1.6 Energy-based Solution Concepts for a Geophysical Fluid Model

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The motion of tectonic plates, which results from convective processes within the Earth's mantle, is nowadays a widely accepted theory to explain the formation of many of the Earth's geological structures like the Himalaya mountain range in Asia or the San Andreas Fault in California, USA. These tectonic plates as a whole form the Earth's lithosphere, which consists of the Earth's crust and the uppermost part of its mantle (see Figure 1). Geophysicists at the *German Research Centre for Geosciences* in Potsdam describe this process of rock deformation as the flow of a fluid, which happens on large time scales and usually moves only several centimeters per year. Initiated from discussions within the project B01 of the Collaborative Research Center 1114: *Scaling Cascades in Complex Systems*, a group at WIAS started an analytical study of the corresponding models.

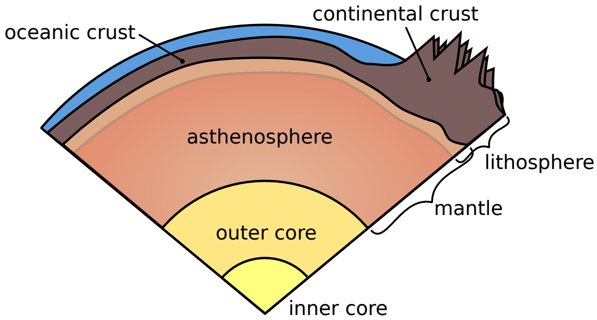

Fig. 1: *A schematic diagram of the Earth's internal structure*

A suitable model for such kind of geophysical flow has to take into account viscous, elastic, and plastic properties. These terms describe the resistance of the flow to deformations: viscosity in terms of deformation rates, but elasticity and plasticity in terms of the actual displacement, which differ in the property whether or not the system returns to the original state when forces are removed (see Figure 2). The plastic effects reflect the brittle nature of rocks and lead to highly nonlinear and nonsmooth terms in the corresponding partial differential equations (PDEs), so that solutions can neither be expected to be smooth nor to satisfy the equations in a pointwise sense. Therefore, a rigorous analytic investigation of the model has to overcome the limitations of classical smooth solutions and needs to be founded on generalized solution concepts. Specifying a suitable notion of generalized solutions to a nonlinear PDE can already be a nontrivial task. Besides the accessibility via mathematical tools, a good choice should also reflect physical principles like the laws of thermodynamics.

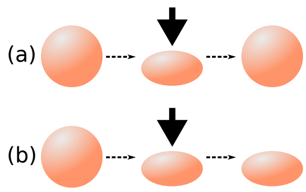

Fig. 2: *After removing forces, an elastic material returns to the original state (a), while a plastic material keeps its shape (b)*

Based on these ideas, we investigated solution concepts and combined energetic considerations with a variational approach suitable for treating the nonsmooth plasticity term and for establishing existence of global-in-time solutions. However, as is also the case for much simpler fluid models, the uniqueness of solutions cannot be guaranteed. Instead, we established a *weak-strong uniqueness property* for these generalized solutions, which means that a generalized solution coincides with the (unique) smooth solution as long as the latter exists. This result is based on a *relative energy inequality*, which expresses the difference of a generalized solution to any function in terms of a *relative energy*.

In a first step, we introduced a diffusive term for the stress evolution. There are viscoelastic models where this term can be justified from physical principles, but it also improved the mathematical properties of the problem and enabled us to obtain existence of solutions in a generalized sense [1]. To also treat the case without stress diffusion, which corresponds to the scenario of the numerical investigations in [5], we introduced the notion of *energy-variational solutions* based on the aforementioned relative energy inequality. In this inequality, a passage to vanishing stress diffusion was possible, and we could show existence of energy-variational solutions in this case [2]. Additionally, we obtained a weak-strong uniqueness principle as an immediate consequence of the relative energy inequality. The idea to base a solution concept on a relative energy inequality goes back to Pierre-Louis Lions, who used a similar approach to define the so-called *dissipative solutions* to the Euler equations.

Modeling of viscoelastoplastic fluids

The fundamental equations for modeling fluid flow as a continuum are the *Navier–Stokes equations*. They describe the evolution of the *Eulerian velocity field* $\boldsymbol{v}$ that represents the velocity $\boldsymbol{v}(t,x)$ of a fluid particle at point x in space at time t in terms of two physical principles from classical mechanics: conservation of mass and balance of momentum. In particular, the latter contains the internal forces, described in terms of the *Cauchy stress tensor* $\mathbb{T}$. The physical properties represented by the model depend on the relation between the stress tensor $\mathbb{T}$ and the *strain rate* $\mathbb{D}=\frac{1}{2}(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^\top)$, which is the symmetric part of the velocity gradient and describes the relative motion between the particles. For example, if $\mathbb{T}$ is independent of the strain rate $\mathbb{D}$, then the model does not consider friction between fluid particles, and we obtain the Euler equations for a perfect fluid.

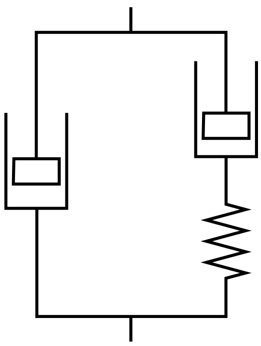

Fig. 3: A model for Jeffreys rheology. Two linear viscous elements (dashpots) are combined with a linear elastic element (spring).

Viscoelastic fluids. The simplest models for viscoelastic rheology combine linear viscous and elastic effects. In our work, we decomposed the Cauchy stress into a radial part, governed by the pressure p, and a *deviatoric part*, which is a symmetric matrix with vanishing trace. Similarly to [5], we considered a rheology model of Jeffreys type, where the deviatoric stress consists of a linear viscous part and an additional part $\mathbb{S}$ satisfying a stress-strain relation of Maxwell type, *i.e.*, the strain rate decomposes into an elastic and a viscous contribution (see Figure 3). The elastic part is determined by the *objective stress rate* $\overset{\triangledown}{\mathbb{S}}$, which is a notion of time derivative independent of the reference frame. In the literature, there are different choices for this stress rate, and we used the *Zaremba–Jaumann rate*, which is a common choice in geophysical flow models, cf. [5]. In summary, we obtain

$$\mathbb{T}=\mathbb{S}+2\mu\mathbb{D}-p\mathbb{I}, \qquad \frac{1}{\eta}\overset{\triangledown}{\mathbb{S}}+a\mathbb{S}=\mathbb{D}, \qquad \text{with} \quad \overset{\triangledown}{\mathbb{S}}=\partial_t\mathbb{S}+\boldsymbol{v}\cdot\nabla\mathbb{S}+\mathbb{S}\mathbb{W}-\mathbb{W}\mathbb{S}, \tag{1}$$

for viscosity constants $\mu, a > 0$, an elastic shear modulus $\eta > 0$, and for the *spin tensor* $\mathbb{W}=\frac{1}{2}(\nabla\boldsymbol{v}-\nabla\boldsymbol{v}^\top)$, *i.e.*, the skew-symmetric part of the velocity gradient.

Modeling plasticity. To model plasticity, the stress-strain relation (1) needs to be further modified. For so-called *perfect plasticity*, one fixes a *yield stress* $\sigma_{\text{yield}} > 0$, which is an upper bound for the stress magnitude, and plastic deformation sets in when this threshold is reached. We include this behavior in the previous model by replacing the evolution equation for $\mathbb{S}$ in (1) with

$$|\mathbb{S}| \le \sigma_{\text{yield}} \quad \text{and} \quad \tfrac{1}{\eta}\overset{\nabla}{\mathbb{S}} + a\mathbb{S} + \lambda\mathbb{S} = \mathbb{D} \quad \text{for} \quad \lambda \ge 0 \quad \text{with} \quad \lambda(\sigma_{\text{yield}} - |\mathbb{S}|) = 0,$$

such that the parameter $\lambda \ge 0$ can be arbitrarily large in the case $|\mathbb{S}| = \sigma_{\text{yield}}$. This allows for arbitrarily large strain rates $\mathbb{D}$, which can be physically observed in the form of the onset of sudden motion that may result in earthquakes. In particular, this nonsmooth stress-strain relation can lead to a highly nonlinear behavior. We can abbreviate the previous relation as

$$\tfrac{1}{\eta}\overset{\nabla}{\mathbb{S}} + \partial\mathcal{P}(\mathbb{S}) \ni \mathbb{D} \quad \text{with} \quad \mathcal{P}(\mathbb{S}) = \int_\Omega \mathfrak{P}(\mathbb{S}(x))\,\mathrm{d}x, \tag{2}$$

where $\partial\mathcal{P}$ denotes the subdifferential of a convex dissipation potential $\mathcal{P}$ induced by the density $\mathfrak{P}$ given by $\mathfrak{P}(\mathbb{S}) = \frac{a}{2}|\mathbb{S}|^2$ if $|\mathbb{S}| \le \sigma_{\text{yield}}$ and $\mathfrak{P}(\mathbb{S}) = \infty$ else (see Figure 4). For the carried-out analysis, the exact form of $\mathcal{P}$ was not relevant, and we studied a general convex, lower semicontinuous dissipation potential $\mathcal{P}\colon L^2(\Omega)^{3\times 3} \to [0,\infty]$ with $\mathcal{P}(0) = 0$.

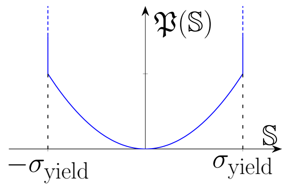

***Fig. 4:** The density $\mathfrak{P}$ of the dissipation potential $\mathcal{P}$ in one dimension*

Generalized solution concepts

We consider the viscoelastoplastic fluid inside a bounded domain $\Omega \subset \mathbb{R}^3$ and in a time interval $(0,T)$ with $T > 0$. Combining the above stress-strain relation with the Navier–Stokes equations for incompressible fluids, we obtain the system

$$\rho(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) = \operatorname{div}\left(\mathbb{S} + 2\mu\mathbb{D} - p\mathbb{I}\right), \qquad \operatorname{div}\boldsymbol{v} = 0, \qquad \tfrac{1}{\eta}\overset{\nabla}{\mathbb{S}} + \partial\mathcal{P}(\mathbb{S}) - \gamma\,\Delta\mathbb{S} \ni \mathbb{D} \tag{3}$$

in $\Omega \times (0,T)$ equipped with appropriate initial and boundary conditions, where $\rho > 0$ denotes the constant density. Observe that the stress-strain relation $(3)_3$ does not coincide with (2), but we introduced a term for *stress diffusion* with coefficient $\gamma > 0$. This makes the problem a fully parabolic problem, while for $\gamma = 0$, system (3) is of mixed parabolic-hyperbolic type.

The total energy of the system is given by $\mathcal{E}(\boldsymbol{v},\mathbb{S}) = \frac{\rho}{2}\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \frac{1}{2\eta}\|\mathbb{S}\|^2_{L^2(\Omega)}$ and consists of the kinetic energy and the stored elastic energy. By formally multiplying $(3)_1$ and $(3)_3$ with $\boldsymbol{v}$ and $\mathbb{S}$, respectively, and integrating in space and time, we obtain the *energy-dissipation balance*

$$\mathcal{E}(\boldsymbol{v}(t),\mathbb{S}(t)) + \mu\int_0^t \|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)}\mathrm{d}s + \int_0^t\int_\Omega \partial\mathcal{P}(\mathbb{S}) : \mathbb{S}\,\mathrm{d}x\mathrm{d}s + \gamma\int_0^t \|\nabla\mathbb{S}\|^2_{L^2(\Omega)}\mathrm{d}s = \mathcal{E}(\boldsymbol{v}_0,\mathbb{S}_0), \tag{4}$$

where we used the matrix scalar product $\mathbb{A} : \mathbb{B} := \mathrm{Tr}(\mathbb{A}^\top\mathbb{B})$ and that $\operatorname{div}\boldsymbol{v} = 0$ implies the relation $\|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)} = 2\|\mathbb{D}\|^2_{L^2(\Omega)}$. This shows that the total energy decreases along solutions due to the dissipative effects of the viscosity, the dissipation potential $\mathcal{P}$, and the stress diffusion. Note that (4) only holds formally since the dissipation potential may be nonsmooth, and the subdifferential $\partial\mathcal{P}$ is multi-valued in general. In this case, we can estimate the respective term in (4) by using the definition of the subdifferential

$$\mathbb{E} \in \partial \mathcal{P}(\mathbb{S}) \iff \forall \Phi \in L^2(\Omega)^{3\times 3}: \int_\Omega \mathbb{E} : (\mathbb{S} - \Phi)\, \mathrm{d}x \geq \mathcal{P}(\mathbb{S}) - \mathcal{P}(\Phi). \tag{5}$$

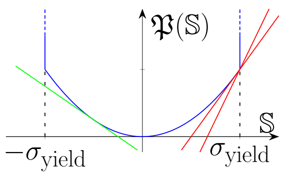

Fig. 5: *While the subdifferential is single-valued at points where $\mathcal{P}$ is smooth (green), it can be multi-valued where this is not the case (red)*

Finally, (4) suggests that for $\gamma > 0$, a solution $(\boldsymbol{v}, \mathbb{S})$ should be searched within the class defined by

$$(\boldsymbol{v}, \mathbb{S}) \in L^\infty(0, T; L^2(\Omega)^3 \times L^2(\Omega)^{3\times 3}) \cap L^2(0, T; H^1(\Omega)^3 \times H^1(\Omega)^{3\times 3}). \tag{6}$$

Generalized solutions. A first attempt to find a suitable notion of weak solutions is to multiply the equations $(3)_1$ and $(3)_3$ with smooth test functions and to integrate. However, for $(3)_3$, we again meet the problem that $\partial\mathcal{P}$ can be multi-valued. To circumvent this problem by means of (5), we instead formally multiply $(3)_3$ with $\mathbb{S} - \Phi$ for a test function Φ, which leads to a variational formulation. We proceed similarly for the evolution of $\boldsymbol{v}$. In summary, we call a pair $(\boldsymbol{v}, \mathbb{S})$ with (6) a *generalized solution* to (3) if it satisfies

$$\frac{\rho}{2}\|\boldsymbol{v} - \boldsymbol{\varphi}\|^2_{L^2(\Omega)}\Big|_0^t + \int_0^t\int_\Omega \Big[\rho\partial_t\boldsymbol{\varphi}\cdot(\boldsymbol{v} - \boldsymbol{\varphi}) - \rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\varphi} + (\mathbb{S} + \mu\nabla\boldsymbol{v}) : \nabla(\boldsymbol{v} - \boldsymbol{\varphi})\Big]\,\mathrm{d}x\mathrm{d}s \leq 0, \tag{7}$$

$$\frac{1}{2\eta}\|\mathbb{S} - \Phi\|^2_{L^2(\Omega)}\Big|_0^t + \int_0^t\Big[\int_\Omega \Big(\frac{1}{\eta}\overset{\triangledown}{\Phi} - \mathbb{D}\Big) : (\mathbb{S} - \Phi) + \gamma\nabla\mathbb{S} : \nabla(\mathbb{S} - \Phi)\,\mathrm{d}x + \mathcal{P}(\mathbb{S}) - \mathcal{P}(\Phi)\Big]\,\mathrm{d}s \leq 0 \tag{8}$$

for a.a. $t \in (0, T)$ and all suitable test functions $(\boldsymbol{\varphi}, \Phi)$. Note that the nonlinear terms are hidden in the Zaremba–Jaumann rate $\overset{\triangledown}{\Phi} : \mathbb{S}$ (see $(1)_3$). In particular, choosing both test functions equal to 0, we obtain separate energy inequalities for the kinetic and the elastic energy. Summing up and applying (5), these resemble (4) with an inequality. This energy estimate serves as an *a priori* bound in our analysis. In [1], we showed existence of generalized solutions to (3) for $\gamma > 0$ by regularization of the dissipation potential $\mathcal{P}$. This makes it possible to work with a weak formulation and to show existence via a Galerkin approximation. As usual, this approach only yields weak convergence of an approximating sequence, but to pass to the limit in the nonlinear terms in (8), the strong convergence with respect to $\mathbb{S}$ is necessary. Within the function class from (6), this follows from the famous Aubin–Lions lemma, which explains the necessity of stress diffusion, that is, of the assumption $\gamma > 0$.

Relative energy inequality. We further derived an inequality for the *relative energy* $\mathcal{R}(\boldsymbol{v}, \mathbb{S} \mid \boldsymbol{\varphi}, \Phi) = \mathcal{E}(\boldsymbol{v} - \boldsymbol{\varphi}, \mathbb{S} - \Phi)$, which provides an energy-based way to measure distances. From (7), (8), and Gronwall's inequality, one concludes that the generalized solution satisfies the *relative energy inequality*

$$\begin{aligned}&\mathcal{R}(\boldsymbol{v}(t), \mathbb{S}(t) \mid \boldsymbol{\varphi}(t), \Phi(t)) - \mathcal{R}(\boldsymbol{v}_0, \mathbb{S}_0 \mid \boldsymbol{\varphi}(0), \Phi(0))\mathrm{e}^{\int_0^t \mathcal{K}(\boldsymbol{\varphi},\Phi)\,\mathrm{d}s}\\ &+ \int_0^t \Big(\mathcal{W}^{(\mathcal{K})}(\boldsymbol{v} - \boldsymbol{\varphi}, \mathbb{S} - \Phi \mid \boldsymbol{\varphi}, \Phi) + \mathcal{P}(\mathbb{S}) - \mathcal{P}(\Phi) + \Big\langle \mathcal{A}(\boldsymbol{\varphi}, \Phi), \begin{pmatrix}\boldsymbol{v} - \boldsymbol{\varphi}\\ \mathbb{S} - \Phi\end{pmatrix}\Big\rangle\Big)\mathrm{e}^{\int_s^t \mathcal{K}(\boldsymbol{\varphi},\Phi)\,\mathrm{d}\tau}\,\mathrm{d}s \leq 0\end{aligned} \tag{9}$$

for a.e. $t \in (0,T)$ and all sufficiently regular $(\boldsymbol{\varphi},\Phi)$; see [2]. The weight $\mathcal{K} \geq 0$ determines the regularity of test functions $(\boldsymbol{\varphi},\Phi)$, the function $\mathcal{W}^{(\mathcal{K})}$ denotes the relative dissipation-like quantity

$$\begin{aligned}\mathcal{W}^{(\mathcal{K})}(\tilde{\boldsymbol{v}},\tilde{\mathbb{S}}|\boldsymbol{\varphi},\Phi) &:= \mathcal{K}(\boldsymbol{\varphi},\Phi)\,\mathcal{E}(\tilde{\boldsymbol{v}},\tilde{\mathbb{S}}) \\ &+ \int_\Omega \mu|\nabla\tilde{\boldsymbol{v}}|^2 + \gamma|\nabla\tilde{\mathbb{S}}|^2 - \rho\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}}\cdot\boldsymbol{\varphi} + \tfrac{1}{\eta}(\tilde{\boldsymbol{v}}\otimes\tilde{\mathbb{S}}) : \nabla\Phi - \tfrac{1}{\eta}(\tilde{\mathbb{S}}(\nabla\tilde{\boldsymbol{v}} - \nabla\tilde{\boldsymbol{v}}^\top)) : \Phi\,\mathrm{d}x,\end{aligned} \tag{10}$$

and $\mathcal{A} = (\mathcal{A}_1,\mathcal{A}_2)$ is the system operator defined in such a way that a smooth pair $(\boldsymbol{\varphi},\Phi)$ satisfies (3) if and only if $\mathcal{A}_1(\boldsymbol{\varphi},\Phi) = 0$ and $\mathcal{A}_2(\boldsymbol{\varphi},\Phi) \in \partial\mathcal{P}(\Phi)$ in $(0,T)$. In particular, if we choose $\mathcal{K}$ such that $\mathcal{W}^{(\mathcal{K})}$ is nonnegative, e.g., $\mathcal{K}(\boldsymbol{\varphi},\Phi) = C\big(\|\boldsymbol{\varphi}\|^s_{L^r(\Omega)} + \|\Phi\|^q_{L^p(\Omega)} + \|\Phi\|^2_{L^p(\Omega)}\big)$ for a certain constant $C > 0$ and suitable parameters p,q,r,s, then for any sufficiently regular solution $(\boldsymbol{\varphi},\Phi)$, the inequality (9) reduces to $\mathcal{R}(\boldsymbol{v}(t),\mathbb{S}(t) \mid \boldsymbol{\varphi}(t),\Phi(t)) \leq \mathcal{R}(\boldsymbol{v}_0,\mathbb{S}_0 \mid \boldsymbol{\varphi}(0),\Phi(0))\mathrm{e}^{\int_0^t \mathcal{K}(\boldsymbol{\varphi},\Phi)\,\mathrm{d}s}$. When the initial values of $(\boldsymbol{v},\mathbb{S})$ and $(\boldsymbol{\varphi},\Phi)$ coincide, we thus conclude $\boldsymbol{v} = \boldsymbol{\varphi}$ and $\mathbb{S} = \Phi$. This shows that generalized solutions satisfy the weak-strong uniqueness principle.

Energy-variational solutions. The notion of energy-variational solutions is based on the relative energy inequality (9). As explained above, stress diffusion was necessary to show existence of generalized solutions in the sense of (7), (8), since weak convergence in the function class (6) was sufficient to perform the limit in the nonlinear terms. Without stress diffusion, we lose control of $\nabla\mathbb{S}$, so that a passage to the limit in (8) is not possible for $\gamma = 0$, so that the above notion of generalized solution seems not suitable in this case. Instead, we proposed a different solution concept and called a pair $(\boldsymbol{v},\mathbb{S})$ an *energy-variational solution* for the *regularity weight* $\mathcal{K}$ if it satisfies the relative energy inequality (9) for all test functions $(\boldsymbol{\varphi},\Phi)$ in a suitable class. If we take

$$\mathcal{K}(\boldsymbol{\varphi},\Phi) = C\big(\|\Phi\|_{L^\infty(\Omega)} + \|\nabla\Phi\|_{L^3(\Omega)}\big) \tag{11}$$

for a suitable constant $C > 0$ such that $\mathbb{S}$ appears in the function $\mathcal{W}^{(\mathcal{K})}$ in (10) in a convex way, then weak convergence is sufficient to pass to the limit $\gamma \to 0$ in (9). In this way, we showed existence of energy-variational solutions in the case $\gamma = 0$. Since the relative energy inequality (9) holds by definition, nonnegativity of $\mathcal{W}^{(\mathcal{K})}$ implies a weak-strong uniqueness principle for energy-variational solutions as above. This can not be deduced for $\mathcal{K}$ as in (11), but for

$$\mathcal{K}(\boldsymbol{\varphi},\Phi) = C\big(\|\boldsymbol{\varphi}\|^s_{L^r(\Omega)} + \|\Phi\|^2_{L^\infty(\Omega)} + \|\nabla\Phi\|^2_{L^3(\Omega)}\big) \tag{12}$$

for sufficiently large $C > 0$ and suitable parameters r,s. Due to monotonicity properties with respect to the regularity weight, a weak-strong uniqueness principle also follows for energy-variational solutions with the weight $\mathcal{K}$ as in (11).

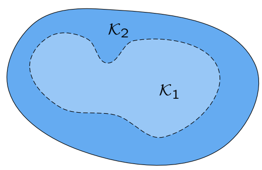

Fig. 6: *The set of energy-variational solutions depends on the regularity weight $\mathcal{K}$. If $\mathcal{K}_1 \leq \mathcal{K}_2$, every energy-variational solution for $\mathcal{K}_1$ is an energy-variational solution for $\mathcal{K}_2$.*

Conclusions and outlook

We studied a model that describes the deformation of rocks in the lithosphere as the flow of a fluid with viscous, elastic, and plastic properties. Since solutions are not smooth in general, we proposed a generalized solution concept based on a variational formulation that takes into account the energy of the system. In this framework, we could only show an existence result after introducing stress diffusion. To omit this term, we introduced the notion of energy-variational solution, where a passage to the limit of vanishing stress diffusion was possible. This solution concept is based on a relative energy inequality, which also ensures a weak-strong uniqueness principle for energy-variational solutions.

Since the concept of energy-variational solutions is rather new, an extension to other systems might also be of interest, which was done for a general class of hyperbolic conservation laws in the recent preprint [3]. Although energy-variational solutions are not unique in general, a suitable choice of the regularity weight may lead to a convex structure of the set of solutions, which may allow to choose a unique physically relevant solution by a minimization procedure. For example, this idea was successfully realized for simpler fluid models in [4].

References

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