On time-splitting methods for gradient flows with two dissipation mechanisms

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A gradient systems $(\mathbf{X},\mathcal{E},\mathcal{R})$ is a triple consisting of a state space $\mathbf{X}$ (typically a separable, reflexive Banach space), an energy or entropy functional $\mathcal{E}:\mathbf{X}\to\mathbb{R}\cup\{\infty\}$, and a dissipation potential $\mathcal{R}:\mathbf{X}_1{\times}\mathbf{X}\to[0,\infty]$. The associated gradient-flow equation is reads

$$0\in\partial_{\dot u}\mathcal{R}(u,\dot u)+\mathrm{D}\mathcal{E}(u)\quad\text{or equivalently}\quad \dot u\in\partial_\xi\mathcal{R}^*\big(u,-\mathrm{D}\mathcal{E}(u)\big).$$

We are interested in the case where $\mathcal{R}^*$ is a sum, namely $\mathcal{R}^*=\mathcal{R}^*_1+\mathcal{R}^*_2$. A simple linear application occurs for reaction-diffusion systems $\dot u=\mathrm{div}(\mathbb{D}\nabla u)-\mathbb{A}u$, where $\mathcal{E}(u)=\int_\Omega\frac12|u|^2\,\mathrm{d}x$ and $\mathcal{R}^*(\xi)=\int_\Omega\big(\frac12\nabla\xi:\mathbb{D}\nabla\xi+\frac12\xi\cdot\mathbb{A}\xi\big)\mathrm{d}x$. Other applications occur in the mechanics of dissipative materials e.g. in viscoelastic-viscoplastic materials with displacement u and plastic tensor p:

$$\mathcal{E}(u,p)=\int_\Omega\big(\tfrac12\mathbf{e}(u){:}\mathbb{C}\mathbf{e}(u)+\tfrac12p{:}\mathbb{H}p\big)\mathrm{d}x\ \text{ and }\ \mathcal{R}(\dot u,\dot p)=\int_\Omega\big(\mathbf{e}(\dot u){:}\mathbb{V}_{\mathrm{el}}\mathbf{e}(\dot u)+\sigma_{\mathrm{yield}}|\dot p|+\tfrac12p{:}\mathbb{V}_{\mathrm{pl}}p\big)\mathrm{d}x.$$

We investigate the question under what assumption the time-splitting algorithm

$$\begin{aligned}\dot u&\in2\,\partial_\xi\mathcal{R}^*_1\big(-\mathrm{D}\mathcal{E}(u)\big)&&\text{for } Nt\in[0,1/2[\quad\text{mod }1,\\ \dot u&\in2\,\partial_\xi\mathcal{R}^*_2\big(-\mathrm{D}\mathcal{E}(u)\big)&&\text{for } Nt\in[1/2,1[\quad\text{mod }1,\end{aligned}$$

converges for $N\to\infty$. We give a finite-dimensional example showing that convergence does not hold if $\mathcal{E}$ is only convex but not differentiable. Under natural conditions we provide convergence results, also for the case where only time-discrete minimization steps are done.

This is ongoing, joint work with Riccarda Rossi (U Brescia) and Artur Stephan (WIAS).