1.1 Extremely Short Optical Pulses

"You must go on and find out all about that light, and what it is for, and if all is perfectly safe..." J.R.R. Tolkien

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Light signals have been used to transmit information since the earliest days of human history, for instance, in the form of "signal bombs" (flares), which were used by the Chinese to report on the movements of the Mongol army during the siege of Yangzhou in 1276. The Mongols also mastered the technique of light signals: They used large lanterns to control their troops at nighttime. Better command and control was one of the reasons why the Mongols' invasion of the rest of the Asian world was so successful. At about the same time, attempts to defend against the Viking attacks on the other side of the continent in Europe suffered from the slow exchange of information between the unprotected coastal settlements and the military authorities.

Mankind has come a long way since the time of signal fires and nowadays information is transmitted in a more effective manner: via short pulses consisting of electromagnetic waves, so-called *wave packets*, like in Figure 1. The Morse code distress signal $\overline{\text{SOS}}$, established in 1906, is a combination of nine such pulses of just two types (short and long or, mathematically speaking, 0 and 1), which is sufficient to encode any message. The most ambitious project that employs short pulses to exchange information is without a doubt the World Wide Web, where a typical data center network is capable to transmit 40 Gb/s. In plain language, one would need one hour to create a full backup of the United States Library of Congress, of course, only after its complete digitization. As the digitization is progressing slowly, and the technology is rapidly evolving, the actual time will be much less than one hour.

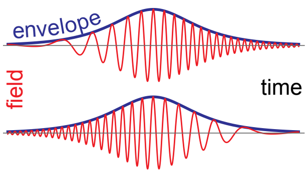

Fig. 1: *Two wave packets with the same envelope. To distinguish between them, a complex-valued envelope is used.*

Both old-fashioned signal lights and modern wave packets, the latter of which are invisible to the naked eye, transmit information using the same electromagnetic waves. All such waves are subject to one common set of equations discovered by James Clerk Maxwell in the mid-19th century. The main practical difference between these pulses is their duration: The modern pulses in optical networks are, to put it mildly, much shorter than the signals from the Chinese signal bombs. It is no wonder that especially the shortest possible pulses are ideally suited for the quick transfer of information, even taking into account all the difficulties associated with their *generation*, made possible by the invention of lasers, and *transmission*, made possible by the invention of optical fibers. This is how the *ultrashort pulses* came into play.

"...the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics." R. Feynman

It is important to note that both short (with the duration of one trillionth of a second) and ultrashort (up to 1,000 times shorter) optical pulses have numerous practical and potential applications in addition to the transmission of information. In the field of popular science, for instance, they allow for the filming of a bullet piercing an apple, as in a famous sequence of photos made by Harold E. Edgerton. To give a more practical example, let an external electromagnetic pulse with the duration t_0 hit and go through a small target, which reacts by emitting its own radiation. On the time scale $t > t_0$, the emitted target's radiation is separated from the initial pulse, which is already gone, what is not the case for $t < t_0$. Ultrashort pulses make it possible to initiate and observe certain very fast processes, those that are fully accomplished and thus unobservable on a larger time scale. A close analogy is found in mathematics: If we know how a (linear) system reacts on a short-but-strong perturbation, which physicists associate with Paul Dirac and call *Dirac's delta function*, we can calculate how the system reacts on any perturbation. A prominent application of this approach was the observation of a chemical reaction on molecular level (the 1999 Nobel Prize in chemistry).

One more possibility to make use of ultrashort optical pulses is to take advantage of similarities between different physical systems, whose behavior is determined by the propagation and interaction of waves. It may happen that *mathematically* such systems are described by the same equation. *Optical black holes*, for instance, appear in a specially designed optical system that simulates solutions of Einstein's equations of general relativity. Such an analog black hole is represented by one edge of a short, extremely intense optical pulse propagating in a fiber. Just as a true black hole absorbs the matter from the surrounding space, the optical one absorbs the energy from other pulses, with the main pulse gradually becoming shorter and stronger. The analogy is destroyed when the hole-presenting pulse becomes too extreme and certain optical effects, which have no equivalent in the theory of gravity, come into play. Before that happens, the optical black hole gives laboratory access to mysterious phenomena such as event horizons and Hawking radiation, which are believed to play their part somewhere in deep space [1].

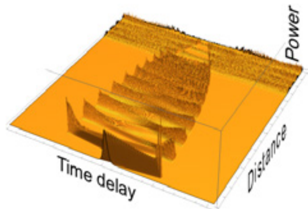

Fig. 2: *A trapped pulse that propagates bouncing between two invisible extreme pulses, like between two mirrors in a laser cavity*

Optical white holes also exist; they are represented by the second edge of the same ultrashort pulse, and they, in turn, feed other pulses with energy. Both edges of the seed ultrashort pulse serve as impermeable barriers for the ordinary radiation (in a range of wavelengths that, by the way, was calculated at WIAS, see [2] and the references cited therein). Among other things, the impermeable extreme pulses might replace physical mirrors in certain fantastical devices such as an all-optical laser cavity in Figure 2. Alternatively, the external radiation can be applied to the "black" and "white" sides of an extreme optical pulse to switch the pulse on and off, just like it happens with the electric current in a transistor. This is a possible approach to all-optical switching and optical transistors.

Another example comes from fluid dynamics. Ocean waves have much in common with electromagnetic waves in fibers: To a certain extent, both wave systems are governed by the same nonlinear Schrödinger equation (NLSE). Here, an important issue is the statistical distribution of the heights of individual ocean waves and especially the probability with which the most dangerous killer waves appear. Sailors often reported on the spontaneous appearance of huge ocean waves, which were two and even three times larger than their neighbors; most accidents with large ships were attributed to such waves. Physicists have argued that the extreme waves are also extremely rare. Their assumed distribution, derived by Lord Rayleigh, had a Gaussian tail. The probability of a large wave was then considered to be negligibly small. In plain language, an observer of stormy weather would have to wait about 27 years before a wave that is three times larger than the average wave height appears.

By the way, the nonlinear Schrödinger equation has no relation to Erwin Schrödinger and his famous equation. Being different, they just look similar and the name has stuck.

The first fully recorded 25.6 m killer wave, which was more than twice as tall as its neighbors and which damaged a sea platform in the North Sea in 1995, put an end to the arguments. The question arose about the practical measurement of an unknown statistical distribution. If wave statistics is difficult to collect in a rough sea, why not to collect it in a fiber with its millions of pulses per second? It was done by Solli et al., and indeed demonstrated that the extreme waves appear much more often than previously thought by Lord Rayleigh and his followers; the scientists were wrong, and the sailors were telling the truth [3].

"Waves that appear from nowhere and disappear without a trace." N. Akhmediev

Direct observations of ocean waves were made using satellites covering large areas simultaneously. As a result, the equivalent of 27 years for an individual observer could be made in 14 minutes for one million of the digital observations. These observations confirmed both the existence of killer waves and the non-Gaussian distribution of the wave heights. Needless to say, the use of the satellites was much more expensive and yet necessary because human lives were at stake.

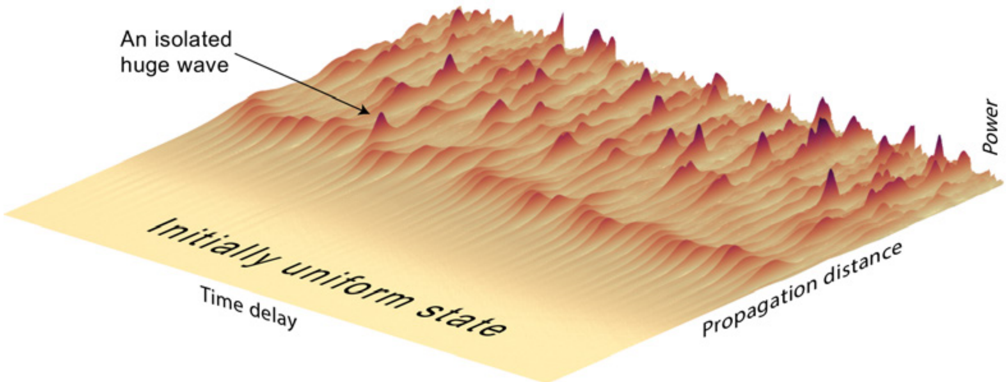

Fig. 3: *A numerical solution of the nonlinear Schrödinger equation that demonstrates how a small-amplitude uniform wave is destroyed and replaced by a turbulent state with the spontaneous extreme events*

What is important from a purely scientific standpoint: Solli's experiments demonstrated universality of extreme waves, which should be expected and indeed have been found in many nonlinear systems later on. Like the theory of catastrophes, killer waves have become very popular and are used to explain just about anything, even economic shocks. Staying on solid scientific ground, an exemplary numerical solution of the above-mentioned NLSE, which equally applies to both fibers and oceans, is shown in Figure 3. What is actually plotted is the squared magnitude of the envelope from Figure 1, which is proportional to the power. One can see how a complex turbulent "rough sea" develops from an initially uniform state and how huge isolated waves appear here and there. To prove their non-Gaussian nature, one has to make several thousands of such calculations to collect statistics of the potential extreme events.

Considering all of the above, there is little doubt that physics of ultra-fast phenomena is a fascinating area of modern science and that the generation of short pulses is a complex technical task. Are there any mathematical problems with their propagation? An accurate description of pulses in fibers seems to be simple. As opposed to string theory or quantum gravity, the fundamental equations governing all electromagnetic waves in nature have been known for almost 200 years. Moreover, pulses in optical fibers propagate in one spatial dimension, making their mathematical description even simpler. Given the equations and availability of powerful computers, why not solve the pulse propagation problem by brute force? The devil is in the separation of scales.

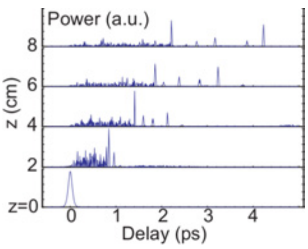

Fig. 4: *As pulses propagate differently, the numerical solution domain may become too small*

The smallest spatial scale for an optical pulse is its carrier wavelength with the typical value around 1.4 μm, which is close to the minimal attenuation in a silica fiber. This spatial scale is much smaller than the propagation distance, because the maximal length of a fiber optic cable is around 100 km. A direct calculation that resolves each wavelength all along the fiber is impossible. All useful pulse propagation models, including NLSE in the first place, employ approximations, namely:

(I) A pulse or a sequence of pulses is considered on a relatively small and comoving computational domain. As all pulses change their velocities and do it *differently* depending on their parameters, the domain may become too narrow and the calculation will have to be repeated (Figure 4).

(II) All normal evolution problems in physics predict a system's state at (time) $t > 0$ from the initial state at $t = 0$. In fiber optics, a system's state is "known" at (position along the fiber) $z = 0$ for all t and should be calculated for the evolution coordinate $z > 0$ for all t. Here, the causality principle is sacrificed (!!) to get the most simple propagation equation.

(III) All pulses move in one common direction along the fiber, and the backward waves are ignored. Moreover, these pulses are not too intense and propagate in a weakly nonlinear limit. We then have a small parameter that makes it possible to expand and simplify the equations.

(IV) As a typical picosecond pulse at $1.4\,\mu$m contains about 200 field oscillations, it is properly described by its carrier frequency ω_0 and envelope ψ. The *slowly varying envelope approximation* (SVEA) assumes that ψ is smooth and does not change much on the time scale $1/\omega_0$.

In the simplest case, the envelope (like in Figure 1) does not change at all and just moves along the fiber with the velocity $V = \text{const}$. The envelope ψ depends only on the delay variable $\tau = t - z/V$, and the propagation equation simply states that $\partial_z \psi(z,\tau) = 0$. If the simplest scenario is destroyed (by dispersion, nonlinearity, attenuation, and so on), the propagation equation takes the form

$$\partial_z \psi(z,\tau) = \text{dispersionOperator}(\psi) + \text{nonlinearOperator}(\psi) + \cdots, \tag{1}$$

where each term on the right-hand side accounts for a certain physical process. They act independently of each other because of the weak nonlinearity and SVEA; the list of the involved operators can be found in any textbook on fiber optics.

The simplest representative of (1) is the (normalized, focusing) NLSE

$$\partial_z \psi = i\partial_\tau^2 \psi + i|\psi|^2 \psi, \tag{2}$$

which is integrable. This is why numerous explicit solutions for all kinds of solitary pulses and spontaneous killer waves were found. By choosing both the dispersion and nonlinear operators in a special and very sophisticated manner, one can get further integrable envelope equations [4], but in practice, the general applications-relevant Eq. (1) is solved numerically. This is usually done by the split-step method, where the change from $\psi(z,\tau)$ to $\psi(z+\Delta z,\tau)$ occurs by the successive accounting of contributions of all involved operators one by one (Figure 5).

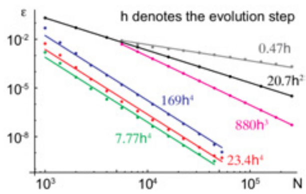

Fig. 5: *Relative error* ε *versus discretization* N *for different splitting methods. The two best methods (green and red lines) were found at WIAS [5].*

RG 2's research in the context of the application topic "Optical pulses in nonlinear media" is focused on pulses, which are so extreme that at least one of the assumptions from the list (I–IV) is violated. In the first place, this applies to the ultra-short few-cycle pulses with a duration of several femtoseconds, such that instead of 200 field oscillations within the pulse, we have only 1 or 2 of them.

Speaking of few-cycle pulses, is there any useful replacement of the envelope concept, one that avoids the costly solution of the full Maxwell system? The SVEA-independent definition of the envelope that is currently accepted in optics employs the so-called *analytic signal* and applies to any pulse, whether short or not. And yet the definition (a signal that is analytic in the upper half-plane of complex times) looks like a trick and comes from nowhere. Instead, we employ the so-called *classical creation-annihilation fields*, which are borrowed from the continuous Hamiltonian mechanics. The complex envelope is chosen to transform the Hamiltonian to its normal form. The definition is equivalent to the standard one for a longer pulse and is different from and better than the standard one for a few-cycle pulse. Among other things, the use of the creation-annihilation fields reduces the costs of the split-step solution.

Another area of research that is closely related to ultrashort pulses deals with an accurate description of medium dispersion, which is encoded in the $\mathrm{dispersionOperator}(\psi)$ in (1). Most commonly, the dispersion effect is approximated by a differential operator, e.g., by $i\partial_\tau^2$ in the simplest Eq. (2) and by a higher-order differential operator in a more general Eq. (1). Being unbounded, these operators lead to stiff numerical solutions. The situation with the ultrashort pulses is especially dangerous, as they have wide spectra such that higher-order derivatives of quickly oscillating frequency components ($e^{-i\omega\tau}$ with large ω) come into play. The difficulty is considerably relaxed by replacing the polynomial approximations with rational ones.

Throughout the lifetime of the application topic "Optical pulses in nonlinear media", a considerable effort was invested in the numerical solutions of (1) and more specific pulse propagation models that avoid the use of the envelope. In addition to the conventional splitting methods, special attention was given to additive methods, such as the Burstein & Mirin splitting

$$e^{h(L+N)} = \frac{2}{3}\left(e^{\frac{h}{2}L}e^{hN}e^{\frac{h}{2}L} + e^{\frac{h}{2}N}e^{hL}e^{\frac{h}{2}N}\right) - \frac{1}{6}\left(e^{hN}e^{hL} + e^{hL}e^{hN}\right) + O(h^4), \qquad (3)$$

where the evolution operator e^{hL} (e^{hN}) yields the solution of the linear (nonlinear) subproblem in (1) on $z \in (0, h)$. The splitting (3) consists of four threads that can be calculated in parallel. We found a new class of the additive splittings, a far-reaching generalization of (3); see [5].

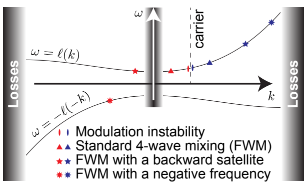

Fig. 6: *Four possible wave mixing scenarios. One with a backward wave and one with a negative frequency wave are new, see [6]*

Last but not least, we studied situations where excitation of backward waves cannot be ignored because it takes place in a resonant way in the course of wave mixing. We found a kind of Brillouin's scattering that takes place due to optical nonlinearities without any involvement of the material waves. The new theory describes non-envelope pulses propagating in both directions ([6], Figure 6).

References

[1] T.G. Philbin, C. Kuklewicz, S. Robertson, S. Hill, F. Konig, U. Leonhardt, *Fiber-optical analog of the event horizon*, Science, **319** (2008), pp. 1367–1370.

[2] U. Bandelow, S. Amiranashvili, S. Pickartz, *Stabilization of optical pulse transmission by exploiting fiber nonlinearities*, J. Lightwave Technol., **38** (2020), pp. 5743–5747.

[3] D.R. Solli, C. Ropers, P. Koonath, B. Jalali, *Optical rogue waves*, Nature, **450** (2007), pp. 1054–1057.

[4] U. Bandelow, A. Ankiewicz, S. Amiranashvili, N. Akhmediev, *Sasa–Satsuma hierarchy of integrable evolution equations*, Chaos, **28** (2018), pp. 053108/1–053108/11.

[5] S. Amiranashvili, M. Radziunas, U. Bandelow, K. Busch, R. Čiegis, *Additive splitting methods for parallel solutions of evolution problems*, J. Comp. Phys., **436** (2021), pp. 1–14.

[6] S. Amiranashvili, U. Bandelow, *Unusual scenarios in four-wave-mixing instability*, Phys. Rev. A, **105** (2022), pp. 063519/1–063519/6.