Singular perturbations of infinite-dimensional gradient flows

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We address the asymptotic behavior, as $\varepsilon \downarrow 0$, of the solutions to the (Cauchy problem for the) gradient flow equation

$$\varepsilon u'(t) + \mathrm{D}\mathcal{E}(t, u(t)) \ni 0 \qquad \text{in } \mathscr{H}, \quad t \in (0, T), \tag{1}$$

where $\mathscr{H}$ is a (separable) Hilbert space, $\mathcal{E} : (0, T) \times \mathscr{H} \to (-\infty, +\infty]$ is a time-dependent energy functional with $u \mapsto \mathcal{E}(t, u)$ possibly nonconvex.

The main difficulty attached to the analysis as $\varepsilon \downarrow 0$ for a family of solutions $(u_\varepsilon)_\varepsilon$ resides in the lack of estimates for u'_ε.

We develop a variational approach to this problem, based on the study of the limit of the energy identity

$$\frac{\varepsilon}{2} \int_s^t |u'_\varepsilon(r)|^2 \,\mathrm{d}r + \frac{1}{2\varepsilon} \int_s^t |\mathrm{D}\mathcal{E}(r, u_\varepsilon(r))|^2 \,\mathrm{d}r + \mathcal{E}(t, u_\varepsilon(t)) = \mathcal{E}(s, u_\varepsilon(s)) + \int_s^t \partial_t \mathcal{E}(r, u_\varepsilon(r)) \,\mathrm{d}r$$

for all $0 \leq s \leq t \leq T$, and on a fine analysis of the asymptotic properties of the quantity

$$\int_s^t |u'_\varepsilon(r)||\mathrm{D}\mathcal{E}(r, u_\varepsilon(r))| \,\mathrm{d}r.$$

In this context, the crucial hypothesis is that for every $t \in (0, T)$ the critical points of $\mathcal{E}(t, \cdot)$ are isolated, a condition of which we discuss the genericity.