Global existence result for rate-independent processes in viscous solids with heat transfer

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This note deals with three-dimensional models for rate-independent processes describing materials undergoing phase transformations with heat transfer. The problem is formulated within the framework of generalized standard solids by the coupling of the momentum equilibrium equation and the flow rule with the heat transfer equation. The existence of a global solution for this thermodynamically consistent problem is obtained by using a fixed-point argument combined with global energy estimates.

1 The mathematical formulation

We consider a reference configuration $\Omega \subset \mathbb{R}^3$. We assume that Ω is a bounded domain such that $\partial\Omega$ is of class $\mathrm{C}^{2+\rho}$. We will denote by $\mathbb{R}^{3\times3}_{\text{sym}}$ the space of symmetric 3×3 tensors endowed with the natural scalar product $v{:}w \overset{\text{def}}{=} \text{tr}(v^{\mathsf{T}}w)$ and the corresponding norm $|v|^2 \overset{\text{def}}{=} v{:}v$ for all $v, w \in \mathbb{R}^{3\times3}_{\text{sym}}$. Given a function $\ell : \Omega\times(0,T) \to \mathbb{R}^3$, we look for a *displacement* $u : \Omega\times(0,T) \to \mathbb{R}^3$, a matrix of *internal variables* $z : \Omega\times(0,T) \to \mathbb{R}^{3\times3}_{\text{dev}}$ and a *temperature* $\theta : \Omega\times(0,T) \to \mathbb{R}$ satisfying the following system:

$$-\,\mathrm{div}(\mathbb{E}(\mathrm{e}(u){-}z){+}\alpha\theta\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot u)) = \ell, \tag{1.1a}$$

$$\partial\Psi(\dot z) + \mathbb{M}\dot z - \mathbb{E}(\mathrm{e}(u){-}z) + \mathrm{D}_z H_1(z) + \theta \mathrm{D}_z H_2(z) - \nu\Delta z \ni 0, \tag{1.1b}$$

$$c(\theta)\dot\theta - \mathrm{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta) = \mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u) + \theta(\alpha\mathrm{tr}(\mathrm{e}(\dot u)){+}\mathrm{D}_z H_2(z){:}\dot z) + \Psi(\dot z) + \mathbb{M}\dot z{:}\dot z, \tag{1.1c}$$

where $\nu > 0$ is a coefficient that measures some nonlocal interaction effect for the internal variable z, $\alpha \geq 0$ is the isotropic thermal expansion coefficient, $\mathbb{E}$ denotes the *elastic tensor*, H_i, $i = 1, 2$, are two *hardening functionals*, $\mathbb{L}$ and $\mathbb{M}$ are two viscosity tensors, $\mathrm{e}(u) \overset{\text{def}}{=} \frac{1}{2}(\nabla u{+}\nabla u^{\mathsf{T}})$ is the *infinitesimal strain tensor*, $c(\theta)$ is the *heat capacity*, $\kappa(\mathrm{e}(u), z, \theta)$ is the *conductivity* and Ψ denotes the dissipation potential, which is assumed to be positively homogeneous of degree 1, i.e., $\Psi(\gamma z) = \gamma\Psi(z)$ for all $\gamma \geq 0$. Finally $(\dot{\,})$ and $\partial\Psi$ denote the time derivative $\frac{\partial}{\partial t}$ and the subdifferential of Ψ in the sense of convex analysis (see [1]), respectively. Observe that (1.1a), (1.1b) and (1.1c) are usually called the momentum equilibrium equation, the flow rule and the heat-transfer equation, respectively. The problem is completed with boundary and initial conditions

$$u_{|\partial\Omega} = 0, \quad \nabla z{\cdot}\eta_{|\partial\Omega} = 0, \quad \kappa\nabla\theta{\cdot}\eta_{|\partial\Omega} = 0, \quad u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \quad \theta(\cdot,0) = \theta^0. \tag{1.2}$$

Here η denotes the outward normal to the boundary $\partial\Omega$ of Ω. Note that the problem (1.1) is thermodynamically consistent.

The original problem (1.1) can be rewritten in terms of enthalpy instead of temperature by employing the so-called enthalpy transformation $g(\theta) = \vartheta \overset{\text{def}}{=} \int_0^\theta c(s)\,\mathrm{d}s$, which is a crucial ingredient to prove the existence result. We observe that g is the unique primitive of the function c, which is supposed to be continuous, such that $g(0) = 0$. Furthermore, we assume that there exists $\beta_1 \geq 2$ such that for all $s \geq 0$, $c(s) \geq c^c(1{+}s)^{\beta_1-1} > 0$ where c^c is a positive constant. Hence we deduce that g is a bijection from $[0,\infty)$ into $[0,\infty)$. We define $\zeta(\vartheta) \overset{\text{def}}{=} g^{-1}(\vartheta)$ if $\vartheta \geq 0$ and $\zeta(\vartheta) \overset{\text{def}}{=} 0$ otherwise, g^{-1} is the inverse of g, and $\kappa^c(\mathrm{e}(u), z, \vartheta) \overset{\text{def}}{=} \frac{\kappa(\mathrm{e}(u),z,\zeta(\vartheta))}{c(\zeta(\vartheta))}$. For more details on the enthalpy transformation, the reader is referred to [5]. Therefore the system (1.1) is transformed into the following form

$$-\,\mathrm{div}(\mathbb{E}(\mathrm{e}(u){-}z){+}\alpha\zeta(\vartheta)\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot u)) = \ell, \tag{1.3a}$$

$$\partial\Psi(\dot z) + \mathbb{M}\dot z - \mathbb{E}(\mathrm{e}(u){-}z) + \mathrm{D}_z H_1(z) + \zeta(\vartheta)\mathrm{D}_z H_2(z) - \nu\Delta z \ni 0, \tag{1.3b}$$

$$\dot\vartheta - \mathrm{div}(\kappa^c(\mathrm{e}(u),z,\vartheta)\nabla\vartheta) = \mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u) + \zeta(\vartheta)(\alpha\mathrm{tr}(\mathrm{e}(\dot u)){+}\mathrm{D}_z H_2(z){:}\dot z) + \Psi(\dot z) + \mathbb{M}\dot z{:}\dot z, \tag{1.3c}$$

with boundary and initial conditions

$$u_{|\partial\Omega} = 0, \quad \nabla z{\cdot}\eta_{|\partial\Omega} = 0, \quad \kappa^c\nabla\vartheta{\cdot}\eta_{|\partial\Omega} = 0, \quad u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \quad \vartheta(\cdot,0) = \vartheta^0 = g(\theta^0). \tag{1.4}$$

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2 Global existence result

As a first step we establish a local existence result for the coupled problem (1.3)–(1.4) by using a fixed point argument. More precisely, for any given $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ with $\bar{q} > 4$ and $\bar{p} = 2$, we define $\theta \stackrel{\text{def}}{=} \zeta(\widetilde{\vartheta})$ and we solve first the system composed by the momentum equilibrium equation and the flow rule (1.1a)–(1.1b), then we solve the enthalpy equation (1.3c) with $\kappa^c \stackrel{\text{def}}{=} \kappa^c(\mathrm{e}(u), z, \zeta(\widetilde{\vartheta}))$. Since we have

$$|\theta| = |\zeta(\widetilde{\vartheta})| \leq \big(\tfrac{\beta_1}{c^c}\max(0,\widetilde{\vartheta})\big)^{\frac{1}{\beta}} \quad \text{a.e. in} \quad \Omega\times(0,T) \tag{2.1}$$

for all $\beta \in [1,\beta_1]$, we infer that $\theta = \zeta(\widetilde{\vartheta}) \in \mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ with $q = \beta_1\bar{q}$ and $p \in \big[4, \min(\beta_1\bar{p}, 6)\big]$. This allows us to define a mapping $\phi : \widetilde{\vartheta} \mapsto \vartheta$, which satisfies the assumptions of Schauder's fixed point theorem. Let us introduce now some notations; let $\mathcal{A} : \mathrm{H}^1(\Omega) \to (\mathrm{H}^1(\Omega))'$ be the linear continuous mapping defined by $\langle \mathcal{A}u, v\rangle_{(\mathrm{H}^1(\Omega))',\mathrm{H}^1(\Omega)} \stackrel{\text{def}}{=} \int_\Omega \nu\mathbb{M}^{-1}\nabla u{:}\nabla v\,\mathrm{d}x$ for all $(u,v) \in (\mathrm{H}^1(\Omega))^2$. We denote by $\mathcal{A}_r$ the realization of its generator in $\mathrm{L}^r(\Omega)$ and by $\mathrm{X}_{q,p}(\Omega) \stackrel{\text{def}}{=} (\mathrm{L}^p(\Omega), \mathcal{D}(\mathcal{A}_p))_{1-\frac{2}{q},\frac{q}{2}} \cap (\mathrm{L}^{p/2}(\Omega), \mathcal{D}(\mathcal{A}_{\frac{p}{2}}))_{1-\frac{1}{q},q}$ (for further details see [2,4]), where $\mathcal{D}(\mathcal{A}_r)$ denotes the domain of $\mathcal{A}_r$, $r = \frac{p}{2}, p$. Let $\mathrm{V}^p_0(\Omega) \stackrel{\text{def}}{=} \{u \in \mathrm{L}^p(\Omega) :\ \nabla u \in \mathrm{L}^p(\Omega),\ u_{|\partial\Omega} = 0\}$. Therefore under the assumption that $u^0 \in \mathrm{V}^p_0(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$ and $\vartheta^0 \in \mathrm{L}^2(\Omega)$ we prove that the problem (1.3)–(1.4) possesses a local solution (u,z,ϑ) defined on $[0,\tau]$ with $0 < \tau \leq T$, such that $u \in \mathrm{W}^{1,q}(0,\tau;\mathrm{V}^p_0(\Omega))$, $z \in \mathrm{L}^\infty(0,\tau;\mathrm{H}^1(\Omega)) \cap \mathrm{H}^1(0,\tau;\mathrm{L}^2(\Omega)) \cap \mathrm{C}^0([0,\tau];\mathrm{X}_{q,p}(\Omega)) \cap \mathrm{L}^q(0,\tau;\mathrm{H}^2(\Omega))$, $\dot{z}, \Delta z \in \mathrm{L}^{q/2}(0,\tau;\mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0,\tau;\mathrm{L}^{p/2}(\Omega))$ and $\vartheta \in \mathcal{W}_\tau \stackrel{\text{def}}{=} \{\vartheta \in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega)) \cap \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega)) :\ \dot{\vartheta} \in \mathrm{L}^2(0,\tau;(\mathrm{H}^1(\Omega))')\}$. Then we assume that there exists $\bar{\vartheta} > 0$ such that $\vartheta^0(x) \geq \bar{\vartheta}$ almost everywhere in Ω. By using the Stampacchia's truncation method we obtain that the enthalpy ϑ remains positive almost everywhere in $\Omega\times(0,\tau)$.

As a second step we establish some a priori estimates for the solutions of the problem (1.3)–(1.4), which are relied on an energy balance combined with Grönwall's lemma and then the global existence result follows by using a contradiction argument. Indeed on the one hand, we multiply the momentum equilibrium equation (1.1a) by $\dot{u}$ that we add to the heat equation (1.1c), then we integrate this expression over $\Omega\times(0,\widetilde{\tau})$ with $\widetilde{\tau} \in [0,\tau]$, on the other hand, using the definition of the subdifferential $\partial\Psi(\dot{z})$, we may easily deduce the following proposition:

Proposition 2.1 *Assume that $u^0 \in \mathrm{V}^p_0(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$ and $\vartheta^0 \in \mathrm{L}^2(\Omega)$ such that $\vartheta^0 \geq \bar{\vartheta}$ almost everywhere in Ω with $\bar{\vartheta} > 0$. Then, there exists a constant $\widetilde{C} > 0$, depending only on $\|u^0\|_{\mathrm{H}^1(\Omega)}$, $\|z^0\|_{\mathrm{H}^1(\Omega)}$, $\|\vartheta^0\|_{\mathrm{L}^1(\Omega)}$ and the data such that for any solution (u,z,ϑ) of problem* (1.3)–(1.4) *defined on $[0,\tau]$, we have $\|u(\cdot,\widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \|z(\cdot,\widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \|\vartheta(\cdot,\widetilde{\tau})\|_{\mathrm{L}^1(\Omega)} \leq \widetilde{C}$ for all $\widetilde{\tau} \in [0,\tau]$.*

Note that (2.1) and Proposition 2.1 imply that $\|\theta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^{\beta_1}(\Omega))} \leq \big(\frac{\beta_1}{c^c}\widetilde{C}\big)^{\frac{1}{\beta_1}}$ for any solution (u,z,θ) of problem (1.1)–(1.2) defined on $[0,\tau]$. We assume now that $\beta_1 \geq 4$ and we define $\bar{R}^\theta \stackrel{\text{def}}{=} T^{\frac{1}{q}}|\Omega|^{\frac{\beta_1-4}{4\beta_1}}\big(\frac{\beta_1}{c^c}\widetilde{C}\big)^{\frac{1}{\beta_1}}$. Let $R^0 > 0$ such that $\max\big(\|u^0\|_{\mathrm{V}^p(\Omega)}, \|z^0\|_{\mathrm{X}_{q,p}(\Omega)}\big) \leq R^0$. Moreover we may prove that there exists $\bar{R}$ depending only on $\|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}$, R^0 and $\bar{R}^\theta$ such that $\|\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \theta(\alpha\mathrm{tr}(\mathrm{e}(\dot{u})) + \mathrm{D}_z H_2(z){:}\dot{z}) + \Psi(\dot{z}) + \mathbb{M}\dot{z}{:}\dot{z}\|_{\mathrm{L}^{q/4}(0,\tau;\mathrm{L}^2(\Omega))} \leq \bar{R}$ for any solution (u,z,θ) of problem (1.1)–(1.2) defined on $[0,\tau]$. It follows that there exists a generic constant $C > 0$ such that $\|\vartheta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega))} \leq \bar{R}^\vartheta_\infty \stackrel{\text{def}}{=} C\big(T^{\frac{q-8}{2q}}\bar{R} + \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big)$. We define $\bar{R}^\vartheta \stackrel{\text{def}}{=} T^{\frac{1}{\bar{q}}}\bar{R}^\vartheta_\infty + 1$. Then we may infer by a contradiction argument that the mapping ϕ admits a fixed point in the closed ball $\bar{B}_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^2(\Omega))}(0,\bar{R}^\vartheta)$. For technical details, the reader is referred to [3].

Theorem 2.2 *Let $\beta_1 \geq 4$. Under the assumptions of Proposition 2.1, the problem* (1.3)–(1.4) *possesses a global solution (u,z,ϑ) such that $u \in \mathrm{W}^{1,q}(0,T;\mathrm{V}^p_0(\Omega))$, $z \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)\cap\mathrm{X}_{q,p}(\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega))$, $\dot{z}, \Delta z \in \mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega))$ and $\vartheta \in \mathcal{W}_T$. Moreover ϑ remains strictly positive and $(u,z,\theta=\zeta(\vartheta))$ is a solution of problem* (1.1)–(1.2) *on $[0,T]$.*

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