On the existence of weak solutions to Prandtl's (1945) one-equation model of turbulence

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Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. In $Q_T = \Omega \times]0,T[$ we consider the following system of PDEs:

$$
\begin{aligned}
&(1) \qquad & \operatorname{div} \boldsymbol{u} &= 0\,, \\
&(2) \qquad & \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= \operatorname{div}(\ell\sqrt{k}\,\boldsymbol{D}(\boldsymbol{u})) - \nabla p + \boldsymbol{f}\,, \\
&(3) \qquad & \partial_t k + \boldsymbol{u}\cdot\nabla k &= \operatorname{div}(\ell\sqrt{k}\,\nabla k) + \ell\sqrt{k}\,|\boldsymbol{D}(\boldsymbol{u})|^2 - \frac{k\sqrt{k}}{\ell}\,,
\end{aligned}
$$

where: $\boldsymbol{u} = (u_1, u_2, u_3)$ mean velocity, $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top)$; p mean pressure, $k =$ mean kinetic turbulence energy. The function ℓ represents the mixing length of the fluid motion.

In 1945, L. Prandtl postulated eq. (3) as a model for the transport of the mean kinetic turbulence theory for the fully developed turbulent motion of an incompressible fluid.

In this talk we consider mixing lengths ℓ which satisfy the conditions

$$\ell \in C(\overline{\Omega}), \quad \ell(x) > 0 \;\forall\, x \in \Omega, \quad \{x \in \partial\Omega;\, \ell(x) = 0\} \neq \emptyset\,.$$

We present an existence result for a weak solution to (1)–(3) under initial conditions in $\Omega \times \{0\}$ and boundary conditions on $\{x \in \partial\Omega;\, \ell(x) > 0\} \times]0,T[$.