Analysis of a compressible Stokes-flow with degenerating and singular viscosity

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Abstract

In this paper we show the existence of a weak solution for a compressible single-phase Stokes flow with mass transport accounting for the degeneracy and the singular behavior of a density-dependent viscosity. The analysis is based on an implicit time-discrete scheme and a Galerkin-approximation in space. Convergence of the discrete solutions is obtained thanks to a diffusive regularization of p-Laplacian type in the transport equation that allows for refined compactness arguments on subdomains.

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1 Introduction

Suspensions, i.e., flows of solid particles immersed in a viscous fluid appear in many aspects in nature and their understanding is of great importance for many technological processes, e.g., in the food, pharmaceutical, printing and oil industries. For suspension flows two major regimes with substantially different rheological properties can be observed. In the dilute regime the volume fractions of the solid particles are very small and mutual interaction between particles is negligible. Instead, in the dense regime large volume fractions lead to rheological behaviors like shear thinning and discontinuous shear thickening, see [BJ14]. When a critical volume fraction of solid, rigid particles is reached, jamming occurs, which means that the rheological behavior of the suspension turns into that of a solid.

The development of a continuum model for binary suspensions of solid and liquid phase applicable across different concentration regimes with substantially different rheology is of great importance to understand the applications but also very challenging from a mathematical point of view. In [PTA$^+$19], the authors construct a PDE model also suited for dense suspensions using a gradient flow structure featuring a dissipative coupling between fluid and solid phase as well as different driving forces. This approach leads to a general mathematical structure of variational type which is able to model the different suspension regimes, from dilute to highly concentrated states up to jamming. This is done by taking into account physically realistic but mathematically non-standard density-dependent constitutive relations, which degenerate for dilute suspensions as the density of solid particles tends to zero and which get singular when reaching a critical value that stands for jamming. Due to these degeneracy and singularity properties in these two extremal situations, the mathematical analysis of the derived model in [PTA$^+$19] requires significant mathematical efforts and is very challenging. Concerning the mathematical analysis of

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compressible viscous fluid models with density-dependent viscosities, we mention here a series of papers [BDL03, BD06, BD07, BMZ19], where the authors investigate with great effort compressible Navier-Stokes fluid models with density-dependent viscosities, which also appear in shallow water and lubrication models. The well-posedness of the models is shown by introducing a new mathematical entropy identity, which is derived either by assuming a linear dependence of the viscosity with respect to the density or by assuming a power-law structure of the viscosity $\mu(\varrho)$ on the mass density ϱ. However, the non-standard form of the viscosity in [PTA+19] cannot be treated by the mathematical methods developed in [BDL03, BD06, BD07, BMZ19, LT18a, LT18b], and references therein.

Our long-term goal is to investigate the full two-phase model for concentrated suspensions proposed in [PTA+19]. As a first step towards this analysis, a single-phase model, which captures the above-described degeneracy and the singularity properties, is investigated in this paper to understand the main difficulties and to pave the road for forthcoming analysis of the full two-phase Stokes and Navier-Stokes system. Let $[0,T]\times\Omega\subset\mathbb{R}^d$ denote the space-time cylinder with space dimension $d\in\{2,3\}$ and final time $T>0$ general but fixed. For the density $\varrho:\Omega\to\mathbb{R}$ and the velocity $u:\Omega\to\mathbb{R}^d$ as unknowns, the bulk equations of our model are given by

$$\partial_t\varrho+\operatorname{div}(\varrho u)-\varepsilon\operatorname{div}(|\nabla\varrho|^{p-2}\nabla\varrho)=0 \quad \text{in } [0,T]\times\Omega, \tag{1.1a}$$

$$-\operatorname{div}(\mu(\varrho)e(u))+M(\varrho)u+\kappa|u|^{s-2}u+\varrho\nabla\mathrm{D}_\varrho\mathcal{E}(\varrho)=0 \quad \text{in } [0,T]\times\Omega, \tag{1.1b}$$

with the symmetric strain tensor $e(u):=\frac{1}{2}(\nabla u+\nabla u^\top)$ and the energy

$$\mathcal{E}(\varrho)=\int_\Omega\left(x_2\varrho+\frac{\tilde\varepsilon}{2}|\varrho|^2\right)\mathrm{d}x. \tag{1.1c}$$

The bulk problem (1.1) is complemented by suitable boundary and initial conditions, which are specified in Section 2.1. By $\mathrm{D}_\varrho\mathcal{E}(\varrho)$ we indicate in (1.1b) the variational derivative of $\mathcal{E}(\varrho)$ with respect to ϱ. Moreover, the shear viscosity μ and the friction M are material parameters, which are modeled as density-dependent functions. Equation (1.1a) is a continuity equation with a diffusive regularization of p-Laplacian type. The Stokes equation (1.1b) contains a regularization in terms of an L^s-nonlinearity, whose exponent s is closely connected to the exponent p appearing in p-Laplacian in the continuity equation. Moreover, the first component in the energy functional is given by the gravitational force and the second convex term of lower order is chosen as a further regularizing term. The prefactors $\varepsilon,\tilde\varepsilon,\kappa>0$ of the regularizing terms may be arbitrarily small but positive. The density ϱ can be understood as the mass density of the system. Yet, setting $\varrho:=\varrho_0\phi$ with constant mass density ϱ_0 it can be seen in direct relation with a phase indicator ϕ when extending the model (1.1) to suspensions in the future. In both situations it will be important to obtain that a solution ϱ of (1.1a) is non-negative and bounded from above by a critical value ϱ_{crit}.

Structure of the paper. In Section 2, we state the precise assumptions and give the Definition of a weak solution to system (1.1) with a corresponding, first existence result. Moreover the diffusive regularization of p-Laplacian type in the transport equation allows for refined compactness arguments on subdomains, discussed in Theorem 2.4. In Section 3 an implicit time-discrete scheme combined with a Galerkin-approximation in space (3.2) is introduced and solved in Proposition 3. Moreover, a priori estimates are investigated in Proposition 3.4 which lead to convergence results, given by Proposition 4.1 in the final Section 4. There, we also establish the non-negativity and boundedness of weak solutions ϱ of (1.1a).

2 Basic assumptions and main results

2.1 Notation and basic assumptions

Let $\Omega\subset\mathbb{R}^d$ with $d\in\{2,3\}$ be a bounded Lipschitz domain with boundary $\Gamma=\partial\Omega$ and $\vec{n}$ the outer unit normal on Γ. By $\mathcal{L}^m(B)$, we denote the m-dimensional Lebesgue measure of a set $B\subset\mathbb{R}^m$, $m\in\mathbb{N}$. Furthermore, we denote by $L^p(\Omega)$, resp. $W^{1,p}(\Omega)$ for $1\le p\le\infty$ the Lebesgue-, resp. Sobolev-spaces on Ω. For a time interval $(0,T)$, $T>0$ and a Banach space X we denote by $L^p(0,T;X)$ the spaces of

Bochner-integrable functions with values in X. Moreover, for a Banach space X we denote its dual by X^* and the duality pairing by $\langle \cdot, \cdot \rangle_X$. For L^p-spaces, we denote by p' the dual index defined by $\frac{1}{p} + \frac{1}{p'} = 1$. "Generic" positive constants are denoted by C. Moreover, for subsets of $\mathbb{R}^m$ defined by the values of a function $f : \mathbb{R}^m \to \mathbb{R}$, $m \in \mathbb{N}$ we shall use the following notation

$$[f \preceq a] := \{x \in \mathbb{R}^m,\ f(x) \preceq a\}\,, \tag{2.1}$$

where the symbol $\preceq$ is here used as a placeholder for one of the relation symbols, i.e., $\preceq \in \{<, >, =, \leq, \geq\}$. Throughout this work, we adopt the following assumptions:

(A1) The shear viscosity is given by

$$\mu : \mathbb{R} \to [0, \infty],\ \mu(\varrho) = |\varrho|\eta(\varrho), \tag{2.2a}$$

where

$$\eta(\varrho) = \begin{cases} \infty & \text{if } \varrho < 0, \\ \frac{\widetilde{\nu}}{(\varrho_{\text{crit}} - \varrho)^\alpha} & \text{if } 0 \leq \varrho < \varrho_{\text{crit}}, \\ \infty & \text{if } \varrho \geq \varrho_{\text{crit}}, \end{cases} \qquad \text{for } \alpha \geq 2 \text{ and } \widetilde{\nu} > \varrho_{\text{crit}}. \tag{2.2b}$$

As we shall see in Theorem 2.7 and lateron in Section 4.3 the singularity of power $\alpha \geq 2$ prevents solutions ϱ of (2.5a) to exceed the value ϱ_{crit}.

(A2) The friction coefficient $M(\varrho)$ is defined by a continuous function $M : \mathbb{R} \to \mathbb{R}_0^+$ with the following growth property: There exist constants $\underline{M}, \overline{M} > 0$ such that

$$\underline{M}|\varrho|^2 \leq M(\varrho) \leq \overline{M}|\varrho|^2 \quad \text{for all } \varrho \in \mathbb{R}. \tag{2.2c}$$

(A3) The initial condition for the density fulfills

$$\varrho_0 \in L^2(\Omega). \tag{2.2d}$$

(A4) The boundary conditions are given by

$$u = 0, \quad \text{and} \quad \varepsilon|\nabla\varrho|^{p-2}\nabla\varrho \cdot \vec{n} = 0 \text{ on } \Gamma. \tag{2.2e}$$

(A5) Assumptions on the exponents p and s in (1.1):

$$\{2, 3\} \ni d < p < 6, \quad \text{and} \quad \frac{1}{2} + \frac{1}{s} + \frac{1}{p} = 1. \tag{2.2f}$$

(A6) Assumptions on the regularization parameters ε, $\tilde{\varepsilon}$ and κ in (1.1):

$$0 < \varepsilon, \tilde{\varepsilon} < 1, \quad \text{and} \quad \kappa > 0. \tag{2.2g}$$

Remark 2.1. 1. *In [PTA+19] the real constitutive material law for the solid phase is given by*

$$\eta(\varrho) = 1 + \frac{5}{2}\frac{\varrho_{crit}}{\varrho_{crit} - \varrho} + \left(\mu_1 + \frac{\mu_2 - \mu_1}{1 + I_0(\varrho_{crit} - \varrho)^{-2}}\right)\frac{\varrho}{(\varrho_{crit} - \varrho)^2} \tag{2.3}$$

with the non-dimensional parameters, $\mu_2 \geq \mu_1$ and I_0. Our choice in (2.2b) *captures the essential behavior of* (2.3) *and is well tailored for the mathematical analysis in this paper.*

2. *By* (2.2f)*, the dual indices p' and s' are given by $\frac{1}{s'} = 1 - \frac{1}{s} = \frac{1}{2} + \frac{1}{p}$ and $\frac{1}{p'} = 1 - \frac{1}{p} = \frac{1}{2} + \frac{1}{s}$, respectively.*

2.2 Basic notion of solution

Here, we specify our notion of solution for the system (1.1).

Definition 2.2 (Basic notion of weak solution)**.** *Suppose that the general assumptions (A1)-(A6) are fulfilled and let the final time $T > 0$ general but fixed. A weak solution of system* (1.1) *is a quadruplet* $(\varrho, u, B_\mu, \zeta)$ *with the regularity*

$$\varrho \in W^{1,p'}(0,T;W^{1,p}(\Omega)^*) \cap L^p(0,T;W^{1,p}(\Omega)), \qquad u \in L^s(0,T;L^s(\Omega;\mathbb{R}^d)), \tag{2.4a}$$
$$B_\mu \in L^{s'}(0,T;W^{1,2}(\Omega;\mathbb{R}^d)^*), \qquad \zeta \in L^{s'}(0,T;L^{s'}(\Omega;\mathbb{R}^d)), \tag{2.4b}$$

that satisfies

$$\langle \partial_t \varrho, \psi\rangle_{L^p(0,T;W^{1,p}(\Omega))} - \int_0^T \int_\Omega (\varrho u - \varepsilon|\nabla\varrho|^{p-2}\nabla\varrho)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.5a}$$
$$\textit{for all } \psi \in L^p(0,T;W^{1,p}(\Omega)),$$
$$\langle B_\mu, v\rangle_{L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d))} + \int_0^T \int_\Omega (M(\varrho)u + \kappa\zeta + \varrho\nabla \mathrm{D}_\varrho \mathcal{E}(\varrho))\cdot v \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.5b}$$
$$\textit{for all } v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d)).$$

We will show in Theorem 2.3 the existence of a weak solution in the sense of Definition 2.2. Moreover, in Theorem 2.4, it will be possible to identify the limit elements B_μ and ζ in (2.5b) as the viscous stress and the L^s-nonlinearity generated by the limit pair (ϱ, u) on Lipschitz subdomains $\Omega_\nu^{\mathrm{Lip}}(t)$, where $\nu < \varrho(t) < \varrho_{\mathrm{crit}} - \nu$ for a.a. $t \in (0,T)$ and for all $\nu > 0$, cf. (2.6a). A further refinement of the notion of weak solution will be deduced in Theorem 2.7: Given that the initial datum ϱ_0 satisfies $0 \le \varrho_0 \le \varrho_{\mathrm{crit}}$ a.e. in Ω one finds that also the weak solution ϱ satisfies $0 \le \varrho \le \varrho_{\mathrm{crit}}$ a.e. in Ω for all $t \in [0,T]$. In this situation the identification of the limit elements B_μ and ζ as the viscous stress and the L^s-nonlinearity generated by (ϱ, u) can be shown to be valid even a.e. in $[0 < \varrho < \varrho_{\mathrm{crit}}]$.

2.3 Main results

In this section we present and discuss our main results.

Theorem 2.3. *Suppose that the general assumptions (A1)-(A6) are fulfilled. Then* (1.1) *has a weak solution in the sense of Definition 2.2.*

The proof of Theorem 2.3 is carried out in Sections 3 and 4. In Section 3 a fully discrete (time-discrete Galerkin) scheme together with a suitable regularization of the shear viscosity μ is devised and investigated. A priori estimates based on a discrete energy estimate are derived. In Section 4 the a priori estimates are used to perform the limit from the discrete to the continuous problem.

In the following Theorem we will provide, on subdomains, an identification for the objects B_μ and ζ appearing in (2.5b). To this end, we define subdomains

$$\Omega_\nu(t) := \{x \in \Omega \mid \nu < \varrho(t) < \varrho_{\mathrm{crit}} - \nu\} \text{ for any } \nu > 0 \text{ and for a.a. } t \in (0,T), \tag{2.6a}$$

and consider any

$$\text{Lipschitz-subdomain} \quad \Omega_\nu^{\mathrm{Lip}}(t) \subset \Omega_\nu(t) \text{ for a.a. } t \in (0,T). \tag{2.6b}$$

We note that $\Omega_\nu(t)$ is an open set in Ω due to the compact embedding $W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ thanks to $p > d$ by assumption (2.2f). Moreover, on the Lipschitz-domains $\Omega_\nu^{\mathrm{Lip}}(t)$ (reflexive) Sobolev spaces such as $W^{1,2}(\Omega_\nu^{\mathrm{Lip}}(t))$ are well defined and embedding theorems are valid, cf. [AF03]. Now, we have

Theorem 2.4. *Let $(\varrho, u, B_\mu, \zeta)$ be a weak solution of system (1.1) obtained in Theorem 2.3. Then, for every $\delta > 0$, there exists a measurable set $I_\delta \subset (0,T)$ such that $\mathcal{L}^1((0,T)\backslash I_\delta) < \delta$ and non-cylindrical domains*

$$Q_\nu^\delta := \bigcup_{t\in I_\delta} \{t\} \times \Omega_\nu^{\mathrm{Lip}}(t) \subset (0,T)\times\Omega \quad \text{with } \Omega_\nu^{\mathrm{Lip}}(t) \text{ as in (2.6), for all } \nu > 0, \tag{2.7}$$

such that for all $v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$ with supp$(v) \subset Q_\nu^\delta$ there holds

$$\int_0^T \int_\Omega \mu(\varrho(t))e(u) : e(v)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega (M(\varrho)u + \kappa|u|^{s-2}u + \varrho\nabla \mathrm{D}_\varrho\mathcal{E}(\varrho))\cdot v\,\mathrm{d}x\,\mathrm{d}t = 0\,, \tag{2.8}$$

i.e., it is

$$\langle B_\mu, v\rangle_{L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d))} = \int_0^T\int_\Omega \mu(\varrho(t))e(u):e(v)\,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \tag{2.9a}$$

$$\int_0^T\int_\Omega \kappa\,\zeta\cdot v\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \kappa|u|^{s-2}u\cdot v\,\mathrm{d}x\,\mathrm{d}t \tag{2.9b}$$

for all $v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$ with supp$(v) \subset Q_\nu^\delta$, for any Q_ν^δ from (2.7), for all $\delta, \nu > 0$. Consequently, relations (2.8) and (2.9) hold true even for all test functions

$$v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \quad \text{such that } \mathrm{supp}(v(t)) \subset [0 < \varrho(t) < \varrho_{\mathrm{crit}}] \text{ for a.a. } t \in (0,T)\,. \tag{2.10}$$

The proof of Theorem 2.4 is given in detail in Section 4.2; we adress here the main ideas: The identification of B_μ and ζ in terms of the limit velocity u can be achieved on subsets of the space-time cylinder $(0,T)\times\Omega$ where the values of the limit density ϱ and its approximants are strictly bounded away from zero and away from the singularity ϱ_{crit}. In space this is ensured by retreating to the subdomains $\Omega_\nu^{\mathrm{Lip}}(t)$ from (2.6). Again, thanks to the assumption $p > d$ there holds $W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ compactly and the approximants converge uniformly in space to the limit density ϱ. In this way, it can be ensured for a.a. $t \in (0,T)$ that also the approximants are strictly bounded away from zero and away from ϱ_{crit} from a particular index $n(t)$ on. We point out that the regularity in time $\varrho \in W^{1,p'}(0,T;W^{1,p}(\Omega)^*) \cap L^p(0,T;W^{1,p}(\Omega))$ given in (2.4), and similarly also for the approximating solutions, is too low in order to ensure continuity and uniform convergence in time. Instead, one can only make use of almost uniform convergence, which will be deduced from strong $L^p(0,T;L^p(\Omega))$-convergence together with Egorov's theorem in Section 4. The almost uniform convergence in time induces the measurable sets $I_\delta \subset (0,T)$ from (2.7) where the sequences converge uniformly and this will allow it to find both the limit density and the approximants strictly bounded away from zero and away from ϱ_{crit} on the non-cylindrical domains Q_ν^δ. In order to carry out the identification argument we need compactness results in Banach spaces defined on non-cylindrical sets. To this end, for Q_ν^δ from (2.7) we introduce the normed vector space

$$L^s(Q_\nu^\delta) := \left\{ f : Q_\nu^\delta \to \mathbb{R} \text{ measurable}, \|f\|_{L^s(Q_\nu^\delta)} := \left(\int_{I_\delta} \|f(t)\|^s_{L^s(\Omega_\nu^{\mathrm{Lip}}(t))}\,\mathrm{d}t\right)^{1/s} < \infty \right\}. \tag{2.11}$$

Lemma 2.5. *Consider a non-cylindrical domain Q_ν^δ as in (2.7) and let $s \in (1,\infty)$. The normed vector space $L^s(Q_\nu^\delta)$ defined in (2.11) is a reflexive, separable Banach space.*

Proof. Let Q_ν^δ be a non-cylindrical domain as in (2.7) and let $s \in (1,\infty)$.

1. We show that the normed vector space $L^s(Q_\nu^\delta)$ is complete: Let $(f_n)_{n\in\mathbb{N}} \subset L^s(Q_\nu^\delta)$ be a Cauchy sequence. We show that there exists an element $f \in L^s(Q_\nu^\delta)$ such that $f_n \to f$ in $L^s(Q_\nu^\delta)$. To this end, we define for a.a. $t \in I_\delta$

$$\widehat{f}_n(t) := \begin{cases} f_n(t) & \text{on } \Omega_\nu^{\mathrm{Lip}}(t), \\ 0 & \text{on } \Omega\setminus\Omega_\nu^{\mathrm{Lip}}(t). \end{cases}$$

Indeed, we have for all $n \in \mathbb{N}$ that $\widehat{f}_n \in L^s(I_\delta; L^s(\Omega))$ with $\|\widehat{f}_n\|_{L^s(I_\delta;L^s(\Omega))} = \|f_n\|_{L^s(Q^\delta_\nu)}$ and it follows that $(\widehat{f}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^s(I_\delta; L^s(\Omega))$. Due to the completeness of $L^s(I_\delta; L^s(\Omega))$, we conclude that there exists an element $\widehat{f} \in L^s(I_\delta; L^s(\Omega))$ such that

$$\widehat{f}_n \to \widehat{f} \text{ in } L^s(I_\delta; L^s(\Omega)). \tag{2.12}$$

For a.a. $t \in I_\delta$ we set $f(t) := \widehat{f}|_{\Omega^{\mathrm{Lip}}_\nu(t)}$ and show for a.a. $t \in I_\delta$ that $\widehat{f}(t)|_{\Omega\setminus\Omega^{\mathrm{Lip}}_\nu(t)} = 0$. In view of (2.12), it follows that there exists a subsequence $(\widehat{f}_{n_l})_{l\in\mathbb{N}}$ such that $\widehat{f}_{n_l}(t,x) \to f(t,x)$ pointwise for a.a. $(t,x) \in I_\delta \times \Omega$. Because of $\widehat{f}_{n_l}(t,x) = 0$ for a.a. $(t,x) \in \{t\} \times (\Omega \setminus \Omega^{\mathrm{Lip}}_\nu(t))$ and for all $l \in \mathbb{N}$, it follows $\widehat{f}(t,x) = 0$ for a.a. $(t,x) \in \{t\} \times (\Omega \setminus \Omega^{\mathrm{Lip}}_\nu(t))$. We finally obtain

$$\|f_n - f\|_{L^s(Q^\delta_\nu)} = \|\widehat{f}_n - \widehat{f}\|_{L^s(I_\delta;L^s(\Omega))} \to 0,$$

which proves that $L^s(Q^\delta_\nu)$ is a Banach space.

2. We show that $L^s(Q^\delta_\nu)$ is reflexive and separable: To this end, we define

$$V := \{\widehat{f} \in L^s(I_\delta; L^s(\Omega)) \mid \text{ for a.a. } t \in I_\delta : \widehat{f}(t) \equiv 0 \text{ on } \Omega \setminus \Omega^{\mathrm{Lip}}_\nu(t)\},$$

which is a closed subspace of $L^s(I_\delta; L^s(\Omega))$, and thus, a reflexive and separable Banach space. Moreover, we have $L^s(Q^\delta_\nu) \equiv V$ and due to isomorphism of the norms, we obtain that $L^s(Q^\delta_\nu)$ is reflexive and separable, too. Further, we note that the dual space of V is given by $V^* := L^{s'}(I_\delta; L^{s'}(\Omega))|_V$. Due to $L^s(Q^\delta_\nu) \equiv V$, we also have $V^* \equiv L^s(Q^\delta_\nu)^* \equiv L^{s'}(Q^\delta_\nu)$. □

Remark 2.6. *Thanks to the properties of $L^s(Q^\delta_\nu)$ verified in Lemma 2.5 and Eberlein-Šmuljan's theorem, each bounded sequence in $L^s(Q^\delta_\nu)$ contains a subsequence that converges weakly to a limit in $L^s(Q^\delta_\nu)$.*

The above results for Banach spaces on the non-cylindrical domains Q^δ_ν will allow us to carry out the identification argument in Section 4.2. We mention that our use of non-cylindrical domains is motivated by the works [Sal85, Sal88], where Banach spaces on non-cylindrical domains are introduced at first using very general sets $\widehat{\Omega} := \bigcup_{t\in(0,T)}\{t\} \times \Omega(t)$. Yet, lateron, in the course of the analysis, higher regularity assumptions, i.e., C^3-regularity for the boundary of $\widehat{\Omega}$, are required in [Sal85]. This is in line with the fact that many works dealing with the analysis of PDEs related to fluid flow on moving domains [SS07, NRL16, AET18, Saa07] postulate higher regularity assumptions for the flow map, which is used to map the current domain $\Omega(t)$ to a fixed reference domain Ω_0. Translating these assumptions to our situation shows that higher temporal regularity would be needed for the density ϱ and its approximating solutions and in some cases also volume conservation for the set $\{x \in \mathbb{R}^d, \varrho(t,x) \in (0, \varrho_{\mathrm{crit}})\}$ in order to apply the methods of the works mentioned above. Instead here, we can only expect $\varrho \in W^{1,p'}(0,T; W^{1,p}(\Omega)^*) \cap L^p(0,T; W^{1,p}(\Omega))$. In difference to the above mentioned works, the non-cylindrical domains Q^δ_ν used in our context are induced by the solution ϱ, so that good regularity of the sets cannot be expected in general. As approved by Remark 2.6, we shall solely use the $L^s(Q^\delta_\nu)$-spaces for a compactness argument in the a posteriori identification of the limit elements.

With a suitable adaption of the non-cylindrical domains and via a contradiction argument, it is possible to deduce the non-negativity of the density ϱ and its boundedness in terms of the critical value ϱ_{crit}. We point out that $\varrho = 0$ is not excluded on subsets of $(0,T) \times \Omega$ of positive measure.

Theorem 2.7 (Non-negativity & boundedeness of the limit density ϱ, refinement of (2.8) & (2.9))**.** *Let the assumptions of Theorem 2.4 be valid and let $(\varrho, u, B_\mu, \zeta)$ be a weak solution obtained in Theorems 2.3 and 2.4. Further assume for the initial datum*

$$\varrho_0 \in L^2(\Omega) \quad \textit{such that } 0 \le \varrho_0 \le \varrho_{\mathrm{crit}} \textit{ a.e. in } \Omega\,. \tag{2.13}$$

Then the density ϱ from (2.5a) *also satisfies*

$$0 \le \varrho(t) \le \varrho_{\mathrm{crit}} \quad \textit{a.e. in } \Omega\,, \textit{ for all } t \in [0,T]. \tag{2.14}$$

Moreover, identification relations (2.8) *and* (2.9) *hold true for all test functions*

$$v \in L^s(0,T; W^{1,2}(\Omega;\mathbb{R}^d)) \textit{ such that } \mathrm{supp}(v(t)) \subset [0 < \varrho(t)] \textit{ for a.a. } t \in (0,T)\,. \tag{2.15}$$

3 Existence of discrete solutions

For final time T general but fixed, we define a partition of the time interval $[0,T]$

$$0 = t^0_{N_\tau} < \ldots < t^{N_\tau}_{N_\tau} = T \quad \text{with} \quad t^k_{N_\tau} - t^{k-1}_{N_\tau} = \frac{T}{N_\tau} =: \tau,$$

and a sequence of finite-dimensional subspaces $\mathbf{U}_n$, $\mathbf{X}_n$ such that, for all $n \in \mathbb{N}$

$$\mathbf{X}_n \subset \mathbf{X}_{n+1} \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} \mathbf{X}_n \text{ dense in } \mathbf{X} := W^{1,p}(\Omega) \text{ with } \operatorname{span}\{e_j; j = 1,\ldots,n\} = \mathbf{X}_n, \tag{3.1a}$$

$$\mathbf{U}_n \subset \mathbf{U}_{n+1} \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} \mathbf{U}_n \text{ dense in } \mathbf{U} := W^{1,2}(\Omega;\mathbb{R}^d) \text{ with } \operatorname{span}\{\mathbf{e}_j; j = 1,\ldots,dn\} = \mathbf{U}_n, \tag{3.1b}$$

where $e_j \in \mathbf{X}$ for $j = 1,\ldots,n$, resp. $\mathbf{e}_j \in \mathbf{U}$ for $j = 1,\ldots,dn$, are linearly independent.

Remark 3.1. *This motivates us to define two different projectors: The projector $P^{\mathbf{X}}_n : \mathbf{X} \to \mathbf{X}_n$ such that $P^{\mathbf{X}}_n(\mathbf{X}) = \mathbf{X}_n$ and the projector $P^{\mathbf{U}}_n : \mathbf{U} \to \mathbf{U}_n$ such that $P^{\mathbf{U}}_n(\mathbf{U}) = \mathbf{U}_n$. For $P^{\mathbf{U}}_n$ we claim that it is selfadjoint (note that $\mathbf{U}$ is a Hilbert space) and $\|P^{\mathbf{U}}_n\|_{\mathcal{L}(\mathbf{U},\mathbf{U})}$ is bounded independently of n. Hence, such a projector $P^{\mathbf{U}}_n$ with the mentioned properties exists, cf. [Rou05, Remark 8.41, p. 238]. Above, we use the notation $\mathcal{L}(\mathbf{U},\mathbf{U}) := \{A : \mathbf{U} \to \mathbf{U}$ linear and continuous$\}$.*

Now, we consider the fully discrete Galerkin scheme corresponding to (1.1) for $k \in \{1,\ldots,N_\tau\}$ and $n \in \mathbb{N}$

$$\int_\Omega \frac{\varrho^k_{\tau n} - \varrho^{k-1}_{\tau n}}{\tau} \psi \,\mathrm{d}x - \int_\Omega (\varrho^k_{\tau n} u^k_{\tau n} - \varepsilon |\nabla \varrho^k_{\tau n}|^{p-2} \nabla \varrho^k_{\tau n}) \cdot \nabla \psi \,\mathrm{d}x = 0, \tag{3.2a}$$
$$\text{for all } \psi \in \mathbf{X}_n,$$

$$\int_\Omega \mu_\tau(\varrho^k_{\tau n}) e(u^k_{\tau n}) : e(v) \,\mathrm{d}x + \int_\Omega (M(\varrho^k_{\tau n}) u^k_{\tau n} + \kappa |u^k_{\tau n}|^{s-2} u^k_{\tau n} + \varrho^k_{\tau n} \nabla \mathrm{D}_\varrho \mathcal{E}(\varrho^k_{\tau n})) \cdot v \,\mathrm{d}x = 0, \tag{3.2b}$$
$$\text{for all } v \in \mathbf{U}_n,$$

where the regularized viscosity μ_τ is defined by

$$\mu_\tau(\varrho) = |\varrho| \eta_\tau(\varrho) + \tau^\beta \text{ for } \beta > 0 \text{ and with } \eta_\tau(\varrho) = \begin{cases} \frac{\widetilde{\nu}}{\varrho^\alpha_{\text{crit}}} + \frac{1}{\tau^\alpha}|\varrho| & \text{if } \varrho < 0, \\ \frac{\widetilde{\nu}}{(\varrho_{\text{crit}} - \varrho)^\alpha} & \text{if } 0 \le \varrho < \varrho_{\text{crit}} - \tau, \\ \frac{\widetilde{\nu}|\varrho|}{\tau^\alpha |\varrho_{\text{crit}} - \tau|} & \text{if } \varrho \ge \varrho_{\text{crit}} - \tau, \end{cases} \tag{3.3a}$$

for $\alpha \ge 2$ and $\widetilde{\nu} > \varrho_{\text{crit}}$.
In contrast to μ from (2.2), μ_τ is continuous for every fixed τ and can be estimated from below, i.e.,

$$\text{there is a constant } c_\mu > 0 \text{ such that } \quad \mu_\tau(\varrho) \ge c_\mu |\varrho|^2 + \tau^\beta \quad \text{for all } \varrho \in \mathbb{R} \text{ and for all } \tau > 0. \tag{3.3b}$$

Moreover, we have

$$\mu_\tau(\varrho) \to \mu(\varrho) \text{ as } \tau \to 0, \quad \text{for all } \varrho \in \mathbb{R}. \tag{3.3c}$$

Proposition 3.2 (Existence of discrete solutions)**.** *Let the assumptions (A1)-(A6) be satisfied and $T > 0$ general but fixed. Also keep $\tau > 0$ and $n \in \mathbb{N}$ fixed. Then the following statements hold true:*

1. *For all $k \in \{1,\ldots,N_\tau\}$ there exists a solution $(\varrho^k_{\tau n}, u^k_{\tau n}) \in \mathbf{X}_n \times \mathbf{U}_n$ for problem (3.2).*

2. *For all $K \in \{1,\dots,N_\tau\}$ the discrete solutions $(\varrho^k_{\tau n}, u^k_{\tau n})_{k=1}^{N_\tau}$ satisfy the following discrete energy-dissipation estimate:*

$$\begin{aligned}
&\mathcal{E}(\varrho^K_{\tau n}) + \sum_{k=1}^K \tau \int_\Omega \mu_\tau(\varrho^k_{\tau n}) e(u^k_{\tau n}) : e(u^k_{\tau n})\,\mathrm{d}x + \sum_{k=1}^K \tau \int_\Omega M(\varrho^k_{\tau n})|u^k_{\tau n}|^2\,\mathrm{d}x \\
&\qquad + \kappa \sum_{k=1}^K \tau \int_\Omega |u^k_{\tau n}|^s\,\mathrm{d}x + \frac{\varepsilon\tilde{\varepsilon}}{p} \sum_{k=1}^K \tau \|\nabla \varrho^k_{\tau n}\|^p_{L^p(\Omega;\mathbb{R}^d)} \\
&\le \mathcal{E}(\varrho^0_{\tau n}) + \frac{\varepsilon\tilde{\varepsilon}^{1-p}}{p}\mathcal{L}^d(\Omega\times(0,T)).
\end{aligned} \tag{3.4}$$

The proof of Proposition 3.2 will be carried out in Section 3.1 below.

Using the discrete solutions $(\varrho^k_{\tau n}, u^k_{\tau n})_{k=1}^{N_\tau}$ obtained in Prop. 3.2, Item 1., we define suitable approximating solutions by interpolation in time. More precisely, we introduce the piecewise constant left-continuous interpolants

$$\overline{\varrho}_{\tau n}(t) := \varrho^k_{\tau n}, \qquad \overline{u}_{\tau n}(t) := u^k_{\tau n}, \qquad \text{for } t \in (t^{k-1}_{N_\tau}, t^k_{N_\tau}],\ k = 1,\dots,N_\tau, \tag{3.5a}$$

as well as the piecewise linear interpolant

$$\varrho_{\tau n}(t) := \frac{t - t^{k-1}_{N_\tau}}{\tau}\varrho^k_{\tau n} + \frac{t^k_{N_\tau} - t}{\tau}\varrho^{k-1}_{\tau n}, \qquad \text{for } t \in (t^{k-1}_{N_\tau}, t^k_{N_\tau}],\ k = 1,\dots,N_\tau. \tag{3.5b}$$

Moreover, by (3.5b), the time derivative of the piecewise linear interpolant is given by

$$\mathrm{D}_\tau \varrho_{\tau n}(t) := \frac{\varrho^k_{\tau n} - \varrho^{k-1}_{\tau n}}{\tau}, \qquad \text{for } t \in (t^{k-1}_{N_\tau}, t^k_{N_\tau}],\ k = 1,\dots,N_\tau. \tag{3.5c}$$

Additionally, we will also use the following notation for the time

$$\bar{t}_\tau(t) := t^k_{N_\tau} \qquad \text{for } t \in (t^{k-1}_{N_\tau}, t^k_{N_\tau}],\ k = 1,\dots,N_\tau. \tag{3.5d}$$

Using (3.5), we will rewrite (3.2). For this, also note that $\bigcup_{n\in\mathbb{N}} C^0([0,T];\mathbf{X}_n)$ is dense in $L^p(0,T;\mathbf{X})$ for any $1 \le p \le \infty$, cf. [GGZ74, Lemma 1.12., p. 144]. Hence, for any $\psi \in L^p(0,T;\mathbf{X})$ there exists a sequence $(\psi_n)_n$ such that $\psi_n \in L^p(0,T;\mathbf{X}_n)$ for each $n \in \mathbb{N}$ and $\psi_n \to \psi$ in $L^p(0,T;\mathbf{X})$ as $n \to \infty$. For any $\psi \in C^0([0,T];\mathbf{X})$ we use nodal projection and subsequent constant interpolation in time, i.e., we introduce the operator $P_\tau : C^0([0,T];\mathbf{X}) \to L^p(0,T;\mathbf{X})$, $P_\tau\psi(t) := \psi(t^k_{N_\tau}) = \overline{\psi}(t)$ for all $t \in (t^{k-1}_{N_\tau}, t^k_{N_\tau}]$, where we used the notation (3.5a) for the piecewise constant, left-continuous interpolant. Based on this, we define a projector for the space $L^p(0,T;\mathbf{X})$ to piecewise contant functions in time with values in the finite-dimensional subspaces $\mathbf{X}_n$ by making use of the approximating sequences $(\psi_n)_n$ with $\psi_n \in C^0([0,T],\mathbf{X}_n)$ for all $n \in \mathbb{N}$. More precisely, we introduce

$$P_{\tau n} : L^p(0,T;\mathbf{X}) \to L^p(0,T;\mathbf{X}_n), \quad \text{where } P_{\tau n}(\psi) := P^{\mathbf{X}}_n(P_\tau(\psi)) := \overline{\psi}_n\,, \tag{3.6}$$

with the notation from (3.5a). In a similar manner we also define a projection for the space $L^s(0,T;\mathbf{U})$ and we denote the corresponding projector by $P^{\mathbf{U}}_n(P_\tau(\psi)) : L^s(0,T;\mathbf{U}) \to L^s(0,T;\mathbf{U}_n)$. Now, we have

$$\langle \mathrm{D}_\tau\varrho_{\tau n}, P_{\tau l}(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)} - \int_0^T\int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla P_{\tau l}(\psi)\,\mathrm{d}x\,\mathrm{d}t + \langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_{\tau l}(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)} = 0, \quad \text{for all } n \ge l, \text{ for all } \psi \in L^p(0,T;\mathbf{X}), \tag{3.7a}$$

$$\begin{aligned}
&\big\langle \mathcal{A}_\mu(\overline{\varrho}_{\tau n}, \overline{u}_{\tau n}), P^{\mathbf{U}}_n(P_\tau(v))\big\rangle_{L^s(0,T;\mathbf{U}_n)} + \int_0^T\int_\Omega M(\overline{\varrho}_{\tau n})\overline{u}_{\tau n}\cdot P^{\mathbf{U}}_n(P_\tau(v))\,\mathrm{d}x\,\mathrm{d}t \\
&+\kappa \int_0^T\int_\Omega |\overline{u}_{\tau n}|^{s-2}\overline{u}_{\tau n}\cdot P^{\mathbf{U}}_n(P_\tau(v))\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \overline{\varrho}_{\tau n}\nabla\mathrm{D}_\varrho\mathcal{E}(\overline{\varrho}_{\tau n})\cdot P^{\mathbf{U}}_n(P_\tau(v))\,\mathrm{d}x\,\mathrm{d}t = 0, \\
&\qquad \text{for all } v \in L^s(0,T;\mathbf{U}),
\end{aligned} \tag{3.7b}$$

where we abbreviated

$$\langle \mathrm{D}_\tau \varrho_{\tau n}, P_{\tau l}(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)} := \int_0^T \int_\Omega \mathrm{D}_\tau \varrho_{\tau n} P_{\tau l}(\psi)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.7c}$$

$$\langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_{\tau l}(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)} := \varepsilon \int_0^T \int_\Omega |\nabla \overline{\varrho}_{\tau n}|^{p-2} \nabla \overline{\varrho}_{\tau n} \cdot \nabla P_{\tau l}(\psi)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.7d}$$

$$\big\langle \mathcal{A}_\mu(\overline{\varrho}_{\tau n}, \overline{u}_{\tau n}), P_n^{\mathbf{U}}(P_\tau(v))\big\rangle_{L^s(0,T;\mathbf{U}_n)} := \int_0^T \int_\Omega \mu_\tau(\overline{\varrho}_{\tau n}) e(\overline{u}_{\tau n}) : e(P_n^{\mathbf{U}}(P_\tau(v)))\,\mathrm{d}x\,\mathrm{d}t. \tag{3.7e}$$

Remark 3.3. *Using the projector $P_{\tau l}$ (instead of $P_{\tau n}$) with $l \le n$ in (3.7a), gives us more flexibility in showing a priori estimates and convergence results. This is carried out by first sending n to infinity by holding l fixed and in a second step letting l to ∞.*

In a similar fashion also the discrete energy-dissipation estimate (3.4) can be rewritten in the notation of the interpolants (3.5), i.e., for all $t \in [0,T]$ there holds

$$\begin{aligned}
&\mathcal{E}(\overline{\varrho}_{\tau n}(t)) + \int_0^{\bar{t}_\tau(t)} \int_\Omega \mu_\tau(\overline{\varrho}_{\tau n}(r)) e(\overline{u}_{\tau n}(r)) : e(\overline{u}_{\tau n}(r))\,\mathrm{d}x\,\mathrm{d}r + \int_0^{\bar{t}_\tau(t)} \int_\Omega M(\overline{\varrho}_{\tau n}(r)) |\overline{u}_{\tau n}(r)|^2\,\mathrm{d}x\,\mathrm{d}r \\
&\qquad + \int_0^{\bar{t}_\tau(t)} \int_\Omega |\overline{u}_{\tau n}(r)|^s\,\mathrm{d}x\,\mathrm{d}r + \frac{\varepsilon\tilde{\varepsilon}}{p} \int_0^{\bar{t}_\tau(t)} \|\nabla \overline{\varrho}_{\tau n}(r)\|^p_{L^p(\Omega;\mathbb{R}^d)}\,\mathrm{d}r \\
&\le \mathcal{E}(\varrho^0_{\tau n}) + \frac{\varepsilon\tilde{\varepsilon}^{1-p}}{p} \mathcal{L}^d(\Omega \times (0,T)).
\end{aligned} \tag{3.8}$$

Based on this, we establish a priori estimates for discrete solutions given by Proposition 3.2.

Proposition 3.4 (A priori estimates)**.** *Let the assumptions of Proposition 3.2 be satisfied and consider a sequence $(\overline{\varrho}_{\tau n}, \overline{u}_{\tau n})_{\tau n}$ of solutions for system (3.7). Then there exists a constant $C > 0$ such that the following statements hold true uniformly with respect to $n \in \mathbb{N}$ and $\tau > 0$*

$$\|\sqrt{\mu_\tau(\overline{\varrho}_{\tau n})}\, e(\overline{u}_{\tau n})\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \le C, \tag{3.9a}$$

$$\tau^\beta \|e(\overline{u}_{\tau n})\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \le C, \tag{3.9b}$$

$$c_\mu \|\overline{\varrho}_{\tau n} e(\overline{u}_{\tau n})\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \le C, \tag{3.9c}$$

$$\underline{M} \|\overline{\varrho}_{\tau n} \overline{u}_{\tau n}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \le C, \tag{3.9d}$$

$$\|\overline{u}_{\tau n}\|_{L^s(0,T;L^s(\Omega;\mathbb{R}^d))} \le C, \tag{3.9e}$$

$$\|\overline{\varrho}_{\tau n}(t)\|_{L^2(\Omega)} \le C \quad \textit{for all } t \in [0,T], \qquad , \tag{3.9f}$$

$$\|\nabla \overline{\varrho}_{\tau n}\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))} \le C. \tag{3.9g}$$

In addition, also the following estimates are valid uniformly with respect to $n \in \mathbb{N}$ and $\tau > 0$

$$\|\overline{\varrho}_{\tau n}\|_{L^p(0,T;W^{1,p}(\Omega))} \le C, \tag{3.9h}$$

$$\|\mathrm{div}(\overline{\varrho}_{\tau n} \overline{u}_{\tau n})\|_{L^2(0,T;L^2(\Omega))} \le C, \tag{3.9i}$$

$$\|\mathrm{D}_\tau \varrho_{\tau n}\|_{(L^p(0,T;\mathbf{X}_l))^*} \le C \quad \textit{for any } l \in \mathbb{N},\ \textit{for all } n \ge l, \tag{3.9j}$$

$$\|\mathcal{A}_p(\overline{\varrho}_{\tau n})\|_{(L^p(0,T;\mathbf{X}))^*} \le C \tag{3.9k}$$

$$\|\mathrm{D}_\tau \varrho_{\tau n}\|_{L^{p'}(0,T;W^{2,2}(\Omega)^*)} \le C, \tag{3.9l}$$

$$\|\mathcal{A}_\mu(\overline{\varrho}_{\tau n}, \overline{u}_{\tau n})\|_{L^{s'}(0,T;W^{1,2}(\Omega;\mathbb{R}^d)^*)} \le C, \tag{3.9m}$$

$$\||\overline{u}_{\tau n}|^{s-2} \overline{u}_{\tau n}\|_{L^{s'}(0,T;L^{s'}(\Omega;\mathbb{R}^d))} \le C. \tag{3.9n}$$

3.1 Proof of Proposition 3.2

Throughout the proof, $\tau > 0$ and $n \in \mathbb{N}$ are kept fixed.

To Proposition 3.2, Item 1.: To show the existence of discrete solutions, we observe that the Galerkin scheme (3.2) can be rewritten as a system of nonlinear equations for the coefficient vectors $\overrightarrow{\varrho}^k_{\tau n} = (\varrho^k_{\tau n,i})_{i=1}^n \in \mathbb{R}^n$, $\overrightarrow{u}^k_{\tau n} = (u^k_{\tau n,i})_{i=1}^{dn} \in \mathbb{R}^{dn}$:

$$\frac{\mathbb{M}}{\tau}(\overrightarrow{\varrho}^k_{\tau n} - \overrightarrow{\varrho}^{k-1}_{\tau n}) + \overrightarrow{u}^k_{\tau n}\mathbb{B}\overrightarrow{\varrho}^k_{\tau n} + \varepsilon\mathbb{M}_p(\varrho^k_{\tau n})\overrightarrow{\varrho}^k_{\tau n} = 0, \tag{3.10a}$$

$$\mathbb{M}_\mu(\varrho^k_{\tau n})\overrightarrow{u}^k_{\tau n} + \mathbb{M}_\varrho(\varrho^k_{\tau n})\overrightarrow{u}^k_{\tau n} + \kappa\mathbb{M}_s(u^k_{\tau n})\overrightarrow{u}^k_{\tau n} + \mathbb{X}\overrightarrow{\varrho}^k_{\tau n} + \mathbb{B}\overrightarrow{\varrho}^k_{\tau n} \otimes \overrightarrow{\varrho}^k_{\tau n} = 0, \tag{3.10b}$$

and the matrices appearing in (3.10) are defined with the aid of the basis elements $\{e_i,\, i = 1,\dots,n\}$ and $\{\mathbf{e}_i,\, i = 1,\dots,dn\}$ from (3.1) as follows

$$\mathbb{M} := \left(\int_\Omega e_i e_j \,\mathrm{d}x\right)_{i,j} \in \mathbb{R}^{n\times n}, \tag{3.10c}$$

$$\mathbb{B} := \left(\int_\Omega e_i \mathbf{e}_j \cdot \nabla e_l \,\mathrm{d}x\right)_{i,j,l} \in \mathbb{R}^{dn\times n\times n}, \tag{3.10d}$$

$$\mathbb{M}_p(\varrho^k_{\tau n}) := \left(\int_\Omega \nabla e_i \cdot \nabla e_j |\nabla \varrho^k_{\tau n}|^{p-2} \,\mathrm{d}x\right)_{i,j} \in \mathbb{R}^{n\times n}, \tag{3.10e}$$

$$\mathbb{M}_\mu(\varrho^k_{\tau n}) := \left(\int_\Omega \mu_\tau(\varrho^k_{\tau n}) e(\mathbf{e}_i) : e(\mathbf{e}_j) \,\mathrm{d}x\right)_{i,j} \in \mathbb{R}^{dn\times dn}, \tag{3.10f}$$

$$\mathbb{M}_\varrho(\varrho^k_{\tau n}) := \left(\int_\Omega M(\varrho^k_{\tau n}) \mathbf{e}_i \cdot \mathbf{e}_j \,\mathrm{d}x\right)_{i,j} \in \mathbb{R}^{dn\times dn}, \tag{3.10g}$$

$$\mathbb{X} := \left(\int_\Omega \nabla x_2 e_i \cdot \mathbf{e}_j \,\mathrm{d}x\right)_{i,j} \in \mathbb{R}^{dn\times n}, \tag{3.10h}$$

$$\mathbb{M}_s(u^k_{\tau n}) := \left(\int_\Omega |u^k_{\tau n}|^{s-2} \mathbf{e}_i \cdot \mathbf{e}_j \,\mathrm{d}x\right)_{i,j} \in \mathbb{R}^{dn\times dn}. \tag{3.10i}$$

We show in the following that for every $k \in \{1,\dots,N_\tau\}$ the nonlinear system of equations given by (3.10) has a solution $(\overrightarrow{\varrho}^k_{\tau n}, \overrightarrow{u}^k_{\tau n}) \in \mathbb{R}^n \times \mathbb{R}^{dn}$. For this, we make use of the following classical result:

Proposition 3.5 ([Zei86, Prop. 2.8, p. 53])**.** *Consider the system of equations*

$$\vec{g}(\vec{z}) = (g_i(\vec{z}))_{i=1}^m = \vec{0} \quad \text{for } \vec{z} \in \mathbb{R}^m. \tag{3.11}$$

Let $\overline{B}_R(0) := \{\vec{z} \in \mathbb{R}^m,\, \|\vec{z}\| \le R\}$ for fixed $R > 0$ and $\|\cdot\|$ a norm on $\mathbb{R}^m$. Let $g_i : \overline{B}_R(0) \to \mathbb{R}$ be continuous for $i = 1,\dots,m$. Further assume that

$$\vec{g}(\vec{z}) \cdot \vec{z} \ge 0 \qquad \text{for all } \vec{z} \text{ with } \|\vec{z}\| = R. \tag{3.12}$$

Then (3.11) *has a solution $\vec{z}$ with $\|\vec{z}\| \le R$.*

In the following we verify that the nonlinear system (3.10) satisfies the assumptions of Proposition 3.5.

We first show the **continuity** of $\vec{g}$ given by (3.10). For this, let $\vec{z} := (\overrightarrow{\varrho}^k_{\tau n}, \overrightarrow{u}^k_{\tau n}) \in \mathbb{R}^m$ with $m := n + dn$ and consider a sequence $(\vec{z}_\ell)_{\ell\in\mathbb{N}}$ with $\vec{z}_\ell := (\overrightarrow{\varrho}^k_{\tau n\ell}, \overrightarrow{u}^k_{\tau n\ell}) \in \mathbb{R}^m$ and such that

$$\vec{z}_\ell \to \vec{z} \quad \text{as } \ell \to \infty\,. \tag{3.13a}$$

We aim to show that also

$$\vec{g}(\vec{z}_\ell) \to \vec{g}(\vec{z}) \quad \text{as } \ell \to \infty\,. \tag{3.13b}$$

A close perusal of (3.10) reveals, that the maps $(\overrightarrow{\varrho}^k_{\tau n\ell}, \overrightarrow{u}^k_{\tau n\ell}) \mapsto \frac{\mathbb{M}}{\tau}(\overrightarrow{\varrho}^k_{\tau n\ell} - \overrightarrow{\varrho}^{k-1}_{\tau n}) + \overrightarrow{u}^k_{\tau n\ell}\mathbb{B}\overrightarrow{\varrho}^k_{\tau n\ell}$ and $\overrightarrow{\varrho}^k_{\tau n\ell} \mapsto \mathbb{X}\overrightarrow{\varrho}^k_{\tau n\ell} + \mathbb{B}\overrightarrow{\varrho}^k_{\tau n\ell} \otimes \overrightarrow{\varrho}^k_{\tau n\ell}$ can be rewritten as polynomials of the components of $\vec{z}_\ell$. Hence, these terms constitute continuous functions. We now discuss the continuity properties of the remaining terms

$\mathbb{M}_p(\varrho^k_{\tau n})\overrightarrow{\varrho}^k_{\tau n}$, $\mathbb{M}_\mu(\varrho^k_{\tau n})\overrightarrow{u}^k_{\tau n}$, $\mathbb{M}_\varrho(\varrho^k_{\tau n})\overrightarrow{u}^k_{\tau n}$, and $\mathbb{M}_s(u^k_{\tau n})\overrightarrow{u}^k_{\tau n}$. For this, we first observe that convergence (3.13a) implies that as $\ell\to\infty$

$$\varrho^k_{\tau n\ell}=\sum_{i=1}^n \varrho^k_{\tau n,i,\ell}e_i \to \varrho^k_{\tau n}=\sum_{i=1}^n \varrho^k_{\tau n,i}e_i \ \text{ in } W^{1,p}(\Omega)\,, \tag{3.14a}$$

$$u^k_{\tau n\ell}=\sum_{i=1}^{nd} u^k_{\tau n,i,\ell}\mathbf{e}_i \to u^k_{\tau n}=\sum_{i=1}^{nd} u^k_{\tau n,i}\mathbf{e}_i \ \text{ in } W^{1,2}(\Omega;\mathbb{R}^d)\,, \tag{3.14b}$$

again with the basis elements $\{e_i,\, i=1,\ldots,n\}$ and $\{\mathbf{e}_i,\, i=1,\ldots,dn\}$ from (3.1). Next, we note that the p-Laplacian $\mathcal{A}_p: W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$, defined in (3.7d), is a Nemyckii operator, hence continuous. In view of (3.14a), this yields that $\mathbb{M}_p(\varrho^k_{\tau n\ell})\overrightarrow{\varrho}^k_{\tau n,\ell}\to\mathbb{M}_p(\varrho^k_{\tau n})\overrightarrow{\varrho}^k_{\tau n}$. In a similar manner, we also see that $u\mapsto\int_\Omega |u|^{s-2}u\,\mathrm{d}x$ as a map from $L^s(\Omega)$ to $L^{s'}(\Omega)$ is a Nemyckii operator, and thus continuous. This provides that $\mathbb{M}_s(u^k_{\tau n\ell})\overrightarrow{u}^k_{\tau n\ell}\to\mathbb{M}_s(u^k_{\tau n})\overrightarrow{u}^k_{\tau n}$. Finally, to conclude that $\mathbb{M}_\mu(\varrho^k_{\tau n\ell})\overrightarrow{u}^k_{\tau n\ell}\to\mathbb{M}_\mu(\varrho^k_{\tau n})\overrightarrow{u}^k_{\tau n}$ and that $\mathbb{M}_\varrho(\varrho^k_{\tau n\ell})\overrightarrow{u}^k_{\tau n\ell}\to\mathbb{M}_\varrho(\varrho^k_{\tau n\ell})\overrightarrow{u}^k_{\tau n\ell}$, we observe that (3.14a) implies that the sequence $(\varrho^k_{\tau n\ell})_\ell$ is uniformly bounded in Ω and hence the dominated convergence theorem provides the result. This verifies (3.13b) and thus we conclude the continuity of the map $\vec{g}$.
Now we deduce (3.12). Testing (3.10) by $\vec{z}=(\overrightarrow{\varrho}^k_{\tau n},\overrightarrow{u}^k_{\tau n})$ we obtain

$$\begin{aligned}
&\vec{g}(\vec{z})\cdot\vec{z}\\
&=\int_\Omega\frac{\varrho^k_{\tau n}-\varrho^{k-1}_{\tau n}}{\tau}\varrho^k_{\tau n}\,\mathrm{d}x+\varepsilon\|\nabla\varrho^k_{\tau n}\|^p_{L^p(\Omega;\mathbb{R}^d)}+\int_\Omega\mu_\tau(\varrho^k_{\tau n})e(u^k_{\tau n}):e(u^k_{\tau n})\,\mathrm{d}x+\int_\Omega M(\varrho^k_{\tau n})|u^k_{\tau n}|^2\,\mathrm{d}x\\
&\quad+\kappa\|u^k_{\tau n}\|^s_{L^s(\Omega;\mathbb{R}^d)}-|1-\tilde\varepsilon|\int_\Omega\varrho^k_{\tau n}u^k_{\tau n}\cdot\nabla\varrho^k_{\tau n}\,\mathrm{d}x+\int_\Omega\varrho^k_{\tau n}u^k_{\tau n}\cdot\nabla x_2\,\mathrm{d}x\\
&\geq\frac{\|\varrho^k_{\tau n}\|^2_{L^2(\Omega)}}{2\tau}-\frac{\|\varrho^{k-1}_{\tau n}\|^2_{L^2(\Omega)}}{2\tau}+\varepsilon\|\nabla\varrho^k_{\tau n}\|^p_{L^p(\Omega;\mathbb{R}^d)}+\tau^\beta\|e(u^k_{\tau n})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}+c_\mu\|\varrho^k_{\tau n}e(u^k_{\tau n})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\\
&\quad+\underline{M}\|\varrho^k_{\tau n}u^k_{\tau n}\|^2_{L^2(\Omega;\mathbb{R}^d)}+\kappa\|u^k_{\tau n}\|^s_{L^s(\Omega;\mathbb{R}^d)}-|1-\tilde\varepsilon|\int_\Omega|\varrho^k_{\tau n}u^k_{\tau n}||\nabla\varrho^k_{\tau n}|\,\mathrm{d}x-\int_\Omega|\varrho^k_{\tau n}u^k_{\tau n}|\underbrace{|\nabla x_2|}_{=1}\,\mathrm{d}x\\
&\geq\tfrac{\|\varrho^k_{\tau n}\|^2_{L^2(\Omega)}}{2\tau}-\tfrac{\|\varrho^{k-1}_{\tau n}\|^2_{L^2(\Omega)}}{2\tau}+\varepsilon\left(1-\tfrac{2}{p}\right)\|\nabla\varrho^k_{\tau n}\|^p_{L^p(\Omega;\mathbb{R}^d)}+\tau^\beta\|e(u^k_{\tau n})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}+c_\mu\|\varrho^k_{\tau n}e(u^k_{\tau n})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\\
&\quad+\tfrac{\underline{M}}{2}\|\varrho^k_{\tau n}u^k_{\tau n}\|^2_{L^2(\Omega;\mathbb{R}^d)}+\kappa\|u^k_{\tau n}\|^s_{L^s(\Omega;\mathbb{R}^d)}+\left(\tfrac{p-2}{p}\right)\left(\tfrac{|1-\tilde\varepsilon|^2}{\underline{M}}\right)^{p/(p-2)}\varepsilon^{-2/(p-2)}\mathcal{L}^d(\Omega)+\tfrac{1}{\underline{M}}\mathcal{L}^d(\Omega)\,.
\end{aligned}\tag{3.15}$$

Above, in the first inequality we have exploited the convexity of the L^2-norm together with (3.3b) and (2.2c). Moreover, to arrive at the second estimate, by virtue of (2.2f), (2.2g), Hölder's and Young's inequality, we have

$$\begin{aligned}
|1-\tilde\varepsilon|\int_\Omega|\varrho^k_{\tau n}u^k_{\tau n}||\nabla\varrho^k_{\tau n}|\,\mathrm{d}x&\leq\tfrac{\underline{M}}{4}\|\varrho^k_{\tau n}u^k_{\tau n}\|^2_{L^2(\Omega)}+\tfrac{2\varepsilon}{p}\|\nabla\varrho^k_{\tau n}\|^p_{L^p(\Omega;\mathbb{R}^d)}\\
&\quad+\left(\tfrac{p-2}{p}\right)\left(\tfrac{|1-\tilde\varepsilon|^2}{\underline{M}}\right)^{p/(p-2)}\varepsilon^{-2/(p-2)}\mathcal{L}^d(\Omega),\\
\text{and}\quad\int_\Omega|\varrho^k_{\tau n}u^k_{\tau n}|\underbrace{|\nabla x_2|}_{=1}\,\mathrm{d}x&\leq\tfrac{\underline{M}}{4}\|\varrho^k_{\tau n}u^k_{\tau n}\|^2_{L^2(\Omega)}+\tfrac{1}{\underline{M}}\mathcal{L}^d(\Omega).
\end{aligned}$$

Using that $\mathrm{span}\{e_j; j=1,\ldots,n\}=\mathbf{X}_n$ and $\mathrm{span}\{\mathbf{e}_j; j=1,\ldots,dn\}=\mathbf{U}_n$, we further estimate the norms in (3.15) via Young's inequality as follows

$$\begin{aligned}
\|\nabla\varrho^k_{\tau n}\|^p_{L^p(\Omega;\mathbb{R}^d)}&\geq\min_{i=1,\ldots,n}\|\nabla e_i\|^p_{L^p(\Omega;\mathbb{R}^d)}\|\overrightarrow{\varrho}^k_{\tau n}\|^p_{l^p}\geq n^{1-p}\min_{i=1,\ldots,n}\|\nabla e_i\|^p_{L^p(\Omega;\mathbb{R}^d)}\|\overrightarrow{\varrho}^k_{\tau n}\|^p_{l^1},\\
-\|\varrho^{k-1}_{\tau n}\|^2_{L^2(\Omega)}&\geq-\max_{i=1,\ldots,n}\|e_i\|^2_{L^2(\Omega;\mathbb{R}^d)}\|\overrightarrow{\varrho}^{k-1}_{\tau n}\|^p_{l^2}\geq-\frac{1}{n}\max_{i=1,\ldots,n}\|e_i\|^2_{L^2(\Omega;\mathbb{R}^d)}\|\overrightarrow{\varrho}^{k-1}_{\tau n}\|^2_{l^1},\\
\|u^k_{\tau n}\|^s_{L^s(\Omega;\mathbb{R}^d)}&\geq\min_{i=1,\ldots,dn}\|\mathbf{e}_i\|^s_{L^s(\Omega;\mathbb{R}^d)}\|\overrightarrow{u}^k_{\tau n}\|^s_{l^s}\geq(dn)^{1-s}\min_{i=1,\ldots,dn}\|\mathbf{e}_i\|^s_{L^s(\Omega;\mathbb{R}^d)}\|\overrightarrow{u}^k_{\tau n}\|^s_{l^1},
\end{aligned}$$

and finally obtain

$$
\begin{aligned}
&\vec{g}(\vec{z})\cdot\vec{z}\\
&\geq \min\left\{\varepsilon\left(1-\tfrac{1}{p}\right)n^{1-p}\min_{i=1,\dots,n}\|\nabla e_i\|^p_{L^p(\Omega;\mathbb{R}^d)},\ \kappa(dn)^{1-s}\min_{i=1,\dots,dn}\|\mathbf{e}_i\|^s_{L^s(\Omega;\mathbb{R}^d)}\right\}\left(\|\overrightarrow{\varrho}^{\,k}_{\tau n}\|^p_{l^1}+\|\overrightarrow{u}^{\,k}_{\tau n}\|^s_{l^1}\right)\\
&\quad-\frac{1}{2\tau n}\max_{i=1,\dots,n}\|e_i\|^2_{L^2(\Omega;\mathbb{R}^d)}\|\overrightarrow{\varrho}^{\,k-1}_{\tau n}\|^2_{l^1}.
\end{aligned}
$$

For this we see that condition (3.12) is satisfied when choosing

$$
R:=\frac{\frac{1}{2\tau n}\max_{i=1,\dots,n}\|e_i\|^2_{L^2(\Omega;\mathbb{R}^d)}\|\overrightarrow{\varrho}^{\,k-1}_{\tau n}\|^2_{l^1}}{\min\left(\varepsilon(1-\frac{1}{p})n^{1-p}\min_{i=1,\dots,n}\|\nabla e_i\|^p_{L^p(\Omega;\mathbb{R}^d)},\kappa(dn)^{1-s}\min_{i=1,\dots,dn}\|\mathbf{e}_i\|^s_{L^s(\Omega;\mathbb{R}^d)}\right)}.
$$

To Proposition 3.2, Item 2.: Testing (3.2a) by $\mathrm{D}_\varrho\mathcal{E}(\varrho^k_{\tau n})$ and (3.2b) by $u^k_{\tau n}$, respectively, gives

$$
\begin{aligned}
0&=\int_\Omega\mu_\tau(\varrho^k_{\tau n})e(u^k_{\tau n}):e(u^k_{\tau n})\,\mathrm{d}x+\int_\Omega M(\varrho^k_{\tau n})|u^k_{\tau n}|^2\,\mathrm{d}x+\kappa\int_\Omega|u^k_{\tau n}|^s\,\mathrm{d}x\\
&\quad+\int_\Omega\frac{\varrho^k_{\tau n}-\varrho^{k-1}_{\tau n}}{\tau}\mathrm{D}_\varrho\mathcal{E}(\varrho^k_{\tau n})\,\mathrm{d}x+\varepsilon\int_\Omega|\nabla\varrho^k_{\tau n}|^{p-2}\nabla\varrho^k_{\tau n}\cdot\nabla(x_2+\tilde\varepsilon\varrho^k_{\tau n})\,\mathrm{d}x\\
&\geq\int_\Omega\mu_\tau(\varrho^k_{\tau n})e(u^k_{\tau n}):e(u^k_{\tau n})\,\mathrm{d}x+\int_\Omega M(\varrho^k_{\tau n})|u^k_{\tau n}|^2\,\mathrm{d}x+\kappa\int_\Omega|u^k_{\tau n}|^s\,\mathrm{d}x+\frac{1}{\tau}(\mathcal{E}(\varrho^k_{\tau n})-\mathcal{E}(\varrho^{k-1}_{\tau n}))\\
&\quad+\frac{\varepsilon\tilde\varepsilon}{p}\|\nabla\varrho^k_{\tau n}\|^p_{L^p(\Omega;\mathbb{R}^d)}-\frac{\varepsilon\tilde\varepsilon^{1-p}}{p}\mathcal{L}^d(\Omega),
\end{aligned}
\tag{3.16}
$$

where we exploited the convexity of $\mathcal{E}(\varrho^k_{\tau n})$, and Hölder's and Young's inequalities in the form

$$
\varepsilon\int_\Omega\Big(|\nabla\varrho^k_{\tau n}|^{p-2}\nabla\varrho^k_{\tau n}\cdot\nabla(x_2+\tilde\varepsilon\varrho^k_{\tau n})\Big)\,\mathrm{d}x\geq\frac{\varepsilon\tilde\varepsilon}{p}\int_\Omega|\nabla\varrho^k_{\tau n}|^p\,\mathrm{d}x-\frac{\varepsilon\tilde\varepsilon^{1-p}}{p}\mathcal{L}^d(\Omega).
$$

Now we multiply (3.16) by τ und sum up from $k=0$ to K to find (3.4). □

3.2 Proof of Proposition 3.4

To a priori estimates (3.9a)–(3.9g)**:** The estimates (3.9a), (3.9d)–(3.9g) are immediate consequences of the discrete energy-dissipation inequality (3.8) for the interpolated solutions of system (3.7). Furthermore, we also deduce estimates (3.9b) and (3.9c) from (3.8) by exploiting the growth property (3.3b) of μ_τ, i.e.,

$$
\sum_{k=1}^{N_\tau}\tau\int_\Omega\mu_\tau(\varrho^k_{\tau n})e(u^k_{\tau n}):e(u^k_{\tau n})\,\mathrm{d}x\geq\tau^\beta\sum_{k=1}^{N_\tau}\tau\|e(u^k_{\tau n})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}+c_\mu\sum_{k=1}^{N_\tau}\tau\|\varrho^k_{\tau n}e(u^k_{\tau n})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}.
$$

To a priori estimates (3.9h)–(3.9n)**:** Estimate (3.9h) follows from (3.9f) and (3.9g) together with a generalized Poincaré inequality, see [Rou05, Theorem 1.32, p. 21]. Thanks to (3.9c), (3.9e), (3.9g), Hölder's and Young's inequalities, and the relations (2.2f) for the exponents p and s, we obtain (3.9i):

$$
\begin{aligned}
&\|\mathrm{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})\|^2_{L^2(0,T;L^2(\Omega))}\leq\|e(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\\
&\leq2\|\varrho^k_{\tau n}e(u^k_{\tau n})\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}+2\left\|\frac{u^k_{\tau n}\otimes\nabla\varrho^k_{\tau n}+\nabla\varrho^k_{\tau n}\otimes u^k_{\tau n}}{2}\right\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}\\
&\leq2\|\varrho^k_{\tau n}e(u^k_{\tau n})\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}+\frac{2}{p}\|\nabla\varrho^k_{\tau n}\|^p_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))}+\frac{2}{s}\|u^k_{\tau n}\|^s_{L^s(0,T;L^s(\Omega;\mathbb{R}^d))}\leq C,
\end{aligned}
$$

where the bounds on the above three terms are provided by the immediate estimates (3.9c), (3.9g) and (3.9e). To find (3.9j) we argue by comparison in the discrete transport equation (3.7a), which reads

$$\begin{aligned}\langle \mathrm{D}_\tau \varrho_{\tau n}, P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)} &= \int_0^T \int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla P_l(\psi)\,\mathrm{d}x\,\mathrm{d}t - \langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)} \\ &=: \langle \mathrm{RH} - \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}\ ,\end{aligned} \tag{3.17}$$

for any test function $\psi \in L^p(0,T;\mathbf{X})$ projected to $\mathbf{X}_l$ by P_l. Now, to find that $\|\mathrm{D}_\tau\varrho_{\tau n}\|_{(L^p(0,T;\mathbf{X}_l))^*}$ is bounded, we show that each of the terms on the right-hand side of (3.17) is uniformly bounded. In particular, we have for all $n \geq l$:

$$\|\mathrm{D}_\tau\varrho_{\tau n}\|_{(L^p(0,T;\mathbf{X}_l))^*} = \sup_{P_l(\psi)\in L^p(0,T;\mathbf{X}_l)} \frac{\left|\langle \mathrm{D}_\tau \varrho_{\tau n}, P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}\right|}{\|P_l(\psi)\|_{L^p(0,T;\mathbf{X}_l)}}, \tag{3.18}$$

and by (3.17)

$$\left|\langle \mathrm{D}_\tau \varrho_{\tau n}, P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}\right| = \left|\langle(\mathrm{RH} - \mathcal{A}_p(\overline{\varrho}_{\tau n})), P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}\right|.$$

Now, the terms on the right-hand side are estimated via Hölder's inequality as follows

$$\begin{aligned}\left|\langle \mathrm{RH}, P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}\right| &= \left|\int_0^T\int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla P_l(\psi)\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \|\overline{\varrho}_{\tau n}\overline{u}_{\tau n}\|_{L^2(0,T;L^2(\Omega))}\|\nabla P_l(\psi)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \\ &\leq C\|P_l(\psi)\|_{L^p(0,T;\mathbf{X}_l)},\end{aligned} \tag{3.19}$$

$$\begin{aligned}\left|\langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}\right| &= \left|\varepsilon\int_0^T\int_\Omega |\nabla\overline{\varrho}_{\tau n}|^{p-2}\nabla\overline{\varrho}_{\tau n}\cdot\nabla P_l(\psi)\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \varepsilon\|\nabla\overline{\varrho}_{\tau n}\|^{p-1}_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))}\|\nabla P_l(\psi)\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))} \\ &\leq C\|P_l(\psi)\|_{L^p(0,T;\mathbf{X}_l)}\,.\end{aligned} \tag{3.20}$$

For the last inequality of (3.19) and (3.20) we used (3.9d) and (3.9g), respectively. This finally yields (3.9j). Estimate (3.9k) then follows by repeating the calculations of (3.20) for $\langle\mathcal{A}_p(\overline{\varrho}_{\tau n}),\psi\rangle_{L^p(0,T;\mathbf{X})}$, realizing that this expression is well-defined for any $\psi \in L^p(0,T;\mathbf{X})$ thanks to a priori estimate (3.9g).
In preparation of showing (3.9l), we define a sequence of finite-dimensional subspaces $\mathbf{Y}_n$ such that, for all $n \in \mathbb{N}$

$$\mathbf{Y}_n \subset \mathbf{Y}_{n+1} \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} \mathbf{Y}_n \text{ dense in } \mathbf{Y} := W^{2,2}(\Omega).$$

By [Rou05, Remark 8.41, p. 238], we define a selfadjoint projector

$$P_n^{\mathbf{Y}} : \mathbf{Y} \to \mathbf{Y} \text{ such that } P_n^{\mathbf{Y}}(\mathbf{Y}) = \mathbf{Y}_n \text{ and } \|P_n^{\mathbf{Y}}\|_{\mathcal{L}(\mathbf{Y},\mathbf{Y})} \text{ is bounded independently of } n. \tag{3.21}$$

In view of [GGZ74, Lemma 1.12., p. 144], there also holds that $\bigcup_{n\in\mathbb{N}} C^0([0,T];\mathbf{Y}_n)$ is dense in $L^p(0,T;\mathbf{Y})$ for any $1 \leq p \leq \infty$. We will show that the operator norm given here below is uniformly bounded, again by testing (3.17), now with functions $\psi \in L^p(0,T;\mathbf{Y})$.

$$\|\mathrm{D}_\tau\varrho_{\tau n}\|_{(L^p(0,T;\mathbf{Y}))^*} = \sup_{\psi\in L^p(0,T;\mathbf{Y})} \frac{\left|\langle \mathrm{D}_\tau\varrho_{\tau n},\psi\rangle_{L^p(0,T;\mathbf{Y})}\right|}{\|\psi\|_{L^p(0,T;\mathbf{Y})}}\,.$$

Since $P_n^{\mathbf{Y}}$ is idempotent and selfadjoint $P_n^{\mathbf{Y}*} = P_n^{\mathbf{Y}}$, it is

$$\begin{aligned}\left|\langle \mathrm{D}_\tau\varrho_{\tau n},\psi\rangle_{L^p(0,T;\mathbf{Y})}\right| &= \left|\left\langle P_n^{\mathbf{Y}}\mathrm{D}_\tau\varrho_{\tau n},\psi\right\rangle_{L^p(0,T;\mathbf{Y})}\right| \\ &= \left|\left\langle P_n^{\mathbf{Y}*}\mathrm{D}_\tau\varrho_{\tau n},\psi\right\rangle_{L^p(0,T;\mathbf{Y})}\right| = \left|\langle \mathrm{D}_\tau\varrho_{\tau n}, P_n^{\mathbf{Y}}(\psi)\rangle_{L^p(0,T;\mathbf{Y}_n)}\right|.\end{aligned}$$

To find (3.9l), we now repeat the lines of (3.19) and (3.20) for the uniform estimates of the right-hand side of (3.17), also using that

$$\|\nabla P_n^{\mathbf{Y}}(\psi)\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))} \leq \|P_n^{\mathbf{Y}}(\psi)\|_{L^p(0,T;\mathbf{Y})} \leq \|P_n^{\mathbf{Y}}\|_{\mathcal{L}(\mathbf{Y},\mathbf{Y})}\|\psi\|_{L^p(0,T;\mathbf{Y})},$$

where $\|P_n^{\mathbf{Y}}\|_{\mathcal{L}(\mathbf{Y},\mathbf{Y})} \leq C$ by (3.21). This together with the isomorphism $(L^p(0,T;\mathbf{Y}))^* \cong L^{p'}(0,T;\mathbf{Y}^*)$ proves estimate (3.9l).

To deduce (3.9m) we also argue by comparison based on (3.7b). In particular, we have

$$\|\mathcal{A}_\mu(\overline{\varrho}_{\tau n},\overline{u}_{\tau n})\|_{(L^s(0,T;\mathbf{U}))^*} = \sup_{v\in L^s(0,T;\mathbf{U})} \frac{\left|\langle\mathcal{A}_\mu(\overline{\varrho}_{\tau n},\overline{u}_{\tau n}),v\rangle_{L^s(0,T;\mathbf{U})}\right|}{\|v\|_{L^s(0,T;\mathbf{U})}}.$$

Now we exploit (3.7b) and also make use of the selfadjoint, idempotent projector $P_n^{\mathbf{U}}:\mathbf{U}\to\mathbf{U}_n$, cf. Remark 3.1. More precisely, since $\mathcal{A}_\mu(\overline{\varrho}_{\tau n}(t),\cdot):\mathbf{U}\to\mathbf{U}^*$ is a linear operator for a.a. $t\in(0,T)$, cf. (3.7e), we have

$$\begin{aligned}\left|\langle\mathcal{A}_\mu(\overline{\varrho}_{\tau n},\overline{u}_{\tau n}),v\rangle_{L^s(0,T;\mathbf{U})}\right| &= \left|\langle\mathcal{A}_\mu(\overline{\varrho}_{\tau n},P_n^{\mathbf{U}}\overline{u}_{\tau n}),v\rangle_{L^s(0,T;\mathbf{U})}\right|\\ &= \left|\left\langle P_n^{\mathbf{U}*}\mathcal{A}_\mu(\overline{\varrho}_{\tau n},\overline{u}_{\tau n}),v\right\rangle_{L^s(0,T;\mathbf{U})}\right| = \left|\langle\mathcal{A}_\mu(\overline{\varrho}_{\tau n},\overline{u}_{\tau n}),P_n^{\mathbf{U}}v\rangle_{L^s(0,T;\mathbf{U}_n)}\right|,\end{aligned}$$

thanks to the fact that $P_n^{\mathbf{U}}$ is idempotent and selfadjoint, with $P_n^{\mathbf{U}*}$ denoting the adjoint operator. Based on this, we further estimate

$$\begin{aligned}\left|\langle\mathcal{A}_\mu(\overline{\varrho}_{\tau n},\overline{u}_{\tau n}),v\rangle_{L^s(0,T;\mathbf{U})}\right| &= \left|\langle\mathcal{A}_\mu(\overline{\varrho}_{\tau n},\overline{u}_{\tau n}),P_n^{\mathbf{U}}(v)\rangle_{L^s(0,T;\mathbf{U}_n)}\right|\\ &= \left|\int_0^T\int_\Omega \mu_\tau(\overline{\varrho}_{\tau n})e(\overline{u}_{\tau n}):e(P_n^{\mathbf{U}}(v))\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\leq \underbrace{\left|\int_0^T\int_\Omega M(\overline{\varrho}_{\tau n})\overline{u}_{\tau n}\cdot P_n^{\mathbf{U}}(v)\,\mathrm{d}x\,\mathrm{d}t\right|}_{\mathrm{RHS}_1}\\ &\quad+\underbrace{\left|\int_0^T\int_\Omega \overline{\varrho}_{\tau n}\nabla \mathrm{D}_\varrho\mathcal{E}(\overline{\varrho}_{\tau n})\cdot P_n^{\mathbf{U}}(v)\,\mathrm{d}x\,\mathrm{d}t\right|}_{\mathrm{RHS}_2}\\ &\quad+\underbrace{\left|\kappa\int_0^T\int_\Omega |\overline{u}_{\tau n}|^{s-2}\overline{u}_{\tau n}\cdot P_n^{\mathbf{U}}(v)\,\mathrm{d}x\,\mathrm{d}t\right|}_{\mathrm{RHS}_3}.\end{aligned}\tag{3.22}$$

We show that each of the three terms on the right-hand side of (3.22) is bounded. For the first term it is

$$\begin{aligned}\mathrm{RHS}_1 &\leq \overline{M}\|\overline{\varrho}_{\tau n}\overline{u}_{\tau n}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\|P_n^{\mathbf{U}}(v)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\\ &\leq \overline{M}\|\overline{\varrho}_{\tau n}\overline{u}_{\tau n}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\|P_n^{\mathbf{U}}\|_{\mathcal{L}(\mathbf{U},\mathbf{U})}\|v\|_{L^s(0,T;\mathbf{U})}\\ &\leq \overline{M}\|\overline{\varrho}_{\tau n}\overline{u}_{\tau n}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\tilde{C}\|v\|_{L^s(0,T;\mathbf{U})},\end{aligned}$$

by virtue of the growth property (2.2c) of $M(\overline{\varrho}_{\tau n})$, the immediate bound (3.9e) for $\overline{\varrho}_{\tau n}\overline{u}_{\tau n}$ and boundedness assumption for $P_n^{\mathbf{U}}$ given by Remark 3.1.

Thanks to the relations (2.2f) for the exponents p,s we find for the second term

$$\begin{aligned}\mathrm{RHS}_2 &\leq \|\overline{\varrho}_{\tau n}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\left[\tilde{\varepsilon}\|\nabla\overline{\varrho}_{\tau n}\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))}+\mathcal{L}^d(Q)^{1/p}\right]\|P_n^{\mathbf{U}}(v)\|_{L^s(0,T;L^s(\Omega;\mathbb{R}^d))}\\ &\leq \|\overline{\varrho}_{\tau n}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\left[\tilde{\varepsilon}\|\nabla\overline{\varrho}_{\tau n}\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))}+\mathcal{L}^d(Q)^{1/p}\right]\|P_n^{\mathbf{U}}\|_{\mathcal{L}(\mathbf{U},\mathbf{U})}\|v\|_{L^s(0,T;\mathbf{U})}\\ &\leq \tilde{C}\|v\|_{L^s(0,T;\mathbf{U})}.\end{aligned}$$

Here, we abbreviated $Q := (0,T) \times \Omega$ and we also made use of the immediate bounds (3.9f), (3.9g) and boundedness assumption for $P_n^{\mathbf{U}}$ given by Remark 3.1. For the third term we have

$$\mathrm{RHS}_3 \leq \kappa \left(\tfrac{s-1}{s} \|\overline{u}_{\tau n}\|^s_{L^s(0,T;L^s(\Omega;\mathbb{R}^d))} + \tfrac{1}{s} \right) \|P_n^{\mathbf{U}}\|_{\mathcal{L}(\mathbf{U},\mathbf{U})} \|v\|_{L^s(0,T;\mathbf{U})} \leq \kappa \tilde{C} \|v\|_{L^s(0,T;\mathbf{U})},$$

where we used Hölder's inequality with the exponent s together with (3.9e) and (3.21). Finally, (3.22) together with the isomorphism $(L^s(0,T;\mathbf{U}))^* \cong L^{s'}(0,T;\mathbf{U}^*)$ provides (3.9n). □

4 Limit passage from discrete to continuous

In this Section we give the proof of our main results, the existence Theorem 2.3 and the identification Theorems 2.4 and 2.7.

4.1 Proof of Theorem 2.3: Existence of solutions in the sense of Def. 2.2

Based on the a priori bounds deduced in Proposition 3.4, we are now in the position to extract a subsequence of solutions of the discrete problems that converges to a limit (ϱ, u) in suitable topologies.

Proposition 4.1 (Convergence of the discrete approximants to a solution of (1.1))**.** *Let the assumptions of Theorem 2.3 be fulfilled. Then the following statements hold true:*

1. *There exists a (not relabeled) subsequence of discrete solutions $(\overline{\varrho}_{\tau n}, \overline{u}_{\tau n})_{\tau n}$ and a limit quadruplet $(\varrho, u, \zeta, B_\mu)$ as well as limit objects (ξ_l, B_p, b) such that as $\tau \to 0$ and $n \to \infty$, the following convergence results are valid*

$$\overline{u}_{\tau n} \rightharpoonup u \qquad \text{in } L^s(0,T;L^s(\Omega;\mathbb{R}^d)), \tag{4.1a}$$

$$|\overline{u}_{\tau n}|^{s-2}\overline{u}_{\tau n} \rightharpoonup \zeta \qquad \text{in } L^{s'}(0,T;L^{s'}(\Omega;\mathbb{R}^d)), \tag{4.1b}$$

$$\overline{\varrho}_{\tau n} \rightharpoonup \varrho \qquad \text{in } L^p(0,T;W^{1,p}(\Omega)), \tag{4.1c}$$

$$\mathrm{D}_\tau \varrho_{\tau n} \rightharpoonup \xi_l \qquad \text{in } (L^p(0,T;\mathbf{X}_l))^*, \tag{4.1d}$$

$$\mathcal{A}_p(\overline{\varrho}_{\tau n}) \rightharpoonup B_p \qquad \text{in } (L^p(0,T;\mathbf{X}))^*, \tag{4.1e}$$

$$\mathrm{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n}) \rightharpoonup b \qquad \text{in } L^2(0,T;L^2(\Omega)), \tag{4.1f}$$

$$\overline{\varrho}_{\tau n}(t) \rightharpoonup \varrho(t) \qquad \text{in } L^2(\Omega), \quad \text{for all } t \in [0,T], \tag{4.1g}$$

$$\mathcal{A}_\mu(\overline{\varrho}_{\tau n}, \overline{u}_{\tau n}) \rightharpoonup B_\mu \qquad \text{in } L^{s'}(0,T;W^{1,2}(\Omega;\mathbb{R}^d)^*). \tag{4.1h}$$

In addition to (4.1a)–(4.1h)*, as $\tau \to 0$ and $n \to \infty$ also the following convergence results hold true:*

$$\overline{\varrho}_{\tau n} \to \varrho \qquad \text{in } L^p(0,T;L^p(\Omega)), \tag{4.1i}$$

$$\overline{\varrho}_{\tau n} \to \varrho \qquad \text{in } L^p(0,T;W^{1,p}(\Omega)). \tag{4.1j}$$

2. *The limit quadruplet $(\varrho, u, \zeta, B_\mu)$ extracted by convergences* (4.1) *is a weak solution of system* (1.1) *in the sense of Definition 2.2.*

The proof of convergence result (4.1i) will rely on the following (discrete) Aubin-Lions type result:

Proposition 4.2 ([DJ12, Thm. 1])**.** *Assume $T > 0, N \in \mathbb{N}, \tau = T/N$, and set $t_k = k\tau, k = 0, \ldots, N$. Let X, B and Y be Banach spaces such that the embedding $X \hookrightarrow B$ is compact and the embedding $B \hookrightarrow Y$ is continuous. Furthermore, let either $1 \leq p < \infty, r = 1$ or $p = \infty, r > 1$, and let (u_τ) be a sequence of functions, which are constant on each subinterval (t_{k-1}, t_k), satisfying*

$$\tau^{-1}\|u_\tau - u_\tau(\cdot - \tau)\|_{L^r(\tau,T;Y)} + \|u_\tau\|_{L^r(0,T;X)} \leq C_0 \text{ for all } \tau > 0, \tag{4.2}$$

where $C_0 > 0$ is a constant which is independent of τ. If $p < \infty$, then (u_τ) is relatively compact in $L^p(0,T;B)$. If $p = \infty$, there exists a subsequence of (u_τ) which converges in each space $L^q(0,T;B)$, $1 \leq q < \infty$, to a limit which belongs to $C^0([0,T];B)$.

Proof of Prop. 4.1: The convergence results (4.1a)–(4.1h) are direct consequences of the a priori estimates (3.9) derived in Proposition 3.4.

To verify convergence statement (4.1i) we observe that (3.9h) and (3.9l) imply the bound

$$\|\mathrm{D}_\tau \varrho_{\tau n}\|_{L^{p'}(0,T;W^{2,2}(\Omega)^*)} + \|\overline{\varrho}_{\tau n}\|_{L^p(0,T;W^{1,p}(\Omega))} \leq C_0, \text{ for all } \tau > 0. \tag{4.3}$$

Now, we apply Proposition 4.2 with $X = W^{1,p}(\Omega)$, $B = L^p(\Omega)$ and $Y = W^{2,2}(\Omega)^*$. Indeed, $W^{1,p}(\Omega)$ embeds compactly into $L^p(\Omega)$ and $L^p(\Omega)$ embeds continuously into $W^{2,2}(\Omega)^*$. Moreover, by (2.2f) and Remark 2.1, Item 2., we have $p, p' > 1 = r$ and therefore, estimate (4.3) fits with assumption (4.2). Hence, Proposition 4.2 provides (4.1i).

In order to verify Prop. 4.1, Item 2, we will pass to the limit $n \to \infty$, $\tau \to 0$ in system (3.7) for the interpolated solutions $(\overline{\varrho}_{\tau n}, \overline{u}_{\tau n})_{\tau n}$. We carry out this procedure separately in Sections 4.1.1 and 4.1.2. To deduce the limit transport equation (2.5a) requires to identify that $\langle B_p, \psi\rangle_{L^p(0,T;\mathbf{X})} = \langle \mathcal{A}_p(\varrho), \psi\rangle_{L^p(0,T;\mathbf{X})}$ for all $\psi \in L^p(0,T;\mathbf{X})$ for the p-Laplacian term defined in (3.7d) and the limit B_p obtained by convergence (4.1e). This identification is also carried out in Section 4.1.1 based on tools from convex analysis and monotone operator theory. As a result of the identification procedure, we will also conclude the strong $L^p(0,T;W^{1,p}(\Omega))$-convergence (4.1j) for the densities. □

Remark 4.3. *(Preparations for the identification argument in Theorems 2.4 & 2.7)*

1. *Using Proposition 4.2 with $X = W^{1,p}(\Omega)$, $B = C(\overline{\Omega})$ and $Y = W^{2,2}(\Omega)^*$, where $W^{1,p}(\Omega)$ embeds compactly into $C(\overline{\Omega})$ for $p > d$, and following the above line of arguments, also yields*
$$\overline{\varrho}_{\tau n} \to \varrho \text{ in } L^p(0,T;C(\overline{\Omega})). \tag{4.4}$$
2. *The above strong convergence (4.4) together with Egorov's theorem provides the existence of a further (not relabelled) subsequence that converges almost uniformly in $(0,T)$. More precisely, for every $\delta > 0$, there exists a measurable subset I_δ^c of $(0,T)$ such that $\mathcal{L}^1(I_\delta^c) < \delta$, and such that*
$$\|\overline{\varrho}_{\tau n}(t) - \varrho(t)\|_{C(\overline{\Omega})} \to 0 \quad \text{uniformly for all } t \in I_\delta := (0,T) \setminus I_\delta^c\,,$$
which also yields
$$\overline{\varrho}_{\tau n} \to \varrho \text{ uniformly in } I_\delta \times \overline{\Omega}. \tag{4.5}$$
The above uniform convergence on $I_\delta \times \overline{\Omega}$ will be exploited subsequently in Section 4.2 for the identification of B_μ and ζ as functions of the limit pair (ϱ, u) on the non-cylindrical domains Q_ν^δ, cf. Thm. 2.4 and in Section 4.3 for proving the non-negativity and boundedness of the limit density ϱ, cf. Prop. 2.7.

4.1.1 Limit passage in the continuity equation and convergence result (4.1j)

In the following, we carry out the limit passage in the discrete transport equation (3.7a) by discussing each of the three apprearing terms separately. We start with the time derivative $\langle \mathrm{D}_\tau \varrho_{\tau n}, P_{\tau l}(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}$ and subsequently address the two remaining terms defined in (3.7c) and (3.7d).

The limit passage in the time derivative is carried out in two steps. For this, we follow the strategy of [Rou05, Proof of Thm. 8.27, p. 225ff]: Firstly, from convergence statement (4.1d), we obtain

$$\langle \xi_l, P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)} = \lim_{\substack{\tau\to 0\\ n\to\infty}} \langle \mathrm{D}_\tau \varrho_{\tau n}, P_{\tau l}(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)} \quad \text{for each } l \in \mathbb{N} \text{ fixed.} \tag{4.6a}$$

Secondly, we let $l \to \infty$. For this, we note that ξ_{l+1} can be regarded as an extension of ξ_l from $L^p(0,T;\mathbf{X}_l)$ to $L^p(0,T;\mathbf{X}_{l+1})$. By (4.1d), there holds $\|\xi_l\|_{(L^p(0,T;\mathbf{X}_l))^*} \leq C$ independently of $l \in \mathbb{N}$. Hence, by density of $\bigcup_{l\in\mathbb{N}} L^p(0,T;\mathbf{X}_l)$ in $L^p(0,T;\mathbf{X})$ and since Hahn-Banach's theorem guarantees the existence and uniqueness of a continuous extension, we conclude the existence of a functional $\dot\varrho \in (L^p(0,T;\mathbf{X}))^*$ such that also $\|\dot\varrho\|_{(L^p(0,T;\mathbf{X}))^*} \leq C$. In addition, it is $\dot\varrho|_{L^p(0,T;\mathbf{X}_l)} = \xi_l = \partial_t\varrho|_{L^p(0,T;\mathbf{X}_l)}$ for each $l \in \mathbb{N}$. Hence, we obtain for any $\psi \in L^p(0,T;\mathbf{X})$ as $l \to \infty$

$$\langle \partial_t \varrho, \psi\rangle_{L^p(0,T;\mathbf{X})} = \lim_{l\to\infty} \langle \xi_l, P_l(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)} = \lim_{l\to\infty}\left(\lim_{\substack{\tau\to 0\\ n\to\infty}} \langle \mathrm{D}_\tau \varrho_{\tau n}, P_{\tau l}(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}\right). \tag{4.6b}$$

This result also allows us to conclude the regularity stated for ϱ in (2.4), i.e., in particular

$$\partial_t \varrho \in (L^p(0,T;\mathbf{X}))^* \cong L^{p'}(0,T;\mathbf{X}^*)\,. \tag{4.6c}$$

For the limit passage in the remaining terms we can simultaneously send $\tau \to 0$, $n \to \infty$, and $l \to \infty$.

We proceed with the drift term given by (3.7c). Due to (4.1i), in particular $\overline{\varrho}_{\tau n} \to \varrho$ in $L^2(0,T;L^2(\Omega))$ as $\tau \to 0$, $n \to \infty$, and by $P_{\tau l}(\psi) \to \psi$ in $L^p(0,T;\mathbf{X})$ as $\tau \to 0$, $l \to \infty$, we obtain

$$\begin{aligned}
&\|\varrho \nabla \psi - \overline{\varrho}_{\tau n} \nabla P_{\tau l}(\psi)\|_{L^{s'}(0,T;L^{s'}(\Omega;\mathbb{R}^d))}\\
&\le \|(\overline{\varrho}_{\tau n} - \varrho)\nabla P_{\tau l}(\psi)\|_{L^{s'}(0,T;L^{s'}(\Omega;\mathbb{R}^d))} + \|\varrho \nabla (P_{\tau l}(\psi) - \psi)\|_{L^{s'}(0,T;L^{s'}(\Omega;\mathbb{R}^d))}\\
&\le \underbrace{\|(\overline{\varrho}_{\tau n} - \varrho)\|_{L^2(0,T;L^2(\Omega))}}_{\to 0} \|\nabla P_{\tau l}(\psi)\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))} + \|\varrho\|_{L^2(0,T;L^2(\Omega))} \underbrace{\|\nabla(P_{\tau l}(\psi) - \psi)\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))}}_{\to 0},
\end{aligned}$$

where we applied Hölder's inequality and exploited the relations (2.2f) and Remark 2.1, Item 2., for the exponents p, s. Hence, we have as $\tau \to 0$ and $n, l \to \infty$

$$\overline{\varrho}_{\tau n} \nabla P_{\tau l}(\psi) \to \varrho \nabla \psi \text{ in } L^{s'}(0,T;L^{s'}(\Omega;\mathbb{R}^d))\,. \tag{4.7}$$

Together with the weak $L^s(0,T;L^s(\Omega))$-convergence (4.1a) for the velocities we conclude

$$\int_0^T \int_\Omega \varrho u \nabla \psi \,\mathrm{d}x\,\mathrm{d}t = \lim_{\substack{\tau \to 0\\ n,l \to \infty}} \int_0^T \int_\Omega \overline{\varrho}_{\tau n} \overline{u}_{\tau n} \nabla P_{\tau l}(\psi)\,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.8}$$

which gives the desired convergence result for the drift term. In addition, we here also deduce an alternative limit expression, which is obtained by performing integration by parts on the drift term and by exploiting convergence relation (4.1f); this expression will be useful for the identification of the term B_p lateron:

$$\lim_{\substack{\tau \to 0\\ n,l \to \infty}} \int_0^T\int_\Omega \overline{\varrho}_{\tau n} \overline{u}_{\tau n} \nabla P_{\tau l}(\psi)\,\mathrm{d}x\,\mathrm{d}t = \lim_{\substack{\tau \to 0\\ n,l \to \infty}} \int_0^T\int_\Omega -\mathrm{div}(\overline{\varrho}_{\tau n} \overline{u}_{\tau n}) P_{\tau l}(\psi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega b\,\psi\,\mathrm{d}x\,\mathrm{d}t\,. \tag{4.9}$$

Moreover, from convergence statement (4.1e) we directly read

$$\langle B_p, \psi\rangle_{L^p(0,T;\mathbf{X})} = \lim_{\substack{\tau \to 0\\ n,l \to \infty}} \langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_{\tau l}(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}\,. \tag{4.10}$$

Putting together (4.6b), (4.8), and (4.10) yields

$$\langle \partial_t \varrho, \psi\rangle_{L^p(0,T;\mathbf{X})} - \int_0^T \int_\Omega \varrho u \nabla \psi\,\mathrm{d}x\,\mathrm{d}t + \langle B_p, \psi\rangle_{L^p(0,T;\mathbf{X})} = 0 \quad \text{for all } \psi \in L^p(0,T;\mathbf{X}). \tag{4.11}$$

Similarly, when putting together (4.6b), (4.9), and (4.10), we obtain

$$\langle \partial_t \varrho, \psi\rangle_{L^p(0,T;\mathbf{X})} + \int_0^T \int_\Omega b\psi\,\mathrm{d}x\,\mathrm{d}t + \langle B_p, \psi\rangle_{L^p(0,T;\mathbf{X})} = 0 \quad \text{for all } \psi \in L^p(0,T;\mathbf{X}). \tag{4.12}$$

Hence, it remains to identify in (4.10), resp. (4.11), that

$$\langle B_p, \psi\rangle_{L^p(0,T;\mathbf{X})} \overset{!}{=} \langle \mathcal{A}_p(\varrho), \psi\rangle_{L^p(0,T;\mathbf{X})} := \int_0^T \int_\Omega \varepsilon\,|\nabla \varrho|^{p-2} \nabla \varrho \cdot \nabla \psi\,\mathrm{d}x\,\mathrm{d}t \tag{4.13}$$

for the limit density $\varrho \in L^p(0,T;\mathbf{X})$. For this, we will carry out a Minty-type argument from convex analysis and monotone operator theory and in the course of this argument we will make use of the limit continuity equation in the form (4.12). In preparation, we introduce the proper, lower semicontinuous, and convex functional

$$\mathcal{F} : L^p(0,T;\mathbf{X}) \to [0,\infty),\ \mathcal{F}(\tilde\varrho) := \int_0^T \int_\Omega \frac{\varepsilon}{p} |\nabla \tilde\varrho|^p\,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.14}$$

and observe that $\mathcal{F}$ is Gâteaux-differentiable with the Gâteaux-derivative $\mathrm{D}\mathcal{F} : L^p(0,T;\mathbf{X}) \to L^{p'}(0,T;\mathbf{X}^*)$,

$$\langle \mathrm{D}\mathcal{F}(\tilde{\varrho}), \psi\rangle_{L^p(0,T;\mathbf{X})} = \langle \mathcal{A}_p(\tilde{\varrho}), \psi\rangle_{L^p(0,T;\mathbf{X})} = \int_0^T \int_\Omega \varepsilon|\nabla\varrho|^{p-2}\nabla\varrho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\,. \tag{4.15}$$

Hence, for each $\tilde{\varrho} \in L^p(0,T;\mathbf{X})$ the convex subdifferential $\partial\mathcal{F}(\tilde{\varrho})$ of the functional $\mathcal{F}$ in $\tilde{\varrho}$ is single-valued and coincides with the Gâteaux-derivative, i.e., it is $\partial\mathcal{F}(\tilde{\varrho}) = \{\mathcal{A}_p(\tilde{\varrho})\}$ for all $\tilde{\varrho} \in L^p(0,T;\mathbf{X})$. By definition of the convex subdifferential, cf. e.g. [ABM06], any subgradient $\xi \in \partial\mathcal{F}(\tilde{\varrho})$ is characterized by the inequality $\langle \xi, \psi - \tilde{\varrho}\rangle_{L^p(0,T;\mathbf{X})} \le \mathcal{F}(\psi) - \mathcal{F}(\tilde{\varrho})$ for all $\psi \in L^p(0,T;\mathbf{X})$. In view of these reasonings the identification (4.13) can be achieved by verifying that

$$\langle B_p, \psi - \varrho\rangle_{L^p(0,T;\mathbf{X})} \overset{!}{\le} \mathcal{F}(\psi) - \mathcal{F}(\varrho) \quad \text{for all } \psi \in L^p(0,T;\mathbf{X})\,. \tag{4.16}$$

To this end, in correspondence to (4.14), we also introduce for the approximating problem (3.7a) the proper, lower semicontinuous, and convex functionals $\mathcal{F}_n : L^p(0,T;\mathbf{X}) \to [0,\infty]$,

$$\mathcal{F}_n(\tilde{\varrho}) := \begin{cases} \int_0^T \int_\Omega \frac{\varepsilon}{p}|\nabla\tilde{\varrho}|^p\,\mathrm{d}x\,\mathrm{d}t & \text{if } \tilde{\varrho} \in L^p(0,T;\mathbf{X}_n). \\ \infty & \text{else.} \end{cases} \tag{4.17}$$

Since $L^p(0,T;\mathbf{X}_n)$ is a closed subspace of $L^p(0,T;\mathbf{X})$, the convex subdifferential of $\mathcal{F}_n$ is given by

$$\partial\mathcal{F}_n(\tilde{\varrho}) := \begin{cases} \{\mathrm{D}\mathcal{F}_n(\tilde{\varrho})\} = \{\mathcal{A}_p(\tilde{\varrho})\}, & \text{if } \tilde{\varrho} \in L^p(0,T;\mathbf{X}_n), \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.18}$$

Hence, for all $\tau > 0$, all $n \ge l \in \mathbb{N}$, and for all $\psi \in L^p(0,T;\mathbf{X})$ there holds

$$\begin{aligned} \langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_{\tau l}(\psi) - \overline{\varrho}_{\tau n}\rangle_{L^p(0,T;\mathbf{X}_n)} &= \langle \mathrm{D}_\varrho\mathcal{F}_n(\overline{\varrho}_{\tau n}), P_{\tau l}(\psi) - \overline{\varrho}_{\tau n}\rangle_{L^p(0,T;\mathbf{X}_n)} \\ &\le \mathcal{F}_n(P_{\tau l}(\psi)) - \mathcal{F}_n(\overline{\varrho}_{\tau n}). \end{aligned} \tag{4.19}$$

We will exploit relation (4.19) in order to deduce (4.16). More precisely, we shall verify the following chain of inequalities

$$\begin{aligned} \mathcal{F}(\psi) - \mathcal{F}(\varrho) &\overset{(4.20\text{-}1)}{\ge} \limsup_{\substack{l\to\infty \\ l\le n}} \Big(\limsup_{\substack{\tau\to 0 \\ n\to\infty}} \mathcal{F}_n(P_{\tau l}(\psi))\Big) - \liminf_{\substack{\tau\to 0 \\ n\to\infty}} \mathcal{F}_n(\overline{\varrho}_{\tau n}) \\ &\ge \limsup_{\substack{l\to\infty \\ l\le n}} \left(\limsup_{\substack{\tau\to 0 \\ n\to\infty}} \big(\mathcal{F}_n(P_{\tau l}(\psi)) - \mathcal{F}_n(\overline{\varrho}_{\tau n})\big)\right) \\ &\overset{(4.19)}{\ge} \liminf_{\substack{l\to\infty \\ l\le n}} \left(\liminf_{\substack{\tau\to 0 \\ n\to\infty}} \langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_{\tau l}(\psi) - \overline{\varrho}_{\tau n}\rangle_{L^p(0,T;\mathbf{X}_n)}\right) \\ &\overset{(4.20\text{-}2)}{\ge} \langle B_p, \psi - \varrho\rangle_{L^p(0,T;\mathbf{X})}\,. \end{aligned} \tag{4.20}$$

In the following we will verify that above in (4.20) the first inequality marked as (4.20-1) and the fourth inequality marked as (4.20-2) indeed hold true, while the second inequality and the third inequality directly follow from the properties of the limit superior and the limit inferior applied to relation (4.19). Alltogether, the chain of inequalities (4.20) will thus prove (4.16), so that (4.13) can be concluded.

To inequality (4.20-1): Since $\bigcup_{n\in\mathbb{N}} L^p(0,T;\mathbf{X}_n)$ is dense in $L^p(0,T;\mathbf{X})$ we may deduce that the sequence of functionals $(\mathcal{F}_n)_n$ Mosco-converges to $\mathcal{F}$, i.e., that (4.21) below holds true, cf. [Mos67, ABM06]. For this, consider a sequence $(\psi_n)_n \subset L^p(0,T;\mathbf{X})$ such that $\psi_n \rightharpoonup \psi$ in $L^p(0,T;\mathbf{X})$; we show that

$$\liminf_{n\to\infty} \mathcal{F}_n(\psi_n) \ge \mathcal{F}(\psi)\,. \tag{4.21a}$$

First assume that $\psi_n \notin L^p(0,T;\mathbf{X}_n)$ for all $n \in \mathbb{N}$. Then, clearly $\infty = \mathcal{F}_n(\psi_n) > \mathcal{F}(\psi)$ so that (4.21a) holds true. Assume now that there is a (not relabelled) subsequence with $\psi_n \in L^p(0,T;\mathbf{X}_n)$. Along this subsequence, it is $\mathcal{F}_n(\psi_n) = \mathcal{F}(\psi_n)$ and now we may exploit the weak sequential lower semicontinuity of $\mathcal{F}$ to find (4.21a).

Consider now any function $\psi \in L^p(0,T;\mathbf{X})$. In order to conclude Mosco-convergence, we verify that there exists a recovery sequence $(\hat{\psi}_n)_n \subset L^p(0,T;\mathbf{X})$ such that

$$\hat{\psi}_n \to \psi \text{ in } L^p(0,T;\mathbf{X}) \text{ and } \limsup_{n\to\infty} \mathcal{F}_n(\hat{\psi}_n) \leq \mathcal{F}(\psi)\,. \tag{4.21b}$$

Indeed, by the density of $\bigcup_{n\in\mathbb{N}} L^p(0,T;\mathbf{X}_n)$ in $L^p(0,T;\mathbf{X})$ it is ensured that for each element $\psi \in L^p(0,T;\mathbf{X})$ there exists a sequence $(\hat{\psi}_n)_n$ such that $\hat{\psi}_n \in \mathbf{X}_n$ and $\hat{\psi}_n \to \psi$ in $L^p(0,T;\mathbf{X})$. The first property ensures that $\mathcal{F}_n(\hat{\psi}_n) = \mathcal{F}(\hat{\psi}_n) < \infty$ for all $n \in \mathbb{N}$ and the strong $L^p(0,T;\mathbf{X})$-convergence then in particular provides that $\mathcal{F}_n(\hat{\psi}_n) = \mathcal{F}(\hat{\psi}_n) \to \mathcal{F}(\psi)$ as $n \to \infty$, which proves (4.21b). In fact, the construction (3.6) of the projectors $P_{\tau l}$ is based on the density of $\bigcup_{n\in\mathbb{N}} C^0([0,T];\mathbf{X}_n)$ in $L^p(0,T;\mathbf{X})$ and hence the sequence $(P_{\tau l}\psi)_{\tau l}$ has the property (4.21b). This ensures that $\mathcal{F}_n(P_{\tau l}\psi) \to \mathcal{F}(\psi)$ as $\tau \to 0$, and $l \leq n \to \infty$. Moreover, we see that (4.21a) provides that $-\liminf_{\tau\to 0, n\to\infty} \mathcal{F}_n(\overline{\varrho}_{\tau n}) \leq -\mathcal{F}(\varrho)$. This finishes the proof of inequality (4.20-1).

To inequality (4.20-2): In order to verify inequality (4.20-2) we exploit the transport equation (3.7a) for the approximants $\overline{\varrho}_{\tau n}$ in order to rewrite the term $\langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_{\tau l}(\psi) - \overline{\varrho}_{\tau n}\rangle_{L^p(0,T;\mathbf{X}_n)}$ in (4.20), i.e., by (3.7a) we have

$$\langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_{\tau l}(\psi) - \overline{\varrho}_{\tau n}\rangle_{L^p(0,T;\mathbf{X}_n)} = -\int_0^T \int_\Omega (\mathrm{D}_\tau \varrho_{\tau n} + \operatorname{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n}))(P_{\tau l}(\psi) - \overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t, \tag{4.22}$$

where we have used integration by parts for the drift term. Now we perform the limit procedure used in (4.20) with the terms on the right-hand side of (4.22). In particular, it is

$$\begin{aligned}
&\liminf_{\substack{l\to\infty\\ l\leq n}} \Big(\liminf_{\substack{\tau\to 0\\ n\to\infty}} \langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), P_{\tau l}(\psi) - \overline{\varrho}_{\tau n}\rangle_{L^p(0,T;\mathbf{X}_n)} \Big)\\
&= \liminf_{\substack{l\to\infty\\ l\leq n}} \Big(\liminf_{\substack{\tau\to 0\\ n\to\infty}} \int_0^T \int_\Omega -(\mathrm{D}_\tau \varrho_{\tau n} + \operatorname{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n}))(P_{\tau l}(\psi) - \overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t \Big)\\
&\geq \liminf_{\substack{l\to\infty\\ l\leq n}} \Big(\liminf_{\substack{\tau\to 0\\ n\to\infty}} \int_0^T \int_\Omega -\mathrm{D}_\tau \varrho_{\tau n}(P_{\tau l}(\psi) - \overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t \Big)\\
&\quad + \liminf_{\substack{l\to\infty\\ l\leq n}} \Big(\liminf_{\substack{\tau\to 0\\ n\to\infty}} \int_0^T \int_\Omega -\operatorname{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})(P_{\tau l}(\psi) - \overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t \Big)
\end{aligned} \tag{4.23}$$

and we now discuss the limit passage separately for each of the two terms on the right-hand side of (4.23).

We start with the first term on the right-hand side of (4.23) that involves the discrete time derivative. To pass to the limit in the first contribution therein, we repeat the arguments along with (4.6) to find

$$\liminf_{\substack{l\to\infty\\ l\leq n}} \Big(\liminf_{\substack{\tau\to 0\\ n\to\infty}} \int_0^T \int_\Omega -\mathrm{D}_\tau \varrho_{\tau n} P_{\tau l}(\psi)\,\mathrm{d}x\,\mathrm{d}t \Big) = \liminf_{l\to\infty} \int_0^T \int_\Omega -\xi_l P_l(\psi)\,\mathrm{d}x\,\mathrm{d}t = \langle -\partial_t \varrho, \psi\rangle_{L^p(0,T;\mathbf{X})}\,. \tag{4.24}$$

To handle the second contribution we use integration in time and subsequently exploit the weak sequential lower semicontinuity of the L^2-norm together with convergence result (4.1g). In this way, we deduce

$$\begin{aligned}
\liminf_{\substack{\tau\to 0\\ n\to\infty}} \int_0^T \int_\Omega -\mathrm{D}_\tau \varrho_{\tau n}(-\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t &= \liminf_{\substack{\tau\to 0\\ n\to\infty}} \sum_{k=1}^{N_\tau} \int_\Omega \frac{\varrho_{\tau n}^k - \varrho_{\tau n}^{k-1}}{\tau} \varrho_{\tau n}^k\,\mathrm{d}x\\
&\geq \liminf_{\substack{\tau\to 0\\ n\to\infty}} \Big(\frac{1}{2}\|\overline{\varrho}_{\tau n}(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\overline{\varrho}_{\tau n}(0)\|^2_{L^2(\Omega)} \Big)\\
&\geq \frac{1}{2}\|\varrho(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varrho(0)\|^2_{L^2(\Omega)} = -\int_0^T \int_\Omega -\partial_t \varrho\, \varrho\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned} \tag{4.25}$$

Here, the last equality in (4.25) follows by integration in time, which is feasible thanks to regularity property (4.6c). Putting together (4.24) and (4.25) allows us to deduce that

$$\begin{aligned}
&\lim_{l\to\infty}\left(\liminf_{\substack{\tau\to 0\\ n\to\infty}}\int_0^T\int_\Omega -\mathrm{D}_\tau\varrho_{\tau n}(P_{\tau l}(\psi)-\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t\right)\\
&\geq \lim_{l\to\infty}\left(\lim_{\substack{\tau\to 0\\ n\to\infty}}\langle -\mathrm{D}_\tau\varrho_{\tau n},P_{\tau l}(\psi)\rangle_{L^p(0,T;\mathbf{X}_l)}\right)-\liminf_{\substack{\tau\to 0\\ n\to\infty}}\int_0^T\int_\Omega -\partial_t\varrho_{\tau n}\overline{\varrho}_{\tau n}\,\mathrm{d}x\,\mathrm{d}t\\
&\geq \langle -\partial_t\varrho,\psi-\varrho\rangle_{L^p(0,T;\mathbf{X})}\,.
\end{aligned}\tag{4.26}$$

Now we turn to the second term on the right-hand side of (4.23), i.e., the drift term. Thanks to $P_{\tau l}\psi\to\psi$ strongly in $L^p(0,T;\mathbf{X})$ and $\overline{\varrho}_{\tau n}\to\varrho$ in $L^p(0,T;L^p(\Omega))$ by (4.1i) together with $\mathrm{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})\rightharpoonup b$ in $L^2(0,T;L^2(\Omega))$ by (4.1f), we immediately conclude by weak-strong convergence arguments that

$$\begin{aligned}
&\liminf_{\substack{l\to\infty\\ l\leq n}}\left(\liminf_{\substack{\tau\to 0\\ n\to\infty}}\int_0^T\int_\Omega -\mathrm{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})(P_{\tau l}(\psi)-\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t\right)\\
&=\lim_{\substack{l\to\infty\\ l\leq n}}\left(\lim_{\substack{\tau\to 0\\ n\to\infty}}\int_0^T\int_\Omega -\mathrm{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})P_{\tau l}(\psi)\,\mathrm{d}x\,\mathrm{d}t\right)+\lim_{\substack{l\to\infty\\ l\leq n}}\left(\lim_{\substack{\tau\to 0\\ n\to\infty}}\int_0^T\int_\Omega \mathrm{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})\overline{\varrho}_{\tau n}\,\mathrm{d}x\,\mathrm{d}t\right)\\
&=\int_0^T\int_\Omega -b(\psi-\varrho)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{4.27}$$

Now we collect the results of (4.23), (4.26), and (4.27), and exploit the limit continuity equation (4.12) to find for all $\psi\in L^p(0,T;\mathbf{X})$

$$\begin{aligned}
&\liminf_{\substack{l\to\infty\\ l\leq n}}\left(\liminf_{\substack{\tau\to 0\\ n\to\infty}}\langle\mathcal{A}_p(\overline{\varrho}_{\tau n}),P_{\tau l}(\psi)-\overline{\varrho}_{\tau n}\rangle_{L^p(0,T;\mathbf{X}_n)}\right)\\
&\geq\langle -\partial_t\varrho,\psi-\varrho\rangle_{L^p(0,T;\mathbf{X})}-\int_0^T\int_\Omega b(\psi-\varrho)\,\mathrm{d}x\,\mathrm{d}t=\langle B_p,\psi-\varrho\rangle_{L^p(0,T;\mathbf{X})}\,.
\end{aligned}\tag{4.28}$$

This finishes the proof of inequality (4.20-2).

We now recall that the deduced inequalities (4.20-1) and (4.20-2) prove the validity of the chain of inequalities (4.20), which in turn establishes the identification (4.13), i.e., that

$$\langle B_p,\psi\rangle_{L^p(0,T;\mathbf{X})}=\langle\mathcal{A}_p(\varrho),\psi\rangle_{L^p(0,T;\mathbf{X})}=\int_0^T\int_\Omega\varepsilon|\nabla\varrho|^{p-2}\nabla\varrho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t$$

for all $\psi\in L^p(0,T;\mathbf{X})$ in the continuity equation (4.11), resp. in (4.12) of the limit.

Proof of the strong $L^p(0,T;W^{1,p}(\Omega))$-convergence (4.1j)**:** Result (4.1i) already provides the strong $L^p(0,T;L^p(\Omega))$-convergence of the densities $(\overline{\varrho}_{\tau n})_{\tau n}$. To conclude (4.1j) it remains to prove the strong $L^p(0,T;L^p(\Omega;\mathbb{R}^d))$-convergence of the gradients $(\nabla\overline{\varrho}_{\tau n})_{\tau n}$. This can be concluded from the following chain of inequalities

$$\int_0^T\!\!\int_\Omega\varepsilon|\nabla\varrho|^p\,\mathrm{d}x\,\mathrm{d}t\leq\liminf_{\substack{\tau\to 0\\ n\to\infty}}\int_0^T\!\!\int_\Omega\varepsilon|\nabla\overline{\varrho}_{\tau n}|^p\,\mathrm{d}x\,\mathrm{d}t\leq\limsup_{\substack{\tau\to 0\\ n\to\infty}}\int_0^T\!\!\int_\Omega\varepsilon|\nabla\overline{\varrho}_{\tau n}|^p\,\mathrm{d}x\,\mathrm{d}t\leq\int_0^T\!\!\int_\Omega\varepsilon|\nabla\varrho|^p\,\mathrm{d}x\,\mathrm{d}t\,.\tag{4.29}$$

We note that the first inequality in (4.29) follows from the weak sequential lower semicontinuity of the functional $\mathcal{F}:L^p(0,T;\mathbf{X})$ from (4.14) and the weak convergence (4.1c), while the second inequality is a property of the limit inferior and the limit superior. The third inequality in (4.29) will be verified now by once more exploiting the transport equations of the approximating solutions (3.7a) and of the limit (4.12), and by making use of the already above deduced convergence relations for the time-derivative

(4.25) and for the drift term (4.27). In this way, we deduce the following chain of inequalities

$$\begin{aligned}
\liminf_{\substack{\tau\to0\\ n\to\infty}} \int_0^T \int_\Omega \varepsilon|\nabla\overline{\varrho}_{\tau n}|^p \,\mathrm{d}x\,\mathrm{d}t &\le \limsup_{\substack{\tau\to0\\ n\to\infty}} \langle \mathcal{A}_p(\overline{\varrho}_{\tau n}), \overline{\varrho}_{\tau n}\rangle_{L^p(0,T;\mathbf{X})} \\
&\overset{(3.7a)}{=} \limsup_{\substack{\tau\to0\\ n\to\infty}} \int_0^T \int_\Omega -\big(\mathrm{D}_\tau\varrho_{\tau n} + \mathrm{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})\overline{\varrho}_{\tau n}\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le -\liminf_{\substack{\tau\to0\\ n\to\infty}} \int_0^T \int_\Omega \mathrm{D}_\tau\varrho_{\tau n}\overline{\varrho}_{\tau n}\,\mathrm{d}x\,\mathrm{d}t - \liminf_{\substack{\tau\to0\\ n\to\infty}} \int_0^T \int_\Omega \mathrm{div}(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})\overline{\varrho}_{\tau n}\,\mathrm{d}x\,\mathrm{d}t \\
&\overset{(4.25),(4.27)}{\le} -\int_0^T \int_\Omega \partial_t\varrho\,\varrho\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega b\,\varrho\,\mathrm{d}x\,\mathrm{d}t \\
&\overset{(4.12)}{=} \langle \mathcal{A}_p(\varrho),\varrho\rangle_{L^p(0,T;\mathbf{X})} = \int_0^T \int_\Omega \varepsilon|\nabla\varrho|^p\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned} \tag{4.30}$$

Putting together (4.29) and (4.30) yields the strong convergence result (4.1j). □

4.1.2 Limit passage in the momentum balance

In the following we carry out the limit passage in the weak momentum balance (3.7b). For this, we discuss each of the terms individually. For the viscous stress defined in (3.7e), in view of its weak convergence (4.1h) in $L^{s'}(0,T;\mathbf{U}^*)$ and the strong convergence $P_n^{\mathbf{U}}(P_\tau(v)) \to v$ in $L^s(0,T;\mathbf{U})$, we obtain

$$\langle B_\mu, v\rangle_{L^s(0,T;\mathbf{U})} = \lim_{\substack{\tau\to0\\ n\to\infty}} \big\langle \mathcal{A}_\mu(\overline{\varrho}_{\tau n},\overline{u}_{\tau n}), P_n^{\mathbf{U}}(P_\tau(v))\big\rangle_{L^s(0,T;\mathbf{U}_n)}\,. \tag{4.31}$$

For the non-quadratic viscosity, in view of the weak $L^{s'}(0,T;L^{s'}(\Omega;\mathbb{R}^d))$-convergence from (4.1b) together with the strong convergence $P_n^{\mathbf{U}}(P_\tau(v)) \to v$ in $L^s(0,T;L^s(\Omega;\mathbb{R}^d))$, we deduce that

$$\kappa\int_0^T\int_\Omega \zeta\cdot v\,\mathrm{d}x\,\mathrm{d}t = \kappa \lim_{\substack{\tau\to0\\ n\to\infty}} \int_0^T\int_\Omega |\overline{u}_{\tau n}|^{s-2}\overline{u}_{\tau n}\cdot P_n^{\mathbf{U}}(P_\tau(v))\,\mathrm{d}x\,\mathrm{d}t. \tag{4.32}$$

Next, we investigate the convergence of the pressure term, i.e., we aim to show that

$$\lim_{\substack{\tau\to0\\ n\to\infty}} \int_0^T\int_\Omega \overline{\varrho}_{\tau n}\nabla\mathrm{D}_\varrho\mathcal{E}(\overline{\varrho}_{\tau n})\cdot P_n^{\mathbf{U}}(P_\tau(v))\,\mathrm{d}x\,\mathrm{d}t \overset{!}{=} \int_0^T\int_\Omega \varrho\nabla\mathrm{D}_\varrho\mathcal{E}(\varrho)\cdot v\,\mathrm{d}x\,\mathrm{d}t\,. \tag{4.33}$$

Recall that $\nabla\mathrm{D}_\varrho\mathcal{E}(\overline{\varrho}_{\tau n}) = (\nabla x_2 + \hat{\varepsilon}\nabla\overline{\varrho}_{\tau n}) \rightharpoonup (\nabla x_2 + \hat{\varepsilon}\nabla\varrho) = \nabla\mathrm{D}_\varrho\mathcal{E}(\varrho)$ in $L^p(0,T;L^p(\Omega;\mathbb{R}^d))$ by (4.1c). Hence, (4.33) will follow from the strong convergence

$$\overline{\varrho}_{\tau n}P_n^{\mathbf{U}}(P_\tau(v)) \overset{!}{\to} \varrho v \text{ in } L^{p'}(0,T;L^{p'}(\Omega,\mathbb{R}^d))\,. \tag{4.34}$$

To verify (4.34), in view of assumption (2.2f) on p,s, we apply Hölder's inequality with the exponents $q := \frac{2}{p'} = \frac{s+2}{s}$ for $\overline{\varrho}_{\tau n}$ and $q' = \frac{s+2}{2}$ for $P_n^{\mathbf{U}}(P_\tau(v))$ to find

$$\begin{aligned}
&\|\overline{\varrho}_{\tau n}P_n^{\mathbf{U}}(P_\tau(v)) - \varrho v\|_{L^{p'}(0,T;L^{p'}(\Omega,\mathbb{R}^d))} \\
&\le \|(\overline{\varrho}_{\tau n}-\varrho)P_n^{\mathbf{U}}(P_\tau(v))\|_{L^{p'}(0,T;L^{p'}(\Omega,\mathbb{R}^d))} + \|\varrho(P_n^{\mathbf{U}}(P_\tau(v)) - v)\|_{L^{p'}(0,T;L^{p'}(\Omega,\mathbb{R}^d))} \\
&\le \|(\overline{\varrho}_{\tau n}-\varrho)\|_{L^2(0,T;L^2(\Omega))}\|P_n^{\mathbf{U}}(P_\tau(v))\|_{L^s(0,T;L^s(\Omega,\mathbb{R}^d))} + \|\varrho\|_{L^2(0,T;L^2(\Omega))}\|(P_n^{\mathbf{U}}(P_\tau(v)) - v)\|_{L^s(0,T;L^s(\Omega,\mathbb{R}^d))} \\
&\le \underbrace{\|(\overline{\varrho}_{\tau n}-\varrho)\|_{L^2(0,T;L^2(\Omega))}}_{\to0}\|P_n^{\mathbf{U}}(P_\tau(v))\|_{L^s(0,T;\mathbf{U})} + \|\varrho\|_{L^2(0,T;L^2(\Omega))}\underbrace{\|(P_n^{\mathbf{U}}(P_\tau(v)) - v)\|_{L^s(0,T;\mathbf{U})}}_{\to0}\,,
\end{aligned}$$

where the convergence of the above terms follows from $P_n^{\mathbf{U}}(P_\tau(v)) \to v$ in $L^s(0,T;\mathbf{U})$ and from the strong $L^p(0,T;L^p(\Omega))$-convergence (4.1i). Thus, both (4.34) and (4.33) are verified.

It remains to discuss the convergence of the term stemming from the quadratic, lower order viscosity, i.e., we show now that also

$$\lim_{\substack{\tau\to 0\\ n\to\infty}} \int_0^T \int_\Omega M(\overline{\varrho}_{\tau n})\overline{u}_{\tau n}\cdot P_n^{\mathbf{U}}(P_\tau(v))\,\mathrm{d}x\,\mathrm{d}t \overset{!}{=} \int_0^T\int_\Omega M(\varrho)u\cdot v\,\mathrm{d}x\,\mathrm{d}t\,. \tag{4.35}$$

By (4.1a) we have $\overline{u}_{\tau n}\rightharpoonup u$ in $L^s(0,T;L^s(\Omega;\mathbb{R}^d))$. Thus (4.35) will follow from the strong convergence

$$M(\overline{\varrho}_{\tau n})P_n^{\mathbf{U}}(P_\tau(v)) \overset{!}{\to} M(\varrho)v \text{ in } L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))\,. \tag{4.36}$$

To deduce (4.36) we shall apply Lebesgue's dominated convergence theorem. For this, we observe that, along a subsequence,

$$M(\overline{\varrho}_{\tau n})P_n^{\mathbf{U}}(P_\tau(v)) \to M(\varrho)v \;\text{ pointwise a.e. in } (0,T)\times\Omega, \tag{4.37}$$

by strong convergence in Lebesgue-spaces. Since this limit is obtained for any convergent subsequence, we conclude that (4.37) holds true even along the whole sequence. Moreover, by the growth properties (2.2c) of $M(\cdot)$ we see that

$$|M(\overline{\varrho}_{\tau n})P_n^{\mathbf{U}}(P_\tau(v))| \le \overline{M}\,\overline{\varrho}_{\tau n}^2|P_n^{\mathbf{U}}(P_\tau(v))| \quad \text{a.e. in } (0,T)\times\Omega, \text{ for all } \tau>0,\, n\in\mathbb{N},$$

and we need to show that the obtained sequence of majorants $(\overline{M}\,\overline{\varrho}_{\tau n}^2|P_n^{\mathbf{U}}(P_\tau(v))|)_{\tau n}$ satisfies

$$\overline{\varrho}_{\tau n}^2 P_n^{\mathbf{U}}(P_\tau(v)) \overset{!}{\to} \varrho^2 v \;\text{ in } L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))\,. \tag{4.38}$$

To deduce (4.38), we now establish suitable estimates by applying Hölder's inequality separately in space and time, again taking into account assumption (2.2f) on the exponents p,s. More precisely, consider $\hat\varrho\in L^\infty(0,T;L^2(\Omega))$, $\tilde\varrho\in L^p(0,T;W^{1,p}(\Omega))$, and $v\in L^s(0,T;\mathbf{U})$. Then also $\tilde\varrho\in L^p(0,T;L^\infty(\Omega))$ and $v\in L^2(0,T;L^p(\Omega))$, the latter because of (2.2f). In this way we find

$$\begin{aligned}
&\|\hat\varrho\tilde\varrho v\|^{s'}_{L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))} = \int_0^T\int_\Omega |\hat\varrho\tilde\varrho v|^{s'}\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^T \Big(\|\tilde\varrho(t)\|^{s'}_{L^\infty(\Omega)}\int_\Omega|\hat\varrho(t)|^{s'}|v|^{s'}\,\mathrm{d}x\Big)\,\mathrm{d}t\\
&\overset{(1)}{\le} \int_0^T\Big(\|\tilde\varrho(t)\|^{s'}_{L^\infty(\Omega)}\Big(\int_\Omega|\hat\varrho(t)|^2\,\mathrm{d}x\Big)^{s'/2}\Big(\int_\Omega|v(t)|^p\,\mathrm{d}x\Big)^{2/(p+2)}\Big)\,\mathrm{d}t\\
&= \int_0^T\Big(\|\tilde\varrho(t)\|^{s'}_{L^\infty(\Omega)}\|\hat\varrho(t)\|^{s'}_{L^2(\Omega)}\|v(t)\|^{s'}_{L^p(\Omega)}\Big)\,\mathrm{d}t\\
&\overset{(2)}{\le} \|\tilde\varrho\|^{s'}_{L^p(0,T;L^\infty(\Omega))}\|\hat\varrho\|^{s'}_{L^\infty(0,T;L^2(\Omega))}\|v\|^{s'}_{L^2(0,T;L^p(\Omega))}\,.
\end{aligned} \tag{4.39}$$

Above, we applied Hölder's inequality with the exponents $q_1 := \frac{2}{s'}$ and $q_1' = \frac{2}{2-s'} = \frac{p+2}{2}$ to find estimate (1), and estimate (2) followed by Hölder's inequality with the exponents $q_2 := \frac{p}{s'}$ and $q_2' = \frac{p}{p-s'} = \frac{p+2}{p}$. Using estimate (4.39) we further deduce

$$\begin{aligned}
&\|\overline{\varrho}_{\tau n}^2 P_n^{\mathbf{U}}(P_\tau(v)) - \varrho^2 v\|_{L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))}\\
&\le \|(\overline{\varrho}_{\tau n}^2-\varrho^2)P_n^{\mathbf{U}}(P_\tau(v))\|_{L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))} + \|\varrho^2(P_n^{\mathbf{U}}(P_\tau(v))-v)\|_{L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))}\\
&= \|(\overline{\varrho}_{\tau n}-\varrho)(\overline{\varrho}_{\tau n}+\varrho)P_n^{\mathbf{U}}(P_\tau(v))\|_{L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))} + \|\varrho\varrho(P_n^{\mathbf{U}}(P_\tau(v))-v)\|_{L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))}\\
&\le \underbrace{\|\overline{\varrho}_{\tau n}-\varrho\|_{L^p(0,T;L^\infty(\Omega))}}_{\to 0}\;\underbrace{\|\overline{\varrho}_{\tau n}+\varrho\|_{L^\infty(0,T;L^2(\Omega))}}_{\le C}\;\underbrace{\|P_n^{\mathbf{U}}(P_\tau(v))\|_{L^2(0,T;L^p(\Omega))}}_{\le C}\\
&\qquad + \|\varrho\|_{L^p(0,T;L^\infty(\Omega))}\|\varrho\|_{L^\infty(0,T;L^2(\Omega))}\;\underbrace{\|P_n^{\mathbf{U}}(P_\tau(v))-v\|_{L^2(0,T;L^p(\Omega))}}_{\to 0}\,.
\end{aligned} \tag{4.40}$$

In (4.40), convergence of the first factor follows from the strong $L^p(0,T;W^{1,p}(\Omega))$-convergence (4.1j) when taking into account the compact embedding $W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ thanks to $p > d$. The boundedness $\|\overline{\varrho}_{\tau n} + \varrho\|_{L^\infty(0,T;L^2(\Omega))} \le C$ is due to the a priori bound (3.9f) which holds for all $t \in [0,T]$. Moreover, the convergence $\|P_n^{\mathbf{U}}(P_\tau(v)) - v\|_{L^2(0,T;L^p(\Omega))} \to 0$ follows from the strong convergence of the interpolants in $L^s(0,T;\mathbf{U})$ using that $L^s(0,T;\mathbf{U}) \subset L^2(0,T;L^p(\Omega))$ by the relations (2.2f) for p, s. This proves (4.38) and hence Lebesgue's dominated convergence theorem yields (4.36), which allows us to conclude (4.35).

Putting together (4.31), (4.32), (4.33), and (4.35) ultimately results in the weak momentum balance of the limit system

$$\langle B_\mu, v\rangle_{L^s(0,T;\mathbf{U})} + \int_0^T \int_\Omega M(\varrho)u \cdot v \,\mathrm{d}x\,\mathrm{d}t + \kappa \int_0^T \int_\Omega \zeta \cdot v \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \varrho \nabla \mathrm{D}_\varrho \mathcal{E}(\varrho) \cdot v \,\mathrm{d}x\,\mathrm{d}t = 0 \quad \text{for all } v \in L^s(0,T;\mathbf{U}). \tag{4.41}$$

The identification of $\langle B_\mu, v\rangle_{L^s(0,T;\mathbf{U})}$ in terms of the viscous stress of the limit pair (ϱ, u) and of ζ as the $L^{s'}$-nonlinearity evaluated in the limit u is the topic of Theorem 2.4 whose proof is carried out subsequently in Section 4.2. □

4.2 Proof of Theorem 2.4: Identification of B_μ and ζ

In preparation of the proof of Theorem 2.4, we first state the following lemma, which results from the isomorphism $L^2((0,T)\times\Omega) \cong L^2(0,T;L^2(\Omega))$.

Lemma 4.4. *Let $m \in \mathbb{N}$ and $(u_n)_{n\in\mathbb{N}} \subset L^2(0,T;L^2(\Omega,\mathbb{R}^m))$, $u \in L^2(0,T;L^2(\Omega,\mathbb{R}^m))$ such that $u_n \rightharpoonup u$ in $L^2(0,T;L^2(\Omega,\mathbb{R}^m))$. Then for almost all $t \in (0,T)$ there holds $u_n(t) \rightharpoonup u(t)$ in $L^2(\Omega,\mathbb{R}^m)$.*

Proof. We consider any test function $\phi = \phi_t\phi_x \in L^2(0,T;L^2(\Omega,\mathbb{R}^m))$ such that $\phi_t \in L^2(0,T)$ and $\phi_x \in L^2(\Omega,\mathbb{R}^m)$, and show that

$$\lim_{n\to\infty}\left|\int_0^T\int_\Omega (u_n - u)\phi \,\mathrm{d}x\,\mathrm{d}t\right| = 0 \quad \Longrightarrow \quad \text{for a.e. } t \in (0,T): \lim_{n\to\infty}\left|\int_\Omega (u_n(t) - u(t))\phi_x \,\mathrm{d}x\right| = 0\,.$$

We proceed by contradiction. Assume, there is a measurable set $B \subset (0,T)$ with $\mathcal{L}^1(B) > 0$ such that

$$\text{for a.a. } t \in B, \text{for all } \phi_x \in L^2(\Omega,\mathbb{R}^m): \quad \lim_{n\to\infty}\int_\Omega (u_n(t) - u(t))\phi_x \,\mathrm{d}x \neq 0\,.$$

Then there exists some $\phi_x \in L^2(\Omega,\mathbb{R}^m)$ and a measurable set $B_+ \subset B$ with $\mathcal{L}^1(B_+) > 0$, such that

$$\text{for a.a. } t \in B_+: \quad \lim_{n\to\infty}\int_\Omega (u_n(t) - u(t))\phi_x \,\mathrm{d}x > 0.$$

Let χ_{B_+} be the characteristic function of the set B_+. Then we use $\phi = \phi_t\phi_x := \chi_{B_+}\phi_x \in L^2(0,T;L^2(\Omega,\mathbb{R}^m))$ as a test function. By Fatou's lemma, we obtain a contradiction:

$$\begin{aligned} 0 < \int_0^T \liminf_{n\to\infty}\left(\chi_{B_+}\int_\Omega (u_n(t) - u(t))\phi_x \,\mathrm{d}x\right)\mathrm{d}t &\le \liminf_{n\to\infty}\int_0^T\int_\Omega (u_n(t) - u(t))\chi_{B_+}\phi_x \,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{n\to\infty}\int_0^T\int_\Omega (u_n(t) - u(t))\phi \,\mathrm{d}x\,\mathrm{d}t = 0. \end{aligned}$$

This proves the statement of Lemma 4.4. □

We now turn to the proof of Theorem 2.4, i.e., to the identification of the terms B_μ and ζ appearing in the weak momentum balance of the limit system (2.5b), resp. (4.41) above. The identification can be carried out by restricting the set of test functions for (2.5b) to those $v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$ supported in non-cylindrical domains Q_ν^δ, i.e., to the functions

$$v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \quad \text{with } \mathrm{supp}(v) \subset Q_\nu^\delta\,. \tag{4.42}$$

For the readers' convenience we here recall the definition of the non-cylindrical domains Q_ν^δ first introduced in (2.7): With the aid of Remark 4.3, based on Aubin-Lions' theorem and Egorov's theorem we have

$$\forall\,\delta>0\;\exists\,I_\delta^c\subset(0,T)\text{ with }\mathcal{L}^1(I_\delta^c)<\delta:\quad \overline{\varrho}_{\tau n}\to\varrho\text{ uniformly in }I_\delta\times\overline{\Omega},\text{ where }I_\delta:=(0,T)\backslash I_\delta^c\,. \tag{4.43a}$$

For every $\delta>0$ we then define

$$Q_\nu^\delta:=\bigcup_{t\in I_\delta}\{t\}\times\Omega_\nu^{\mathrm{Lip}}(t)\subset(0,T)\times\Omega\,,\qquad\text{with} \tag{4.43b}$$

$$\Omega_\nu^{\mathrm{Lip}}(t)\text{ any Lipschitz-domain such that }\Omega_\nu^{\mathrm{Lip}}(t)\subset\Omega_\nu(t)\qquad\text{and} \tag{4.43c}$$

$$\Omega_\nu(t):=\{x\in\Omega,\,\nu<\varrho(t)<\varrho_{\mathrm{crit}}-\nu\}\text{ for any }\nu>0\,. \tag{4.43d}$$

Thanks to the uniform convergence of the sequence $(\overline{\varrho}_{\tau n})_{\tau n}$ on the non-cylindrical domains Q_ν^δ, it is possible to find a uniform bound from below for the viscosity $(\mu(\overline{\varrho}_{\tau n}))_{\tau n}$ for all $(t,x)\in Q_\nu^\delta$ and thus, to deduce the following result, which will be used to ultimately verify the statements of Theorem 2.4:

Lemma 4.5. *Let the assumptions of Theorem 2.4 be valid. For every $\delta>0$, for every $\nu>0$ consider the non-cylindrical domain Q_ν^δ as in (4.43). Then, for every test function $v\in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$ with* $\mathrm{supp}(v)\subset Q_\nu^\delta$ *there holds:*

$$\int_0^T\int_\Omega e(\overline{u}_{\tau n}):e(v)\,\mathrm{d}x\,\mathrm{d}t\;\to\;\int_0^T\int_\Omega e(u):e(v)\,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.44a}$$

$$\int_0^T\int_\Omega \mu_\tau(\overline{\varrho}_{\tau n})e(\overline{u}_{\tau n}):e(v)\,\mathrm{d}x\,\mathrm{d}t\;\to\;\int_0^T\int_\Omega \mu(\varrho)e(u):e(v)\,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.44b}$$

$$\int_0^T\int_\Omega \kappa|\overline{u}_{\tau n}|^{s-2}\overline{u}_{\tau n}\cdot v\,\mathrm{d}x\,\mathrm{d}t\;\to\;\int_0^T\int_\Omega \kappa|u|^{s-2}u\cdot v\,\mathrm{d}x\,\mathrm{d}t \tag{4.44c}$$

as $\tau\to$ and $n\to\infty$.

Proof. Consider the sequence $(\overline{\varrho}_{\tau n},\overline{u}_{\tau n})_{\tau n}$ of approximating solutions converging to the limit pair (ϱ,u) in the topologies (4.1). To simplify the arguments, but without loss of generality, for the index $\tau>0$ we fix here a subsequence

$$\tau=\tau(n)\to0\quad\text{as }n\to\infty\quad\text{such that approximation property (4.1) is valid.} \tag{4.45}$$

Since our arguments will be true for any such subsequence $(\tau(n))_n$ with property (4.45), they will hold true along the original, full sequence. We write $(\overline{\varrho}_{\tau n},\overline{u}_{\tau n})_{\tau n}$ also for this subsequence of approximating solutions, i.e., without indicating this explicitely, we have in mind that $\tau=\tau(n)$ as in (4.45).

From now on, for any $\delta>0$ and $\nu>0$ general but fixed, consider the non-cylindrical domain Q_ν^δ defined in (4.43).

Proof of (4.44a)**:** Thanks to the uniform convergence (4.43a) on Q_ν^δ, we have

$$\exists\,n(\nu/2)\in\mathbb{N}\;\;\forall n\geq n(\nu/2)\;\;\forall(t,x)\in Q_\nu^\delta:\quad \frac{\nu}{2}<\overline{\varrho}_{\tau n}(t,x)<\varrho_{\mathrm{crit}}-\frac{\nu}{2}\,. \tag{4.46}$$

In view of the uniform a priori bounds (3.9c) and (3.9d) this yields

$$\forall n\geq n(\nu/2):\quad \tfrac{M\nu}{2}\|\overline{u}_{\tau n}\|^2_{L^2(Q_\nu^\delta;\mathbb{R}^d)}+\tfrac{c_\mu\nu}{2}\|e(\overline{u}_{\tau n})\|^2_{L^2(Q_\nu^\delta;\mathbb{R}^{d\times d})}\leq C\,.$$

Thus, since the spaces $L^2(Q_\nu^\delta;\mathbb{R}^d)$ and $L^2(Q_\nu^\delta;\mathbb{R}^{d\times d})$ are reflexive, separable Banach spaces by Lemma 2.5, according to Remark 2.6 there exists a (not relabelled) subsequence and a limit pair $(\tilde{u},E)\in L^2(Q_\nu^\delta;\mathbb{R}^d)\times L^2(Q_\nu^\delta;\mathbb{R}^{d\times d})$ such that

$$\overline{u}_{\tau n}\rightharpoonup\tilde{u}\quad\text{ in }L^2(Q_\nu^\delta;\mathbb{R}^d), \tag{4.47a}$$

$$e(\overline{u}_{\tau n})\rightharpoonup E\quad\text{ in }L^2(Q_\nu^\delta;\mathbb{R}^{d\times d}). \tag{4.47b}$$

By convergence result (4.1a) we already know that $\overline{u}_{\tau n} \rightharpoonup u$ in $L^s(0,T;L^s(\Omega))$ and hence we conclude that $\tilde{u} = u$. Verifying that

$$E = e(u) \quad \text{in } L^2(Q_\nu^\delta;\mathbb{R}^{d\times d}) \tag{4.48}$$

will then provide the assertion (4.44a). To deduce (4.48) we now set

$$f_n(t,x) := \begin{cases} \overline{u}_{\tau n}(t,x) & \text{for } (t,x)\in Q_\nu^\delta, \\ 0 & \text{otherwise,} \end{cases} \qquad f(t,x) := \begin{cases} u(t,x) & \text{for } (t,x)\in Q_\nu^\delta, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \tag{4.49a}$$

$$e_n(t,x) := \begin{cases} e(\overline{u}_{\tau n}(t,x)) & \text{for } (t,x)\in Q_\nu^\delta, \\ 0 & \text{otherwise,} \end{cases} \qquad e(t,x) := \begin{cases} E(t,x) & \text{for } (t,x)\in Q_\nu^\delta, \\ 0 & \text{otherwise.} \end{cases} \tag{4.49b}$$

Then, by (4.47) there holds

$$f_n \rightharpoonup f \;\text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)) \quad \text{and} \quad e_n \rightharpoonup e \;\text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))\,, \tag{4.50}$$

and Lemma 4.4 further implies that

$$\text{for a.a. } t\in(0,T): \quad f_n(t) \rightharpoonup f(t) \;\text{ in } L^2(\Omega;\mathbb{R}^d) \quad \text{and} \quad e_n(t) \rightharpoonup e(t) \;\text{ in } L^2(\Omega;\mathbb{R}^{d\times d})\,. \tag{4.51}$$

In view of (4.49) and the definition of Q_ν^δ from (4.43), this is equivalent to

$$\text{for a.a. } t\in I_\delta: \; \overline{u}_{\tau n}(t) \rightharpoonup u(t) \text{ in } L^2(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d) \;\text{ and }\; e(\overline{u}_{\tau n}(t)) \rightharpoonup E(t) \text{ in } L^2(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^{d\times d})\,. \tag{4.52}$$

Using (4.52) and the fact that weak convergence implies boundedness, we find

$$\text{for a.a. } t\in I_\delta \;\; \exists\,\tilde{C}(t)>0: \quad \|\overline{u}_{\tau n}(t)\|_{W^{1,2}(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)} \le \tilde{C}(t)\,. \tag{4.53}$$

Hence, for a.a. $t\in I_\delta$, there exist a further t-dependent subsequence and a limit $\hat{u}(t)\in W^{1,2}(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)$ such that

$$\overline{u}_{\tau n}(t) \rightharpoonup \hat{u}(t) \text{ in } W^{1,2}(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d). \tag{4.54}$$

In view of (4.52), due to the uniqueness of the weak limit we thus conclude that

$$\text{for a.a. } t\in I_\delta: \quad \hat{u}(t) = u(t) \;\text{ in } L^2(\Omega;\mathbb{R}^d) \;\text{ and }\; e(\hat{u}(t)) = e(u(t)) = E(t) \;\text{ in } L^2(\Omega;\mathbb{R}^{d\times d})\,. \tag{4.55}$$

This proves (4.48) and thus finishes the proof of assertion (4.44a).

Proof of (4.44b): By virtue of the uniform bound (4.46) for $(\overline{\varrho}_{\tau n})_n$ on Q_ν^δ there are constants $\underline{\nu},\overline{\nu}>0$, such that

$$\forall n\ge n(\nu/2) \;\; \forall (t,x)\in Q_\nu^\delta: \quad \underline{\nu} < \mu_\tau(\overline{\varrho}_{\tau n}^{\nu}(t,x)) < \overline{\nu}\,. \tag{4.56}$$

Moreover, by the continuity of μ_τ, cf. (3.3a), the convergence property (3.3c) of $(\mu_\tau(\varrho))_\tau$, and by the uniform convergence (4.43a) of $(\overline{\varrho}_{\tau n})_n$ on Q_ν^δ, we have

$$\mu(\varrho(t,x)) = \lim_{n\to\infty} \mu_\tau(\overline{\varrho}_{\tau n}(t,x)) \text{ for all } (t,x)\in Q_\nu^\delta. \tag{4.57}$$

Consider now any test function $v\in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \subset L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$ with the property (4.42), i.e., such that $\mathrm{supp}(v)\subset Q_\nu^\delta$. Then, in view of (4.57) and (4.56), Lebesgue's dominated convergence theorem implies

$$\mu_\tau(\overline{\varrho}_{\tau n})e(v) \to \mu(\varrho)e(v) \text{ in } L^2(Q_\nu^\delta;\mathbb{R}^{d\times d})\,.$$

Now, exploiting convergence (4.47b) and identification (4.55), we obtain in particular

$$\int_{I_\delta}\int_{\Omega_\nu^{\mathrm{Lip}}(t)} \mu(\varrho)e(u):e(v)\,\mathrm{d}x\,\mathrm{d}t = \lim_{n\to\infty}\int_0^T\int_{\Omega_\nu^{\mathrm{Lip}}(t)} \mu_\tau(\overline{\varrho}_{\tau n})e(\overline{u}_{\tau n}):e(v)\,\mathrm{d}x\,\mathrm{d}t.$$

for all $v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$ such that $\mathrm{supp}(v) \subset Q_\nu^\delta$. This concludes the proof of assertion (4.44b).

Proof of (4.44c)**:** By Rellich-Kondrachov's embedding theorem, the space $W^{1,2}(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)$ is compactly imbedded in $L^s(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)$ for the exponent s as in (2.2f). In view of convergence (4.52) and identification (4.55) this provides

$$\text{for a.a. } t \in I_\delta: \quad \overline{u}_{\tau n}(t) \to u(t) \text{ in } L^s(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)\,. \tag{4.58}$$

For every $n \in \mathbb{N}$, we introduce the functions

$$f_n : I_\delta \to [0,\infty), \quad f_n(t) := \big\| |\overline{u}_{\tau n}(t)|^{s-2}\overline{u}_{\tau n}(t) - |u(t)|^{s-2}u(t)\big\|_{L^{s'}(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)} \tag{4.59}$$

and for the sequence $(f_n)_n$ we are now going to show that

$$f_n(t) \to 0 \text{ for a.a. } t \in I_\delta\,, \tag{4.60a}$$

$$f_n \rightharpoonup 0 \text{ in } L^{s'}(I_\delta)\,. \tag{4.60b}$$

To (4.60a)*:* Making use of the estimate, cf. [Kne04, (A.7)],

$$\text{for each } s>2 \text{ fixed } \exists\, c>0\ \forall A,B \in \mathbb{R}^d: \quad \big||A|^{s-2}A - |B|^{s-2}B\big| \le c(|A|+|B|)^{s-2}|A-B|\,, \tag{4.61}$$

we infer by (4.58) for a.a. $t \in I_\delta$ that

$$\begin{aligned} f_n(t)^{s'} &= \big\| |\overline{u}_{\tau n}(t)|^{s-2}\overline{u}_{\tau n}(t) - |u(t)|^{s-2}u(t)\big\|^{s'}_{L^{s'}(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)} \\ &\le c^{s'}\big\|(|\overline{u}_{\tau n}(t)| + |u(t)|)^{s-2}|\overline{u}_{\tau n}(t) - u(t)|\big\|^{s'}_{L^{s'}(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)} \\ &\le \big\||\overline{u}_{\tau n}(t)| + |u(t)|\big\|^{s'(s-2)}_{L^s(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)}\|\overline{u}_{\tau n}(t) - u(t)\|^{s'}_{L^s(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)} \to 0 \end{aligned} \tag{4.62}$$

as $n \to \infty$. This proves (4.60a).

To (4.60b)*:* By [Els18, Satz 5.9] or [AFP06, Thm. 1.35, p. 17], for $s' \in (1,\infty)$, the weak $L^{s'}$-convergence of a sequence can be concluded from its convergence pointwise a.e. together with the uniform boundedness of the $L^{s'}$-norms. In view of (4.60a) it thus remains to show

$$\exists\, C>0\ \forall n \in \mathbb{N}: \quad \|f_n\|_{L^{s'}(I_\delta)} \le C\,. \tag{4.63}$$

Indeed, setting $g_n(t) := \||\overline{u}_{\tau n}(t)|^{s-2}\overline{u}_{\tau n}(t)\|_{L^{s'}(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)}$ and using the uniform bound (3.9n), we obtain

$$\|g_{\tau n}\|_{L^{s'}(I_\delta)} \le C. \tag{4.64}$$

Moreover, $g_{\tau n} \ge 0$ for a.a. $t \in I_\delta$. Thanks to (4.60a), we further have

$$g_{\tau n}(t) \to g(t) := \||u(t)|^{s-2}u(t)\|_{L^{s'}(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)} \quad \text{for a.a. } t \in I_\delta\,. \tag{4.65}$$

Hence, in view of (4.64), Fatou's lemma implies

$$\|g\|_{L^{s'}(I_\delta)} \le \liminf_{n\to\infty}\|g_{\tau n}\|_{L^{s'}(I_\delta)} \le C\,. \tag{4.66}$$

This yields (4.63), and hence the assertion (4.60b) follows.

It remains to conclude assertion (4.44c) with the aid of weak convergence (4.60b). Indeed, by (4.60b) we infer for every $v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$ with $\mathrm{supp}(v) \subset Q_\nu^\delta$ that

$$\Big|\int_{I_\delta}\int_{\Omega_\nu^{\mathrm{Lip}}(t)}\big(|\overline{u}_{\tau n}(t)|^{s-2}\overline{u}_{\tau n}(t) - |u(t)|^{s-2}u(t)\big)\cdot v\,\mathrm{d}x\,\mathrm{d}t\Big| \le \int_{I_\delta} f_n(t)\|v(t)\|_{L^s(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)}\,\mathrm{d}t \to 0 \tag{4.67}$$

as $n \to \infty$, since $\|v(\cdot)\|_{L^s(\Omega_\nu^{\mathrm{Lip}}(t);\mathbb{R}^d)} \in L^s(I_\delta)$. This gives assertion (4.44c). Thus, the proof of Lemma 4.5 is complete. □

Based on Lemma 4.5 we now obtain the identification of the limit objects B_μ and ζ in the following

Corollary 4.6 (Identification of B_μ and ζ)**.** *Let the assumptions of Theorem 2.4 and Lemma 4.5 be satisfied. Then, for all $\delta > 0$, for all $\nu > 0$, all non-cylindrical domains Q_ν^δ, and for every test function $v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$ such that* $\mathrm{supp}(v) \subset Q_\nu^\delta$ *there holds:*

$$\int_0^T \int_\Omega \mu_\tau(\overline{\varrho}_{\tau n}) e(\overline{u}_{\tau n}) : e(P_n^{\mathbf{U}}(P_\tau(v)))\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \mu(\varrho) e(u) : e(v)\,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.68a}$$

$$\int_0^T \int_\Omega \kappa |\overline{u}_{\tau n}|^{s-2} \overline{u}_{\tau n} \cdot P_n^{\mathbf{U}}(P_\tau(v))\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \kappa |u|^{s-2} u \cdot v\,\mathrm{d}x\,\mathrm{d}t\,. \tag{4.68b}$$

Hence, for all $v \in L^s(0,T;W^{1,2}(\Omega),\mathbb{R}^d)$ such that $\mathrm{supp}(v) \subset Q_\nu^\delta$ *it is*

$$\langle B_\mu, v\rangle_{L^s(0,T;W^{1,2}(\Omega):\mathbb{R}^d)} = \int_0^T \int_\Omega \mu(\varrho) e(u) : e(v)\,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.69a}$$

$$\int_0^T \int_\Omega \kappa \zeta \cdot v\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \kappa |u|^{s-2} u \cdot v\,\mathrm{d}x\,\mathrm{d}t \tag{4.69b}$$

for the elements $B_\mu \in L^{s'}(0,T;W^{1,2}(\Omega,\mathbb{R}^d)^*)$ and $\zeta \in L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))$ appearing in the limit momentum balance (2.5b)*.*

Proof. We shall verify convergences (4.68), then statement (4.69) follows by comparison by the uniqueness of weak limits.

To (4.68a)*:* With the aid of convergence result (4.44b), the uniform bound (3.9a), and the strong convergence $P_n^{\mathbf{U}}(P_\tau(v)) \to v$ in $L^s(0,T;W^{1,2}(\Omega,\mathbb{R}^d))$ we obtain

$$\begin{aligned}
&\Big| \int_0^T \int_\Omega \big(\mu_\tau(\overline{\varrho}_{\tau n}) e(\overline{u}_{\tau n}) : e(P_n^{\mathbf{U}}(P_\tau(v))) - \mu(\varrho) e(u) : e(v)\big)\,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\le \Big| \int_0^T \int_\Omega \big(\mu_\tau(\overline{\varrho}_{\tau n}) e(\overline{u}_{\tau n}) - \mu(\varrho) e(u)\big) : e(v)\,\mathrm{d}x\,\mathrm{d}t \Big| + \Big| \int_0^T \int_\Omega \mu_\tau(\overline{\varrho}_{\tau n}) e(\overline{u}_{\tau n}) : \big(e(P_n^{\mathbf{U}}(P_\tau(v))) - e(v)\big)\,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\le \underbrace{\Big| \int_0^T \int_\Omega \big(\mu_\tau(\overline{\varrho}_{\tau n}) e(\overline{u}_{\tau n}) - \mu(\varrho) e(u)\big) : e(v)\,\mathrm{d}x\,\mathrm{d}t \Big|}_{\to 0} \\
&\qquad + \underbrace{\big\| \sqrt{\mu_\tau(\overline{\varrho}_{\tau n})} e(\overline{u}_{\tau n}) \big\|_{L^2(0,T;L^2(\Omega,\mathbb{R}^d))}}_{\le C} \underbrace{\big\| e(P_n^{\mathbf{U}}(P_\tau(v))) - e(v) \big\|_{L^2(0,T;L^2(\Omega,\mathbb{R}^{d\times d}))}}_{\to 0},
\end{aligned}$$

which gives (4.68a).

To (4.68b)*:* Using convergence result (4.44c), the uniform bound (3.9n), and the strong convergence $P_n^{\mathbf{U}}(P_\tau(v)) \to v$ in $L^s(0,T;W^{1,2}(\Omega,\mathbb{R}^d))$ we find

$$\begin{aligned}
&\Big| \int_0^T \int_\Omega \big(|\overline{u}_{\tau n}|^{s-2} \overline{u}_{\tau n} \cdot P_n^{\mathbf{U}}(P_\tau(v)) - |u|^{s-2} u \cdot v\big)\,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\le \Big| \int_0^T \int_\Omega \big(|\overline{u}_{\tau n}|^{s-2} \overline{u}_{\tau n} - |u|^{s-2} u\big) \cdot v\,\mathrm{d}x\,\mathrm{d}t \Big| + \Big| \int_0^T \int_\Omega |\overline{u}_{\tau n}|^{s-2} \overline{u}_{\tau n} \cdot \big(P_n^{\mathbf{U}}(P_\tau(v)) - v\big)\,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\le \underbrace{\Big| \int_0^T \int_\Omega \big(|\overline{u}_{\tau n}|^{s-2} \overline{u}_{\tau n} - |u|^{s-2} u\big) \cdot v\,\mathrm{d}x\,\mathrm{d}t \Big|}_{\to 0} \\
&\qquad + \underbrace{\big\| |\overline{u}_{\tau n}|^{s-2} \overline{u}_{\tau n} \big\|_{L^{s'}(0,T;L^{s'}(\Omega,\mathbb{R}^d))}}_{\le C} \underbrace{\big\| P_n^{\mathbf{U}}(P_\tau(v)) - v \big\|_{L^s(0,T;L^s(\Omega,\mathbb{R}^d))}}_{\to 0}.
\end{aligned}$$

This proves (4.68b). □

We now verify the last statement of Theorem 2.4, i.e., that the identification relations (2.8) and (2.9) holds true even for all test functions with property (2.10), cf. (4.74). This argument is based on a more general statement, which will be applied lateron also in a different context. Therefore, we give the argument in the following remark:

Remark 4.7 (Generalization of the identification result to test functions satisfying (2.10)). *1. The restriction of $\Omega_\nu(t)$ to Lipschitz subdomains $\Omega_\nu^{\mathrm{Lip}}(t) \subset \Omega_\nu(t)$ in the construction* (4.43) *of the non-cylindrical domains Q_ν^δ is needed for the proof of* (4.44c) *in order to ensure that Rellich-Kondrachov's embedding thorem is available to handle the L^s-nonlinearity. Yet, [Vor10, Theorem 1] states that any open set D in $\mathbb{R}^d$ is a union of an ascending sequence of bounded domains D_m with analytic boundary and such that $D_m \subset D$. In this way, for any $\nu > 0$ it is possible to approximate $\Omega_\nu(t)$ from the inside by unions of sets D_m with analytic boundary, which are clearly contained in the class of sets with Lipschitz boundary. Using a partition of unity of constructed from approximating, smooth sets D_m we obtain that* (4.69) *holds true even for all test functions*

$$v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \;\; \textit{such that} \;\; \mathrm{supp}(v) \subset \bigcup_{t\in I_\delta} \{t\} \times \Omega_\nu(t) \;\; \textit{for all } \delta, \nu > 0\,. \tag{4.70}$$

Due to this, we ultimately conclude that (4.69) *holds true even for all test functions*

$$v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \;\; \textit{such that} \;\; \mathrm{supp}(v) \subset \bigcup_{t\in I_\delta} \{t\} \times [0 < \varrho(t) < \varrho_{\mathrm{crit}}] \;\; \textit{for all } \delta > 0\,, \tag{4.71}$$

where we used the notation (2.1).

2. Let us now consider a sequence

$$(\delta_j)_{j\in\mathbb{N}} \;\; \textit{such that } \delta_j > 0 \textit{ for all } j \in \mathbb{N} \textit{ and } \delta_j \to 0 \textit{ as } j \to \infty\,. \tag{4.72}$$

It is possible to apply Egorov's theorem such that the sets $I^c_{\delta_j} \subset (0,T)$ with $\mathcal{L}^1(I^c_{\delta_j}) < \delta_j$ form a nested descending sequence, i.e., such that $I^c_{\delta_{j+1}} \subset I^c_{\delta_j}$. More precisely, by Egorov's theorem, for each $\delta_j > 0$ one finds a set $I^c_{\delta_j} \subset (0,T)$ such that $\mathcal{L}^1(I^c_{\delta_j}) < \delta_j$ and such that $\|\overline{\varrho}_{\tau n} - \varrho\|_{C(\overline{\Omega})} \to 0$ uniformly on I_{δ_j}. Subsequently, for $\delta_{j+1} > 0$ one finds by Egorov's theorem a set $I^c_{\delta_{j+1}} \subset I^c_{\delta_j}$ such that $\mathcal{L}^1(I^c_{\delta_{j+1}}) < \delta_{j+1}$ and such that $\|\overline{\varrho}_{\tau n} - \varrho\|_{C(\overline{\Omega})} \to 0$ uniformly on $I_{\delta_{j+1}}$. In this way one obtains

$$I_{\delta_j} \subset I_{\delta_{j+1}} \;\; \textit{for all } j \in \mathbb{N} \;\; \textit{and } \mathcal{L}^1((0,T)\backslash I_{\delta_j}) \to 0 \textit{ as } j \to \infty\,. \tag{4.73}$$

In view of (4.71) *and* (4.73) *we now conclude that* (4.69) *holds true even for all test functions*

$$v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \;\; \textit{such that} \;\; \mathrm{supp}(v(t)) \subset [0 < \varrho(t) < \varrho_{\mathrm{crit}}] \;\; \textit{for a.a. } t \in (0,T)\,. \tag{4.74}$$

Based on the ideas of Remark 4.7 we also deduce a uniform L^2-bound for $\sqrt{\mu(\varrho)}e(u)$ on the set $[0 < \varrho < \varrho_{\mathrm{crit}}] \subset (0,T) \times \Omega$ and we obtain that $e(u) \in L^2([0 < \varrho])$:

Lemma 4.8. *Let the assumptions of Theorem 2.4 hold true. Then there exists a constant $C > 0$, such that for all $\delta, \nu > 0$ and for every non-cylindrical domain Q_ν^δ it is*

$$\|\sqrt{\mu(\varrho)}e(u)\|_{L^2(Q_\nu^\delta;\mathbb{R}^{d\times d})} \leq C\,. \tag{4.75}$$

Consequently, the pair (ϱ, u) also satisfies

$$\|\sqrt{\mu(\varrho)}e(u)\|_{L^2([0<\varrho<\varrho_{\mathrm{crit}}];\mathbb{R}^{d\times d})} \leq C\,. \tag{4.76}$$

Moreover, there holds

$$e(u) \in L^2([\nu < \varrho]) \;\; \textit{for all } \nu > 0\,. \tag{4.77}$$

This extends the identification result (4.69), *resp.* (2.9), *to hold true even for all test functions*

$$v \in L^s(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \;\; \textit{such that} \; \mathrm{supp}(v(t)) \subset [0 < \varrho(t)] \textit{ for a.a. } t \in (0,T)\,. \tag{4.78}$$

Proof. **To** (4.75)**:** Consider any fixed non-cylindrical domain Q_ν^δ as in (4.43). To show (4.75) we repeat the arguments of (4.46)–(4.55) to find with the aid of the uniform convergence (4.43a) on Q_ν^δ

$$\exists\, n(\nu/2) \in \mathbb{N}\ \forall n \geq n(\nu/2)\ \forall (t,x) \in Q_\nu^\delta : \quad \frac{\nu}{2} < \overline{\varrho}_{\tau n}(t,x) < \varrho_{\mathrm{crit}} - \frac{\nu}{2}\,, \tag{4.79}$$

and to find that $e(\overline{u}_{\tau n}) \rightharpoonup e(u)$ in $L^2(Q_\nu^\delta;\mathbb{R}^{d\times d})$. Then also

$$e_n := \begin{cases} e(\overline{u}_{\tau n}) & \text{on } Q_\nu^\delta, \\ 0 & \text{otherwise} \end{cases} \rightharpoonup e := \begin{cases} e(u) & \text{on } Q_\nu^\delta, \\ 0 & \text{otherwise} \end{cases} \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))\,. \tag{4.80}$$

In addition, we set

$$\hat{\varrho}_n := \begin{cases} \overline{\varrho}_{\tau n} & \text{on } Q_\nu^\delta, \\ 0 & \text{otherwise.} \end{cases} \tag{4.81}$$

Then we also have

$$\hat{\varrho}_n \to \hat{\varrho} := \begin{cases} \varrho & \text{on } Q_\nu^\delta, \\ 0 & \text{otherwise} \end{cases} \quad \text{uniformly in } [0,T]\times\overline{\Omega} \text{ and strongly in } L^p(0,T;L^p(\Omega))\,, \tag{4.82}$$

and $\hat{\varrho}, \hat{\varrho}_n \in [\frac{\nu}{2}, \varrho_{\mathrm{crit}} - \frac{\nu}{2}]$. We further observe that the restricted viscosities $\mu|_{[\frac{\nu}{2},\varrho_{\mathrm{crit}}-\frac{\nu}{2}]}$ and $\mu_\tau|_{[\frac{\nu}{2},\varrho_{\mathrm{crit}}-\frac{\nu}{2}]}$, $\tau = \tau(n)$, are uniformly bounded from above and from below, and continuous, cf. (2.2b) & (3.3a). Moreover, we find an index $n_\nu \geq n(\nu/2)$ such that for every $n \geq n_\nu$ it is $\tau = \tau(n) < \frac{\nu}{2}$. Thus, in view of the definitions (2.2b) & (3.3a), it is $\mu_\tau(\hat{\varrho}_n) = \mu(\hat{\varrho}_n)$ for all $n \geq n_\nu$. Now [Dac12, Thm. 3.4, p. 74] ensures that the functional $(\tilde{\varrho}, \tilde{e}) \mapsto \int_0^T \int_\Omega \mu_{[\frac{\nu}{2},\varrho_{\mathrm{crit}}-\frac{\nu}{2}]}(\tilde{\varrho})\tilde{e} : \tilde{e}\,\mathrm{d}x\,\mathrm{d}t$ is lower semicontinuous with respect to strong $L^p(0,T;L^p(\Omega))$-convergence and weak $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$-convergence. Based on the uniform bound (3.9a) we now conclude by lower semicontinuity that

$$\begin{aligned} C &\geq \liminf_{n\to\infty} \|\sqrt{\mu_\tau}(\overline{\varrho}_{\tau n}) e(\overline{u}_{\tau n})\|_{L^2(Q_\nu^\delta;\mathbb{R}^{d\times d})} = \liminf_{n\to\infty} \|\sqrt{\mu}(\hat{\varrho}_n) e_n\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \\ &\geq \|\sqrt{\mu}(\hat{\varrho}) e\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} = \|\sqrt{\mu}(\varrho) e(u)\|_{L^2(Q_\nu^\delta;\mathbb{R}^{d\times d})}\,. \end{aligned} \tag{4.83}$$

To (4.76)**:** To conclude (4.76) we first apply the argument of Remark (4.7), Item 1, to see that above bound (4.83) also holds true for any noncylindrical domain $\bigcup_{t\in I_\delta}\{t\}\times\Omega_\nu(t)$ for all $\delta, \nu > 0$. Here we also exploit the additivity of the integral to patch together any Lipschitz-sets $\Omega_\nu^{\mathrm{Lip1}}(t)$ and $\Omega_\nu^{\mathrm{Lip2}}(t)$ in order to approximate $\Omega_\nu(t)$ from inside for any $\nu > 0$. Thanks to (4.83) this gives $C \geq \|\sqrt{\mu}(\varrho)e(u)\|_{L^2(\cup_{t\in I_\delta}\{t\}\times\Omega_\nu(t);\mathbb{R}^{d\times d})}$ for all $\nu, \delta > 0$, and thus $C \geq \|\sqrt{\mu}(\varrho)e(u)\|_{L^2(\cup_{t\in I_\delta}\{t\}\times[0<\varrho(t)<\varrho_{\mathrm{crit}}];\mathbb{R}^{d\times d})}$. Subsequently, we apply the argument of Remark (4.7), Item 2, to observe that the bound remains true for a sequence $\delta_j \to 0$ as in (4.72) & (4.73). This yields (4.76).

To (4.77)**:** For every $\nu > 0$ we set $\tilde{\Omega}_\nu(t) := \{x \in \Omega,\ \varrho(t) > \nu\}$ and consider any Lipschitz-subdomain $\tilde{\Omega}_\nu^{\mathrm{Lip}}(t)$. For each $\delta > 0$ we have the sets $I_\delta \subset (0,T)$, where the uniform convergence (4.43a) holds true. Based on this, we further set $\tilde{Q}_\nu^\delta := \cup_{t\in I_\delta}\{t\}\times\tilde{\Omega}_\nu(t)$. By repeating the arguments of (4.46)–(4.55) we find an index $\tilde{n}(\nu/2) \in \mathbb{N}$ such that for all $n \geq \tilde{n}(\nu/2)$ and for all $(t,x) \in \tilde{Q}_\nu^\delta$ we have $\overline{\varrho}_{\tau n}(t,x) > \nu/2$. This provides a weakly convergent subsequence $e(u_n) \rightharpoonup e(u)$ in $L^2(\tilde{Q}_\nu^\delta;\mathbb{R}^{d\times d})$. Using the notation of (4.80), now with $\tilde{Q}_\nu^\delta$, we thus also have $e_n \rightharpoonup e$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$. Hence, like in (4.83), the weak lower semicontinuity of the $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$-norm in combination with the uniform bound (4.76) allows us to conclude

$$\begin{aligned} C &\geq \liminf_{n\to\infty} \int_{\tilde{Q}_\nu^\delta} \mu(\overline{\varrho}_{\tau n}) e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t \geq \liminf_{n\to\infty} \int_{\tilde{Q}_\nu^\delta} \mu(\nu/2) e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \int_{\tilde{Q}_\nu^\delta} \mu(\nu/2) e(u) : e(u)\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \nu > 0\,. \end{aligned}$$

Invoking Remark 4.7 ultimately yields (4.77).

To (4.78)**:** Here we repeat the steps of the proofs of (4.44b) and (4.44c) as well as (4.68) based on the non-cylindrical domains $\tilde{Q}_\nu^\delta$. □

4.3 Proof of Theorem 2.7: Non-negativity & boundedness of ϱ, and refined identification of B_μ and ζ

In this section we verify the statements of Theorem 2.7. First, we shall deduce that the density of the limit system ϱ has the property

$$0 \le \varrho(t) \le \varrho_{\text{crit}} \quad \text{a.e. in } \Omega \text{ for all } t \in [0,T]\,. \tag{4.84}$$

Property (4.84) will be established with the aid of two separate lemmata: In a first step, in Lemma 4.9 we show that the velocity of the limit system satisfies $e(u) = 0$ on subsets $B \subset (0,T) \times \Omega$, where ϱ is strictly negative, i.e., on sets $B \subset [\varrho < 0]$, or where it reaches or even exceeds the critical value ϱ_{crit}, i.e., on sets $B \subset [\varrho \ge \varrho_{\text{crit}}]$, cf. notation (2.1). This result will be deduced by investigating the convergence of the approximating velocities $(\overline{u}_{\tau n})_{\tau n}$ on suitable non-cylindrical domains, where additional information can be drawn from the a priori bound (3.9a) by exploiting the growth properties (3.3) of the viscosities $(\mu_\tau(\overline{\varrho}_{\tau n}))_{\tau n}$. With the aid of the information $e(u) = 0$ on B and suitably devised test functions for the limit transport equation (2.5a), it will be shown in a second step in Lemma 4.10 that the sets $[\varrho(t) < 0]$ and $[\varrho(t) > \varrho_{\text{crit}}]$ are $\mathcal{L}^d$-null sets for all $t \in [0,T]$.

Lemma 4.9. *Let the assumptions of Theorem 2.4 be valid and assume that $(\varrho, u, B_\mu, \zeta)$ is a weak solution of* (1.1) *in the sense of Definition 2.2 and Thm. 2.4, obtained by discrete approximation via scheme* (3.2) *and extracted from discrete solutions $(\overline{\varrho}_{\tau n}, \overline{u}_{\tau n})_{\tau n}$ through convergences* (4.1)*. Then, for all $\nu > 0$ the following convergence information hold true*

$$\|e(\overline{u}_{\tau n})\|_{L^2([\overline{\varrho}_{\tau n} < -\nu];\mathbb{R}^{d\times d})} \to 0 \quad \textit{as } \tau \to 0,\ n \to \infty, \tag{4.85a}$$

$$\|e(\overline{u}_{\tau n})\|_{L^2([\overline{\varrho}_{\tau n} > \varrho_{\text{crit}}+\nu];\mathbb{R}^{d\times d})} \to 0 \quad \textit{as } \tau \to 0,\ n \to \infty. \tag{4.85b}$$

and the velocity u has the property

$$e(u) = 0 \quad \textit{a.e. on } [\varrho < 0] \cup [\varrho \ge \varrho_{\text{crit}}]\,. \tag{4.86}$$

Proof. **To** (4.85)**:** Keep $\nu > 0$ fixed and recall from (3.3a) the definition of μ_τ. Thus, for $[\overline{\varrho}_{\tau n} < -\nu]$ the a priori estimate (3.9a) yields

$$\|e(\overline{u}_{\tau n})\|^2_{L^2([\overline{\varrho}_{\tau n} < -\nu];\mathbb{R}^{d\times d})} \le \tfrac{\tau(n)^\alpha}{\nu} \|\sqrt{\mu_\tau} e(\overline{u}_{\tau n})\|^2_{L^2([\overline{\varrho}_{\tau n} < -\nu];\mathbb{R}^{d\times d})} \le \tfrac{C\tau(n)^\alpha}{\nu} \to 0 \quad \text{as } \tau(n) \to 0\,,$$

where we again used the notation of (4.45). Similarly, one finds for $[\overline{\varrho}_{\tau n} > \varrho_{\text{crit}} + \nu]$

$$\|e(\overline{u}_{\tau n})\|^2_{L^2([\overline{\varrho}_{\tau n} > \varrho_{\text{crit}}+\nu];\mathbb{R}^{d\times d})} \le \tfrac{\tau(n)^\alpha |\varrho_{\text{crit}} - \tau|}{\tilde{\nu}\nu} \|\sqrt{\mu_\tau} e(\overline{u}_{\tau n})\|^2_{L^2([\overline{\varrho}_{\tau n} > \varrho_{\text{crit}}+\nu];\mathbb{R}^{d\times d})} < \tfrac{C\tau(n)^\alpha \varrho_{\text{crit}}}{\tilde{\nu}\nu} \to 0$$

as $\tau(n) \to 0$. Hence (4.85) is verified.

To (4.86)**:** For each $\delta > 0$ and every $\nu > 0$, we define the non-cylindrical domains $B^\bullet_{\delta,\nu}$ with $\bullet \in \{-,+\}$ by

$$B^\bullet_{\delta,\nu} := \bigcup_{t \in I_\delta} \{t\} \times \Omega^\bullet_\nu(t) \subset [0,T] \times \Omega,$$

where $\Omega^-_\nu(t) := \{x \in \Omega \mid \varrho(t) < -\nu\}$ and $\Omega^+_\nu(t) := \{x \in \Omega \mid \varrho(t) > \varrho_{\text{crit}} - \nu\}$ for suitably small $\nu > 0$. By virtue of a priori estimate (3.9a), we obtain in particular for $B^\bullet_{\delta,\nu}$, $\bullet \in \{-,+\}$,

$$\big\|\sqrt{\mu_\tau(\overline{\varrho}_{\tau n})}\, e(\overline{u}_{\tau n})\big\|_{L^2(B^\bullet;\mathbb{R}^{d\times d})} \le C. \tag{4.87}$$

In the following we work again with a subsequence $\tau = \tau(n)$ as defined in (4.45). In this notation, due to the uniform convergence (4.43a), we have

$$\exists\, n(\nu/2) \in \mathbb{N}\ \ \forall n \ge n(\nu/2)\ \ \forall (t,x) \in B^-_{\delta,\nu}: \quad \overline{\varrho}_{\tau n}(t,x) < -\frac{\nu}{2}\,, \tag{4.88a}$$

$$\exists\, n(\nu/2) \in \mathbb{N}\ \ \forall n \ge n(\nu/2)\ \ \forall (t,x) \in B^+_{\delta,\nu}: \quad \overline{\varrho}_{\tau n}(t,x) > \varrho_{\text{crit}} - \frac{\nu}{2}\,. \tag{4.88b}$$

In view of the definition (3.3a) of the viscosity μ_τ, estimate (4.87) implies for $B^-_{\delta,\nu}$

$$\underbrace{\int_{I_\delta}\int_{\Omega_\nu^-(t)} \frac{|\overline{\varrho}_{\tau n}|\widetilde{\nu}}{\varrho_{\text{crit}}^\alpha}\, e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t}_{\geq 0} + \underbrace{\int_{I_\delta}\int_{\Omega_\nu^-(t)} \frac{1}{\tau^\alpha}|\overline{\varrho}_{\tau n}|^2\, e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t}_{\geq 0}$$
$$+ \underbrace{\int_{I_\delta}\int_{\Omega_\nu^-(t)} \tau^\beta\, e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t}_{\geq 0} \leq C\,. \tag{4.89}$$

Applying (4.88a) to the second term of (4.89) we obtain

$$\int_{B^-_{\delta,\nu}} e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t = \int_{I_\delta}\int_{\Omega_\nu^-(t)} e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t \leq \frac{4C}{\nu^2}\tau^\alpha\,. \tag{4.90}$$

With similar ideas we now deduce a uniform L^2-estimate for $e(\overline{u}_{\tau n})$ on $B^+_{\delta,\nu}$. In view of (4.88b), for all $t \in I_\delta$ and for all $n \geq n(\nu/2)$ the set $\Omega_\nu^+(t)$ can be decomposed as follows:

$$\begin{aligned}\Omega_\nu^+(t) &= \big(\Omega_\nu^+(t) \cap [\overline{\varrho}_{\tau n}(t) \geq \varrho_{\text{crit}} - \tau]\big) \cup \big(\Omega_\nu^+(t) \cap [\overline{\varrho}_{\tau n}(t) < \varrho_{\text{crit}} - \tau]\big)\\ &= \big(\Omega_\nu^+(t) \cap [\overline{\varrho}_{\tau n}(t) \geq \varrho_{\text{crit}} - \tau]\big) \cup \big(\Omega_\nu^+(t) \cap [\varrho_{\text{crit}} - \tfrac{\nu}{2} < \overline{\varrho}_{\tau n}(t) < \varrho_{\text{crit}} - \tau]\big)\,.\end{aligned} \tag{4.91}$$

Thus, for all $n \geq n(\nu/2)$ decomposition (4.91) together with the definition (3.3a) of the viscosity μ_τ and estimate (4.87) yields

$$\begin{aligned} C &\geq \int_{B^+_{\delta,\nu}} \mu_\tau(\overline{\varrho}_{\tau n})\, e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t\\ &= \underbrace{\int_{I_\delta}\int_{\Omega_\nu^+(t)\cap[\overline{\varrho}_{\tau n}(t)\geq\varrho_{\text{crit}}-\tau]} \frac{\widetilde{\nu}|\overline{\varrho}_{\tau n}|^2}{\tau^\alpha|\varrho_{\text{crit}}-\tau|}\, e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t}_{\geq 0}\\ &\quad + \underbrace{\int_{I_\delta}\int_{\Omega_\nu^+(t)\cap[\overline{\varrho}_{\tau n}(t)\geq\varrho_{\text{crit}}-\tau]} \tau^\beta\, e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t}_{\geq 0}\\ &\quad + \underbrace{\int_{I_\delta}\int_{\Omega_\nu^+(t)\cap[\varrho_{\text{crit}}-\frac{\nu}{2}<\overline{\varrho}_{\tau n}(t)<\varrho_{\text{crit}}-\tau]} \frac{|\overline{\varrho}_{\tau n}|\tilde{\nu}}{(\varrho_{\text{crit}}-\overline{\varrho}_{\tau n})^\alpha}\, e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t}_{\geq 0}\\ &\quad + \underbrace{\int_{I_\delta}\int_{\Omega_\nu^+(t)\cap[\varrho_{\text{crit}}-\frac{\nu}{2}<\overline{\varrho}_{\tau n}(t)<\varrho_{\text{crit}}-\tau]} \tau^\beta\, e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t}_{\geq 0}\,.\end{aligned} \tag{4.92}$$

To estimate (4.92) from below we disregard the second and the fourth term on its right-hand side. For the first term we see that $\frac{\tilde{\nu}|\overline{\varrho}_{\tau n}(t)|^2}{\tau^\alpha|\varrho_{\text{crit}}-\tau|} \geq \frac{\tilde{\nu}|\varrho_{\text{crit}}-\tau|}{\tau^\alpha} > \frac{\tilde{\nu}|\varrho_{\text{crit}}-\frac{\nu}{2}|}{\tau^\alpha}$ on $\Omega_\nu^+ \cap [\overline{\varrho}_{\tau n}(t) \geq \varrho_{\text{crit}} - \tau]$. For the third term on the right-hand side of (4.92) we have that $\frac{|\overline{\varrho}_{\tau n}(t)|\tilde{\nu}}{(\varrho_{\text{crit}}-\overline{\varrho}_{\tau n}(t))^\alpha} > \frac{\tilde{\nu}|\varrho_{\text{crit}}-\frac{\nu}{2}|}{\tau^\alpha}$ on $\Omega_\nu^+ \cap [\varrho_{\text{crit}} - \frac{\nu}{2} < \overline{\varrho}_{\tau n}(t) < \varrho_{\text{crit}} - \tau]$. Altogether this gives

$$\int_{B^+_{\delta,\nu}} e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t = \int_{I_\delta}\int_{\Omega_\nu^+(t)} e(\overline{u}_{\tau n}) : e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t < \frac{C\tau^\alpha}{\widetilde{\nu}|\varrho_{\text{crit}} - \frac{\nu}{2}|}\,. \tag{4.93}$$

Based on the uniform estimates (4.90) and (4.93) we may repeat the arguments of (4.47)–(4.55) in the proof of Lemma 4.5 in order to extract a (not relabelled) subsequence $(\overline{u}_{\tau n})_n$ with the properties

$$\overline{u}_{\tau n} \rightharpoonup u \;\text{ in } L^2(B^\bullet_{\delta,\nu};\mathbb{R}^d) \quad\text{and}\quad e(\overline{u}_{\tau n}) \rightharpoonup e(u) \;\text{ in } L^2(B^\bullet_{\delta,\nu};\mathbb{R}^{d\times d})\,. \tag{4.94}$$

Moreover, in the notation of (4.49b) we have that

$$e_n \rightharpoonup e := \begin{cases} e(u) & \text{on } B^{\bullet}_{\delta,\nu}, \\ 0 & \text{on } ((0,T)\times\Omega)\backslash B^{\bullet}_{\delta,\nu}, \end{cases} \qquad \text{weakly in } L^2(0,T;L^2(\Omega))\,.$$

Hence, we find by weak sequential lower semicontinuity of the $L^2(0,T;L^2(\Omega))$-norm

$$\begin{aligned} &\int_{I_\delta}\int_{\Omega^{\bullet}_{\nu}(t)} e(u):e(u)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega e:e\,\mathrm{d}x\,\mathrm{d}t \\ &\le \liminf_{n\to\infty}\int_0^T\int_\Omega e_n:e_n\,\mathrm{d}x\,\mathrm{d}t = \liminf_{n\to\infty}\int_{I_\delta}\int_{\Omega^{\bullet}_{\nu}(t)} e(\overline{u}_{\tau n}):e(\overline{u}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t \le \lim_{n\to\infty} L(\bullet)\,\tau(n)^{\alpha} = 0\,, \end{aligned}$$

where for $\bullet\in\{-,+\}$, we have $L(-) := \frac{4C}{\nu^2}$ and $L(+) := \frac{C}{\overline{\nu}|\varrho_{\mathrm{crit}}-\frac{\nu}{2}|}$. This gives $e(u)=0$ a.e. on $B^{\bullet}_{\delta,\nu}$ for $\bullet\in\{-,+\}$, for all $\delta>0$ and all $\nu>0$. For every $\delta>0$ and all $t\in I_\delta$ we further notice that $[\varrho(t)<0] = \bigcup_{\nu>0}\Omega^-_\nu(t)$ and $[\varrho(t)\ge\varrho_{\mathrm{crit}}] = \bigcap_{\nu>0}\Omega^+_\nu(t)$. Choosing now a sequence $(\delta_j)_{j\in\mathbb{N}}$ such that $\delta_j\to0$ as $j\to\infty$, as outlined in Remark 4.7, Item 2, we see that $[\varrho<0] = \bigcup_{j\in\mathbb{N}}\bigcup_{t\in I_{\delta_j}}[\varrho(t)<0]$ in $\mathcal{L}^1$-measure as well as $[\varrho\ge\varrho_{\mathrm{crit}}] = \bigcup_{j\in\mathbb{N}}\bigcup_{t\in I_{\delta_j}}[\varrho(t)\ge\varrho_{\mathrm{crit}}]$ in $\mathcal{L}^1$-measure. In this way we conclude (4.86). □

Information (4.85) in particular provides that

$$\|\operatorname{div}\overline{u}_{\tau n}\|_{L^2(A^{\nu}_{\tau n})}\to 0 \quad \text{for } A^{\nu}_{\tau n}\in\{[\overline{\varrho}_{\tau n}<-\nu],[\overline{\varrho}_{\tau n}\ge\varrho_{\mathrm{crit}}+\nu]\},\ \text{for any } \nu>0\,. \tag{4.95}$$

This will be used to ultimately infer the statements of Proposition 2.7. More precisely, we will show in Lemma 4.10 and Lemma **??** that the sets $[\varrho<0]$ and $[\varrho\ge\varrho_{\mathrm{crit}}]$ are $\mathcal{L}^{d+1}$-null sets. This will be achieved by testing the discrete transport equation (3.7a) with suitably devised test functions and by exploiting information (4.95) together with the strong $L^p(0,T;W^{1,p}(\Omega))$-convergence (4.1j) of the approximating sequence $(\overline{\varrho}_{\tau n})_{\tau n}$ when letting $\tau=\tau(n)\to0$ and $n\to\infty$.

Lemma 4.10. *Let the assumptions of Lemma 4.9 be valid. Further assume that the initial datum has the property*

$$\varrho_0\in L^2(\Omega)\,,\quad \text{such that } 0\le\varrho_0\le\varrho_{\mathrm{crit}} \text{ a.e. in } \Omega\,. \tag{4.96}$$

Then, for all $t\in[0,T]$, the sets $[\varrho(t)<0]$ and $[\varrho(t)>\varrho_{\mathrm{crit}}]$ are $\mathcal{L}^d$-null sets, i.e.,

$$\mathcal{L}^d([\varrho(t)<0]) = 0 \ \text{ for all } t\in[0,T]\,, \tag{4.97a}$$

$$\mathcal{L}^d([\varrho(t)>\varrho_{\mathrm{crit}}]) = 0 \ \text{ for all } t\in[0,T]\,. \tag{4.97b}$$

Proof. In order to verify the assertion (4.97) one would like to test the transport equation (2.5a), here

$$\langle\partial_t\varrho,\psi\rangle_{L^p(0,T;W^{1,p}(\Omega))} - \int_0^T\int_\Omega(\varrho u)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega\varepsilon|\nabla\varrho|^{p-2}\nabla\varrho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{4.98}$$

by functions of the type $\psi:=\max\{\varrho,a\}$ or $\psi:=\min\{\varrho,a\}$ for a constant $a\in\mathbb{R}$ and $\varrho\in L^p(0,T;W^{1,p}(\Omega))$. We note that the functions $\max\{\cdot,a\}:\mathbb{R}\to\mathbb{R}$ and $\min\{\cdot,a\}:\mathbb{R}\to\mathbb{R}$ are Lipschitz-continuous functions. Thus [MM79] ensures that their composition with a $L^p(0,T;W^{1,p}(\Omega))$-function again results in a $L^p(0,T;W^{1,p}(\Omega))$-function. However, to handle the drift term in (4.98) would require an integration by parts in space, which is not admissible for the limit problem because it is not clear that $(\varrho u)\in H^1(\Omega;\mathbb{R}^d)$ for all of Ω. Therefore we instead resort to the discrete equation (3.7a), where, for all $\tau=\tau(n)>0$ and $n\in\mathbb{N}$ fixed μ_τ from (3.3a) provides the needed information $(\overline{\varrho}_{\tau n}\overline{u}_{\tau n})\in H^1(\Omega;\mathbb{R}^d)$. Yet, in the space-discrete setting the above described functions involving the cut-off by max or min, denoted here for brevity by $\Psi'_{t_*}(\varrho)$, are not admissible test functions for the discrete transport equation (3.7a). To make it admissible, we shall apply the projector $P_{\tau l}:L^p(0,T;\mathbf{X})\to L^p(0,T;\mathbf{X}_l)$ from (3.6) to such a function and then we use the projected function $P_{\tau l}\Psi'_{t_*}(\varrho)$ as a test function for (3.7a) for any $n \geq l \in \mathbb{N}$. For ϱ a solution of (2.5a), this results in

$$\begin{aligned}\big\langle \mathrm{D}_\tau \varrho_{\tau n}, P_{\tau l}(\Psi'_{t_*}(\varrho))\big\rangle_{L^p(0,T;\mathbf{X}_l)} - \int_0^T \int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot \nabla P_{\tau l}(\Psi'_{t_*}(\varrho))\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T \int_\Omega \varepsilon |\nabla \overline{\varrho}_{\tau n}|^{p-2}\nabla\overline{\varrho}_{\tau n}\cdot \nabla P_{\tau l}(\Psi'_{t_*}(\varrho))\,\mathrm{d}x\,\mathrm{d}t = 0\,.\end{aligned} \tag{4.99}$$

Yet, in (4.99) it is neither possible to integrate the first term in time nor to judge about the sign of the p-Laplacian term. Both steps would be possible if $P_{\tau l}(\Psi'_{t_*}(\varrho))$ was replaced by $\Psi'_{t_*}(\overline{\varrho}_{\tau n})$. Therefore, we shall add to (4.99) the desired terms and subtract them again for compensation. Morally, we would test each of the terms in (3.7a) by $\Psi'(\overline{\varrho}_{\tau n})$, then add and subtract the resulting terms to (4.99). At this point, we note that $\langle \mathrm{D}_\tau\varrho_{\tau n}, \Psi'_{t_*}(\overline{\varrho}_{\tau n})\rangle_{L^p(0,T;\mathbf{X}_l)}$ is well-defined for all $\tau > 0$, $n \in \mathbb{N}$ fixed, since $\Psi'(\overline{\varrho}_{\tau n}) \in L^p(0,T,W^{1,p}(\Omega))$. But we cannot obtain any information about the convergence of this term, because a priori estimate (3.9j) only provides uniform boundedness of $(\mathrm{D}_\tau\varrho_{\tau n})_{\tau n}$ in $(L^p(0,T;\mathbf{X}_l))^*$ but not in $(L^p(0,T;\mathbf{X}))^*$, where $\mathbf{X} = W^{1,p}(\Omega)$. Therefore, instead of adding and subtracting to (4.99) the term $\langle \mathrm{D}_\tau\varrho_{\tau n}, \Psi'_{t_*}(\overline{\varrho}_{\tau n})\rangle_{L^p(0,T;\mathbf{X}_l)}$, we will use the time-integrated version of it. To be more specific about this, let us suppose that $\Psi'_{t_*}(\rho)$ has the following properties for all $\rho \in L^p(0,T;W^{1,p}(\Omega))$:

$$\Psi'_{t_*}(\rho(t)) = 0 \quad \text{a.e. in } \Omega, \text{ for all } t \in (t_*,T], \text{ for any } t_* \in (0,T) \text{ general but fixed,} \tag{4.100a}$$

$$\Psi'_{t_*} : L^p(0,T;W^{1,p}(\Omega)) \to L^p(0,T;W^{1,p}(\Omega)) \quad \text{is continuous}\,, \tag{4.100b}$$

$$\Psi'_{t_*} \text{ is the derivative of the convex, continuous function } \Psi_{t_*} : \mathbb{R}\to\mathbb{R}\,, \tag{4.100c}$$

$$\Psi''_{t_*}(\rho(t,x)) \geq 0 \text{ for all } x \in A(\rho(t)) \quad \text{and } \Psi'_{t_*}(\rho(t,x)) = 0 \text{ otherwise}\,. \tag{4.100d}$$

Hence, by properties (4.100a) and (4.100c) we find that

$$\begin{aligned}\big\langle \mathrm{D}_\tau\varrho_{\tau n}, \Psi'_{t_*}(\overline{\varrho}_{\tau n})\big\rangle_{L^p(0,T;\mathbf{X}_n)} &\geq \|\Psi_{t_*}(\overline{\varrho}_{\tau n}(\bar{t}_\tau(t_*)))\|_{L^1(\Omega)} - \|\Psi_{t_*}(\overline{\varrho}_{\tau n}(0))\|_{L^1(\Omega)} \\ &= \|\Psi_{t_*}(\overline{\varrho}_{\tau n}(t_*))\|_{L^1(\Omega)} - \|\Psi_{t_*}(\overline{\varrho}_{\tau n}(0))\|_{L^1(\Omega)}\,,\end{aligned} \tag{4.101}$$

where we exploited the convexity of Ψ_{t_*} and used the notation (3.5d). Thanks to the well-preparedness of the initial data and the strong $L^2(\Omega)$-convergence (4.1g) of $(\overline{\varrho}_{\tau n})_{\tau n}$ pointwise for all $t \in [0,T]$ together with (4.100c) we further deduce that

$$\|\Psi_{t_*}(\overline{\varrho}_{\tau n}(t_*))\|_{L^1(\Omega)} - \|\Psi_{t_*}(\overline{\varrho}_{\tau n}(0))\|_{L^1(\Omega)} \;\to\; \|\Psi_{t_*}(\varrho(t_*))\|_{L^1(\Omega)} - \|\Psi_{t_*}(\varrho(0))\|_{L^1(\Omega)} \tag{4.102}$$

as $n\to\infty$. In view of this, we will add and subtract to (4.99) directly the limit terms on the right-hand side of (4.102). In this way, we get

$$\begin{aligned}\|\Psi_{t_*}(\varrho(t_*))\|_{L^1(\Omega)} - \|\Psi_{t_*}(\varrho(0))\|_{L^1(\Omega)} - \int_0^{t_*}\int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla\Psi'_{t_*}(\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^{t_*}\int_\Omega \varepsilon|\nabla\overline{\varrho}_{\tau n}|^{p-2}\nabla\overline{\varrho}_{\tau n}\cdot\nabla\Psi'_{t_*}(\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t + \sum_{j=1}^3 R_{j,nl} = 0, \quad \text{where}\end{aligned} \tag{4.103a}$$

$$R_{1,nl} := \big\langle \mathrm{D}_\tau\varrho_{\tau n}, P_{\tau l}(\Psi'_{t_*}(\varrho))\big\rangle_{L^p(0,T;\mathbf{X}_l)} - \big(\|\Psi_{t_*}(\varrho(t_*))\|_{L^1(\Omega)} - \|\Psi_{t_*}(\varrho(0))\|_{L^1(\Omega)}\big)\,, \tag{4.103b}$$

$$R_{2,nl} := -\int_0^{t_*}\int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla\big(P_{\tau l}(\Psi'_{t_*}(\varrho)) - \Psi'_{t_*}(\overline{\varrho}_{\tau n})\big)\,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.103c}$$

$$R_{3,nl} := \int_0^{t_*}\int_\Omega \varepsilon|\nabla\overline{\varrho}_{\tau n}|^{p-2}\nabla\overline{\varrho}_{\tau n}\cdot\nabla\big(P_{\tau l}(\Psi'_{t_*}(\varrho)) - \Psi'_{t_*}(\overline{\varrho}_{\tau n})\big)\,\mathrm{d}x\,\mathrm{d}t\,. \tag{4.103d}$$

For the sum of the error terms in (4.103a) we shall verify below that

$$\sum_{j=1}^3 R_{j,nl} \to 0 \quad \text{as } n \geq l \to \infty\,. \tag{4.104}$$

We now discuss the treatment of the remaining two terms in (4.103a). In view of (4.100d) and (4.100d) we readily observe for the p-Laplacian term in (4.103a)

$$\begin{aligned}\int_0^{t_*}\int_\Omega \varepsilon|\nabla\overline{\varrho}_{\tau n}|^{p-2}\nabla\overline{\varrho}_{\tau n}\cdot\nabla\Psi'_{t_*}(\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t &= \int_0^{t_*}\int_{A(\overline{\varrho}_{\tau n}(t))}\varepsilon|\nabla\overline{\varrho}_{\tau n}|^{p-2}\nabla\overline{\varrho}_{\tau n}\cdot\Psi''_{t_*}(\overline{\varrho}_{\tau n})\nabla\overline{\varrho}_{\tau n}\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^{t_*}\int_{A(\overline{\varrho}_{\tau n}(t))}\varepsilon|\nabla\overline{\varrho}_{\tau n}|^{p}\Psi''_{t_*}(\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t\geq 0\,.\end{aligned}\tag{4.105}$$

For the drift term we are now in the position to apply integration by parts in space and with the function $\Psi'(\overline{\varrho}_{\tau n})$ precisely tailored to the two cases (4.97a) and (4.97b) it will be the goal to show that

$$\Big|-\int_0^{t_*}\int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla\Psi'_{t_*}(\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t\Big|\leq c\,\|\mathrm{div}(\overline{u}_{\tau n})\|_{L^2(A(\overline{\varrho}_{\tau n}))}\;\to 0\,,\tag{4.106}$$

where the convergence $\|\mathrm{div}(\overline{u}_{\tau n})\|_{L^2(A(\overline{\varrho}_{\tau n}))}\to 0$ follows from information (4.85) for $A(\overline{\varrho}_{\tau n})=[\overline{\varrho}_{\tau n}<-\nu]$ or $A(\overline{\varrho}_{\tau n})=[\overline{\varrho}_{\tau n}>\varrho_{\mathrm{crit}}+\nu]$ for any $\nu>0$.

Now, putting together (4.103a) and (4.104)–(4.106) leads to the following estimate

$$\begin{aligned}\|\Psi_{t_*}(\varrho(t_*))\|_{L^1(\Omega)} &\leq \|\Psi_{t_*}(\varrho(0))\|_{L^1(\Omega)}+\Big|\sum_{j=1}^3 R_{j,nl}\Big|+c\,\|\mathrm{div}(\overline{u}_{\tau n})\|_{L^2(A(\overline{\varrho}_{\tau n}))}\\ &=\Big|\sum_{j=1}^3 R_{j,nl}\Big|+c\,\|\mathrm{div}(\overline{u}_{\tau n})\|_{L^2(A(\overline{\varrho}_{\tau n}))}\;\longrightarrow 0\quad\text{as } n\geq l\to\infty\,.\end{aligned}\tag{4.107}$$

given that the initial datum satisfies $\|\Psi_{t_*}(\varrho(0))\|_{L^1(\Omega)}=0$. In the limit we thus have

$$\|\Psi_{t_*}(\varrho(t_*))\|_{L^1(\Omega)}=0\,,\tag{4.108}$$

and for suitably tailored Ψ_{t_*} this will result in the information (4.97a) and (4.97b).

Proof of the convergence (4.104) *of the error terms:* We have $\overline{\varrho}_{\tau n}\to\varrho$ strongly in $L^p(0,T;W^{1,p}(\Omega))$ by (4.1j). Since the function $\Psi'_{t_*}:L^p(0,T;W^{1,p}(\Omega))\to L^p(0,T;W^{1,p}(\Omega))$ is continuous, there also holds $\Psi'_{t_*}(\overline{\varrho}_{\tau n})\to\Psi'_{t_*}(\varrho)$ in $L^p(0,T;W^{1,p}(\Omega))$. Furthermore, by the approximation properties of the projector we then also have $P_{\tau n}\Psi'(\varrho)\to\Psi'(\varrho)$ in $L^p(0,T;W^{1,p}(\Omega))$. Then, the subsequent estimates will allow us to conclude (4.104). We start with R_1. For this, we may first repeat the arguments of (4.6)–(4.6c) and subsequently apply integration by parts in time to arrive that

$$\lim_{l\to\infty}\lim_{\substack{n\to\infty\\ n\geq l}}\langle \mathrm{D}_\tau\overline{\varrho}_{\tau n},P_{\tau l}(\Psi'_{t_*}(\varrho))\rangle_{L^p(0,T;\mathbf{X}_l)}=\langle\partial_t\varrho,\Psi'_{t_*}(\varrho)\rangle_{L^p(0,T;\mathbf{X})}=\|\Psi_{t_*}(\varrho(t_*))\|_{L^1(\Omega)}-\|\Psi_{t_*}(\varrho(0))\|_{L^1(\Omega)}\,.$$

In view of (4.103b) this shows that $R_{1,nl}\to 0$ as $n\geq l\to\infty$.

For the error term $R_{2,nl}$ we deduce via Hölder's inequality and with the aid of a priori estimate (3.9d) together with the strong $L^p(0,T;W^{1,p}(\Omega))$-convergence of $P_{\tau n}\Psi'(\varrho)-\Psi'(\overline{\varrho}_{\tau n})$ that

$$\begin{aligned}|R_{2,nl}| &= \Big|-\int_0^{t_*}\int_\Omega\overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla\big(P_{\tau l}(\Psi'_{t_*}(\varrho))-\Psi'_{t_*}(\overline{\varrho}_{\tau n})\big)\,\mathrm{d}x\,\mathrm{d}t\Big|\\ &\leq\|\overline{\varrho}_{\tau n}\overline{u}_{\tau n}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\big\|\nabla\big(P_{\tau l}(\Psi'_{t_*}(\varrho))-\Psi'_{t_*}(\overline{\varrho}_{\tau n})\big)\big\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\\ &\leq C\big\|\nabla\big(P_{\tau l}(\Psi'_{t_*}(\varrho))-\Psi'_{t_*}(\overline{\varrho}_{\tau n})\big)\big\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}\to 0\quad\text{as } n\geq l\to 0\,.\end{aligned}$$

With similar arguments also the convergence of $R_{3,nl}$ follows, now exploiting a priori estimate (3.9k), resp. (3.9g)

$$\begin{aligned}|R_{3,nl}| &\leq\|\nabla\overline{\varrho}_{\tau n}\|^{p-1}_{L^p(0,T;\mathbf{X})}\big\|\nabla\big(P_{\tau l}(\Psi'_{t_*}(\varrho))-\Psi'_{t_*}(\overline{\varrho}_{\tau n})\big)\big\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))}\\ &\leq C\big\|\nabla\big(P_{\tau l}(\Psi'_{t_*}(\varrho))-\Psi'_{t_*}(\overline{\varrho}_{\tau n})\big)\big\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))}\to 0\quad\text{as } n\geq l\to 0\,.\end{aligned}$$

Thus, (4.104) is verified.

To (4.97a)**:** We apply the reasoning (4.99)–(4.108) with the function

$$\Psi'_{t_*}(\rho(t)) = (\rho(t))^\nu_{-,t_*} := \begin{cases} (\rho(t))^\nu_- & \text{for } t \in [0,t_*], \\ 0 & \text{for } t \in (t_*,T], \end{cases} \qquad \text{where } (\rho(t))^\nu_- := \min\{\rho(t)+\nu, 0\} \tag{4.109}$$

for any $\nu > 0$. We note that $(\rho)^\nu_{-,t_*} \in L^p(0,T;W^{1,p}(\Omega))$ for any $\rho \in L^p(0,T;W^{1,p}(\Omega))$, for all $t_* \in [0,T]$ and all $\nu > 0$. As outlined in (4.99) we test the discrete transport equation (3.7a) by $P_{\tau l}\Psi'_{t_*}(\varrho)$ with $\varrho \in L^p(0,T;W^{1,p}(\Omega))$ a solution of (2.5a). Moreover, we carry out the steps (4.99)–(4.108) also with $\Psi'_{t_*}(\overline{\varrho}_{\tau n}) = (\overline{\varrho}_{\tau n})_{-,t_*}$. To arrive at (4.108) we have to verify for the function $\Psi'_{t_*}(\rho(t)) := (\rho(t))^\nu_{-,t_*}$ defined in (4.109) that properties (4.100) hold true and that the drift term can be estimated as claimed in (4.106).

To properties (4.100) *for* Ψ'_{t_*} *from* (4.109)*:* Clearly, by definition (4.109) property (4.100a) is valid. Moreover, the function $(\bullet)^\nu_- = \min\{\bullet + \nu, 0\}$ is Lipschitz continuous and thus $\Psi'_{t_*} : L^p(0,T;W^{1,p}(\Omega)) \to L^p(0,T;W^{1,p}(\Omega))$ from (4.109) is continuous, which is (4.100b). We further observe that the primitive and the derivative of Ψ'_{t_*} are given by

$$\Psi_{t_*}(\rho(t)) := \begin{cases} \frac{1}{2}((\rho(t))^\nu_-)^2 & \text{for } t \in [0,t_*], \\ 0 & \text{for } t \in (t_*,T], \end{cases} \tag{4.110a}$$

$$\Psi''_{t_*}(\rho(t)) := \begin{cases} 1 & \text{if } \rho(t) < -\nu \text{ and for } t \in [0,t_*], \\ 0 & \text{if } \rho(t) \geq -\nu \text{ and for } t \in [0,t_*], \\ 0 & \text{for } t \in (t_*,T]\,. \end{cases} \tag{4.110b}$$

From (4.110a) we see that Ψ_{t_*} is continuous for $t \in [0,t_*]$. Moreover (4.110b) yields (4.100d) with $A(\rho(t)) := [\rho(t) < -\nu]$, which then also provides the convexity of Ψ_{t_*} and thus finishes the proof of property (4.100c).

To estimate (4.106) *for the drift term:* For $\Psi'_{t_*}(\overline{\varrho}_{\tau n}(t)) := (\overline{\varrho}_{\tau n}(t))^\nu_{-,t_*}$ from (4.109) the expression (4.106) can now be handled using integration by parts in space, also exploiting that $\overline{u}_{\tau n}(t) = 0$ on $\partial\Omega$ for a.a. $t \in (0,T)$, i.e.,

$$\begin{aligned}
&\Big| -\int_0^{t_*}\int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla\Psi'_{t_*}(\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t\Big| = \Big| -\int_0^{t_*}\int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla(\overline{\varrho}_{\tau n})^\nu_-\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&= \Big| -\int_0^{t_*}\int_\Omega (\overline{\varrho}_{\tau n}+\nu)\overline{u}_{\tau n}\cdot\nabla(\overline{\varrho}_{\tau n})^\nu_-\,\mathrm{d}x\,\mathrm{d}t + \int_0^{t_*}\int_\Omega \nu\overline{u}_{\tau n}\cdot\nabla(\overline{\varrho}_{\tau n})^\nu_-\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&= \Big| -\int_0^{t_*}\int_\Omega \overline{u}_{\tau n}\cdot\nabla\frac{((\overline{\varrho}_{\tau n})^\nu_-)^2}{2}\,\mathrm{d}x\,\mathrm{d}t + \int_0^{t_*}\int_\Omega \nu\overline{u}_{\tau n}\cdot\nabla(\overline{\varrho}_{\tau n})^\nu_-\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leq \Big| \int_0^{t_*}\int_\Omega \mathrm{div}(\overline{u}_{\tau n})\frac{((\overline{\varrho}_{\tau n})^\nu_-)^2}{2}\,\mathrm{d}x\,\mathrm{d}t - \int_0^{t_*}\int_{\partial\Omega}(\overline{u}_{\tau n}\cdot\vec{n})\,\frac{((\overline{\varrho}_{\tau n})^\nu_-)^2}{2}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\Big| \\
&\quad + \Big| \int_0^{t_*}\int_\Omega \nu\,\mathrm{div}(\overline{u}_{\tau n})(\overline{\varrho}_{\tau n})^\nu_-\,\mathrm{d}x\,\mathrm{d}t - \int_0^{t_*}\int_{\partial\Omega}\nu(\overline{u}_{\tau n}\cdot\vec{n})(\overline{\varrho}_{\tau n})^\nu_-\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\Big| \\
&= \Big| \int_0^{t_*}\int_{[\overline{\varrho}_{\tau n}(t)<-\nu]} \mathrm{div}(\overline{u}_{\tau n})\frac{((\overline{\varrho}_{\tau n})^\nu_-)^2}{2}\,\mathrm{d}x\,\mathrm{d}t\Big| + \Big| \int_0^{t_*}\int_{[\overline{\varrho}_{\tau n}(t)<-\nu]} \nu\,\mathrm{div}(\overline{u}_{\tau n})(\overline{\varrho}_{\tau n})^\nu_-\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leq \|\,\mathrm{div}(\overline{u}_{\tau n})\|_{L^2([\overline{\varrho}_{\tau n}(t)<-\nu])}\big(\tfrac{1}{2}\|((\overline{\varrho}_{\tau n})^\nu_-)^2\|_{L^2(0,T;L^2(\Omega))} + \nu\|(\overline{\varrho}_{\tau n})^\nu_-\|_{L^2(0,T;L^2(\Omega))}\big) \\
&= c\|\,\mathrm{div}(\overline{u}_{\tau n})\|_{L^2([\overline{\varrho}_{\tau n}(t)<-\nu])} \to 0 \text{ as } n \to \infty
\end{aligned} \tag{4.111}$$

by information (4.85a) and thanks to the assumptions $p \geq 4$ and $\varrho_0 \geq 0$ a.e. in Ω by (4.96). This proves (4.106). Now the validity of properties (4.100) and (4.106) provides (4.108), i.e., that $\|\Psi_{t_*}(\varrho(t_*))\|_{L^1(\Omega)} = \|(\varrho(t_*))^\nu_-\|_{L^1(\Omega)} = 0$, which gives (4.97a).

To (4.97b)**:** Now, we apply the reasoning (4.99)–(4.108) using the function

$$\Psi_{t_*}(\rho(t)) = (\varrho(t))^\nu_{\mathrm{crit}+,t_*} := \begin{cases} (\rho(t))^\nu_{\mathrm{crit}+} & \text{for } t \in [0,t_*], \\ 0 & \text{for } t \in (t_*,T], \end{cases} \qquad \text{where } (\rho(t))^\nu_{\mathrm{crit}+} := \max\{\rho(t) - \varrho_{\mathrm{crit}} - \nu, 0\} \tag{4.112}$$

and to arrive at (4.108) we have to show that $\Psi_{t_*}(\rho(t)) = (\varrho(t))^\nu_{\text{crit}+,t_*}$ satisfies properties (4.100) and that estimate (4.106) holds true.

To properties (4.100) *for* Ψ_{t_*} *from* (4.112)*:* Here, the primitive and the derivative are given by

$$\Psi_{t_*}(\rho(t)) := \begin{cases} \frac{1}{2}((\rho(t))^\nu_{\text{crit}+})^2 & \text{for } t \in [0,t_*], \\ 0 & \text{for } t \in (t_*,T], \end{cases} \tag{4.113a}$$

$$\Psi''_{t_*}(\rho(t)) := \begin{cases} 1 & \text{if } \rho(t) > \varrho_{\text{crit}} + \nu \text{ and for } t \in [0,t_*], \\ 0 & \text{if } \rho(t) \le \varrho_{\text{crit}} + \nu \text{ and for } t \in [0,t_*], \\ 0 & \text{for } t \in (t_*,T]\,. \end{cases} \tag{4.113b}$$

Now similar arguments as for (4.109) provide properties (4.100) also for Ψ_{t_*} from (4.112).

To estimate (4.106) *for the drift term:* Very similar calculations via integration by parts in space, as in (4.111), also here result in

$$\begin{aligned}
&\Big| -\int_0^{t_*}\int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla\Psi'_{t_*}(\overline{\varrho}_{\tau n})\,\mathrm{d}x\,\mathrm{d}t\Big| = \Big| -\int_0^{t_*}\int_\Omega \overline{\varrho}_{\tau n}\overline{u}_{\tau n}\cdot\nabla(\overline{\varrho}_{\tau n})^\nu_{\text{crit}+}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&= \Big| -\int_0^{t_*}\int_\Omega (\overline{\varrho}_{\tau n}-\varrho_{\text{crit}}-\nu)\overline{u}_{\tau n}\cdot\nabla(\overline{\varrho}_{\tau n})^\nu_{\text{crit}+}\,\mathrm{d}x\,\mathrm{d}t - \int_0^{t_*}\int_\Omega(\varrho_{\text{crit}}+\nu)\overline{u}_{\tau n}\cdot\nabla(\overline{\varrho}_{\tau n})^\nu_{\text{crit}+}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&= \Big| -\int_0^{t_*}\int_\Omega \overline{u}_{\tau n}\cdot\nabla\frac{((\overline{\varrho}_{\tau n})^\nu_{\text{crit}+})^2}{2}\,\mathrm{d}x\,\mathrm{d}t - \int_0^{t_*}\int_\Omega(\varrho_{\text{crit}}+\nu)\overline{u}_{\tau n}\cdot\nabla(\overline{\varrho}_{\tau n})^\nu_{\text{crit}+}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \Big| \int_0^{t_*}\int_\Omega \mathrm{div}(\overline{u}_{\tau n})\frac{((\overline{\varrho}_{\tau n})^\nu_{\text{crit}+})^2}{2}\,\mathrm{d}x\,\mathrm{d}t - \int_0^{t_*}\int_{\partial\Omega}(\overline{u}_{\tau n}\cdot\vec{n})\,\frac{((\overline{\varrho}_{\tau n})^\nu_{\text{crit}+})^2}{2}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\Big| \\
&\quad + \Big| \int_0^{t_*}\int_\Omega(\varrho_{\text{crit}}+\nu)\,\mathrm{div}(\overline{u}_{\tau n})(\overline{\varrho}_{\tau n})^\nu_{\text{crit}+}\,\mathrm{d}x\,\mathrm{d}t + \int_0^{t_*}\int_{\partial\Omega}(\varrho_{\text{crit}}+\nu)(\overline{u}_{\tau n}\cdot\vec{n})(\overline{\varrho}_{\tau n})^\nu_{\text{crit}+}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\Big| \\
&= \Big| \int_0^{t_*}\int_{[\overline{\varrho}_{\tau n}(t)>\varrho_{\text{crit}}+\nu]} \mathrm{div}(\overline{u}_{\tau n})\frac{((\overline{\varrho}_{\tau n})^\nu_{\text{crit}+})^2}{2}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\quad + \Big| \int_0^{t_*}\int_{[\overline{\varrho}_{\tau n}(t)>\varrho_{\text{crit}}+\nu]}(\varrho_{\text{crit}}+\nu)\,\mathrm{div}(\overline{u}_{\tau n})(\overline{\varrho}_{\tau n})^\nu_{\text{crit}+}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \|\,\mathrm{div}(\overline{u}_{\tau n})\|_{L^2([\overline{\varrho}_{\tau n}(t)>\varrho_{\text{crit}}+\nu])}\big(\tfrac{1}{2}\|((\overline{\varrho}_{\tau n})^\nu_{\text{crit}+})^2\|_{L^2(0,T;L^2(\Omega))} + \nu\|(\overline{\varrho}_{\tau n})^\nu_{\text{crit}+}\|_{L^2(0,T;L^2(\Omega))}\big) \\
&= c\|\,\mathrm{div}(\overline{u}_{\tau n})\|_{L^2([\overline{\varrho}_{\tau n}(t)>\varrho_{\text{crit}}+\nu])} \;\to 0 \text{ as } n\to\infty
\end{aligned} \tag{4.114}$$

by information (4.85a) and thanks to $\varrho_0 \le \varrho_{\text{crit}}$ a.e. in Ω by (4.96). This proves (4.106). Now the validity of properties (4.100) and (4.106) provides (4.108), i.e., that $\|\Psi_{t_*}(\varrho(t_*))\|_{L^1(\Omega)} = \|(\varrho(t_*))^\nu_{\text{crit}+}\|_{L^1(\Omega)} = 0$, which yields (4.97b). □

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