An H-convergence-based implicit function theorem for homogenization of nonlinear non-smooth elliptic systems

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Abstract

We consider homogenization of semilinear elliptic PDE systems of the type

$$\partial_{x_i}\Big(a_{ij}^{\alpha\beta}(\varepsilon,x)\partial_{x_j}u^\beta(x)+f_i^\alpha(x,u(x))\Big)=0 \text{ in } \Omega,\ \alpha=1,\dots,n,$$

with homogeneous Dirichlet boundary conditions. Here $\varepsilon>0$ is the small homogenization parameter, $\Omega\subset\mathbb{R}^N$ is a bounded Lipschitz domain, $a_{ij}^{\alpha\beta}(\varepsilon,\cdot)\in L^\infty(\Omega)$ satisfy the Legendre ellipticity condition, and $u\in C(\overline{\Omega};\mathbb{R}^n)\mapsto f_i^\alpha(\cdot,u(\cdot))\in L^{p_0}(\Omega)$ are C^1-smooth with certain $p_0>N$. We suppose that the family of diffusion tensors $[a_{ij}^{\alpha\beta}(\varepsilon,\cdot)]$ H-converges for $\varepsilon\to 0$ and that

$$N=2 \text{ or } a_{ij}^{\alpha\beta}=0 \text{ for } \alpha>\beta.$$

Our result is of implicit function theorem type: If u_0 is a non-degenerate weak solution to the homogenized problem, then for $\varepsilon\approx 0$ there exists exactly one weak solution $u=u_\varepsilon$ with $\|u-u_0\|_\infty\approx 0$, and $\|u_\varepsilon-u_0\|_\infty\to 0$ for $\varepsilon\to 0$.

The main tools of the proofs are gradient estimates of Meyers and Morrey type for solutions to linear elliptic systems with non-smooth data. Neither assumptions about global solution uniqueness are needed nor additional smoothness of $\partial\Omega$ or $a_{ij}^{\alpha\beta}$ or f_i^β or u_0 nor growth restrictions for $f_i^\alpha(x,\cdot)$.