On the viscoelastodynamic problem with Signorini boundary conditions

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This note focuses on a viscoelastodynamic problem being subject to unilateral boundary conditions. Under appropriate regularity assumptions on the initial data, the problem can be reduced to the pseudodifferential linear complementarity problem through Fourier analysis. We prove that this problem possesses a solution, which, is obtained as the limit of a sequence of solutions of penalized problems and we establish that the energy losses are purely viscous.

1 Mathematical formulation

The present work is dedicated to the study of a viscoelastodynamic problem with unilateral boundary condition in the particular case of Kelvin-Voigt material. We assume that the material is homogeneous isotropic and we seek a solution depending only on the coordinate x in a half-space $x \leq 0$. Note that the only interesting equation is that for the first component of the displacement. After adimensionalization, we are interested in the following problem

$$u_{tt} - u_{xx} - \alpha u_{xxt} = \ell, \quad x < 0, \quad t > 0, \quad \alpha > 0, \tag{1.1}$$

with Cauchy initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.2}$$

and Signorini boundary conditions

$$0 \leq u(0,\cdot) \perp (u_x + \alpha u_{xt})(0,\cdot) \geq 0, \tag{1.3}$$

where $(\cdot)_t \stackrel{\text{def}}{=} \frac{\partial}{\partial t}(\cdot)$ and $(\cdot)_x \stackrel{\text{def}}{=} \frac{\partial}{\partial x}(\cdot)$. The orthogonality has the natural meaning: if we have enough regularity, it means that the product $u(u_x + \alpha u_{xt})$ vanishes almost everywhere on the boundary. If we do not have enough regularity, the above inequality is integrated on an appropriate set of test functions, yielding a weak formulation for the unilateral condition.

We suppose that the initial position $u_0(0)$ vanishes; otherwise, we solve the problem with vanishing $\big(u_x + \alpha u_{xt}\big)(0,\cdot)$ until $u(0,t)$ vanishes and we change the origin of times. Let η be the inverse Fourier transform of the causal determination of $i\omega\sqrt{1+\alpha i\omega}$, $\bar{w} \stackrel{\text{def}}{=} -(\bar{u}_x + \alpha \bar{u}_{xt})(0,\cdot)$ with $\bar{u}$ denoting the solution of (1.1) and (1.2) with Dirichlet data and $v \stackrel{\text{def}}{=} u(0,\cdot)$. We assume that ℓ belongs to $\mathrm{H}^1_{\text{loc}}([0,\infty); \mathrm{L}^2(-\infty,0))$, u_0 and u_1 belong to $\mathrm{H}^2(-\infty,0) \cap \mathrm{H}^1_0(-\infty,0)$. Thus we may deduce that $\bar{w}$ belongs to $\mathrm{H}^{1/2}_{\text{loc}}([0,\infty))$. The problem (1.1)–(1.3) can be reduced to pseudodifferential linear complementarity problem:

$$\eta * v = \bar{w} + m, \quad \bar{w} \geq 0, \quad m \geq 0, \quad \langle \bar{w}, m \rangle = 0. \tag{1.4}$$

Note that the principal term of η is a constant times the derivative of order $3/2$ of the Dirac mass. The pseudodifferential character of the convolution with η makes everything difficult; the viscous term does not help as we thought at the beginning of this study and the things are not simpler than for elastodynamics, where the Signorini problem is still open problem. Nevertheless, we prove the existence of solution to (1.4) by the penalty method; the rigid constraint is approximated by a very stiff response, which, vanishes when the constraint is not active and it is linear otherwise. More precisely, we approximate the pseudodifferential linear complementarity problem (1.4) by the following penalty problem:

$$\eta * v^\varepsilon = \bar{w} + (v^\varepsilon)^-/\varepsilon, \tag{1.5}$$

where $(v^\varepsilon)^- \stackrel{\text{def}}{=} -\min(v^\varepsilon, 0)$ and v^ε vanishes for all $t < 0$. The existence of the penalty problem (1.5) is obtained by using the Picard iterations, the reader is referred to [5] for the detailed proof. On the other hand, formally we multiply (1.5) by v^ε_t and we estimate the pseudodifferential term in the Fourier variables, which, allows to deduce that $v \in \mathrm{H}^{5/4}_{\text{loc}}([0,\infty))$. The essential idea consists to use the causality, which, enables us to modify $\bar{w}$ for large times, to validate the desired result on a time interval for which $\bar{w}$ has not been modified, and then to conclude for $\mathbb{R}^+$, since the modification time has been arbitrarily chosen.

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Theorem 1.1 *Assume that $\bar{w}$ belongs to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}) \cap \mathrm{H}^{-1/4}_{\mathrm{loc}}(\mathbb{R})$ and vanishes for $t < 0$. Then there exists a solution $v \in \mathrm{H}^{5/4}_{\mathrm{loc}}([0,\infty))$ of* (1.4).

Let us observe that nothing is known about the uniqueness. On the other hand, we note that the existence results for viscoelastodynamics with contact and with a given friction at the boundary are proved in [1, 2] and the functional properties of all the traces are precisely identified for viscoelastodynamics with contact at the boundary through Fourier analysis in [3].

2 Energy balance

The energy balance is obtained by multiplying (1.1) by u_t and integrating over $(-\infty,0)\times(0,\tau)$, $\tau>0$, we get

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{0}\big(|u_t(\cdot,\tau)|^2+|u_x(\cdot,\tau)|^2\big)\,\mathrm{d}x+\alpha\int_{-\infty}^{0}\int_{0}^{\tau}|u_{xt}|^2\,\mathrm{d}x\,\mathrm{d}t=\frac{1}{2}\int_{-\infty}^{0}|u_t(\cdot,0)|^2\,\mathrm{d}x\\ +\frac{1}{2}\int_{-\infty}^{0}|u_x(\cdot,0)|^2\,\mathrm{d}x+\int_{0}^{\tau}\big((u_x+\alpha u_{xt})u_t\big)(0,\cdot)\,\mathrm{d}t+\int_{0}^{\tau}\int_{-\infty}^{0}\ell u_t\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.6}$$

The losses are purely viscous iff $\langle v_t, m\rangle$ vanishes. This statement looks trivial since v_t vanishes almost everywhere on the support of m. However m is only a measure, and we do not know that its singular part vanishes. Thus this is not enough to conclude that $\langle v_t, m\rangle = 0$ holds. Note that the non local character of the convolution by η prevents any argument based on the signs functions to conclude. Nevertheless it is possible to prove that v_t vanishes on the set $\{u{=}0\}$, except on a countable subset, and that m has no atoms. But these results cannot be deduced from functional estimates on the penalized approximation, since they only imply that $v \in \mathrm{H}^{5/4}_{\mathrm{loc}}([0,\infty))$. Therefore we develop another construction based on the assumption that $\bar{w}$ is a half-integral of a measure μ, namely $\bar{w}(t) \overset{\text{def}}{=} \int_{[0,t[}\big(\pi(t-s)\big)^{-1/2}\mu(s)$. Let v be a solution of (1.4) having a locally finite structure; $\operatorname{supp} v \subset \bigcup_{j\in J}[\sigma_j,\tau_j]$ and $\operatorname{supp} m \subset \bigcup_{j\in J}[\tau_j,\sigma_{j+1}]$ where J is a finite or infinite interval of $\mathbb{N}$ starting at 0 and satisfying $0 \le \sigma_0 < \tau_0 < \sigma_1 < \tau_1 \ldots$ such that the above support conditions hold. Let H be the Heaviside function and let $\omega(t) \overset{\text{def}}{=} H(t)/\big(\pi(t{+}1)\sqrt{t}\big)$, $\nu(t) \overset{\text{def}}{=} H(t)\mathrm{e}^{-t/\alpha}/\alpha$ and $\Omega(t) \overset{\text{def}}{=} \int_0^t \omega(s)\,\mathrm{d}s$. Furthermore, we define $\varphi_0 \overset{\text{def}}{=} \varphi$, $\psi_j \overset{\text{def}}{=} \varphi_j 1_{[\tau_j,\sigma_{j+1})} + \delta(\cdot-\tau_j)\mathrm{e}^{-\tau_j/\alpha}\int_{[\sigma_j,\tau_j)}\mathrm{e}^{s/\alpha}\varphi_j$ and $\varphi_{j+1} \overset{\text{def}}{=} 1_{[\sigma_{j+1},\infty)}\varphi_j + 1_{[\sigma_{j+1},\infty)}\int_{[\tau_j,\sigma_{j+1})}\mathrm{e}^{-(\cdot-s)/\alpha}\omega\big(\frac{\cdot-\sigma_{j+1}}{\sigma_{j+1}-s}\big)\frac{\psi_j(s)}{\sigma_{j+1}-s}$ for all $j \in J\backslash(\max J)$. Then an explicit calculation gives that $v1_{[\sigma_j,\tau_j]} = \big(H{*}\nu{*}\varphi_j\big)1_{[\sigma_j,\tau_j]}$, which, implies that

$$\big(v_t(\sigma{-}0)\big)^- \le \frac{1}{\alpha}\int_{[\sigma_j,\sigma)}|\varphi_0(s)| + \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)}\big(2\mathrm{e}^{(\sigma-\sigma_j)/\alpha}{-}1\big)\big(\Omega\big(\tfrac{\sigma-\sigma_i}{\sigma_i-s}\big){-}\Omega\big(\tfrac{\sigma_j-\sigma_i}{\sigma_i-s}\big)\big)\psi^+_{i-1}(s), \tag{2.7}$$

where $\psi^+_{i-1} \overset{\text{def}}{=} \max(\psi_{i-1},0)$. We do not know a priori whether for a given $\bar{w} = \int_{[0,t[}\big(\pi(t{-}s)\big)^{-1/2}\mu(s)$, there is a solution of (1.5) having the locally finite structure introduced above. Note that probably there is no such solution. Nevertheless, we choose $n \gg 1$ and we can construct φ^n and a solution v^n which has the structure described above and which approximates a solution of (1.5). The construction is recursive, for more details the reader is referred to [4, 5]. The estimates on the approximation give that v^n_t is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}([0,\infty))$ and m^n is locally integrable on $\mathbb{R}$, moreover v^n converges uniformly on compact sets to its limit v which verifies (1.4). These estimates are much better than these obtained via the penalization. In particular, since v_t is essentially bounded, we may deduce that the measure m has no atoms. On the other hand, using (2.7), we may show the following theorem.

Theorem 2.1 *Let N_a be the set of atoms of φ, N_c be a countable set composed of all the points σ_κ and τ_κ and $N_p \overset{\text{def}}{=} \{t \in \mathbb{R} : v(t) > 0\}$ be an open set. Then for all $t \notin N_a \cup N_c \cup N_p$, v is differentiable at t and its derivative vanishes.*

Since v_t vanishes on the set N_p, except on a countable subset, and m has no atoms, it follows that $\langle v_t, m\rangle$ vanishes, which, implies that the energy losses are purely viscous.

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