakashian & Pascal '03, Ainsworth '07]
  - advection–diffusion-reaction [AE, Stephansen & Vohralík '10]
  - heat equation [AE & Vohralík '10]

Advection-reaction

- Continuous problem
- Abstract nonconforming error analysis
- Centered fluxes
- Upwind fluxes
- The material of this section can be generalized to Friedrichs systems [AE & Guermond '06–'08]

Continuous problem I

- Let $\beta \in [W^{1,\infty}(\Omega)]^d$ and $\mu \in L^\infty(\Omega)$
  - a weaker assumption on β can be $\beta \in [L^\infty(\Omega)]^d$, $\nabla\cdot\beta \in L^\infty(\Omega)$
- Inflow and outflow parts of boundary $\partial\Omega$

$$\partial\Omega^\pm = \{x \in \partial\Omega \mid \pm\beta(x)\cdot\mathrm{n}(x) > 0\}$$

- Let $f \in L^2(\Omega)$; the model problem is

$$\begin{cases} \mu u + \beta\cdot\nabla u = f & \text{in } L^2(\Omega) \\ u = 0 & \text{on } \partial\Omega^- \end{cases}$$

Continuous problem II

- Graph space $W = \{v \in L^2(\Omega) \mid \beta{\cdot}\nabla v \in L^2(\Omega)\}$
- Hilbert space with the norm $\|v\|_W^2 = \|v\|_{L^2(\Omega)}^2 + \|\beta{\cdot}\nabla v\|_{L^2(\Omega)}^2$
- Assume that $\partial\Omega^-$ and $\partial\Omega^+$ are well-separated
- Then, there is a continuous trace operator from W onto

$$L^2(|\beta{\cdot}\mathrm{n}|;\partial\Omega) = \{v \text{ is measurable on } \partial\Omega \mid \int_{\partial\Omega} |\beta{\cdot}\mathrm{n}|v^2 < +\infty\}$$

- The separation assumption cannot be circumvented to work with traces in $L^2(|\beta{\cdot}\mathrm{n}|;\partial\Omega)$

Continuous problem III

- Define on $W \times W$ the bilinear form

$$a(v,w) = \int_\Omega [\mu v + (\beta \cdot \nabla v)] w + \int_{\partial\Omega} (\beta \cdot \mathrm{n})^\ominus v w$$

where for $x \in \mathbb{R}$, $x^\oplus = \frac{1}{2}(|x| + x)$ and $x^\ominus = \frac{1}{2}(|x| - x)$

- Assume that

$$\exists \mu_0 > 0, \qquad \mu - \tfrac{1}{2}\nabla \cdot \beta \geq \mu_0 \quad \text{in } \Omega$$

- This implies L^2-coercivity of a on W since

$$\begin{aligned} a(v,v) &= \int_\Omega \left(\mu - \tfrac{1}{2}\nabla \cdot \beta\right) v^2 + \tfrac{1}{2}\int_{\partial\Omega} (\beta \cdot \mathrm{n}) v^2 + \int_{\partial\Omega} (\beta \cdot \mathrm{n})^\ominus v^2 \\ &\geq \mu_0 \|v\|^2_{L^2(\Omega)} + \tfrac{1}{2}\int_{\partial\Omega} |\beta \cdot \mathrm{n}| v^2. \end{aligned}$$

Continuous problem IV

- **Weak formulation** Seek $u \in W$ s.t.

$$a(u, w) = \int_\Omega fw \qquad \forall w \in W$$

  - BCs are weakly enforced
  - same trial and test spaces

- Theorem. This problem is well-posed
  - L^2-coercivity implies uniqueness
  - existence by inf-sup argument (using L^2-coercivity of a)

Nonconforming error analysis I

- Finite-dimensional space W_h
- Discrete bilinear form a_h defined on $W_h \times W_h$
- **Discrete problem** Seek $u_h \in W_h$ s.t.

$$a_h(u_h, w_h) = \int_\Omega f w_h \qquad \forall w_h \in W_h$$

- **Nonconforming setting** $W_h \not\subset W$

Nonconforming error analysis II

- We want to assert strong consistency by plugging the exact solution u into a_h
- This may not be possible in general for $u \in W$; some additional smoothness is required, say

$$u \in W_\dagger \text{ with } W_\dagger \subset W$$

- We assume that a_h can be extended to $W_\dagger \times W_h$
- Approximation error $(u - u_h)$ belongs to $W_{\dagger h} \stackrel{\text{def}}{=} W_\dagger + W_h$
- **We work with two norms:** $|||\cdot|||$ and $|||\cdot|||_*$ both defined on $W_{\dagger h}$
  - the approximation error will be estimated in the $|||\cdot|||$-norm
  - the $|||\cdot|||_*$-norm controls the $|||\cdot|||$-norm

Nonconforming error analysis III

- **Consistency** (dG methods are consistent methods!)

$$\forall w_h \in W_h, \qquad a_h(u, w_h) = \int_\Omega f w_h$$

- **Stability**

$$\forall v_h \in W_h, \qquad |||v_h||| \lesssim \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{|||w_h|||}$$

  - ensures well-posedness of discrete problem
  - a sufficient condition is discrete coercivity

- **Boundedness**

$$\forall v \in W_{\dagger h}, \ \forall w_h \in W_h, \qquad a_h(v, w_h) \lesssim |||v|||_* |||w_h|||$$

Nonconforming error analysis IV

- Error estimate

$$|||u - u_h||| \lesssim \inf_{y_h \in W_h} |||u - y_h|||_*$$

- Proof. Let $y_h \in W_h$.
  - stability, consistency, and boundedness imply

$$\begin{aligned} |||u_h - y_h||| &\lesssim \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(u_h - y_h, w_h)}{|||w_h|||} \\ &= \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(u - y_h, w_h)}{|||w_h|||} \\ &\lesssim |||u - y_h|||_* \end{aligned}$$

  - conclude using the triangle inequality

Nonconforming error analysis V

- Recall that

$$|||u - u_h||| \lesssim \inf_{y_h \in W_h} |||u - y_h|||_*$$

- The estimate is not optimal since different norms are used
- The estimate is quasi-optimal if the upper bound has the same CV order as the optimal bound $\inf_{y_h \in W_h} |||u - y_h|||$; otherwise, the estimate is suboptimal

Centered fluxes I

- DG approximation in $W_h = \mathbb{P}_d^k(\mathcal{T}_h)$
- **Discrete problem** Seek $u_h \in W_h$ s.t.

$$a_h(u_h, w_h) = \int_\Omega f w_h \qquad \forall w_h \in W_h$$

- Guidelines to design the discrete bilinear form a_h
  - consistency
  - discrete L^2-coercivity on W_h
- Assumptions on the exact solution u
  - u has possibly two-valued traces on all mesh faces
  - $\beta \cdot n_F [\![u]\!] = 0$ on all $F \in \mathcal{F}_h^i$ (mesh fitted to possible singularities)

Centered fluxes II

- Step 1: Localize gradient

$$a_h(v_h, w_h) = \int_\Omega [\mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h] + \sum_{F\in\mathcal{F}_h^b} \int_F (\beta\cdot\mathrm{n})^\ominus v_h w_h$$

- a_h is not L^2-coercive on W_h

$$\begin{aligned} a_h(v_h, v_h) &= \int_\Omega [\mu v_h^2 + (\beta\cdot\nabla_h v_h) v_h] + \sum_{F\in\mathcal{F}_h^b} \int_F (\beta\cdot\mathrm{n})^\ominus v_h^2 \\ &= \int_\Omega \left(\mu - \tfrac{1}{2}\nabla\cdot\beta\right) v_h^2 + \sum_{T\in\mathcal{T}_h} \int_{\partial T} \tfrac{1}{2}(\beta\cdot\mathrm{n}_T) v_h^2 + \sum_{F\in\mathcal{F}_h^b} \int_F (\beta\cdot\mathrm{n})^\ominus v_h^2 \\ &= \int_\Omega \left(\mu - \tfrac{1}{2}\nabla\cdot\beta\right) v_h^2 + \sum_{F\in\mathcal{F}_h^i} \int_F \tfrac{1}{2}(\beta\cdot\mathrm{n}_F)[\![v_h^2]\!] + \sum_{F\in\mathcal{F}_h^b} \int_F \tfrac{1}{2}|\beta\cdot\mathrm{n}| v_h^2 \end{aligned}$$

and the second term has no sign a priori

Centered fluxes III

- Step 2: Recover discrete L^2-coercivity in a consistent way by setting

$$a_h^{\mathrm{cf}}(v_h, w_h) \stackrel{\mathrm{def}}{=} \int_\Omega [\mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h] + \sum_{F\in\mathcal{F}_h^b} \int_F (\beta\cdot\mathrm{n})^\ominus v_h w_h$$

$$- \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F)[\![v_h]\!]\{w_h\}$$

since $[\![v_h^2]\!] = 2[\![v_h]\!]\{v_h\}$. This yields

$$a_h^{\mathrm{cf}}(v_h, v_h) \geq \mu_0 \|v_h\|_{L^2(\Omega)}^2 + \sum_{F\in\mathcal{F}_h^b} \int_F \frac{1}{2}|\beta\cdot\mathrm{n}| v_h^2$$

Centered fluxes IV

- For simplicity, assume μ and β of order unity
- Discrete coercivity: $|||v_h|||^2 \lesssim a_h^{\mathrm{cf}}(v_h, v_h)$ with

$$|||v_h|||^2 \stackrel{\mathrm{def}}{=} \|v_h\|_{L^2(\Omega)}^2 + \sum_{F \in \mathcal{F}_h^b} \int_F \tfrac{1}{2} |\beta{\cdot}\mathrm{n}| v_h^2$$

- Boundedness: $a_h^{\mathrm{cf}}(v, w_h) \lesssim |||v|||_* |||w_h|||$ with

$$|||v|||_*^2 = |||v|||^2 + \sum_{T \in \mathcal{T}_h} \|\beta{\cdot}\nabla v\|_\Omega^2 + \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \int_F |\beta{\cdot}\mathrm{n}_F| [\![v]\!]^2$$

- Error estimate

$$|||u - u_h||| \lesssim \inf_{y_h \in W_h} |||u - y_h|||_*$$

Centered fluxes V

- Local polynomial approximation: $\forall z \in H^{k+1}(\mathcal{T}_h)$, $\forall T \in \mathcal{T}_h$,

$$\left.\begin{array}{l} \|z - \pi_h z\|_T \\ h_T^{1/2}\|z - \pi_h z\|_{\partial T} \\ h_T\|\nabla(z - \pi_h z)\|_T \end{array}\right\} \lesssim h_T^{k+1}\|z\|_{H^{k+1}(T)}$$

where π_h is the L^2-orthogonal projection onto W_h

- Convergence rate $|||u - u_h||| \lesssim h^k$ if $u \in H^{k+1}(\mathcal{T}_h)$
- Convergence for $k \geq 1$ and with suboptimal rate

Centered fluxes VI

- Let $T \in \mathcal{T}_h$, let $\xi \in \mathbb{P}_d^k(T)$

$$\int_T [(\mu - \nabla{\cdot}\beta)u_h\xi - u_h(\beta{\cdot}\nabla\xi)] + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h)\xi = \int_T f\xi$$

with $\epsilon_{T,F} = \mathrm{n}_T{\cdot}\mathrm{n}_F$ and the (consistent) numerical fluxes

$$\phi_F(u_h) = \begin{cases} \beta{\cdot}\mathrm{n}_F\{u_h\} & (F \in \mathcal{F}_h^i) \\ (\beta{\cdot}\mathrm{n})^{\oplus} u_h & (F \in \mathcal{F}_h^b) \end{cases}$$

- $\xi \equiv 1$ yields the usual FV formulation

$$\int_T (\mu - \nabla{\cdot}\beta)u_h + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) = \int_T f$$

Upwind fluxes I

- Strengthen discrete stability by penalizing interface jumps in a least-squares sense [Brezzi, Marini & Süli '04]

$$a_h(v_h, w_h) \stackrel{\text{def}}{=} a_h^{\text{cf}}(v_h, w_h) + s_h(v_h, w_h)$$

with (consistent) stabilization bilinear form

$$s_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h^i} \int_F \eta \frac{1}{2} |\beta \cdot \mathrm{n}_F| [\![v_h]\!] [\![w_h]\!]$$

and positive user-dependent parameter η

Upwind fluxes II

- a_h is consistent
- a_h is coercive on W_h: $|||v_h|||_\flat^2 \lesssim a_h(v_h, v_h)$ with

$$|||v|||_\flat^2 \stackrel{\text{def}}{=} \|v\|_{L^2(\Omega)}^2 + \sum_{F\in\mathcal{F}_h^b} \int_F \tfrac{1}{2}|\beta\cdot\mathrm{n}|v^2 + \sum_{F\in\mathcal{F}_h^i} \int_F \eta\tfrac{1}{2}|\beta\cdot\mathrm{n}_F|[\![v]\!]^2$$

- Variant: boundedness on orthogonal subscales (OSS)

$$a_h(v - \pi_h v, w_h) \lesssim |||v - \pi_h v|||_* |||w_h|||_\flat$$

where

$$|||y|||_*^2 = |||y|||_\flat^2 + \sum_{T\in\mathcal{T}_h} \|y\|_{L^2(\partial T)}^2$$

- Rk. With upwinding, the $|||\cdot|||_*$-norm is much "closer" to the $|||\cdot|||_\flat$-norm

Upwind fluxes III

- Key technical point

$$
\begin{aligned}
\int_\Omega (v-\pi_h v)\beta\cdot\nabla_h w_h &= \sum_{T\in\mathcal{T}_h}\int_T (v-\pi_h v)(\beta-\langle\beta\rangle_T)\cdot\nabla w_h \\
&\lesssim \sum_{T\in\mathcal{T}_h} \|v-\pi_h v\|_{L^2(T)} h_T \|\nabla w_h\|_{[L^2(T)]^d} \\
&\lesssim \sum_{T\in\mathcal{T}_h} \|v-\pi_h v\|_{L^2(T)} \|w_h\|_{L^2(T)} \\
&\leq |||v-\pi_h v|||_\flat \, |||w_h|||_\flat
\end{aligned}
$$

- Convergence rate $|||u-u_h|||_\flat \lesssim |||u-\pi_h u|||_* \lesssim h^{k+1/2}$ if $u\in H^{k+1}(\mathcal{T}_h)$
- Convergence for $k\geq 0$ and with quasi-optimal rate

Upwind fluxes IV

Error estimate in the advective derivative

- Discrete stability with stronger norm

$$|||v|||^2 \stackrel{\text{def}}{=} |||v|||_\flat^2 + \sum_{T\in\mathcal{T}_h} h_T \|\beta\cdot\nabla v\|_{L^2(T)}^2$$

- Discrete inf-sup condition [Johnson & Pitkäranta '86]

$$|||v_h||| \lesssim \sup_{w_h\in W_h\setminus\{0\}} \frac{a_h(v_h,w_h)}{|||w_h|||}$$

  - $|||v_h|||_\flat$ is controlled by coercivity
  - advective derivative is controlled by testing with $w_h|_T = h_T\langle\beta\rangle_T\cdot\nabla v_h$

Upwind fluxes V

- (Full) boundedness: $a_h(v, w_h) \lesssim |||v|||_* |||w_h|||$ with

$$|||v|||_*^2 \stackrel{\text{def}}{=} |||v|||^2 + \sum_{T\in\mathcal{T}_h} [\|v\|_{L^2(\partial T)}^2 + h_T^{-1}\|v\|_{L^2(T)}^2]$$

- Convergence rate $|||u - u_h||| \lesssim h^{k+1/2}$ if $u \in H^{k+1}(\mathcal{T}_h)$
- Optimal estimate for advective derivative

Upwind fluxes VI

- New numerical fluxes

$$\phi_F(u_h) = \begin{cases} \beta\cdot\mathrm{n}_F\{u_h\} + \frac{1}{2}\eta|\beta\cdot\mathrm{n}_F|[\![u_h]\!] & (F \in \mathcal{F}_h^i) \\ (\beta\cdot\mathrm{n})^{\oplus} u_h & (F \in \mathcal{F}_h^b) \end{cases}$$

- Particular choice $\eta = 1$ yields the upwind flux

Salient points of this lecture

- Centered fluxes correspond to a basic design ensuring consistency and discrete coercivity
- Upwinding can be interpreted as tightening discrete stability by penalizing jumps
- Error estimates are similar to other stabilized methods with continuous FEM
  - subgrid viscosity [Guermond '99]
  - continuous interior penalty of gradient jumps [Burman & Hansbo '04]
  - local projection [Braack, Burman, John & Lube '07, Knobloch & Tobiska '09]
  - ...

The Laplacian

- Model problem
- Symmetric Interior Penalty (SIP)
- Liftings and discrete gradients
- Diffusive flux reconstruction
- Variations on symmetry and penalty

Model problem I

- Let $f \in L^2(\Omega)$; seek $u : \Omega \to \mathbb{R}$ s.t. $-\triangle u = f$ in Ω and $u|_{\partial\Omega} = 0$
- Weak formulation: $u \in V \stackrel{\text{def}}{=} H^1_0(\Omega)$ s.t.

$$a(u, v) \stackrel{\text{def}}{=} \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \qquad \forall v \in V$$

- u is termed the **potential** and $\sigma = -\nabla u$ the **diffusive flux**
- Since $\nabla \cdot \sigma = f$, the diffusive flux is in

$$H(\mathrm{div}; \Omega) \stackrel{\text{def}}{=} \{\tau \in [L^2(\Omega)]^d \mid \nabla \cdot \tau \in L^2(\Omega)\}$$

Physically, the normal component of σ is continuous across any interface

Model problem II

- Since $u \in H^1_0(\Omega)$, u admits a trace on each face $F \in \mathcal{F}_h$ and

$$[\![u]\!] = 0 \qquad \forall F \in \mathcal{F}_h$$

- We want to consider the normal gradient of u on each face
- $\nabla u \in H(\mathrm{div};\Omega)$ only implies $\nabla u \cdot \mathrm{n}|_{\partial T} \in H^{-1/2}(\partial T)$ for all $T \in \mathcal{T}_h$, which **cannot be simply localized to mesh faces**
- A minimal assumption is $\nabla u \cdot \mathrm{n}|_{\partial T} \in L^1(\partial T)$ for all $T \in \mathcal{T}_h$
- A simple sufficient condition is $u \in V_\dagger \stackrel{\mathrm{def}}{=} H^2(\mathcal{T}_h)$
  - more generally, $u \in W^{2,p}(\mathcal{T}_h)$ with $p > 1$ if $d = 2$ and $p > \frac{6}{5}$ if $d = 3$

Model problem III

- Important property $[\![\nabla u]\!]\cdot \mathrm{n}_F = 0$ for all $F \in \mathcal{F}_h^i$
- **Proof**
  - Let $\varphi \in C_0^\infty(\Omega)$. For all $T \in \mathcal{T}_h$, since $u \in V_\dagger$,

$$\int_T (-\triangle u)\varphi = \int_T \nabla u \cdot \nabla \varphi - \int_{\partial T} (\nabla u \cdot \mathrm{n}_T)\varphi$$

  - Summing over $T \in \mathcal{T}_h$ and using the weak formulation yields

$$\sum_{F \in \mathcal{F}_h^i} \int_F ([\![\nabla u]\!]\cdot \mathrm{n}_F)\varphi = 0$$

  - Choose the support of φ intersecting a single interface and use a density argument

SIP I

- Discrete space $V_h \stackrel{\text{def}}{=} \mathbb{P}_d^k(\mathcal{T}_h)$ with $k \geq 1$
  - see [Di Pietro '10] for cell-centered Galerkin methods with $k = 0$
- Discrete bilinear form [Arnold '82]

$$a_h(v_h, w_h) \stackrel{\text{def}}{=} \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\nabla_h v_h\} \cdot \mathrm{n}_F [\![w_h]\!] - \sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \{\nabla_h w_h\} \cdot \mathrm{n}_F + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!] [\![w_h]\!]$$

for **user-dependent positive parameter** η

- a_h can be extended to $V_{\dagger h} \times V_{\dagger h}$

SIP II

- Elementwise integration by parts yields

$$\int_\Omega \nabla_h v \cdot \nabla_h w = -\sum_{T\in\mathcal{T}_h}\int_T (\triangle v)w + \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla v\cdot \mathrm{n}_T)w$$

$$= -\sum_{T\in\mathcal{T}_h}\int_T (\triangle v)w + \sum_{F\in\mathcal{F}_h}\int_F \{\nabla_h v\}\cdot \mathrm{n}_F [\![w]\!]$$

$$+\sum_{F\in\mathcal{F}_h^i}\int_F [\![\nabla_h v]\!]\cdot \mathrm{n}_F \{w\}$$

- This yields

$$a_h(v,w) = -\sum_{T\in\mathcal{T}_h}\int_T (\triangle v)w + \sum_{F\in\mathcal{F}_h^i}\int_F [\![\nabla_h v]\!]\cdot \mathrm{n}_F \{w\}$$

$$-\sum_{F\in\mathcal{F}_h}\int_F [\![v]\!]\{\nabla_h w\}\cdot \mathrm{n}_F + \sum_{F\in\mathcal{F}_h}\frac{\eta}{h_F}\int_F [\![v]\!][\![w]\!]$$

SIP III

- **Discrete problem** Seek $u_h \in V_h$ s.t.

$$a_h(u_h, w_h) = \int_\Omega f w_h \qquad \forall w_h \in V_h$$

- The discrete problem "weakly enforces"
  - $-\triangle u_h = f$ for all $T \in \mathcal{T}_h$
  - $[\![\nabla_h u_h]\!]{\cdot}\mathrm{n}_F = 0$ for all $F \in \mathcal{F}_h^i$
  - $[\![u_h]\!] = 0$ for all $F \in \mathcal{F}_h$

- **The SIP bilinear form is consistent**

SIP IV

- Basic terminology

$$a_h(v,w) = \int_\Omega \nabla_h v \cdot \nabla w_h - \underbrace{\sum_{F \in \mathcal{F}_h} \int_F \{\nabla_h v\} \cdot \mathrm{n}_F [\![w]\!]}_{\text{consistency term}}$$

$$- \underbrace{\sum_{F \in \mathcal{F}_h} \int_F [\![v]\!] \{\nabla_h w\} \cdot \mathrm{n}_F}_{\text{symmetry term}} + \underbrace{\sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v]\!] [\![w]\!]}_{\text{penalty term}}$$

SIP V

- Discrete stability norm: For all $v \in H^1(\mathcal{T}_h)$

$$|||v|||^2 \stackrel{\text{def}}{=} \|\nabla_h v\|^2_{[L^2(\Omega)]^d} + |v|^2_{\mathrm{J}}$$

with the jump seminorm

$$|v|^2_{\mathrm{J}} = \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[\![v]\!]\|^2_{L^2(F)}$$

- $|||\cdot|||$ is a norm on $H^1(\mathcal{T}_h)$ (direct verification)
- The following Poincaré inequality holds true [Brenner '03]

$$\exists \sigma_2, \qquad \forall v \in H^1(\mathcal{T}_h), \qquad \|v\|_{L^2(\Omega)} \leq \sigma_2 |||v|||$$

SIP VI

- Bound on consistency term For all $(v, w) \in V_{\dagger h} \times V_{\dagger h}$

$$\left| \sum_{F \in \mathcal{F}_h} \int_F \{\nabla_h v\} \cdot \mathrm{n}_F [\![w]\!] \right| \le \left(\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \| \nabla v|_T \cdot \mathrm{n}_F \|^2_{L^2(F)} \right)^{1/2} |w|_{\mathrm{J}}$$

- Discrete trace inequality $\forall T \in \mathcal{T}_h$, $\forall F \in \mathcal{F}_T$

$$h_F^{1/2} \|v_h\|_{L^2(F)} \le C_{\mathrm{tr}} \|v_h\|_{L^2(T)} \qquad \forall v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$$

C_{tr} depends on d, k, and mesh-regularity

- Hence, for all $(v_h, w) \in V_h \times V_{\dagger h}$

$$\left| \sum_{F \in \mathcal{F}_h} \int_F \{\nabla_h v_h\} \cdot \mathrm{n}_F [\![w]\!] \right| \le C_{\mathrm{tr}} N_\partial^{1/2} \|\nabla_h v_h\|_{[L^2(\Omega)]^d} |w|_{\mathrm{J}}$$

SIP VII

- Discrete coercivity: Assume $\eta > C_{\mathrm{tr}}^2 N_\partial$ Then,

$$a_h(v_h, v_h) = \|\nabla_h v_h\|^2_{[L^2(\Omega)]^d} - 2 \sum_{F\in\mathcal{F}_h} \int_F \{\nabla_h v_h\}\cdot \mathrm{n}_F [\![v_h]\!] + \eta|v_h|^2_{\mathrm{J}}$$
$$\geq \|\nabla_h v_h\|^2_{[L^2(\Omega)]^d} - 2C_{\mathrm{tr}} N_\partial^{1/2} \|\nabla_h v_h\|_{[L^2(\Omega)]^d} |v_h|_{\mathrm{J}} + \eta|v_h|^2_{\mathrm{J}}$$
$$\geq C_{\mathrm{stb}} |\!|\!| v_h |\!|\!|^2$$

with $C_{\mathrm{stb}} = \frac{\eta - C_{\mathrm{tr}}^2 N_\partial}{\eta + C_{\mathrm{tr}}^2 N_\partial} \min(1, C_{\mathrm{tr}}^2 N_\partial)$

- **Corollary: The discrete problem is well-posed**

SIP VIII

- The minimal value for η is difficult to determine precisely because of the presence of C_{tr}
- This can be circumvented by **modifying the penalty strategy**
- Discrete inf-sup stability (instead of coercivity) holds **without penalty**
  - in 1D, for $k \geq 2$ [Burman, AE, Mozolevski, Stamm '07]
  - in 2D and 3D, for piecewise affine polynomials supplemented by element bubbles [Burman & Stamm '08]

SIP IX

- Boundedness $\forall(v, w_h) \in V_{\dagger h} \times V_h$, $a_h(v, w_h) \lesssim |||v|||_* |||w_h|||$ with

$$|||v|||_*^2 \stackrel{\text{def}}{=} |||v|||^2 + \sum_{T \in \mathcal{T}_h} h_T \|\nabla v|_T \cdot \mathrm{n}_T\|_{L^2(\partial T)}^2$$

- Error estimate $|||u - u_h||| \lesssim \inf_{y_h \in V_h} |||u - y_h|||_*$
- Convergence rate $|||u - u_h||| \lesssim h^k$ if $u \in H^{k+1}(\mathcal{T}_h)$
  - optimal for the gradient
  - optimal for the jumps and boundary values

SIP X

Error analysis using only the $|||\cdot|||_*$-norm

- The $|||\cdot|||$- and $|||\cdot|||_*$-norms are uniformly equivalent on V_h
- The SIP bilinear form is coercive and bounded using only $|||\cdot|||_*$
- Error estimate $|||u - u_h|||_* \lesssim \inf_{y_h \in V_h} |||u - y_h|||_*$
- Convergence rate $|||u - u_h|||_* \lesssim h^k$ if $u \in H^{k+1}(\mathcal{T}_h)$

SIP XI

L^2-norm error estimate

- Elliptic regularity There is $C_{\rm ell}$ s.t for all $\psi \in L^2(\Omega)$, the unique function $\zeta \in H_0^1(\Omega)$ s.t. $-\triangle\zeta = \psi$ satisfies $\|\zeta\|_{H^2(\Omega)} \le C_{\rm ell}\|\psi\|_{L^2(\Omega)}$
  - Ω convex $\Longrightarrow$ elliptic regularity
- Assume elliptic regularity. Then,

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h|\!|\!|u - u_h|\!|\!|_*$$

so that $\|u - u_h\|_{L^2(\Omega)} \lesssim h^{k+1}$ if $u \in H^{k+1}(\mathcal{T}_h)$

SIP XII

- Let $\zeta \in H_0^1(\Omega) \cap H^2(\Omega)$ be s.t. $-\triangle\zeta = u - u_h$; hence,

$$\|u - u_h\|^2_{L^2(\Omega)} = \int_\Omega (-\triangle\zeta)(u - u_h) = a_h(\zeta, u - u_h)$$

- Exploiting the symmetry of a_h

$$\|u - u_h\|^2_{L^2(\Omega)} = a_h(u - u_h, \zeta)$$

- Owing to consistency, boundedness, affine polynomial approximation, and elliptic regularity

$$\begin{aligned}\|u - u_h\|^2_{L^2(\Omega)} &= a_h(u - u_h, \zeta - \pi_h^1\zeta) \\ &\lesssim \|u - u_h\|_* \|\zeta - \pi_h^1\zeta\|_* \\ &\lesssim \|u - u_h\|_* h \|\zeta\|_{H^2(\mathcal{T}_h)} \\ &\lesssim \|u - u_h\|_* h \|u - u_h\|_{L^2(\Omega)}\end{aligned}$$

where π_h^1 is the L^2-orthogonal projection onto $\mathbb{P}_d^1(\mathcal{T}_h) \subset V_h$

Liftings and discrete gradients I

- Let $l \geq 0$
- For any $F \in \mathcal{F}_h$, $\mathrm{r}_F^l : L^2(F) \longrightarrow [\mathbb{P}_d^l(\mathcal{T}_h)]^d$ is s.t.

$$\int_\Omega \mathrm{r}_F^l(\varphi)\cdot\tau_h = \int_F \{\tau_h\}\cdot\mathrm{n}_F\varphi \qquad \forall\tau_h \in [\mathbb{P}_d^l(\mathcal{T}_h)]^d$$

  - r_F^l **is vector-valued, colinear to** n_F
  - **support of** r_F^l **reduces to the one or two mesh elements sharing** F

- Liftings were introduced by Bassi, Rebay et al ('97) in the context of incompressible flows
- They were analyzed by Brezzi et al ('00) for the Poisson problem

Liftings and discrete gradients II

- Main stability result For all $\varphi \in L^2(F)$

$$\| \mathrm{r}_F^l(\varphi)\|_{[L^2(\Omega)]^d} \leq C_{\mathrm{tr}} h_F^{-1/2} \|\varphi\|_{L^2(F)}$$

- Proof: Use Cauchy–Schwarz and discrete trace inequality

$$\begin{aligned}
\| \mathrm{r}_F^l(\varphi)\|_{[L^2(\Omega)]^d}^2 &= \int_\Omega \mathrm{r}_F^l(\varphi)\cdot \mathrm{r}_F^l(\varphi) = \int_F \{\mathrm{r}_F^l(\varphi)\}\cdot \mathrm{n}_F \varphi \\
&\leq \left(\frac{1}{h_F}\int_F |\varphi|^2\right)^{1/2} \times \left(h_F \int_F |\{\mathrm{r}_F^l(\varphi)\}|^2\right)^{1/2} \\
&\leq h_F^{-1/2} \|\varphi\|_{L^2(F)} \times C_{\mathrm{tr}} \| \mathrm{r}_F^l(\varphi)\|_{[L^2(\Omega)]^d}
\end{aligned}$$

Liftings and discrete gradients III

Global lifting

- For all $v \in H^1(\mathcal{T}_h)$

$$\mathrm{R}_h^l([\![v]\!]) \stackrel{\mathrm{def}}{=} \sum_{F \in \mathcal{F}_h} \mathrm{r}_F^l([\![v]\!]) \in [\mathbb{P}_d^l(\mathcal{T}_h)]^d$$

- Main stability result

$$\| \mathrm{R}_h^l([\![v]\!]) \|_{[L^2(\Omega)]^d} \leq C_{\mathrm{tr}} N_\partial^{1/2} |v|_{\mathrm{J}}$$

Liftings and discrete gradients IV

- Proof

$$
\begin{aligned}
\| \mathrm{R}_h^l([\![v]\!]) \|_{[L^2(\Omega)]^d}^2 &= \sum_{T\in\mathcal{T}_h} \int_T \left| \sum_{F\in\mathcal{F}_T} \mathrm{r}_F^l([\![v]\!]) \right|^2 \\
&\leq \sum_{T\in\mathcal{T}_h} \mathrm{card}(\mathcal{F}_T) \sum_{F\in\mathcal{F}_T} \int_T |\, \mathrm{r}_F^l([\![v_h]\!])|^2 \\
&\leq \max_{T\in\mathcal{T}_h} \mathrm{card}(\mathcal{F}_T) \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_T |\, \mathrm{r}_F^l([\![v_h]\!])|^2 \\
&= N_\partial \sum_{F\in\mathcal{F}_h} \| \mathrm{r}_F^l([\![v]\!]) \|_{[L^2(\Omega)]^d}^2
\end{aligned}
$$

and recall $\| \mathrm{r}_F^l([\![v]\!]) \|_{[L^2(\Omega)]^d} \leq C_{\mathrm{tr}} h_F^{-1/2} \| [\![v]\!] \|_{L^2(F)}$

Liftings and discrete gradients V

Discrete gradient

- Let $l \geq 0$
- $G_h^l : H^1(\mathcal{T}_h) \longrightarrow [L^2(\Omega)]^d$ is s.t.

$$G_h^l(v) \stackrel{\text{def}}{=} \nabla_h v - \mathrm{R}_h^l([\![v]\!])$$

- Main stability result: For all $v \in H^1(\mathcal{T}_h)$

$$\|G_h^l(v)\|_{[L^2(\Omega)]^d} \leq (1 + C_{\mathrm{tr}}^2 N_\partial)^{1/2} |\!|\!| v |\!|\!|$$

- Discrete gradients enjoy important properties (discrete Sobolev embedding, compactness): see 4th lecture and [Di Pietro & AE '10]

Liftings and discrete gradients VI

Reformulation of the SIP bilinear form

- Recall

$$a_h(v_h, w_h) = \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\nabla_h v_h\} \cdot \mathrm{n}_F [\![w_h]\!]$$
$$- \sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \{\nabla_h w_h\} \cdot \mathrm{n}_F + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$

- Observe that for $l \in \{k-1, k\}$

$$\sum_{F \in \mathcal{F}_h} \int_F \{\nabla_h v_h\} \cdot \mathrm{n}_F [\![w_h]\!] = \int_\Omega \nabla_h v_h \cdot \mathrm{R}_h^l([\![w_h]\!])$$

Liftings and discrete gradients VII

- Hence,

$$a_h(v_h, w_h) = \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \int_\Omega \nabla_h v_h \cdot \mathrm{R}_h^l([\![w_h]\!]) - \int_\Omega \mathrm{R}_h^l([\![v_h]\!]) \cdot \nabla_h w_h + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$

that is

$$a_h(v_h, w_h) = \int_\Omega G_h^l(v_h) \cdot G_h^l(w_h) + \hat{s}_h(v_h, w_h)$$

with

$$\hat{s}_h(v_h, w_h) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!] - \int_\Omega \mathrm{R}_h^l([\![v_h]\!]) \cdot \mathrm{R}_h^l([\![w_h]\!])$$

Liftings and discrete gradients VIII

- Discrete coercivity: For all $v_h \in V_h$

$$a_h(v_h, v_h) \geq \|G_h^l(v_h)\|^2_{[L^2(\Omega)]^d} + (\eta - C_{\mathrm{tr}}^2 N_\partial)|v_h|^2_{\mathrm{J}}$$

- The reformulated SIP bilinear form is equivalent to the original one only at the discrete level
  - at the continuous level, a difference appears because liftings are discrete objects
  - the reformulated bilinear form is only weakly consistent
- The importance of discrete gradients has been recognized recently in the context of nonlinear problems
  - nonlinear elasticity [Lew et al. '04], nonlinear variational problems [Buffa & Ortner '09, Burman & AE '08]

Liftings and discrete gradients IX

Numerical fluxes

- Let $T \in \mathcal{T}_h$ and let $\xi \in \mathbb{P}_d^k(T)$
- For the exact solution ($\epsilon_{T,F} = \mathrm{n}_T \cdot \mathrm{n}_F$)

$$\int_T \nabla u \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \Phi_F(u) \xi = \int_T f \xi$$

with the exact flux $\Phi_F(u) = -\nabla u \cdot \mathrm{n}_F$

- For the discrete solution

$$\int_T G_h^l(u_h) \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) \xi = \int_T f \xi$$

with the numerical flux $\phi_F(u_h) = -\{\nabla_h u_h\} \cdot \mathrm{n}_F + \frac{\eta}{h_F} [\![u_h]\!]$

Diffusive flux reconstruction I

- Recall that the exact diffusive flux is $\sigma = -\nabla u$
  - $\nabla\cdot\sigma = f$
  - $\sigma \in H(\mathrm{div};\Omega)$
- We want to postprocess u_h so as to build a discrete vector-valued field σ_h s.t.
  - $\sigma_h \in H(\mathrm{div};\Omega) \iff$ **the normal component of σ_h is continuous across any interface**
  - σ_h is an accurate approximation of $\sigma = -\nabla u$
  - $\nabla\cdot\sigma_h$ is an accurate approximation of $\nabla\cdot\sigma = f$
- Postprocessing should have a negligible cost

Diffusive flux reconstruction II

- Diffusive flux reconstruction has been recently introduced in the context of a posteriori error estimates
  - see [Kim '07, AE, Nicaise & Vohralík '07]
- It is also important in groundwater flow problems to reconstruct the Darcy velocity
- For simplicity, we focus on **matching simplicial meshes**
  - general meshes can be handled by postprocessing the diffusive flux in a matching simplicial submesh and solving local Neumann problems [Ern & Vohralík '09]

Diffusive flux reconstruction III

- Direct diffusive flux reconstruction [Bastian & Rivière '03]
  - local reconstruction using neighboring values of $-\nabla_h u_h$
  - projection onto Brezzi–Douglas–Marini FE space
  - L^2-norm estimate, no estimate on the divergence
- Scheme-oriented diffusive flux reconstruction [Kim '07, AE, Nicaise & Vohralík '07]
  - local reconstruction using dG scheme explicitly
  - projection onto Raviart–Thomas–Nédélec FE space
  - $H(\text{div};\Omega)$-norm estimate

Diffusive flux reconstruction IV

- Raviart–Thomas–Nédélec FE spaces ($l \geq 0$)

$$\mathbb{RTN}_d^l(\mathcal{T}_h) = \left\{\tau_h \in H(\text{div};\Omega) \mid \forall T \in \mathcal{T}_h,\ \tau_h|_T \in [\mathbb{P}_d^l(T)]^d + x\mathbb{P}_d^l(T)\right\}$$

- Examples of dof's for $l \in \{0, 1\}$

$l = 0$ $l = 1$

- More generally, dof's are
  - on each face, moments of normal components against $q \in \mathbb{P}_{d-1}^l(F)$
  - on each element, moments against $r \in [\mathbb{P}_d^{l-1}(T)]^d$

Diffusive flux reconstruction V

- Construction of $\sigma_h \in \mathbb{RTN}_d^l(\mathcal{T}_h)$ $(l \in \{k-1, k\})$
- Direct prescription of dof's
  - on each face $F \in \mathcal{F}_h$,

$$\int_F (\sigma_h \cdot \mathrm{n}_F) q = \int_F \phi_F(u_h) q \qquad \forall q \in \mathbb{P}_{d-1}^l(F)$$

  - in each element $T \in \mathcal{T}_h$,

$$\int_T \sigma_h \cdot r = - \int_T G_h^{k-1}(u_h) \cdot r \qquad \forall r \in [\mathbb{P}_d^{l-1}(T)]^d$$

Diffusive flux reconstruction VI

- $\nabla{\cdot}\sigma_h$ is an optimal approximation of f

$$\int_T (\nabla{\cdot}\sigma_h)\xi = \int_T f\xi \qquad \forall T \in \mathcal{T}_h \; \forall \xi \in \mathbb{P}_d^l(\mathcal{T}_h)$$

- **Proof**

$$\begin{aligned}\int_T (\nabla{\cdot}\sigma_h)\xi &= -\int_T \sigma_h{\cdot}\nabla\xi + \int_{\partial T} (\sigma_h{\cdot}\mathrm{n}_T)\xi \\ &= -\int_T \sigma_h{\cdot}\nabla\xi + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F (\sigma_h{\cdot}\mathrm{n}_F)\xi \\ &= \int_T G_h^{k-1}(u_h){\cdot}\nabla\xi + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h)\xi = \int_T f\xi\end{aligned}$$

Diffusive flux reconstruction VII

- L^2-norm estimate

$$\|\sigma_h - \sigma\|_{[L^2(\Omega)]^d} \lesssim \eta \| u - u_h \| + \mathcal{R}_{\mathrm{osc},\mathcal{T}_h}$$

with the data oscillation term $\mathcal{R}_{\mathrm{osc},\mathcal{T}_h} = h\|f - \pi_h f\|_{L^2(\Omega)}$

- This estimate is optimal if $l = k - 1$ and sub-optimal if $l = k$
  - Mixed FE with $\mathbb{RTN}_d^k(\mathcal{T}_h)/\mathbb{P}_d^k(\mathcal{T}_h)$ yield an $O(h^{k+1})$ L^2-estimate on the flux
  - Mixed FE can often be implemented as a cell-centered method, but with a wider stencil than dG

Variations on symmetry and penalty I

Variations on penalty

- Recall that for SIP

$$a_h(v_h, w_h) = \int_\Omega G_h^l(v_h)\cdot G_h^l(w_h) + \hat{s}_h(v_h, w_h)$$

with

$$\hat{s}_h(v_h, w_h) = \sum_{F\in\mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!] - \int_\Omega \mathrm{R}_h^l([\![v_h]\!])\cdot \mathrm{R}_h^l([\![w_h]\!])$$

Variations on symmetry and penalty II

- The idea of Bassi and Rebay ('97) is to stabilize with

$$\hat{s}_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h} \eta \int_\Omega \mathrm{r}_F^l([\![v_h]\!]) \cdot \mathrm{r}_F^l([\![w_h]\!]) - \int_\Omega \mathrm{R}_h^l([\![v_h]\!]) \cdot \mathrm{R}_h^l([\![w_h]\!])$$

- The key advantage is that discrete coercivity holds true for $\eta > N_\partial$, thereby removing the dependency on C_{tr}

Variations on symmetry and penalty III

- A further alternative is to stabilize with

$$\hat{s}_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$

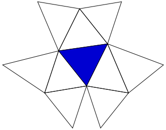
yielding

$$a_h(v_h, w_h) = \int_\Omega G_h^l(v_h) \cdot G_h^l(w_h) + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$

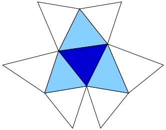
- The advantage is that discrete coercivity holds true for $\eta > 0$
- However, the term $\int_\Omega \mathrm{R}_h^l([\![v_h]\!]) \cdot \mathrm{R}_h^l([\![w_h]\!])$ **widens the stencil to neighbors of neighbors**

Variations on symmetry and penalty IV

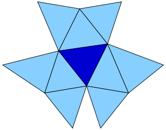
$$\int_\Omega \nabla_h v_h \cdot \nabla_h w_h$$



$$-\int_\Omega \nabla_h v_h \cdot \mathrm{R}_h^l([\![w_h]\!]) - \int_\Omega \mathrm{R}_h^l([\![v_h]\!]) \cdot \nabla_h w_h + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$



$$\int_\Omega \mathrm{R}_h^l([\![v_h]\!]) \cdot \mathrm{R}_h^l([\![w_h]\!])$$



Variations on symmetry and penalty V

Variations on symmetry

- Let $\theta \in \{-1, 0, 1\}$ and set

$$a_h(v_h, w_h) = \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\nabla_h v_h\} \cdot \mathrm{n}_F [\![w_h]\!] - \theta \sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \{\nabla_h w_h\} \cdot \mathrm{n}_F + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!] [\![w_h]\!]$$

- $\theta = 1$ yields SIP
- $\theta = 0$ yields Incomplete IP [Dawson, Sun & Wheeler '04]
  - one motivation can be to use the broken gradient instead of the discrete gradient in the local formulation

$$\int_T \nabla_h u_h \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) \xi = \int_T f \xi$$

Variations on symmetry and penalty VI

- $\theta = -1$ yields Nonsymmetric IP
  - introduced by Oden, Babuška & Baumann ('98) without penalty ($\eta = 0$): numerical experiments
  - analysis with penalty by Rivière, Girault & Wheeler ('99, '01)
  - discrete inf-sup stability without penalty in 2D for $k \geq 2$ [Larson & Niklasson '04]
- Energy-error estimates for {S,I,N}IP are similar
- Optimal L^2-error estimates are not available for {N,I}PG because the duality argument requires symmetry
  - optimal L^2-error estimates can be recovered by using over-penalty [Brenner & Owens '07]

Salient points of this lecture

- Derivation of SIP ensuring consistency
- Energy error analysis of SIP using the $|||\cdot|||$-norm
- The concept of discrete gradient
- The possibility of cheap and accurate diffusive flux reconstruction

PDEs with diffusion

- Darcy flows
- Diffusion-advection-reaction
- Two-phase porous media flows

Darcy flows I

Model problem

- Let $f \in L^2(\Omega)$; seek $u : \Omega \to \mathbb{R}$ s.t. $-\nabla\cdot(\kappa\nabla u) = f$ in Ω and $u|_{\partial\Omega} = 0$
- Weak formulation: $u \in V \stackrel{\text{def}}{=} H_0^1(\Omega)$ s.t.

$$a(u, v) \stackrel{\text{def}}{=} \int_\Omega \kappa\nabla u\cdot\nabla v = \int_\Omega fv \qquad \forall v \in V$$

  - κ is scalar-valued, bounded, and uniformly positive in Ω
  - the model problem is well-posed
- Specific numerical difficulty: κ is highly contrasted

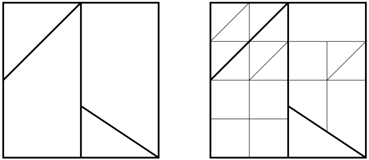
Darcy flows II

- We assume that κ is piecewise constant on a given polyhedral partition $P_\Omega = \{\Omega_i\}_{1\le i\le N_\Omega}$ of Ω
- $\sigma = -\kappa\nabla u$ is the diffusive flux
  - by its definition, $\sigma \in H(\mathrm{div};\Omega)$
  - the normal component of σ is continuous across any interface
  - the normal component of ∇u is not if κ jumps
- Important application: groundwater flows
  - u is the hydraulic head, σ the **Darcy velocity**
  - for each geological layer Ω_i, $\kappa|_{\Omega_i}$ is its **hydraulic conductivity**

Darcy flows III

Discretization

- Compatible mesh with the partition P_Ω



- Discrete space $V_h \stackrel{\text{def}}{=} \mathbb{P}_d^k(\mathcal{T}_h)$ with $k \geq 1$

Darcy flows IV

- A rather natural way to extend SIP to heterogeneous diffusion is to set [Houston, Schwab & Süli '02]

$$a_h(v_h, w_h) = \int_\Omega \kappa \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \nabla_h v_h\} \cdot \mathrm{n}_F [\![w_h]\!]$$
$$- \sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \{\kappa \nabla_h w_h\} \cdot \mathrm{n}_F + \sum_{F \in \mathcal{F}_h} \eta \frac{\gamma_{\kappa,F}}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$

- a_h yields consistency and is symmetric
- To achieve discrete coercivity, **the penalty coefficient must depend on κ**
  - For the above choice, $\gamma_{\kappa,F} = \{\kappa\} = \frac{1}{2}(\kappa_1 + \kappa_2)$, $F = \partial T_1 \cap \partial T_2$
  - For high contrasts, $\gamma_{\kappa,F}$ is controlled by the highest value (the most permeable layer)

Darcy flows V

- We believe instead that for high contrasts, $\gamma_{\kappa,F}$ should be controlled by the lowest value (the least permeable layer)
  - This is the approach encountered in Mixed FE and FV

- Moreover, in the presence of dominant advection, diffusion heterogeneities can trigger internal layers and even solution discontinuities for locally zero diffusion
  - see [Gastaldi & Quarteroni '89, Di Pietro, AE & Guermond '08]
  - penalizing the jump at such interfaces does not make good sense

- One simple choice is **harmonic averaging**

$$\gamma_{\kappa,F} \stackrel{\text{def}}{=} \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}$$

but to achieve discrete coercivity requires modifying the consistency and symmetry terms

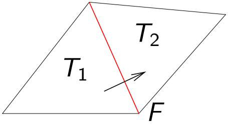
Darcy flows VI

Weighted averages

- To any interface $F \in \mathcal{F}_h^i$ with $F = \partial T_1 \cap \partial T_2$, we assign two nonnegative real numbers $\omega_{T_1,F}$ and $\omega_{T_2,F}$ s.t.

$$\omega_{T_1,F} + \omega_{T_2,F} = 1$$

- Weighted average $\{v\}_{\omega,F} \stackrel{\text{def}}{=} \omega_{T_1,F} v|_{T_1} + \omega_{T_2,F} v|_{T_2}$



- The choice $\omega_{T_1,F} = \omega_{T_2,F} = \frac{1}{2}$ recovers usual arithmetic averages
- On the boundary with $F = \partial T \cap \partial\Omega$, $\{v\}_{\omega,F} = v|_T$

Darcy flows VII

Symmetric Weighted IP (SWIP)

$$a_h(v_h, w_h) = \int_\Omega \kappa \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \nabla_h v_h\}_\omega \cdot \mathrm{n}_F [\![w_h]\!]$$
$$- \sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \{\kappa \nabla_h w_h\}_\omega \cdot \mathrm{n}_F + \sum_{F \in \mathcal{F}_h} \eta \frac{\gamma_{\kappa,F}}{h_F} \int_F [\![v_h]\!] [\![w_h]\!]$$

- Discrete problem Seek $u_h \in V_h$ s.t.

$$a_h(u_h, w_h) = \int_\Omega f w_h \qquad \forall w_h \in V_h$$

Darcy flows VIII

- Diffusion-dependent weighted averages

$$\omega_{T_1,F} \stackrel{\text{def}}{=} \frac{\kappa_2}{\kappa_1 + \kappa_2} \qquad \omega_{T_2,F} \stackrel{\text{def}}{=} \frac{\kappa_1}{\kappa_1 + \kappa_2}$$

  - homogeneous diffusion yields back arithmetic averages
  - dG methods with non-arithmetic averages were considered by Stenberg ('98), Heinrich et al. ('02–'05), Hansbo & Hansbo ('02)
  - diffusion-dependent weighted averages were introduced by Burman & Zunino ('06)
  - the SWIP method was introduced and analyzed by [AE, Stephansen & Zunino '09, Di Pietro, AE & Guermond '08]

Darcy flows IX

- The SWIP bilinear form yields consistency since

$$a_h(v,w) = -\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(\kappa\nabla v)w + \sum_{F\in\mathcal{F}_h^i}\int_F [\![\kappa\nabla_h v]\!]\cdot\mathrm{n}_F\{w\}_{\overline{\omega}} - \sum_{F\in\mathcal{F}_h}\int_F [\![v]\!]\{\kappa\nabla_h w\}_\omega\cdot\mathrm{n}_F + \sum_{F\in\mathcal{F}_h}\eta\frac{\gamma_{\kappa,F}}{h_F}\int_F [\![v]\!][\![w]\!]$$

with $\{w\}_{\overline{\omega}} = \omega_{T_2,F}w|_{T_1} + \omega_{T_1,F}w|_{T_2}$

- As a result, if the exact solution is smooth enough,

$$a_h(u,w_h) = \int_\Omega f w_h \qquad \forall w_h\in V_h$$

Darcy flows X

- Discrete stability norm

$$|||v|||^2 \stackrel{\text{def}}{=} \|\kappa^{1/2}\nabla_h v\|^2_{[L^2(\Omega)]^d} + |v|^2_{\mathrm{J},\kappa}$$

with diffusion-dependent jump seminorm

$$|v|^2_{\mathrm{J},\kappa} \stackrel{\text{def}}{=} \sum_{F\in\mathcal{F}_h} \frac{\gamma_{\kappa,F}}{h_F}\|[\![v]\!]\|^2_{L^2(F)}$$

- **Bound on consistency term** $\forall(v,w_h)\in V_{\dagger h}\times V_{\dagger h}$

$$\left|\sum_{F\in\mathcal{F}_h}\int_F \{\kappa\nabla_h v\}_\omega\cdot \mathrm{n}_F[\![w]\!]\right| \le \left(\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F\|\kappa^{1/2}\nabla v|_T\cdot\mathrm{n}_F\|^2_{L^2(F)}\right)^{1/2}|w|_{\mathrm{J},\kappa}$$

since $2(\omega_1^2\kappa_1+\omega_2^2\kappa_2)=\gamma_{\kappa,F}$

Darcy flows XI

- Discrete coercivity: Assume $\eta > C_{\mathrm{tr}}^2 N_\partial$ Then, for all $v_h \in V_h$

$$C_{\mathrm{stb}} |||v_h|||^2 \leq a_h(v_h, v_h)$$

with C_{stb} independent of κ

- Boundedness For all $(v, w_h) \in V_{\dagger h} \times V_h$

$$a_h(v, w_h) \leq C_{\mathrm{bnd}} |||v|||_* |||w_h|||$$

with C_{bnd} independent of κ and

$$|||v|||_*^2 \stackrel{\mathrm{def}}{=} |||v|||^2 + \sum_{T \in \mathcal{T}_h} h_T \|\kappa^{1/2} \nabla v \cdot \mathrm{n}_T\|_{L^2(\partial T)}^2$$

Darcy flows XII

- Error estimate $|||u - u_h||| \leq C \inf_{y_h \in V_h} |||u - y_h|||_*$ with C independent of κ
- Convergence rate $|||u - u_h||| \lesssim \|\kappa\|^{1/2}_{L^\infty(\Omega)} h^k$ if $u \in H^{k+1}(\mathcal{T}_h)$
  - optimal for the gradient, jumps, and boundary values
- An optimal $O(h^{k+1})$-L^2-norm error estimate can be proven using duality techniques
- A local formulation with numerical fluxes can be derived

Darcy flows XIII

- Local lifting operator $\mathrm{r}^l_{F,\kappa} : L^2(F) \to [\mathbb{P}^l_d(\mathcal{T}_h)]^d$ s.t. $\forall \varphi \in L^2(F)$

$$\int_\Omega \kappa\, \mathrm{r}^l_{F,\kappa}(\varphi) \cdot \tau_h = \int_F \{\kappa \tau_h\}_\omega \cdot \mathrm{n}_F \varphi \qquad \forall \tau_h \in [\mathbb{P}^l_d(\mathcal{T}_h)]^d$$

- Discrete gradient $G^l_{h,\kappa}(v) \stackrel{\mathrm{def}}{=} \nabla_h v - \sum_{F \in \mathcal{F}_h} \mathrm{r}^l_{F,\kappa}([\![v]\!]) \in [\mathbb{P}^l_d(\mathcal{T}_h)]^d$

- Let $T \in \mathcal{T}_h$ and let $\xi \in \mathbb{P}^k_d(T)$; then, for $l \in \{k-1, k\}$

$$\int_T \kappa G^l_{h,\kappa}(u_h) \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) \xi = \int_T f \xi$$

with $\epsilon_{T,F} = \mathrm{n}_T \cdot \mathrm{n}_F$ and numerical flux

$$\phi_F(u_h) \stackrel{\mathrm{def}}{=} -\{\kappa \nabla_h u_h\}_\omega \cdot \mathrm{n}_F + \eta \frac{\gamma_{\kappa,F}}{h_F} [\![u_h]\!]$$

Darcy flows XIV

Regularity of exact solution

- Diffusion heterogeneities can trigger solution singularities
- In 2D, the exact solution $\in W^{2,p}(P_\Omega)$, $p > 1$ [Nicaise & Sändig '94]
- For all $T \in \mathcal{T}_h$, $\nabla u \cdot \mathrm{n}|_{\partial T} \in L^1(\partial T) \Longrightarrow$ consistency can be asserted in the usual way
- Owing to Sobolev embedding, $u \in H^{1+\alpha}(\mathcal{T}_h)$ with $\alpha = 2 - \frac{2}{p} > 0$
- An $O(h^\alpha)$ $|||\cdot|||$-norm error estimate can be proven
  - see [Di Pietro & AE '10]
  - see also [Rivière & Wihler '10] for the Poisson problem

Darcy flows XV

Diffusion anisotropy

- In some applications (e.g., groundwater flow), κ is $\mathbb{R}^{d,d}$-valued, symmetric, bounded, and uniformly PD
- The SWIP method is then designed using the normal component of the diffusion tensor at each interface

Diffusion-advection-reaction I

Model problem

- Let $f \in L^2(\Omega)$; seek $u : \Omega \to \mathbb{R}$ s.t. $\nabla\cdot(-\kappa\nabla u + \beta u) + \tilde{\mu} u = f$ in Ω and $u|_{\partial\Omega} = 0$
- Weak formulation: $u \in V \stackrel{\text{def}}{=} H^1_0(\Omega)$ s.t.

$$a(u, v) \stackrel{\text{def}}{=} \int_\Omega (\kappa\nabla u - u\beta)\cdot\nabla v + \int_\Omega \tilde{\mu} uv = \int_\Omega fv \qquad \forall v \in V$$

  - κ is scalar-valued, bounded, and uniformly positive in Ω
  - β is Lipschitz, $\tilde{\mu} \in L^\infty(\Omega)$, $\tilde{\mu} + \frac{1}{2}\nabla\cdot\beta \geq \mu_0 > 0$ in Ω
  - the model problem is well-posed: For all $v \in V$

$$a(v, v) \geq \|\kappa^{1/2}\nabla v\|^2_{[L^2(\Omega)]^d} + \mu_0\|v\|^2_{L^2(\Omega)}$$

Diffusion-advection-reaction II

- $\Phi(u) \stackrel{\text{def}}{=} -\kappa\nabla u + u\beta$ is the diffusive-advective flux
  - by its construction, $\Phi(u) \in H(\text{div};\Omega)$
  - the normal component of $\Phi(u)$ is continuous across any interface
- Nonconservative form of advective term
  $\nabla\cdot(-\kappa\nabla u) + \beta\cdot\nabla u + \mu u = f$ where $\mu = \tilde{\mu} + \nabla\cdot\beta$
- The fully conservative form is **more natural from a physical viewpoint**

Diffusion-advection-reaction III

Discretization

- κ is piecewise constant on a given polyhedral partition P_Ω of Ω
- Meshes are compatible with this partition
- Discrete space $V_h \stackrel{\text{def}}{=} \mathbb{P}_d^k(\mathcal{T}_h)$ with $k \geq 1$
- Key idea: Combine SWIP with upwinding

Diffusion-advection-reaction IV

$$a_h(v,w) = \int_\Omega (\kappa\nabla_h v - v\beta)\cdot\nabla_h w + \int_\Omega \tilde{\mu} v w$$
$$- \sum_{F\in\mathcal{F}_h} \int_F (\{\kappa\nabla_h v\}_\omega + \{\beta v\})\cdot \mathrm{n}_F [\![w]\!]$$
$$- \sum_{F\in\mathcal{F}_h} \int_F [\![v]\!]\{\kappa\nabla_h w\}_\omega \cdot \mathrm{n}_F + \sum_{F\in\mathcal{F}_h} \int_F \gamma_{\kappa,\beta,F} [\![v]\!][\![w]\!]$$

- for $F \in \mathcal{F}_h^i$, $\gamma_{\kappa,\beta,F} = \eta \frac{\gamma_{\kappa,F}}{h_F} + \frac{1}{2}|\beta\cdot\mathrm{n}_F|$
- for $F \in \mathcal{F}_h^b$, $\gamma_{\kappa,\beta,F} = \eta \frac{\gamma_{\kappa,F}}{h_F} + (\beta\cdot\mathrm{n})^\ominus$

- For dominant diffusion with local Péclet numbers $h_F|\beta\cdot\mathrm{n}_F|/\gamma_{\kappa,F} \lesssim 1$, the amount of penalty introduced by SWIP is sufficient and **centered fluxes can be used for advection** and the boundary penalty term with $(\beta\cdot\mathrm{n})^\ominus$ can be dropped

Diffusion-advection-reaction V

- **Discrete problem** Seek $u_h \in V_h$ s.t.

$$a_h(u_h, w_h) = \int_\Omega f w_h \qquad \forall w_h \in V_h$$

- The exact solution is such that $[\![u]\!] = 0$ for all $F \in \mathcal{F}_h$ and $[\![\Phi(u)]\!]\cdot\mathrm{n}_F = 0$ for all $F \in \mathcal{F}_h^i$
- $[\![u]\!] = 0 \Longrightarrow [\![\beta u]\!]\cdot\mathrm{n}_F = (\beta\cdot\mathrm{n}_F)[\![u]\!] = 0$
- Hence, $[\![\kappa\nabla u]\!]\cdot\mathrm{n}_F = 0$ for all $F \in \mathcal{F}_h^i$

Diffusion-advection-reaction VI

- The previous consistency proofs for SWIP and upwind can be combined

$$
\begin{aligned}
a(u, w_h) = & \int_\Omega \nabla\cdot(-\kappa\nabla u) w_h + \int_\Omega \nabla\cdot(\beta u) w_h + \int_\Omega \tilde{\mu} u w_h \\
& + \sum_{F\in\mathcal{F}_h^i} \int_F [\![\kappa\nabla u]\!]\cdot \mathrm{n}_F \{w_h\}_{\overline{\omega}} - \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot \mathrm{n}_F)[\![u]\!]\{w_h\} \\
= & \int_\Omega f w_h
\end{aligned}
$$

Diffusion-advection-reaction VII

- Recall discrete coercivity norm for SWIP for $\eta > C_{\mathrm{tr}}^2 N_\partial$

$$|||v|||_{\mathrm{swip}}^2 = \|\kappa^{1/2}\nabla_h v\|_{[L^2(\Omega)]^d}^2 + \sum_{F\in\mathcal{F}_h} \frac{\gamma_{\kappa,F}}{h_F}\|[\![v]\!]\|_{L^2(F)}^2$$

- Recall discrete coercivity norm for upwind

$$|||v|||_{\mathrm{upw}}^2 = \|v\|_{L^2(\Omega)}^2 + \sum_{F\in\mathcal{F}_h^b}\int_F \tfrac{1}{2}|\beta\cdot\mathrm{n}|v^2 + \sum_{F\in\mathcal{F}_h^i}\int_F \tfrac{1}{2}|\beta\cdot\mathrm{n}_F|[\![v]\!]^2$$

- Letting $|||\cdot|||^2 \stackrel{\mathrm{def}}{=} |||\cdot|||_{\mathrm{swip}}^2 + |||\cdot|||_{\mathrm{upw}}^2$ yields for all $v_h \in V_h$

$$|||v_h|||^2 \lesssim a_h(v_h, v_h)$$

and therefore **discrete coercivity and well-posedness**

Diffusion-advection-reaction VIII

- Boundedness on OSS for upwind combined with full boundedness for SWIP yields boundedness on OSS for DAR
- Assuming $u \in H^{k+1}(\mathcal{T}_h)$ typically yields the estimate

$$|||u - u_h|||_{\text{swip}} + |||u - u_h|||_{\text{upw}} \lesssim \|\kappa\|_{L^\infty(\Omega)}^{1/2} h^k + \|\beta\|_{[L^\infty(\Omega)]^d}^{1/2} h^{k+1/2}$$

  - in the **dominant diffusion regime**, $|||u - u_h|||_{\text{swip}}$ converges as $O(h^k)$ as for pure diffusion
  - in the **dominant advection regime**, $|||u - u_h|||_{\text{upw}}$ converges as $O(h^{k+1/2})$ as for pure advection-reaction

- An optimal error estimate on the advective derivative can be established by proving discrete inf-sup stability with a stronger norm

Diffusion-advection-reaction IX

Locally vanishing diffusion

- κ is scalar-valued and vanishes locally; more generally, κ is tensor-valued and some of its eigenvalues vanish locally

- elliptic/hyperbolic interface

$$\mathcal{I}_{0,\Omega} \stackrel{\text{def}}{=} \{x \in \partial\Omega_i \cap \partial\Omega_j \mid \mathrm{n}_I^t(\kappa|_{\Omega_i})\mathrm{n}_I > \mathrm{n}_I^t(\kappa|_{\Omega_j})\mathrm{n}_I = 0\}$$

where n_I is a normal to $\partial\Omega_i \cap \partial\Omega_j$

- $\mathcal{I}_{0,\Omega}$ is decomposed into

$$\mathcal{I}_{0,\Omega}^{+} \stackrel{\text{def}}{=} \{x \in \mathcal{I}_{0,\Omega} \mid (\beta{\cdot}\mathrm{n}_I)(x) > 0\}$$
$$\mathcal{I}_{0,\Omega}^{-} \stackrel{\text{def}}{=} \{x \in \mathcal{I}_{0,\Omega} \mid (\beta{\cdot}\mathrm{n}_I)(x) < 0\}$$

and for simplicity we assume that $(\beta{\cdot}\mathrm{n}_I)(x) \neq 0$ in $\mathcal{I}_{0,\Omega}$

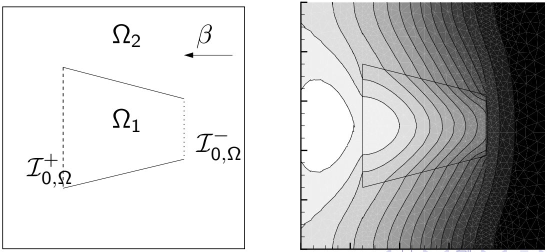
Diffusion-advection-reaction X

- The interface conditions on $\mathcal{I}_{0,\Omega}$ are

$$\begin{aligned} [\![-\kappa\nabla u + \beta u]\!]\cdot \mathrm{n}_I &= 0 \quad \text{on } \mathcal{I}_{0,\Omega} \\ [\![u]\!] &= 0 \quad \text{on } \mathcal{I}_{0,\Omega}^{+} \end{aligned}$$

  so that u can be discontinuous on $\mathcal{I}_{0,\Omega}^{-}$
  - see [Gastaldi & Quarteroni '89, Di Pietro, AE & Guermond '08]

- Example with $\kappa|_{\Omega_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ and $\kappa|_{\Omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$



Diffusion-advection-reaction XI

- Weighted averages are crucial to ensure consistency
- Let $\mathcal{F}_h^{i*}$ collect the interfaces in $\mathcal{I}_{0,\Omega}^-$
- For the SWIP part, since $\{w_h\}_{\overline{\omega}} = w_h|_{\Omega_1}$,

$$a_h^{\mathrm{swip}}(u, w_h) = -\int_\Omega \nabla\cdot(\kappa\nabla u)w_h + \sum_{F\in\mathcal{F}_h^{i*}} \int_F [\![\kappa\nabla_h u]\!]\cdot \mathrm{n}_F w_h|_{\Omega_1}$$

- For the upwind part,

$$a_h^{\mathrm{upw}}(u, w_h) = \int_\Omega \nabla\cdot(\beta u)w_h + \int_\Omega \tilde{\mu} u w_h - \sum_{F\in\mathcal{F}_h^{i*}} \int_F (\beta\cdot\mathrm{n}_F)[\![u]\!]w_h|_{\Omega_1}$$

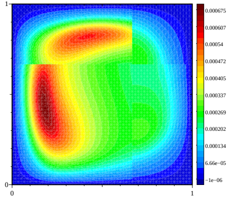
- Owing to **conservation for the diffusive-advective** flux,

$$a_h(u, w_h) = a_h^{\mathrm{swip}}(u, w_h) + a_h^{\mathrm{upw}}(u, w_h) = \int_\Omega f w_h$$

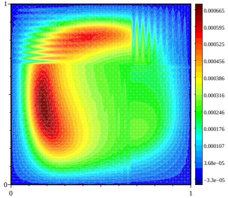
Diffusion-advection-reaction XII

- Example: Unit square divided into 4 subdomains
- Strong x-diffusion in 2 quadrants and strong y-diffusion in the others, anisotropy ratio 10^6
- Rotating advective field

SWIP+upw



SIP+upw



- SIP+upw oscillates because it enforces zero jumps near underresolved layers

Two-phase porous media flows I

- We consider two-phase, immiscible, incompressible flows through isothermal and indeformable porous media
  - motivated by secondary oil recovery and oil trapping effects
  - several dG methods available [Bastian '99, Bastian & Rivière '03, Eslinger '05, Klieber & Rivière '06, Epshteyn & Rivière '07]
- Heterogeneous media with distinct capillary pressure curves lead to discontinuous saturations
  - FV methods designed by [Enchéry, Eymard & Michel '06, Cancès '09, Cancès, Gallouët & Porretta '09]
  - dG method recently designed by [AE, Mozolevski & Schuh '10]

Two-phase porous media flows II

- Mass conservation for each phase

$$\partial_t(\phi S_\alpha) + \nabla\cdot u_\alpha = q_\alpha \qquad \alpha \in \{n, w\}$$

  ϕ: (constant) porosity, S_α: phase saturation, u_α: phase velocity, q_α: source/sink

- $S_n + S_w = 1$, $S := S_n \in [S_{nr}, 1 - S_{wr}]$
- Generalized Darcy's law (no gravity)

$$u_\alpha = -K\lambda_\alpha(S)\nabla p_\alpha$$

  K: absolute permeability, λ_α: phase mobility, p_α: phase pressure

- Capillary pressure

$$\pi(S) = p_n - p_w$$

Two-phase porous media flows III

- Fractional flow formulation
  - Total mobility $\lambda = \lambda_w + \lambda_n$, fractional flow $f = \lambda_n/\lambda$
  - **Global pressure** p (Chavant & Jaffré '86)
- Total velocity $u = u_w + u_n$ s.t.

$$u = -\lambda K \nabla p \qquad \nabla \cdot u = q_w + q_n$$

- Non-wetting phase mass conservation becomes

$$\phi \partial_t S + \nabla \cdot (uf(S)) - \nabla \cdot (\epsilon(S)\pi'(S)\nabla S) = q_n$$

with $\epsilon(S) := \lambda_w(S) f(S) K$

  - **degeneracy** $\epsilon(S_{nr}) = \epsilon(1 - S_{wr}) = 0$

Two-phase porous media flows IV

- Sequential approach to march in time: For $m = 0, 1, \ldots$
  1. **solve elliptic equation for global pressure**

     $$\nabla\cdot(\lambda(S^m)K\nabla p^{m+1}) = q_w^{m+1} + q_n^{m+1}$$

  2. **reconstruct total velocity**

     $$u^{m+1} = -\lambda(S^m)K\nabla p^{m+1}$$

  3. **advance in time saturation equation** (semi-implicit Euler)

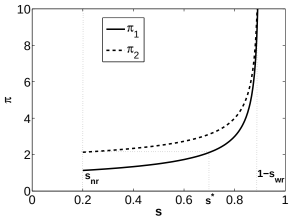
     $$\phi\frac{S^{m+1}-S^m}{\tau^m} + \nabla\cdot(u^{m+1}f(S^{m+1})) - \nabla\cdot(\epsilon(S^m)\pi'(S^m)\nabla S^{m+1}) = q_n^{m+1}$$

- S^0 given by IC
- BC's can be of Dirichlet or Neumann type for both pressure and saturation

Two-phase porous media flows V

Interface conditions

- For simplicity, two subdomains Ω_β, $\beta \in \{1,2\}$, with different rock properties
- Up to rescaling, both S_β's take values in $[S_{nr}, 1 - S_{wr}]$
- Example of capillary pressure curves



- Critical value $S^* = \pi_1^{-1}\pi_2(S_{nr})$

Two-phase porous media flows VI

- **We assume that the wetting phase is present on both sides of interface**
- Jump $[\![a]\!] := a_1 - a_2$ on interface $\Gamma \stackrel{\text{def}}{=} \partial\Omega_1 \cap \partial\Omega_2$
- Interface conditions on saturation
  - flux continuity $[\![uf(S) - \epsilon(S)\pi'(S)\nabla S]\!]\cdot \mathrm{n}_\Gamma = 0$
  - $S_2 = S_{nr}$ if $S_1 \in [S_{nr}, S^*]$
- Interface conditions on pressure
  - flux continuity $[\![-\lambda K \nabla p]\!]\cdot \mathrm{n}_\Gamma = 0$
  - continuity of (some) phase pressures

$$\begin{aligned} [\![p_w]\!] &= 0 \qquad \text{if } S_1 \in [S_{nr}, S^*] \\ [\![p_w]\!] = [\![p_n]\!] &= 0 \qquad \text{if } S_1 \in [S^*, 1 - S_{wr}) \end{aligned}$$

so that $[\![\pi(S)]\!] = 0$ if $S_1 \in [S^*, 1 - S_{wr})$

Two-phase porous media flows VII

- Reformulate interface condition on saturation as $[\![S]\!] = J(S_1)$ with

$$J(S) = \begin{cases} S - S_{nr} & \text{if } S_1 \in [S_{nr}, S^*] \\ S - \pi_2^{-1}(\pi_1(S)) & \text{if } S_1 \in [S^*, 1 - S_{wr}) \end{cases}$$

  - for $S_1 \in [S_{nr}, S^*]$, $[\![S]\!] = J(S_1)$ yields $S_2 = S_{nr}$
  - for $S_1 \in [S^*, 1 - S_{wr})$, $[\![S]\!] = J(S_1)$ yields $\pi_1(S_1) = \pi_2(S_2)$

- Reformulate interface condition on pressure as $[\![p]\!] = G(S)$ with suitable function G depending on S_1 and S_2

Two-phase porous media flows VIII

Step 1: SWIP for pressure equation

- Find $p_h^{m+1} \in V_h$ s.t. for all $z_h \in V_h$ (only Dirichlet BC's)

$$\sum_{T\in\mathcal{T}_h}\int_T -(\nabla\cdot(\lambda(S_h^m)K\nabla p_h^{m+1}) + q_w^{m+1} + q_n^{m+1})z_h$$

$$+ \sum_{F\in\mathcal{F}_h^i}\int_F [\![\lambda(S_h^m)K\nabla p_h^{m+1}]\!]\cdot\mathrm{n}_F\{z_h\}_{\overline{\omega}}$$

$$+ \sum_{F\in\mathcal{F}_h}\int_F [\![p_h^{m+1}]\!]'\left(-n_F\cdot\{\lambda(S_h^m)K\nabla z_h\}_\omega + \eta\frac{\gamma_F}{h_F}[\![z_h]\!]\right) = 0$$

where

$$[\![p_h^{m+1}]\!]' = \begin{cases} [\![p_h^{m+1}]\!] & \text{if } F\in\mathcal{F}_h^i\setminus\Gamma \\ [\![p_h^{m+1}]\!] - G(S_h^m) & \text{if } F\in\Gamma \\ p_h^{m+1} - p_D & \text{if } F\in\mathcal{F}_h^b \end{cases}$$

Two-phase porous media flows IX

- Reference diffusion $\kappa_{T^\pm,F} = \|(\lambda(S_h^m)K)|_{T^\pm}\|_{L^\infty(F)}$
- Penalty coefficient γ_F based on harmonic average
- The pressure interface condition that is weakly enforced is

$$[\![p_h^{m+1}]\!] = G(S_h^m)$$

Step 2: RTN reconstruction of total velocity

- Direct prescription of dof's

Two-phase porous media flows X

Step 3: Saturation equation

- Implicit Euler and semi-linearization of diffusive term

$$\phi \frac{S^{m+1}-S^m}{\tau^m} + \nabla\cdot(u^{m+1} f(S^{m+1})) - \nabla\cdot(\epsilon(S^m)\pi'(S^m)\nabla S^{m+1}) = q_n^{m+1}$$

- SWIP for diffusive term
  - reference diffusion $\kappa_{T\pm,F} = \|(\epsilon(S_h^m)\pi'(S_h^m))|_{T\pm}\|_{L^\infty(F)}$
  - penalty coefficient based on harmonic average

- Upwind for advection by total velocity

Two-phase porous media flows XI

Numerical illustration

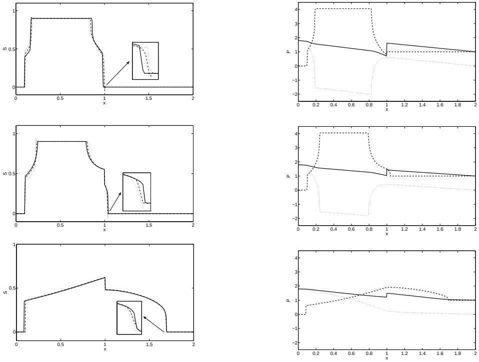
- Pushing a blob of oil
  - $\phi = 0.2$, $S_{nr} = S_{wr} = 0$
  - Brooks–Corey model for mobilities with parameter $\theta = 2$
  - Absolute permeabilities $K_1 = 1$ and $K_2 = 0.1$
  - Capillary pressure curves with $S^* = 5^{-1/2} \simeq 0.45$

$$\pi_1(s) = 5s^2 \qquad \pi_2(s) = 4s^2 + 1$$

- 1D setting with $\Omega_1 = (0, 1)$ and $\Omega_2 = (1, 2)$
  - Dirichlet BC's on the pressure: $p|_{x=0} = 1.8$ and $p|_{x=2} = 1.0$
  - Mixed BC's on saturation: $S|_{x=0} = 0$ and $\epsilon(S)\pi'(S)\frac{dS}{dx}|_{x=2} = 0$
- Discretization with $k = 1$
- No limiters were used

Two-phase porous media flows XII

- **Saturation and pressures at times $\{0.008, 0.015, 0.25\}$**
  - global (solid), capillary (dashed), wetting-phase (dotted) pressures
  - $h^{-1} = 80/120/160$, $\tau = 0.001/0.0005/0.00025$



Salient points of this lecture

- Weighted averages and harmonic penalties for heterogeneous diffusion
- Combining SWIP and upwind for diffusion-advection-reaction, robust even for locally semidefinite diffusion
- These ideas are also important in nonlinear problems with fronts and interface conditions

Incompressible NS

- Discrete functional analysis
- Poisson problem revisited
- Stokes equations: pressure-velocity coupling
- Incompressible NS

Discrete functional analysis I

- For (steady) linear PDEs, the mathematical analysis of dG methods is relatively well-understood
- For nonlinear PDEs, the situation is substantially different
  - FE-based techniques require strong regularity assumptions on the exact solution
  - the analysis of FV schemes proceeds along a different path, avoiding such assumptions [Eymard, Gallouët, Herbin et al '00–08]
- New discrete functional analysis tools in dG spaces are needed
  - discrete **Sobolev embeddings**
  - discrete **Rellich–Kondrachov compactness result**

  see [Buffa & Ortner '09, Di Pietro & AE '10]

Discrete functional analysis II

- Recall discrete stability norm for SIP (and other variants)

$$\|v\|_{\mathrm{dG}}^2 \stackrel{\mathrm{def}}{=} |||v|||^2 = \|\nabla_h v\|_{[L^2(\Omega)]^d}^2 + \underbrace{\sum_{F\in\mathcal{F}_h} \frac{1}{h_F}\int_F |[\![v]\!]|^2}_{|v|_{\mathrm{J}}^2}$$

- Non-Hilbertian setting $(1 \le p < +\infty)$

$$\|v\|_{\mathrm{dG},p}^p \stackrel{\mathrm{def}}{=} \|\nabla_h v\|_{[L^p(\Omega)]^d}^p + \sum_{F\in\mathcal{F}_h} \frac{1}{h_F^{p-1}}\int_F |[\![v]\!]|^p$$

- Broken polynomial space $V_h \stackrel{\mathrm{def}}{=} \mathbb{P}_d^k(\mathcal{T}_h)$ with $k \ge 1$

Discrete functional analysis III

Discrete Sobolev embeddings

- For all q such that

  (i) $1 \leq q \leq p^* \stackrel{\text{def}}{=} \frac{pd}{d-p}$ if $1 \leq p < d$
  (ii) $1 \leq q < +\infty$ if $d \leq p < +\infty$

$$\exists \sigma_{q,p}, \qquad \forall v_h \in V_h, \qquad \|v_h\|_{L^q(\Omega)} \leq \sigma_{p,q} \|v_h\|_{\mathrm{dG},p}$$

- Particular case $p = 2$ and $d \in \{2, 3\}$: For all q such that

  (i) $1 \leq q \leq 6$ if $d = 3$
  (ii) $1 \leq q < +\infty$ if $d = 2$

$$\exists \sigma_q, \qquad \forall v_h \in V_h, \qquad \|v_h\|_{L^q(\Omega)} \leq \sigma_q \|v_h\|_{\mathrm{dG}}$$

Discrete functional analysis IV

- Discrete Poincaré–Friedrichs inequality ($q = 2$, $p = 2$) [Brenner '03]
- $q = 4$, $p = 2$ for NS [Karakashian & Jureidini '98]
- Discrete Sobolev embeddings with $p = 2$ [Lasis & Süli '03]
- Two key differences
  - present technique is much simpler: no elliptic regularity or nonconforming FE interpolation $\Rightarrow$ general meshes can be used
  - embeddings are proven in discrete spaces, not in broken Sobolev spaces

Discrete functional analysis V

Principle of proof

- Inspired from [Eymard, Gallouët & Herbin '08]
- BV estimate $\left(\sum_{i=1}^{d} \sup\{\int_{\mathbb{R}^d} u\partial_i\varphi,\ \varphi \in C_0^\infty(\mathbb{R}^d),\ \|\varphi\|_{L^\infty(\mathbb{R}^d)} \le 1\}\right)$

$$\forall v_h \in V_h, \qquad \|v_h\|_{\mathrm{BV}} \lesssim \|v_h\|_{\mathrm{dG},1} \lesssim \|v_h\|_{\mathrm{dG},p} \quad (p \ge 1)$$

(v_h extended by zero outside Ω)

- Classical result ($1^* \stackrel{\mathrm{def}}{=} \frac{d}{d-1}$): $\|v\|_{L^{1^*}(\mathbb{R}^d)} \le \frac{1}{2d}\|v\|_{\mathrm{BV}}$
- For $1 < p < d$, use $\|\cdot\|_{L^{1^*}(\mathbb{R}^d)}$-estimate for $|v_h|^\alpha$, Hölder's inequality and a trace inequality
- For $p \ge d$, use Hölder's inequality

Discrete functional analysis VI

Compactness for discrete gradients

- Let $l \geq 0$
- Recall that $G_h^l : H^1(\mathcal{T}_h) \longrightarrow [L^2(\Omega)]^d$ is s.t.

$$G_h^l(v) \stackrel{\text{def}}{=} \nabla_h v - \mathrm{R}_h^l([\![v]\!])$$

where

$$\mathrm{R}_h^l([\![v]\!]) = \sum_{F \in \mathcal{F}_h} \mathrm{r}_F^l([\![v]\!])$$

and for any $F \in \mathcal{F}_h$, $\mathrm{r}_F^l : L^2(F) \longrightarrow [\mathbb{P}_d^l(\mathcal{T}_h)]^d$ is s.t. for all $\varphi \in L^2(F)$

$$\int_\Omega \mathrm{r}_F^l(\varphi)\cdot\tau_h = \int_F \{\tau_h\}\cdot\mathrm{n}_F\varphi \qquad \forall \tau_h \in [\mathbb{P}_d^l(\mathcal{T}_h)]^d$$

Discrete functional analysis VII

Main result

- Let $(v_h)_{h\in\mathcal{H}}$ be a sequence in $(V_h)_{h\in\mathcal{H}}$ bounded in the $\|\cdot\|_{\mathrm{dG}}$-norm
- Then, there exists a subsequence of $(v_h)_{h\in\mathcal{H}}$ and a function $v \in H^1_0(\Omega)$ s.t. as $h \to 0$

$$v_h \to v \quad \text{strongly in } L^2(\Omega)$$

and for all $l \geq 0$

$$G_h^l(v_h) \rightharpoonup \nabla v \quad \text{weakly in } [L^2(\Omega)]^d$$

Discrete functional analysis VIII

Principle of proof

- Inspired from [Eymard, Gallouët & Herbin '08]
- Functions extended by zero outside Ω
- Uniform BV estimate on space translates

$$\|v_h(\cdot + \xi) - v_h\|_{L^1(\mathbb{R}^d)} \leq |\xi|_{\ell^1} \|v_h\|_{\mathrm{BV}} \leq C|\xi|_{\ell^1}$$

- Kolmogorov Compactness Criterion in $L^1(\mathbb{R}^d)$
- Sobolev embedding: compactness in $L^q(\mathbb{R}^d)$, $q > 2$
- There is $v \in L^2(\mathbb{R}^d)$ s.t. $v_h \to v$ strongly in $L^2(\mathbb{R}^d)$

Discrete functional analysis IX

- Bound on $\|\cdot\|_{\mathrm{dG}}$-norm $\Longrightarrow$ bound on discrete gradient $\Longrightarrow$ there is $w \in [L^2(\Omega)]^d$ s.t. $G_h^l(v_h) \rightharpoonup w$ in $[L^2(\Omega)]^d$
- For all $\varphi \in [C_0^\infty(\mathbb{R}^d)]^d$

$$\begin{aligned}\int_{\mathbb{R}^d} G_h^l(v_h)\cdot\varphi &= \int_{\mathbb{R}^d} \nabla_h v_h\cdot\varphi - \int_{\mathbb{R}^d} \mathrm{R}_h^l([\![v_h]\!])\cdot\pi_h^l\varphi \\ &= -\int_{\mathbb{R}^d} v_h \nabla\cdot\varphi + \sum_{F\in\mathcal{F}_h}\int_F \{\varphi - \pi_h^l\varphi\}\cdot\mathrm{n}_F[\![v_h]\!]\end{aligned}$$

converges to $-\int_{\mathbb{R}^d} v\nabla\cdot\varphi \Longrightarrow \nabla v = w$

- Thus, $v \in H^1(\mathbb{R}^d)$ and since $v \equiv 0$ outside $\Omega \Longrightarrow v \in H_0^1(\Omega)$

Poisson problem revisited I

- Recall SIP bilinear form for Poisson problem in $V_h = \mathbb{P}_d^k(\mathcal{T}_h)$ $(k \geq 1)$

$$a_h(v_h, w_h) = \int_\Omega G_h^l(v_h) \cdot G_h^l(w_h) + \hat{s}_h(v_h, w_h)$$

with $l \in \{k-1, k\}$ and

$$\hat{s}_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!] - \int_\Omega \mathrm{R}_h^l([\![v_h]\!]) \cdot \mathrm{R}_h^l([\![w_h]\!])$$

- Discrete coercivity $(\eta > C_{\mathrm{tr}}^2 N_\partial)$: For all $v_h \in V_h$,

$$a_h(v_h, v_h) \geq C_{\mathrm{stb}} \|v_h\|_{\mathrm{dG}}^2$$
$$a_h(v_h, v_h) \geq \|G_h^l(v_h)\|_{[L^2(\Omega)]^d}^2 + (\eta - C_{\mathrm{tr}}^2 N_\partial)|v_h|_{\mathrm{J}}^2$$

Poisson problem revisited II

- We no longer assert strong consistency by plugging the exact solution into a_h
- Only discrete arguments are used for a_h
- **Asymptotic consistency** For any sequence $(v_h)_{h\in\mathcal{H}}$ in $(V_h)_{h\in\mathcal{H}}$ bounded in the $\|\cdot\|_{\mathrm{dG}}$-norm and for any smooth function $\varphi \in C_0^\infty(\Omega)$

$$\lim_{h\to 0} a_h(v_h, \pi_h\varphi) = a(v,\varphi) = \int_\Omega \nabla v\cdot\nabla\varphi$$

where $v \in H_0^1(\Omega)$ is the limit of the sequence $(v_h)_{h\in\mathcal{H}}$ given by the compactness theorem

Poisson problem revisited III

Asymptotic consistency for SIP

$$a_h(v_h, \pi_h\varphi) = \int_\Omega G_h^l(v_h)\cdot G_h^l(\pi_h\varphi) + \hat{s}_h(v_h, \pi_h\varphi) = \mathfrak{T}_1 + \mathfrak{T}_2$$

- $\mathfrak{T}_1 \to \int_\Omega \nabla v\cdot\nabla\varphi$ as $h \to 0$ since
  - $G_h^l(v_h) \rightharpoonup \nabla v$ weakly in $[L^2(\Omega)]^d$
  - $G_h^l(\pi_h\varphi) \to \nabla\varphi$ strongly in $[L^2(\Omega)]^d$
- $\mathfrak{T}_2 \to 0$ since $|\mathfrak{T}_2| \lesssim |v_h|_{\mathrm{J}}|\pi_h\varphi|_{\mathrm{J}}$
  - $|v_h|_{\mathrm{J}}$ is bounded and $|\pi_h\varphi|_{\mathrm{J}} \to 0$

Poisson problem revisited IV

Convergence to minimal regularity solutions

Let $(u_h)_{h\in\mathcal{H}}$ be the sequence of discrete solutions. Then, as $h \to 0$, for the whole sequence

$$\begin{aligned} u_h &\to u && \text{strongly in } L^2(\Omega) \\ \nabla_h u_h &\to \nabla u && \text{strongly in } [L^2(\Omega)]^d \\ |u_h|_{\mathrm{J}} &\to 0 \end{aligned}$$

where $u \in H^1_0(\Omega)$ is the exact solution

Poisson problem revisited V

Step 1: A priori bound

$$C_{\mathrm{stb}}\|u_h\|_{\mathrm{dG}}^2 \le a(u_h,u_h) = \int_\Omega f u_h \le \sigma_2 \|f\|_{L^2(\Omega)}\|u_h\|_{\mathrm{dG}}$$

$\Longrightarrow (u_h)_{h\in\mathcal{H}}$ is bounded in the $\|\cdot\|_{\mathrm{dG}}$-norm

Step 2: Compactness

There exists $v \in H_0^1(\Omega)$ such that, as $h \to 0$, up to a subsequence, $u_h \to v$ strongly in $L^2(\Omega)$ and $G_h^l(u_h) \rightharpoonup \nabla v$ weakly in $[L^2(\Omega)]^d$

Poisson problem revisited VI

Step 3: Asymptotic consistency

For all $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega f\varphi \leftarrow \int_\Omega f\pi_h\varphi = a_h(u_h, \pi_h\varphi) \rightarrow \int_\Omega \nabla v\cdot\nabla\varphi$$

$\Longrightarrow$ by density, v solves the Poisson problem

Step 4: Additional properties

- Uniqueness of solution $\Longrightarrow$ the whole sequence $(u_h)_{h\in\mathcal{H}}$ converges

Poisson problem revisited VII

- Strong convergence of the discrete gradient
  - owing to weak convergence

$$\liminf_{h\to 0} a_h(u_h,u_h) \geq \liminf_{h\to 0} \|G_h^l(u_h)\|^2_{[L^2(\Omega)]^d} \geq \|\nabla u\|^2_{[L^2(\Omega)]^d}$$

  - Owing to stability

$$\|G_h^l(u_h)\|^2_{[L^2(\Omega)]^d} \leq a_h(u_h,u_h) = \int_\Omega f u_h$$

yielding

$$\limsup_{h\to 0} \|G_h^l(u_h)\|^2_{[L^2(\Omega)]^d} = \limsup_{h\to 0} \int_\Omega f u_h = \int_\Omega f u = \|\nabla u\|^2_{[L^2(\Omega)]^d}$$

- Convergence of $|u_h|_{\mathrm{J}}$ to zero using stability

Stokes equations I

Model problem

- Let $f \in [L^2(\Omega)]^d$; seek velocity field $u : \Omega \to \mathbb{R}^d$ and pressure field $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
-\triangle u + \nabla p &= f \quad \text{in } \Omega \\
\nabla \cdot u &= 0 \quad \text{in } \Omega
\end{aligned}
$$

with $u|_{\partial\Omega} = 0$ and $\langle p \rangle_\Omega = 0$

- Mass and momentum conservation for a slow, incompressible flow

Stokes equations II

Weak formulation

- Functional spaces

$$U \stackrel{\text{def}}{=} [H_0^1(\Omega)]^d \qquad P \stackrel{\text{def}}{=} L_*^2(\Omega) \stackrel{\text{def}}{=} \{q \in L^2(\Omega) \mid \langle q \rangle_\Omega = 0\}$$

- Bilinear forms

$$a(u,v) \stackrel{\text{def}}{=} \int_\Omega \nabla u \cdot \nabla v \qquad b(v,q) \stackrel{\text{def}}{=} -\int_\Omega q \nabla \cdot v$$

- Find $(u,p) \in U \times P$ s.t.

$$\begin{aligned} a(u,v) + b(v,p) &= \int_\Omega f \cdot v && \forall v \in U \\ -b(u,q) &= 0 && \forall q \in P \end{aligned}$$

Stokes equations III

- Well-posedness hinges on **surjectivity of divergence operator** [Ladyzhenskaya, Nečas, Bogovskiĭ, Solonnikov,...]
- There is β_Ω s.t. for all $q \in P$, there is $v_q \in U$ with

$$q = \nabla{\cdot}v_q \qquad \beta_\Omega \|v_q\|_{[H^1(\Omega)]^d} \leq \|q\|_{L^2(\Omega)}$$

- Equivalent inf-sup condition

$$\forall q \in P \qquad \beta_\Omega \|q\|_{L^2(\Omega)} \leq \sup_{w \in U\setminus\{0\}} \frac{b(w,q)}{\|w\|_{[H^1(\Omega)]^d}}$$

Stokes equations IV

Discrete divergence

- Let $l \geq 0$
- Define $D_h^l : [H^1(\mathcal{T}_h)]^d \rightarrow \mathbb{P}_d^l(\mathcal{T}_h)$ s.t.

$$D_h^l(v) \stackrel{\text{def}}{=} \sum_{i=1}^{d} G_h^l(v_i) \cdot e_i$$

- Bilinear form for discrete divergence

$$b_h(v, q) \stackrel{\text{def}}{=} -\int_\Omega q D_h^l(v)$$

Stokes equations V

- Link with discrete gradient

$$
\begin{aligned}
b_h(v_h, q_h) &= -\int_\Omega q_h D_h^l(v_h) \\
&= -\int_\Omega q_h \nabla_h \cdot v_h + \sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \cdot \mathrm{n}_F \{q_h\} \\
&= \int_\Omega v_h \cdot \nabla_h q_h - \sum_{F \in \mathcal{F}_h^i} \int_F \{v_h\} \cdot \mathrm{n}_F [\![q_h]\!] = \int_\Omega v_h \cdot \mathcal{G}_h^l(q_h)
\end{aligned}
$$

with slightly modified discrete gradient

$$
\mathcal{G}_h^l(q_h) \stackrel{\text{def}}{=} \nabla_h q_h - \sum_{F \in \mathcal{F}_h^i} r_F^l([\![q_h]\!])
$$

Stokes equations VI

Equal-order discontinuous spaces for velocity and pressure

- For $k \geq 1$,

$$U_h \stackrel{\text{def}}{=} [\mathbb{P}_d^k(\mathcal{T}_h)]^d \qquad P_h \stackrel{\text{def}}{=} \mathbb{P}_d^k(\mathcal{T}_h)/\mathbb{R}$$

- Discrete inf-sup condition (LBB) $\forall q_h \in P_h$

$$\beta \|q_h\|_{L^2(\Omega)} \leq \sup_{w_h \in U_h \setminus \{0\}} \frac{b_h(w_h, q_h)}{|\!|\!| w_h |\!|\!|_{\mathrm{vel}}} + |q_h|_p$$

with $|\!|\!| w_h |\!|\!|_{\mathrm{vel}}^2 \stackrel{\text{def}}{=} \sum_{i=1}^d |\!|\!| w_{h,i} |\!|\!|_{\mathrm{dG}}^2$ and

$$|q_h|_p^2 \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h^i} h_F \|[\![q]\!]\|_{L^2(F)}^2$$

[Cockburn, Kanschat, Schötzau, Schwab '02, AE & Guermond '08]

Stokes equations VII

- Discrete problem combines **SIP for velocity components, discrete divergence operator, and pressure jump penalty**
- Find $(u_h, p_h) \in U_h \times P_h$ s.t.

$$a_h(u_h, v_h) + b_h(v_h, p_h) = \int_\Omega f \cdot v_h \qquad \forall v_h \in U_h$$
$$-b_h(u_h, q_h) + j_h(p_h, q_h) = 0 \qquad \forall q_h \in P_h$$

with

$$a_h(v_h, w_h) = \sum_{i=1}^{d} \left(\int_\Omega G_h^l(v_{h,i}) \cdot G_h^l(w_{h,i}) + \hat{s}_h(v_{h,i}, w_{h,i}) \right)$$
$$j_h(q_h, r_h) = \sum_{F \in \mathcal{F}_h^i} h_F \int_F [\![q_h]\!][\![r_h]\!]$$

Stokes equations VIII

Convergence for smooth solutions

- Strong consistency can be asserted if (u, p) smooth enough
- Discrete inf-sup stability with norm

$$|||(v, q)|||^2 \stackrel{\text{def}}{=} |||v|||_{\text{vel}}^2 + \|q\|_{L^2(\Omega)}^2 + |q|_p^2$$

- Boundedness with suitable $|||\cdot|||_*$-norm
- Convergence rate if $(u, p) \in H^{k+1}(\mathcal{T}_h) \times H^k(\mathcal{T}_h)$

$$|||(u - u_h, p - p_h)||| \lesssim h^k$$

  - optimal on velocity gradient, jumps, and boundary values
  - optimal on pressure and its jumps
- Optimal $O(h^{k+1})$-L^2-norm velocity error estimate if Cattabriga's regularity holds true (e.g., if Ω is convex)

Stokes equations IX

Convergence with minimal regularity

- Only assume $(u, p) \in [H_0^1(\Omega)]^d \times L_*^2(\Omega)$
- Let $((u_h, p_h))_{h\in\mathcal{H}}$ be the sequence of discrete solutions. Then, as $h \to 0$, for the whole sequence,

$$
\begin{aligned}
u_h &\to u && \text{in } [L^2(\Omega)]^d \\
\nabla_h u_h &\to \nabla u && \text{in } [L^2(\Omega)]^{d,d} \\
|u_h|_{\mathrm{J}} &\to 0 && \\
p_h &\to p && \text{in } L^2(\Omega) \\
|p_h|_p &\to 0 &&
\end{aligned}
$$

where $(u, p) \in [H_0^1(\Omega)]^d \times L_*^2(\Omega)$ is the exact solution

Stokes equations X

Alternative formulations

- Non-stabilized formulations on affine quadrilateral or hexahedral meshes [Toselli '02]
- Non-stabilized formulations on triangular meshes with $P_h = \mathbb{P}_d^{k-1}(\mathcal{T}_h)$ [Hansbo & Larson '02, Girault, Rivière & Wheeler '05]
- Using continuous pressures
  - mass conservation is expressed less locally
  - earlier related work [Becker, Burman, Hansbo & Larson '01]

Incompressible NS I

Model problem

- Let $f \in [L^2(\Omega)]^d$; seek velocity field $u : \Omega \to \mathbb{R}^d$ and pressure field $p : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\nu\triangle u + (u\cdot\nabla)u + \nabla p = f \quad & \text{in } \Omega \\ \nabla\cdot u = 0 \quad & \text{in } \Omega \end{aligned}$$

  with $u|_{\partial\Omega} = 0$ and $\langle p\rangle_\Omega = 0$ and $d \in \{2,3\}$

- Mass and momentum conservation for an incompressible flow (ν: shear viscosity)

- The convective term can be written in the conservative form $\nabla\cdot(u\otimes u)$ since $(u\cdot\nabla)u = \nabla\cdot(u\otimes u) - (\nabla\cdot u)u$ and $\nabla\cdot u = 0$

Incompressible NS II

Weak formulation

- Functional spaces U and P as for Stokes
- Bilinear forms a and b as for Stokes and trilinear form

$$t(w,u,v) = \int_\Omega ((w\cdot\nabla)u)\cdot v$$

- Find $(u,p) \in U \times P$ s.t.

$$\begin{aligned} \nu a(u,v) + t(u,u,v) + b(v,p) &= \int_\Omega f\cdot v \quad && \forall v \in U \\ -b(u,q) &= 0 && \forall q \in P \end{aligned}$$

Incompressible NS III

- The key property of the trilinear form is that for divergence-free w

$$t(w,u,u) = -\frac{1}{2}\int_\Omega (\nabla\cdot w)u^2 = 0 \qquad \forall u \in U$$

so that the **convective term does not affect the kinetic energy balance**

- Existence of a solution for incompressible NS can be proven by passing to the limit from a conforming FE approximation
- Uniqueness holds true under a smallness condition on the data

Incompressible NS IV

Literature overview

- One key issue is controlling the convective term
  - piecewise divergence-free velocity fields [Karakashian & Jureidini '98]
  - nonconservative method based on Temam's device [Girault, Rivière & Wheeler '05]

$$t'(w,u,v) = t(w,u,v) + \frac{1}{2}\int_\Omega (\nabla\cdot w)u\cdot v$$

  - conservative LDG method using BDM velocity projection [Cockburn, Kanschat & Schötzau '05]

- The analysis of such methods hinges on strong regularity assumptions on the exact solutions and generally uses a smallness assumption on the data

- We want to avoid such assumptions as in recent FV work [Eymard, Herbin et al '07-'10] $\Longrightarrow$ see [Di Pietro & AE '10]

Incompressible NS V

Discrete trilinear form

- Let $k \geq 1$ and take as before $U_h = [\mathbb{P}_d^k(\mathcal{T}_h)]^d$
- Elementwise integration by parts yields

$$\int_\Omega ((w_h \cdot \nabla) v_h) \cdot v_h = -\frac{1}{2} \int_\Omega (\nabla_h \cdot w_h)(v_h \cdot v_h) + \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F [\![w_h]\!] \cdot \mathrm{n}_F \{v_h \cdot v_h\} + \sum_{F \in \mathcal{F}_h^i} \int_F \{w_h\} \cdot \mathrm{n}_F [\![v_h]\!] \cdot \{v_h\}$$

Difficulties

- w_h is not divergence-free
- w_h and v_h have jumps and do not vanish on boundary

Incompressible NS VI

- For all (w_h, u_h, v_h), we set

$$t_h(w_h, u_h, v_h) \stackrel{\text{def}}{=} \int_\Omega ((w_h \cdot \nabla_h) u_h) \cdot v_h - \sum_{F \in \mathcal{F}_h^i} \int_F \{w_h\} \cdot \mathrm{n}_F [\![u_h]\!] \cdot \{v_h\}$$
$$+ \frac{1}{2} \int_\Omega (\nabla_h \cdot w_h)(u_h \cdot v_h) - \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F [\![w_h]\!] \cdot \mathrm{n}_F \{u_h \cdot v_h\}$$

- Key stability property:

$$t_h(w_h, v_h, v_h) = 0 \qquad \forall (w_h, v_h) \in U_h \times U_h$$

- If $u \in U$ is divergence-free and smooth, $t_h(u, u, v_h) = t(u, u, v_h)$ for all $v_h \in U_h$
  - yet, strong consistency will not be used here

Incompressible NS VII

- Alternative expression for t_h

$$t_h(w_h, u_h, v_h) = \int_\Omega \sum_{i=1}^{d} w_h \cdot \mathcal{G}_h^{2k}(u_{h,i})\, v_{h,i} + \frac{1}{2}\int_\Omega D_h^{2k}(w_h) u_h \cdot v_h + \frac{1}{4} \sum_{F \in \mathcal{F}_h^i} \int_F [\![w_h]\!] \cdot \mathrm{n}_F [\![u_h]\!] \cdot [\![v_h]\!]$$

- Boundedness for t_h: using **the discrete Sobolev embedding** for $L^4(\Omega)$, one proves for all $(w_h, u_h, v_h) \in U_h \times U_h \times U_h$

$$t_h(w_h, u_h, v_h) \lesssim |\!|\!| w_h |\!|\!|_{\mathrm{vel}} |\!|\!| u_h |\!|\!|_{\mathrm{vel}} |\!|\!| v_h |\!|\!|_{\mathrm{vel}}$$

Incompressible NS VIII

Discrete problem

- Seek $(u_h, p_u) \in U_h \times P_h$ s.t.

$$\nu a_h(u_h, v_h) + t_h(u_h, u_h, v_h) + b_h(v_h, p_h) = \int_\Omega f \cdot v_h \qquad \forall v_h \in U_h$$
$$-b_h(u_h, q_h) + \nu^{-1} j_h(p_h, q_h) = 0 \qquad \forall q_h \in P_h$$

- Existence of a discrete solution without any smallness assumption on the data
  - topological degree argument
  - use discrete stability and boundedness of t_h

Incompressible NS IX

Convergence with minimal regularity

- Let $((u_h, p_h))_{h\in\mathcal{H}}$ be a sequence of discrete solutions. Then, as $h \rightarrow 0$, up to a subsequence,

$$
\begin{aligned}
u_h &\rightarrow u && \text{in } [L^2(\Omega)]^d \\
\nabla_h u_h &\rightarrow \nabla u && \text{in } [L^2(\Omega)]^{d,d} \\
|u_h|_{\mathrm{J}} &\rightarrow 0 && \\
p_h &\rightarrow p && \text{in } L^2(\Omega) \\
|p_h|_p &\rightarrow 0 &&
\end{aligned}
$$

  where $(u, p) \in [H_0^1(\Omega)]^d \times L_*^2(\Omega)$ is an exact solution

- Convergence of the whole sequence if uniqueness

Incompressible NS X

- Asymptotic consistency for t_h For any sequence $(v_h)_{h\in\mathcal{H}}$ in $(U_h)_{h\in\mathcal{H}}$ bounded in the $|\!|\!|\cdot|\!|\!|_{\mathrm{vel}}$-norm and for any smooth function $\varphi \in [C_0^\infty(\Omega)]^d$

$$\lim_{h\to 0} t_h(v_h, v_h, \pi_h\varphi) = t'(v, v, \varphi) = \int_\Omega ((v\cdot\nabla)v)\cdot\varphi + \frac{1}{2}\int_\Omega (\nabla\cdot v)v\cdot\varphi$$

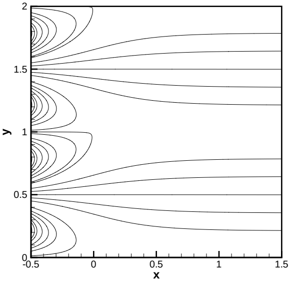
  where $v \in [H_0^1(\Omega)]^d$ is the limit of the sequence $(v_h)_{h\in\mathcal{H}}$ given by the compactness theorem

- A slightly different form of asymptotic consistency is also needed to prove the strong convergence of the pressure

Incompressible NS XI

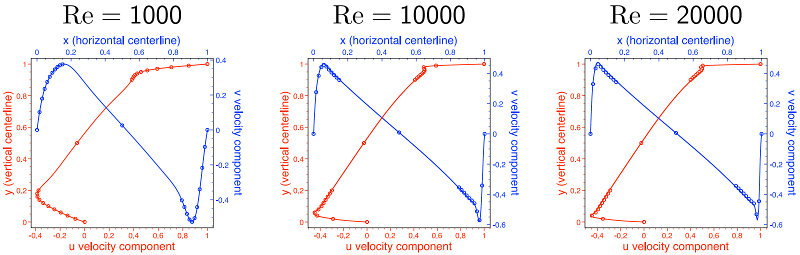
Numerical illustrations

- Kovasznay solution [K. '48] laminar flow behind a 2D grid
  - $k = 1$ for velocity and pressure, both discontinuous, 64×64 grid



Incompressible NS XII

- Lid driven cavity problem
  - $k = 2$ for velocity and pressure, continuous pressure, 120×120 grid
  - calculations from [Botti & Di Pietro '10]
  - ref. solution of [Erturk, Corke & Gökçöl '05]



Salient points of this lecture

- Discrete functional analysis (Sobolev embedding, compactness)
- Asymptotic consistency and convergence with minimal regularity
- Discrete divergence and discrete inf-sup for pressure-velocity coupling for Stokes
- Design conditions for discrete trilinear form in NS
- An existence result and a convergence result for NS with minimal regularity and general data