Thermally driven phase transformation in shape-memory alloys

A. Mielke, A. Petrov

Weierstrass Institute for Applied Analysis and Stochastics,
Mohrenstrasse 39,
10117 Berlin

mielke@wias-berlin.de,petrov@wias-berlin.de

Oberwolfach Workshop Analysis and Numerics for Rate–Independent Processes,
February 25th - March 3rd, 2007

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In this work: a model for thermally driven phase transfomation in shape memory alloys (given temperature) is presented and existence result is obtained.

Previous works:

- A. Souza, E. Mamiya, N. Zouain'98,
- A. Mielke, F. Theil'04,
- A. Mainik, A. Mielke'05,
- G. Francfort, A. Mielke'06,
- A. Mielke'05,
- F. Auricchio, A. Mielke, U. Stefanelli'07.

Mathematical formulation

We consider a body with reference configuration $\Omega \subset \mathbb{R}^d$.

- $u : \Omega \to \mathbb{R}^d$: the phase transformation and deformations,
- $z : \Omega \to \mathbb{R}^{d\times d}_{\text{dev}} := \{z \in \mathbb{R}^{d\times d}_{\text{sym}} : \text{tr}(z) = 0\}$: the internal variable,

Notations

- $\mathbb{R}^{d\times d}_{\text{sym}} := \{z \in \mathbb{R}^{d\times d} : z = z^T\}$
  - $a : b := \text{tr}(ab) = a_{ij} b_{ij}$: the scalar product,
  - $|a|^2 := a : a = a_{ij} a_{ij}$: the norm,
- $(\cdot)^T$: the transpose of the matrix $(\cdot)$,
- $\text{tr}(\cdot)$: the trace of the matrix $(\cdot)$.

The potential energy has the following form:

$$\mathcal{E}(t,u,z) := \int_\Omega W(e(u),z,\theta) + \frac{\sigma}{2}|\nabla z|^2\,dx - \langle l(t),u\rangle,$$

Notations

- The stored energy density is defined by

$$W(e(u),z,\theta) := \frac{1}{2}(e(u)-z):\mathbb{C}(\theta):(e(u)-z) + h(z,\theta),$$

  - $e(u) := \nabla u + \nabla u^T$: the linearized deformation satisfies the Korn's inequality, i.e.

$$\int_\Omega |e(u)|^2\,dx \geq c_{\text{Korn}}\|u\|^2_{W^{1,2}},\ c_{\text{Korn}} > 0,$$

  - $\mathbb{C}(\theta) : \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}_{\text{sym}}$: the elasticity tensor (symmetric positive linear map) that depends on the temperature θ and is defined as follows:

$$\mathbb{C}(\theta) : a := \lambda(\theta)\text{tr}(a) + 2\mu(\theta)a,$$

    - $\lambda(\theta)$, $\mu(\theta)$: the Lamé coefficients depending on the temperature θ.

  - $h(z,\theta) := c_1(\theta)|z|^2 + c_2(\theta)\sqrt{\delta^2+|z|^2} + \left(|z|^2 - c_3(\theta)\right)^3_+,$

- $\sigma > 0$: measures some nonlocal interaction effect for z,

- $l(t)$: the applied mechanical loading is defined as follows:

$$\langle l(t), u \rangle = \int_\Omega f_{\mathrm{appl}}(t,x) \cdot u(x)\, dx + \int_{\partial\Omega} g_{\mathrm{appl}}(t,x) \cdot u(x)\, d\gamma.$$

The dissipation potential is defined by

$$\mathcal{R}(\dot{z}) := \int_\Omega \rho |\dot{z}|\, dx = \rho \|\dot{z}\|_{L^1(\Omega)}, \ \rho > 0.$$

Remark 1.

- We do not solve an associated heat equation,
- This approximation used in engineering models:

  Assumptions:

  - the changes of the loading are slow,
  - the body is small in at least one direction,

  $\Rightarrow$ excess heat can be transported very fast to the surface.

We specify now the set of admissible deformations $\mathcal{F}$ by choosing a suitable Sobolev space $W^{1,2}(\Omega;\mathbb{R}^d)$ and by describing Dirichlet data at the part Γ_{Dir} of the boundary $\partial\Omega$

$$\mathcal{F} := \{u \in W^{1,2}(\Omega;\mathbb{R}^d) : u_{|\Gamma_{\mathrm{Dir}}} = 0\},$$

and the internal variable z live in $\mathcal{Z} := L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$.

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Energetic formulation:

A function $(u, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ is an *energetic solution* of the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{R}$ if for all $t \in [0, T]$, the *global stability condition* (S) and the *global energy conservation* (E) are satisfied, i.e.

$$(S) \quad \forall(\bar{u}, \bar{z}) \in \mathcal{F} \times \mathcal{Z} : \mathcal{E}(t, u(t), z(t)) \le \mathcal{E}(t, \bar{u}, \bar{z}) + \mathcal{R}(\bar{z} - z(t)),$$

$$(E) \quad \mathcal{E}(t, u(t), z(t)) + \int_0^t \mathcal{R}(\dot{z}(s))\, ds = \mathcal{E}(0, u_0, z_0) + \int_0^t \partial_s\mathcal{E}(s, u(s), z(s))\, ds.$$

A priori estimates

We clarify now the assumptions and we establish some preliminary results that we will use in the next section.

Lemma 1. Assume $c_j(\theta)$, $j = 1, \ldots, 3$, belong to $C^1([0, T])$. Then there exist c_j^W, $j = 1, 2$, such that for all $j = 1, 2$,

$$|\partial_\theta^j W(e(u), z, \theta)| \leq c_1^W (W(e(u), z, \theta) + c_0^W).$$

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Lemma 2. Under the assumptions of Lemma 1, for all $\theta_1 \in [\theta_{\min}, \theta_{\max}]$, we have

$$W(e(u), z, \theta_1) + c_0^W \leq \exp(c_1^W |\theta_1 - \theta|)(W(e(u), z, \theta) + c_0^W).$$

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Remark 1. There exist $c > 0$ and $C > 0$ such that

$$W(e(u), z, \theta) \ge c|e(u)|^2 - C.$$

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Proposition 1. Under the above assumptions the following holds:

1. If for some $(t,u,z) \in [0,T] \times \mathcal{F} \times \mathcal{Z}$ we have $\mathcal{E}(t,u,z) < +\infty$, then $\mathcal{E}(\cdot,u,z)$ are bounded in $C^1([0,T])$ and

$$\partial_t \mathcal{E}(t,u,z) = \int_\Omega \partial_\theta W(e(u), z, \theta_{\mathrm{appl}}(t)) \dot{\theta}_{\mathrm{appl}}(t)\, dx - \langle \dot{l}(t), u \rangle. \qquad (1)$$

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2. There exist two constants $c_0^E > 0$ and $c_1^E > 0$ such that $\mathcal{E}(t,u,z) < +\infty$ implies
$$|\partial_t \mathcal{E}(t,u,z)| \le c_1^E(\mathcal{E}(t,u,z) + c_0^E). \qquad (2)$$

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3. For each strictly positive ε and $E \in \mathbb{R}$ there exists δ such that $\mathcal{E}(t_1,u,z) \le E$ and $|t_1 - t_2| < \delta$ imply
$$|\partial_t \mathcal{E}(t_1,u,z) - \partial_t \mathcal{E}(t_2,u,z)| \le \varepsilon. \tag{3}$$

Proof:

1. The Korn's inequality and Remark 1 $\Rightarrow \exists c > 0, C > 0$ such that

$$\mathcal{E}(t, u, z) \geq c_0 \|u\|^2_{W^{1,2}} - C_0.$$

For all $h \neq 0$ and $t + h \in [0, T]$ the mean-value theorem provides that

$$\frac{1}{h}(\mathcal{E}(t+h, u, z) - \mathcal{E}(t, u, z)) = \partial_t \mathcal{E}(t + sh, u, z)$$

$$= \int_\Omega \partial_\theta W(e(u), z, \theta_{\text{appl}})\dot{\theta}_{\text{appl}}(t + sh)\, dx - \langle l(t+h) - l(t), u\rangle, s \in (0, 1).$$

The Lebegue's theorem $\Rightarrow$ the differentiability of $\mathcal{E}(t, u, z)$ $(\partial_t \mathcal{E}(t, u, z))$.
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2. Lemma 1 for $j=1$ and Cauchy-Schwarz's inequality $\Rightarrow$ (2).

3. Observe now that
$$|\partial_t\mathcal{E}(t_1,u,z) - \partial_t\mathcal{E}(t_2,u,z)|$$
$$\leq \int_\Omega |\partial_\theta W(e(u),z,\theta_{\mathrm{appl}}(t_1)) - \partial_\theta W(e(u),z,\theta_{\mathrm{appl}}(t_2))|\,dx\|\dot{\theta}_{\mathrm{appl}}\|_{L^\infty}$$
$$+ \int_\Omega |\partial_\theta W(e(u),z,\theta_{\mathrm{appl}})|\,dx\|\dot{\theta}_{\mathrm{appl}}(t_1) - \dot{\theta}_{\mathrm{appl}}(t_2)\|_{L^\infty}$$
$$+\|\dot{l}(t_1) - \dot{l}(t_2)\|_{(W^{1,2})'}\|u\|_{W^{1,2}}.$$
The mean-value theorem, Lemma 1 and Lemma 2 $\Rightarrow$ (3).

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- $\mathrm{Argmin}\{\varphi(u) : u \in \mathcal{H}\}$: the set of all minimizers of a functional $\varphi : \mathcal{H} \to \mathbb{R}_\infty$,
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We define the incremental problem as follows:

$$(IP)_\Pi \quad \begin{cases} \text{for } k = 1, \ldots, d \text{ find} \\ (u_k, z_k) \in \mathrm{Argmin}\{\mathcal{E}(t_k, \widetilde{u}, \widetilde{z}) + \mathcal{R}(\widetilde{z} - z_k) : (\widetilde{u}, \widetilde{z}) \in \mathcal{F} \times \mathcal{Z}\}. \end{cases}$$

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- $(IP)_\Pi$ has always solutions,
- we are able to define the piecewise constant interpolant $(u^\Pi, z^\Pi) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ with $(u^\Pi(t), z^\Pi(t)) = (u_j, z_j)$ for $t \in [t_{j-1}, t_j)$ for $j = 0, \ldots, N$.

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Assumption: $(u_0, z_0) \in \mathcal{F} \times \mathcal{Z}$ are given stable initial datum, i.e. (u_0, z_0) satisfies the *global stability condition* (S) at $t = 0$.

Theorem 1. Assume that $\mathcal{E}$ and $\mathcal{R}$ satisfy the assumptions from above. Then, for each stable $(u(0), z(0)) = (u_0, z_0)$, there exists an *energetic solution* $(u, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ such that

$$u \in L^\infty([0, T]; W^{1,2}(\Omega; \mathbb{R}^d)),$$
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Moreover, let $\Pi_k = \{0 = t_0^k < t_1^k < \ldots < t_{N_k}^k = T\}$, $k \in \mathbb{N}$, be a sequence of partitions with fineness $\Delta(\Pi_k) := \max\{t_j^k - t_{j-1}^k\, j = 1, \ldots, N_k\}$ tends to zero and $(u^{\Pi_k}, z^{\Pi_k}) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ be piecewise constant interpolants of the solution of the incremental problem $(IP)_{\Pi_k}$, then there exists a subsequence $(\bar{u}_n, \bar{z}_n) := (u^{\Pi_{k_n}}, z^{\Pi_{k_n}})$ such that for all $t \in [0, T]$ the following holds

$$\bar{z}_n(t) \to z(t) \text{ in } \mathcal{Z},$$
$$\mathcal{E}(t, \bar{u}_n(t), \bar{z}_n(t)) \to \mathcal{E}(t, u(t), z(t)),$$
$$\int_0^t \mathcal{R}(\dot{\bar{z}}_n(s))\, ds \to \int_0^t \mathcal{R}(\dot{z}(s))\, ds,$$

there exists a subsequence $(N_l^t)_{l\in\mathbb{N}}$ such that

$$\bar{u}_{N_l^t}(t) \to u(t) \text{ in } \mathcal{F} \text{ for } l \to 0.$$

Conclusion

- Uniqueness result,
- Existence result for the same problem with an associated heat equation.

Assumptions on θ_{appl} and l imply that

$$\|\dot{l}(t_1) - \dot{l}(t_2)\|_{(W^{1,2})'} + \|\dot{\theta}_{\text{appl}}(t_1) - \dot{\theta}_{\text{appl}}(t_2)\|_{L^\infty} \leq \omega(|t_1 - t_2|),$$

where $\omega : [0, +\infty) \to [0, +\infty)$ is a modulus of continuity with $\omega(0) = 0$.