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Change of the type of contrast structures in parabolic Neumann problems

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Abstract

We consider some class of singularly perturbed nonlinear parabolic problems in the case when a solution with an interior layer changes into a solution having only boundary layers. Analytical results on this phenomenon are compared with numerical studies of some examples.

1 Introduction

We consider the scalar singularly perturbed parabolic differential equation

$$\varepsilon^2 \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} \right) = f(u, x, t), \quad x \in (0,1), \quad t > 0, \tag{1.1}$$

where ε is a small parameter, with the initial condition

$$u(x, 0, \varepsilon) = u^0(x, \varepsilon), \quad 0 \leq x \leq 1, \tag{1.2}$$

and the boundary conditions of Neumann's type

$$\frac{\partial u}{dx}(0, t, \varepsilon) = \frac{\partial u}{\partial x}(1, t, \varepsilon) = 0, \quad t > 0. \tag{1.3}$$

It is well known that the boundary value problem (1.1), (1.3) in general has solutions exhibiting for small ε boundary layers (i.e., there are small regions near the boundaries $x = 0$ and $x = 1$, where the solutions rapidly change) and/or interior layers (i.e., there are small regions in the interval $0 < x < 1$, where the solutions rapidly change). We call solutions of (1.1), (1.3), which have only boundary layers as pure boundary layer solutions, solutions possessing an interior layer are said to be contrast structures (see, e.g. [1]).
The case that the type of the solution changes with increasing time has been considered for the first time in [2]. Such solutions has been called alternating contrast structures. The paper [2] contains numerical investigations of alternating contrast structures and some analytical interpretations. The first rigorous results about alternating contrast structures can be found in [3].
The focus of this paper is, after some numerical motivation, on the analytical investigation of the initial-boundary value problem (1.1)–(1.3) with periodic right hand side in the case that a solution with an interior layer exists which moves to the boundary $x = 1$ or $x = 0$ and changes its type to a pure boundary layer solution when the interior layer arrives at the boundary.

2 Formulation of the problem

We consider the initial-boundary value problem (1.1)–(1.3) under the following assumptions:

(A_0) $f : R \times R \times R \to R$ and $u^0 : [0,1] \times [0, \varepsilon_0] \to R$ are sufficiently smooth, where ε_0 is some small positive number.

(A_1) There is a positive number T such that

$$f(u,x,t) = f(u,x,t+T), \quad (u,x,t) \in R \times R \times R.$$

(A_2) The degenerate equation

$$f(u,x,t) = 0$$

has in $R \times R \times R$ exactly three roots $u = \varphi_i(x,t),\ i = 0,1,2$, satisfying

(i) $\varphi_i : R \times R \to R$ is sufficiently smooth and T-periodic in t for $i = 0,1,2$.

(ii) $\varphi_1(x,t) < \varphi_0(x,t) < \varphi_2(x,t) \quad (x,t) \in R \times R$.

(iii) $\frac{\partial f}{\partial u}(\varphi_i(x,t),x,t) > 0 \quad i = 1,2,$ and $(x,t) \in R \times R$.

Now we introduce the T-periodic function $x_0(t)$ as an isolated solution of the equation

$$I(x,t) = 0,$$

where

$$I(x,t) := \int_{\varphi_1(x,t)}^{\varphi_2(x,t)} f(u,x,t)\,du, \tag{2.1}$$

and

$$\frac{\partial I}{\partial x}(x_0(t),t) \neq 0 \quad \forall t \in [0,T].$$

The function $x_0(t)$ plays an important role in studying contrast structures as solutions of the initial-boundary value problem (1.1)–(1.3). As a contrast structure of step-type we denote a solution $u(x,t,\varepsilon)$ of (1.1)–(1.3), which stays for small ε near $\varphi_1(x,t)$ ($\varphi_2(x,t)$) for $x < x_0(t)$ and near $\varphi_2(x,t)$ ($\varphi_1(x,t)$) for $x > x_0(t)$. Thus, the solution $u(x,t,\varepsilon)$ changes rapidly near $x_0(t)$ from $\varphi_1(x,t)$ to $\varphi_2(x,t)$ (from $\varphi_2(x,t)$ to $\varphi_1(x,t)$), that is, there is an interior layer in a neighborhood of $x_0(t)$, and the point $x_0(t)$ itself is called a transition point. It is known [4] that if $x_0(t)$ satisfies

$$0 < x_0(t) < 1 \quad \forall t \in [0,T], \tag{2.2}$$

then under the assumptions $(A_1) - (A_2)$ there is an asymptotically stable periodic contrast structure of the boundary value problem (1.1), (1.3), where the transition point $x_0(t)$ moves periodically in the interval $(0,1)$. Hence, if we assume that the

initial function $u^0(x,\varepsilon)$ has a step form whose transition point $\tilde{x}_0$ is sufficiently close to $x_0(0)$, then the solution of the initial-boundary value problem (1.1)–(1.3) tends to a periodic contrast structure.
In case that the inequality (2.2) is not fulfilled, that is, $x_0(t)$ passes either the boundary $x = 1$ or $x = 0$, then the results obtained under the condition (2.2) in general will fail. Some numerical investigations and formal analytical treatment described e.g. in [2] suggest that if the condition (2.2) is violated, then new processes with different time-scales arise during some transition period and a change of the type of contrast structures.

In this paper we will investigate the initial-boundary value problem (1.1)–(1.3) under the assumptions (A_0)–(A_2), but in contrast to (2.2) we suppose that $x_0(t)$ crosses at some moment the boundary. Especially, we will study the transition of a solution $u(t,x,\varepsilon)$ of (1.1)–(1.3) from a contrast structure of step type to a solution, which has only boundary layers. In the next section we present some numerical studies to our initial-boundary value problem (1.1)–(1.3) which show this change and the occurence of transition processes with different time-scales.

3 Numerical studies

For numerical investigations we consider the following right hand side of (1.1)

$$\begin{aligned} f(u,x,t) &:= [u-\varphi_1(x)]\,[u-\varphi_2(x)]\,[u-\varphi_0(x,t)], && (3.1)\\ \varphi_0(x,t) &:= -0.5x + 0.63\,sin(t+\alpha) + 0.25, && (3.2)\\ \varphi_{1,2}(x) &:= \mp[(x-0.5)^2 + 0.75]. && (3.3) \end{aligned}$$

The constant α will be specified in the following subcases, furthermore we set $\varepsilon^2 = 10^{-3}$.

In each of the following figures the solution $u(x,t,\varepsilon)$ is represented for fixed t by a boldface curve. The graph of the function $\varphi_0(x,t)$ for fixed t is depictured by a thick dashed line, the graph of the functions φ_1 and φ_2 are represented by thin dashed lines.
We note that when the right-hand side of equation (1.1) is given by (3.1), then we have

$$I(x,t) \equiv \frac{4}{3}[\varphi_1(x)]^2\varphi_0(x,t)$$

such that $I(x,t) = 0$ is equivalent to the equation $\varphi_0(x,t) = 0$. Thus, to given t, the point $x_0(t)$ represents in all pictures the intersection point of the curve $u = \varphi_0(x,t)$ (thick dashed line) with the x-axis.

In Fig. 1, where we have $\alpha = 0$, the initial function $u^0(x) = -sin(2\pi x)$ intersects the x-axis in the interval $(0,1)$ in the point $\tilde{x}_0 = x_0(0) = 0.5$. After a very short time interval, the solution takes the form of a step. This process will not be studied

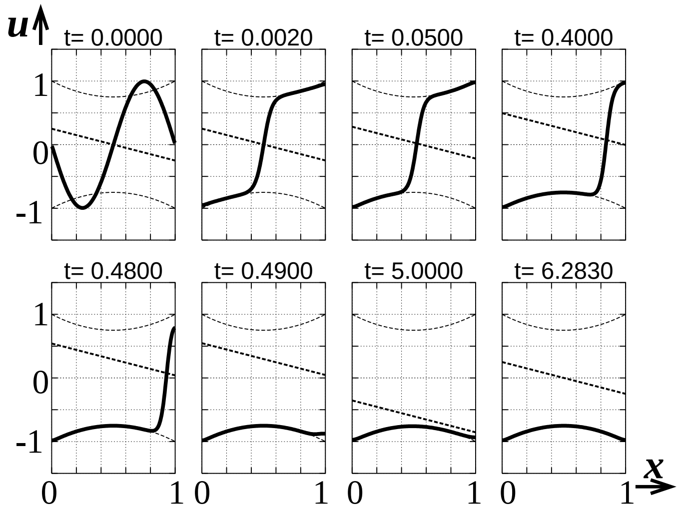

Figure 1: Slow passage.

in that paper. Figure 1 shows that this step moves to the right together with the transition point $x_0(t)$. That means that the velocity of $x_0(t)$ defines the velocity of the step moving to the right. By numerical evidence we denote this process as slow transition. After some time, $x_0(t)$ passes the boundary $x = 1$, and the contrast structure of step type changes into a solution which is located in $[0, 1]$ near $\varphi_1(x)$ and has only boundary layers.

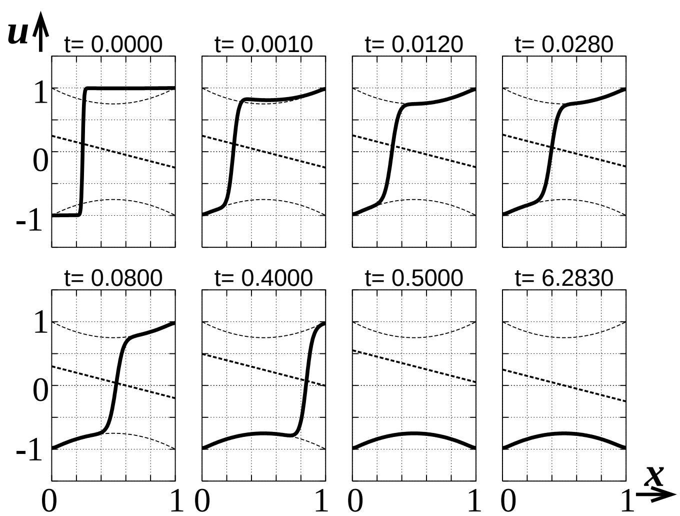

Figure 2: Fast-slow passage.

In Fig. 2, where we also have $\alpha = 0$, the initial function $u^0(x, \varepsilon)$ has the form of a step intersecting the x-axis in the point $\tilde{x}_0 = 0.25$, where the difference $x_0(0) - \tilde{x}_0 = 0.5 - \tilde{x}_0 = 0.25$ is not small. In that case, according to the numerical results, we can distinguish a short time interval, where the solution $u(x, t, \varepsilon)$ moves to the right with a high velocity, which is much larger than the velocity of $x_0(t)$, until its transition point reaches $x_0(t)$ (approximately at $t = 0.08$). Then its behavior can be described as in Fig. 1, that is, it approaches slowly the root φ_1. Thus, we call this case as fast-slow transition.

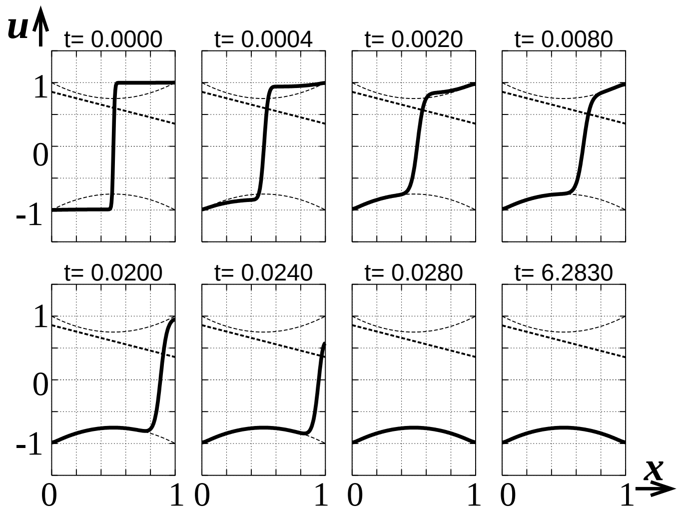

Figure 3: Fast passage.

Fig. 3, where we have $\alpha = 5$, is characterized by the fact that the point $x_0(0)$ is located outside the interval [0,1] in the region $x > 1$. The initial function has the form of a step and intersects the x-axis in the point $\tilde{x}_0 = 0.5$. From Fig. 3 it follows that the step in the solution $u(x, t, \varepsilon)$ moves fast to the right in the direction of $x_0(t)$. When the step has arrived the boundary $x = 1$, then the solution changes to a pure boundary layer solution located near $\varphi_1(x)$. This case is called fast transition.

4 Preliminaries

The proof of our results presented in the sections 5 and 6 is based on the technique of lower and upper solutions. For the convenience of the reader we recall the definition of these functions and the corresponding basic result. Let

$$D := \{(x, t) : 0 < x < 1, 0 < t \leq T\}, \qquad I_{\varepsilon_0} := \{0 < \varepsilon \leq \varepsilon_0\}.$$

Definition 4.1 *Let $\alpha(x, t, \varepsilon)$ and $\beta(x, t, \varepsilon)$ be functions continuously mapping $\overline{D} \times \overline{I}_{\varepsilon_0}$ into R, twice continuously differentiable in x and continuously differentiable in t for $(x, t) \in D$. Then $\alpha(x, t, \varepsilon)$ and $\beta(x, t, \varepsilon)$ are called ordered lower and upper solutions of* (1.1)–(1.3) *for $\varepsilon \in I_{\varepsilon_0}$, if they satisfy for $\varepsilon \in I_{\varepsilon_0}$ the following conditions:*

$$1^\circ \quad \alpha(x, t, \varepsilon) \leq \beta(x, t, \varepsilon), \ (x, t) \in \overline{D}, \tag{4.1}$$

$$2^\circ \quad \varepsilon^2\Big(\frac{\partial^2\alpha}{\partial x^2}-\frac{\partial\alpha}{\partial t}\Big)-f(\alpha,x,t,\varepsilon)\geq 0\geq \varepsilon^2\Big(\frac{\partial^2\beta}{\partial x^2}-\frac{\partial\beta}{\partial t}\Big)-f(\beta,x,t,\varepsilon),\ (x,t)\in D, \quad (4.2)$$

$$3^\circ \quad \frac{\partial\alpha}{\partial x}(0,t,\varepsilon)\geq 0\geq \frac{\partial\beta}{\partial x}(0,t,\varepsilon),\ \frac{\partial\alpha}{\partial x}(1,t,\varepsilon)\leq 0\leq \frac{\partial\beta}{\partial x}(1,t,\varepsilon),\ t\in[0,T], \quad (4.3)$$

$$4^\circ \quad \alpha(x,0,\varepsilon)\leq u^0(x,\varepsilon)\leq \beta(x,0,\varepsilon),\ x\in[0,1]. \quad (4.4)$$

It is known (see, e.g., [8]) that the existence of ordered lower and upper solutions implies the existence of a unique solution $u(x,t,\varepsilon)$ of (1.1)–(1.3) satisfying

$$\alpha(x,t,\varepsilon)\leq u(x,t,\varepsilon)\leq \beta(x,t,\varepsilon).$$

Remark 4.2 *In case that α and β are only picewise twice continuously differentiable with respect to x, we have to ensure that the first derivatives of α and β with respect to x satisfy the following inequalities for $0<\bar{x}(t)<1$*

$$\frac{\partial\alpha}{\partial x}(\bar{x}(t)+0,t,\varepsilon)\geq \frac{\partial\alpha}{\partial x}(\bar{x}(t)-0,t,\varepsilon) \quad (4.5)$$

$$\frac{\partial\beta}{\partial x}(\bar{x}(t)+0,t,\varepsilon)\leq \frac{\partial\beta}{\partial x}(\bar{x}(t)-0,t,\varepsilon) \quad (4.6)$$

Under the assumptions (A_0)–(A_2), the existence of two different asymptotically stable periodic solutions to the boundary value problem (1.1), (1.3) has been proven in [4], where one of these solutions tends to the root $\varphi_1(x,t)$ as ε tends to zero, the other one tends to $\varphi_2(x,t)$. This result has been established by means of the technique of differential inequalities. The following version can be obtained by applying simplified lower and upper solutions than used in [4].

Proposition 4.3 *Suppose the hypotheses (A_0)–(A_2) to be valid. Then there exists a sufficiently small positive number ε_0 such that for $0<\varepsilon\leq\varepsilon_0$ the boundary value problem (1.1), (1.3) has at least two T-periodic solutions $u_i(x,t,\varepsilon)$, $i=1,2$, which are asymptotically stable and satisfy for any fixed $(x,t)\in[0,1]\times R$*

$$\lim_{\varepsilon\to 0} u_i(x,t,\varepsilon)=\varphi_i(x,t),\ \ i=1,2.$$

For $i=1,2$, the region $S^\gamma_{i,\varepsilon}$ defined by

$$S^\gamma_{i,\varepsilon}:=\{(u,x,t):\varphi_i(x,t)-\gamma\varepsilon<u<\varphi_i(x,t)+\gamma\varepsilon,\ 0\leq x\leq 1,\ 0\leq t\leq T\},$$

where γ is some positive constant, belongs to the basin of attraction of $u_i(x,t,\varepsilon)$.

The periodic solutions established in Proposition 4.3 do not have any interior layer for sufficiently small ε though they will have boundary layers (the functions $\varphi_i(x,t)$ will in general not satisfy the boundary conditions (1.3)).
To be able to formulate conditions ensuring the existence of a periodic solution to the boundary value problem (1.1), (1.3) with an interior layer we use the function $I(x,t)$ introduced in (2.1).

(A_3^*) The equation

$$I(x,t)=0$$

has in $[0,1]\times R$ a unique smooth T-periodic solution $x=x_0(t)$ satisfying $\forall t\in[0,T]$

(i) $0<x_0(t)<1$,

(ii) $\frac{\partial I(x_0(t),t)}{\partial x}<0$.

Under the hypotheses of Proposition 4.3 and the additional assumption (A_3^*), the existence of a T-periodic solution to the boundary value problem (1.1), (1.3) with an interior layer has been proven in [4]. The following result is valid.

Proposition 4.4 *Under the assumptions* (A_0)–(A_2), (A_3^*) *there exists a sufficiently small positive number* ε_0 *such that for* $0<\varepsilon\leq\varepsilon_0$ *the boundary value problem* (1.1), (1.3) *has at least three periodic solutions. Two of these solutions are described in Proposition* 4.3, *the third solution* $u_3(x,t,\varepsilon)$ *represents a periodic contrast structure which is asymptotically stable and satisfies for fixed* $(x,t)\in[0,1]\times[0,T]$

$$\lim_{\varepsilon\to 0}u_3(x,t,\varepsilon)=\begin{cases}\varphi_1(x,t) & \textit{for}\quad 0\leq x<x_0(t),\\ \varphi_2(x,t) & \textit{for}\quad x_0(t)<x\leq 1,\end{cases}$$

i.e., it has an interior layer near $x_0(t)$.

The proof of this theorem is also based on the technique of differential inequalities.

Remark In the case that the condition (ii) in assumption (A_3^*) will be replaced by the inequality $\frac{\partial I(x_0(t),t)}{\partial x}>0$, then we can prove analogously the existence of a periodic contrast structure $u_4(x,t,\varepsilon)$ satisfying

$$\lim_{\varepsilon\to 0}u_4(x,t,\varepsilon)=\begin{cases}\varphi_2(x,t) & \text{for}\quad 0\leq x<x_0(t),\\ \varphi_1(x,t) & \text{for}\quad x_0(t)<x\leq 1,\end{cases}$$

In the next section we will consider the initial-boundary value problem (1.1)–(1.3), where we assume that $x_0(t)$ will cross the boundary $x=1$.

5 Slow transition from a step type contrast structure to a pure boundary layer solution

In the sequel we consider the initial-boundary value problem (1.1)–(1.3) under the additional assumption

(A$_3$) The equation

$$I(x,t)=0$$

has in $R\times R$ a unique smooth T-periodic solution $x=x_0(t)$ satisfying

(i) $\frac{\partial I(x_0(t),t)}{\partial x}<0$.

(ii) $0<x_0(0)<1$.

(iii) There is a number $t_1>0$ such that $\frac{dx_0(t)}{dt}\geq 0$ for $t\in[0,t_1]$, and $x_0(t_1)>1$.

Our goal is to prove that the initial-boundary value problem (1.1)–(1.3) has a unique solution $u(x,t,\varepsilon)$ tending to some asymptotically stable periodic solution as $t\to\infty$ if the assumptions (A$_0$)–(A$_3$) hold and if the initial function $u^0(x,\varepsilon)$ has some internal transition layer.

The proof is based on the method of ordered lower and upper solutions. For this purpose, we construct ordered lower and upper solutions $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ respectively for the boundary value problem (1.1), (1.3), which have an internal layer of step-type near $x_0(t)$ as long as $x_0(t)$ belongs to the interval $0<x<1$. The condition on the initial function will be formulated by means of the constructed lower and upper solutions.
According to assumption (A_3) there are a small positive number δ_0 and a number t_0 such that $0<x_0(0)\leq x_0(t)\leq 1-\delta_0$ for $0\leq t\leq t_0$. Now we construct lower and upper solutions to (1.1)–(1.3) for $0\leq t\leq t_0$ as in [4]. The lower and upper solutions are functions connecting the roots $\varphi_1(x,t)$ and $\varphi_2(x,t)$ by means of a transition layer whose position at the moment t is defined by the function $x_0(t)$. Hence, we have the following structure of α and β (see [4]).

$$\begin{aligned}\alpha(x,t,\varepsilon) &= u_{0\alpha}(x,t)-\gamma\varepsilon+Q_{0\alpha}(\xi_\alpha,t)+\varepsilon Q_{1\alpha}(\xi_\alpha,t)-\varepsilon(e^{-\kappa\xi_0}+e^{-\kappa\xi_1}), && (5.1)\\ \beta(x,t,\varepsilon) &= u_{0\beta}(x,t)+\gamma\varepsilon+Q_{0\beta}(\xi_\beta,t)+\varepsilon Q_{1\beta}(\xi_\beta,t)+\varepsilon(e^{-\kappa\xi_0}+e^{-\kappa\xi_1}). && (5.2)\end{aligned}$$

Here, the functions $u_{0\alpha}$ and $u_{0\beta}$ are related to the roots φ_1 and φ_2 as follows

$$u_{0\alpha}:=\begin{cases}\varphi_1(x,t) & \text{for}\quad 0\leq x<x_\alpha(t),\\ \varphi_2(x,t) & \text{for}\quad x_\alpha(t)\leq x\leq 1,\end{cases}$$
$$u_{0\beta}:=\begin{cases}\varphi_1(x,t) & \text{for}\quad 0\leq x<x_\beta(t),\\ \varphi_2(x,t) & \text{for}\quad x_\beta(t)\leq x\leq 1,\end{cases}$$

where the functions x_α and x_β are defined by

$$x_\alpha(t):=x_0(t)+\delta,\ \ x_\beta(t):=x_0(t)-\delta,$$

here δ is a sufficiently small number independent of ε and satisfying $0<\delta<\delta_0$.

The functions $Q_{i\alpha}$ and $Q_{i\beta}, i=1,2$, are interior layer functions characterizing the transition from φ_1 to φ_2. To their definition we have to consider the following boundary value problem for ordinary differential equations depending on the parameter t. $Q_{0\beta}^-(\xi_\beta,t)$, where ξ_β is defined by $\xi_\beta=(x-x_\beta(t))/\varepsilon$, is the solution of the problem

$$
\begin{aligned}
\frac{d^2 Q_{0\beta}^-}{d\xi_\beta^2} &= f(\varphi_1(x_\beta(t),t) + Q_{0\beta}^-, x_\beta(t), t) \quad \text{for} \quad \xi_\beta < 0, \\
Q_{0\beta}^-(0,t) &= \varphi_0(x_\beta(t),t) - \varphi_1(x_\beta(t),t), \quad Q_{0\beta}^-(-\infty,t) = 0.
\end{aligned} \tag{5.3}
$$

Under the assumptions (A_0)–(A_3) this problem has a solution $Q_{0\beta}^-(\xi_\beta,t)$ defined for $\xi_\beta \le 0$ and $t \ge 0$ which exponentially decays for decreasing ξ_β (see, e.g., [6]). The function $Q_{0\beta}^+$ is the solution of the problem

$$
\begin{aligned}
\frac{d^2 Q_{0\beta}^+}{d\xi_\beta^2} &= f(\varphi_2(x_\beta(t),t) + Q_{0\beta}^+, x_\beta(t), t), \quad \text{for} \quad \xi_\beta > 0, \\
Q_{0\beta}^+(0,t) &= \varphi_0(x_\beta(t),t) - \varphi_2(x_\beta(t),t), \quad Q_{0\beta}^+(\infty,t) = 0.
\end{aligned} \tag{5.4}
$$

The functions $Q_{1\beta}^-(\xi_\beta,t)$ and $Q_{1\beta}^+(\xi_\beta,t)$ are defined as solutions of the boundary value problems

$$
\begin{aligned}
\frac{d^2 Q_{1\beta}^-}{d\xi_\beta^2} &= f_u(\varphi_1(x_\beta(t),t) + Q_{0\beta}^-, x_\beta(t), t) Q_{1\beta}^- + h_1^-(\xi_\beta) \quad \text{for} \quad \xi_\beta < 0, \\
Q_{1\beta}^-(0,t) &= -\gamma, \quad Q_{1\beta}^-(-\infty,t) = 0,
\end{aligned} \tag{5.5}
$$

and

$$
\begin{aligned}
\frac{d^2 Q_{1\beta}^+}{d\xi_\beta^2} &= f_u(\varphi_2(x_\beta(t),t) + Q_{0\beta}^+, x_\beta(t), t) Q_{1\beta}^+ + h_1^+(\xi_\beta), \quad \text{for} \quad \xi_\beta > 0, \\
Q_{1\beta}^+(0,t) &= -\gamma, \quad Q_{1\beta}^+(\infty,\tau) = 0,
\end{aligned} \tag{5.6}
$$

where

$$
\begin{aligned}
h_1^-(\xi_\beta) \;:=\; & \Big[f_u(\varphi_1(x_\beta(t),t) + Q_{0\beta}^-(\xi_\beta,t), x_\beta(t), t) \frac{\partial \varphi_1}{\partial x}(x_\beta(t),t) \\
& + f_x(\varphi_1(x_\beta(t),t) + Q_{0\beta}^-(\xi_\beta,t), x_\beta(t), t) \Big] \xi_\beta + \\
& + \gamma \Big[f_u(\varphi_1(x_\beta(t),t) + Q_{0\beta}^-(\xi_\beta,t), x_\beta(t), t) - f_u(\varphi_1(x_\beta(t),t), x_\beta(t), t) \Big],
\end{aligned}
$$

and $h_1^+(\xi_\beta)$ is defined similarly. It is known that these problems have solutions which exponentially decay.

Using (5.3)–(5.6), the functions $Q_{i,\beta}(\xi_\beta,t), i = 1,2$, are defined by

$$
Q_{i\beta}(\xi_\beta,t) := \begin{cases} Q_{i\beta}^-(\xi_\beta,t) & \text{for} \quad \xi_\beta < 0, \\ Q_{i\beta}^+(\xi_\beta,t) & \text{for} \quad \xi_\beta > 0. \end{cases}
$$

Correspondingly, we can define the functions $Q_{i\alpha}(\xi_\alpha,t)$ for $i = 0,1$, where $\xi_\alpha = (x - x_\alpha(t))/\varepsilon$. The terms $e^{-\kappa\xi_0}$ and $e^{-\kappa\xi_1}$ in (5.1) and (5.2), where $\xi_0 = \frac{x}{\varepsilon}$ and

$\xi_1 = \frac{1-x}{\varepsilon}$ are needed to fulfil the inequalities in (4.3) near the boundaries $x = 0$ and $x = 1$, κ and γ are appropriate positive constants.
From the paper [4] it follows that under our assumptions the functions α and β introduced in (5.1) and (5.2) are ordered lower and upper solutions for the initial-boundary value problem (1.1)–(1.3) for $0 \le t \le t_0$, $0 \le x \le 1$. Hence, (1.1)–(1.3) has a unique solution on the interval $[0, t_0]$ with an internal layer near $x_0(t)$.
By means of these lower and upper solutions we can formulate the condition for the initial functions $u^0(x, \varepsilon)$.

(A_4) The function $u^0 : [0, 1] \times [0, \varepsilon_0] \to R$ is sufficiently smooth and satisfies

$$\alpha(x, 0, \varepsilon) \le u^0(x, \varepsilon) \le \beta(x, 0, \varepsilon)$$

where α and β are defined by (5.1), (5.2).

Under the conditions (A_1)–(A_4) we can conclude that the initial-boundary value problem (1.1)–(1.3) has a unique solution $u(x, t, \varepsilon)$ with an interior layer for $0 \le t \le t_0$. In what follows we study this problem in the interval $[0, t_1]$, where t_1 has been introduced in hypothesis (A_3) and is characterized by the relation $x_0(t_1) = 1 + \eta$, where η is some small positive number. For this purpose we introduce for the interval $[t_0, t_1]$ a lower solution $\tilde{\alpha}$ in the form

$$\tilde{\alpha}(x, t, \varepsilon) = \varphi_1(x, t) - \gamma\varepsilon - \varepsilon(e^{-\kappa\xi_0} + e^{-\kappa\xi_1}).$$

Compared with the lower solution $\alpha(x, t, \varepsilon)$ in (5.1), $\tilde{\alpha}$ has no transition layer. It is easy to verify that $\tilde{\alpha}$ satisfies all conditions for a lower solution.
As an upper solution we keep the function $\beta(x, t, \varepsilon)$ as defined in (5.2). Most conditions for an upper solution can be checked easily as demonstrated in [4]. At the boundary $x = 1$ we have

$$\frac{\partial \beta}{\partial x}(1, t, \varepsilon) = \frac{\partial \varphi_2}{\partial x}(1, t) + \frac{1}{\varepsilon}\frac{\partial Q_{0\beta}}{\partial \xi_\beta}\Big|_{x=1} + \kappa + O(\varepsilon) + \frac{\partial Q_{1\beta}}{\partial \xi_\beta}\Big|_{x=1}. \tag{5.7}$$

We note that for sufficiently large κ this expression is not negative, since $\frac{\partial \varphi}{\partial x}(x, t)$ and $\frac{\partial Q_{1\beta}}{\partial \xi}(\xi_\beta, t)$ are bounded, and $\frac{\partial Q_{0\beta}}{\partial \xi_\beta}(\xi_\beta, t)$ is always positive. Therefore, we can conclude that the initial-boundary value problem (1.1)–(1.2) has a unique solution in the region $[0, t_1] \times [0, 1]$. If we choose the constant δ so small that we have $x_0(t_1) - \delta > 1$, then by taking into account that the functions $Q_{i\beta}(\xi_\beta)$, i=0,1, decay exponentially according to Proposition 4.3, for sufficiently small ε the solution $u(x, t, \varepsilon)$ is located in the region of attraction of the asymptotically stable periodic solution $u_1(x, t, \varepsilon)$. Thus, the solution $u(x, t, \varepsilon)$ tends to $u_1(x, t, \varepsilon)$ as t tends to infinity, and has no interior layer for $t \ge t_1$, but in general a boundary layer. Therefore, the following theorem is valid:

Theorem 5.1 *Suppose the hypotheses* (A_0)–(A_4) *are valid. Then there is a sufficiently small positive* ε_0 *such that for* $0 < \varepsilon \leq \varepsilon_0$ *the initial-boundary value problem* (1.1)–(1.3) *has a unique solution* $u(x,t,\varepsilon)$ *satisfying*

$$\lim_{t\to+\infty} \Big[u(x,t,\varepsilon) - u_1(x,t,\varepsilon)\Big] = 0 \quad \textit{for} \quad 0 \leq x \leq 1,$$

where u_1 *is the* T*-periodic solution introduced in Proposition* 4.3.

Remark 5.2 *From the proof of this theorem it follows that the solution* $u(x,t,\varepsilon)$ *has an interior layer for any fixed* t *of the interval* $[0,\tilde{t})$, *where* $\tilde{t}$ *is defined by* $x_0(\tilde{t}) = 1$. *For any fixed* t *satisfying* $t > \tilde{t}$, $u(x,t,\varepsilon)$ *has no interior layer and is a pure boundary layer solution.*

6 Fast transition from a step type contrast structure to a pure boundary layer solution

In this section we consider the initial-boundary value problem (1.1)–(1.3) under the assumption

(H$_1$) $\quad I(x,0) > 0 \quad \text{for} \quad 0 \leq x \leq 1.$

Assumption (H_1) implies that $x_0(0)$ is located outside the interval [0,1]. We note that the case $I(x,0) < 0$ can be treated analogously.

The numerical results represented in Fig. 3 show that under the hypothesis (H_1) the solution $u(x,t,\varepsilon)$ of (1.1)–(1.3) starting from the steplike initial function $u^0(x,\varepsilon)$ with the transition point $\tilde{x}_0 \in (0,1)$ moves very fast to the boundary in the direction of $x_0(t)$. After the interior layer has arrived the boundary, the solution $u(x,t,\varepsilon)$ changes from a contrast structure to a pure boundary layer solution. In what follows we want to prove this behavior. For this purpose, we first construct a formal asymptotic solution for the step-type solution of the boundary value problem (1.1), (1.3). In the second step we use this approximation to construct moving lower and upper solutions yielding the predicted behavior.

6.1 Formal asymptotics

In (1.1), (1.3) we rescale t by setting $t = \varepsilon\tau$ and consider the corresponding boundary value problem on a finite τ-interval

$$\begin{aligned} \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon\, \frac{\partial u}{\partial \tau} &= f(u,x,\varepsilon\tau),\ \tau \in (0,\tilde{\tau}),\ 0 < x < 1, && (6.1)\\ \frac{\partial u}{\partial x}(0,\varepsilon\tau,\varepsilon) &= \frac{\partial u}{\partial x}(1,\varepsilon\tau,\varepsilon) = 0 \quad \text{for} \quad \tau \in (0,\tilde{\tau}), && (6.2) \end{aligned}$$

where $\tilde{\tau}$ is some positive number. We construct the formal asymptotics of a step-type contrast structure of this boundary value problem. For this purpose we denote by $\hat{x}(\tau,\varepsilon)$ the transition point of the internal layer of the solution $u(x,\tau,\varepsilon)$ of (6.1), (6.2). We look for an asymptotic expansion of $\hat{x}(\tau,\varepsilon)$ in the form

$$\hat{x}(\tau,\varepsilon) = \hat{x}_0(\tau) + \varepsilon\ \hat{x}_1(\tau) + \dots \tag{6.3}$$

For the following we introduce the notation

$$\begin{aligned}
&\varrho(x,\tau,\varepsilon) := \frac{x-\hat{x}(\tau,\varepsilon)}{\varepsilon}, \quad \varrho_0 = x/\varepsilon, \quad \varrho_1 = (1-x)/\varepsilon,\\
&\overline{D}^{(-)} := \{(x,\tau) \in R^2 : 0 \le x \le \hat{x}(\tau,\varepsilon), \quad 0 \le \tau \le \tilde{\tau}\},\\
&\overline{D}^{(+)} := \{(x,\tau) \in R^2 : \hat{x}(\tau,\varepsilon) \le x \le 1, \quad 0 \le \tau \le \tilde{\tau}\}.
\end{aligned} \tag{6.4}$$

First, in $\overline{D}^{(-)}$ we consider the boundary value problem

$$\varepsilon^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon\ \frac{\partial u}{\partial \tau} \ = \ f(u,x,\varepsilon\tau), \quad \tau \in (0,\tilde{\tau}),\ 0 < x \le \hat{x}(\tau,\varepsilon), \tag{6.5}$$

$$\frac{\partial u}{\partial x}(0,\varepsilon\tau,\varepsilon) \ = \ 0, \quad u(\hat{x}(\tau,\varepsilon),\varepsilon\tau,\varepsilon) = \varphi_0(\hat{x}(\tau,\varepsilon),\varepsilon\tau), \quad \tau \in (0,\tilde{\tau}). \tag{6.6}$$

We seek the formal asymptotic expansion of the solution to this boundary value problem in the form

$$\begin{aligned}
U^{(-)}(x,\tau,\varepsilon) \ &= \ \overline{u}^{(-)}(x,\tau,\varepsilon) + Q^{(-)}(\varrho,\tau,\varepsilon) + \Pi(\varrho_0,\tau,\varepsilon) = \\
&= \sum_{i=0}^{\infty} \varepsilon^i [\overline{u}_i^{(-)}(x,\tau) + Q_i^{(-)}(\varrho,\tau) + \Pi_i(\varrho_0,\tau)],
\end{aligned} \tag{6.7}$$

where $\overline{u}^{(-)}, Q^{(-)}$ and Π denote the regular, internal layer and boundary layer parts of the asymptotic expansion of $U^{(-)}$ in the region $\overline{D}^{(-)}$.

Next, in $\overline{D}^{(+)}$ we consider the boundary value problem

$$\varepsilon^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon\ \frac{\partial u}{\partial \tau} \ = \ f(u,x,\varepsilon\tau),\ \tau \in (0,\tilde{\tau}),\ \hat{x}(\tau,\varepsilon) < x \le 1, \tag{6.8}$$

$$\frac{\partial u}{\partial x}(1,\varepsilon\tau,\varepsilon) \ = \ 0, \quad u(\hat{x}(\tau,\varepsilon),\varepsilon\tau,\varepsilon) = \varphi_0(\hat{x}(\tau,\varepsilon),\varepsilon\tau), \quad \tau \in (0,\tilde{\tau}) \tag{6.9}$$

and its formal asymptotic expansion

$$\begin{aligned}
U^{(+)}(x,\tau,\varepsilon) \ &= \ \overline{u}^{(+)}(x,\tau,\varepsilon) + Q^{(+)}(\varrho,\tau,\varepsilon) + R(\varrho_1,\tau,\varepsilon) = \\
&= \sum_{i=0}^{\infty} \varepsilon^i [\overline{u}_i^{(+)}(x,\tau) + Q_i^{(+)}(\varrho,\tau) + R_i(\varrho_1,\tau)],
\end{aligned} \tag{6.10}$$

where $\overline{u}^{(+)}, Q^{(+)}$ and R denote the regular, internal layer and boundary layer parts of the asymptotic expansion in the region $\overline{D}^{(+)}$.

To determine the zero-th order regular asymptotic terms we put $\varepsilon = 0$ in (6.5) and (6.8) and obtain

$$f(\overline{u}_0(x,\tau), x, 0) = 0.$$

Thus, according to assumption (A$_1$) we set

$$\overline{u}_0(x,\tau) = \begin{cases} \overline{u}_0^{(-)}(x,\tau) = \varphi_1(x,0) & \text{for} \quad (x,\tau) \in \overline{D}^{(-)}, \\ \overline{u}_0^{(+)}(x,\tau) = \varphi_2(x,0) & \text{for} \quad (x,\tau) \in \overline{D}^{(+)}. \end{cases}$$

In order to determine the coefficients $\hat{x}_k(\tau)$ of the expansion (6.3) we use the C^1-matching condition for the formal solutions $U^{(-)}$ and $U^{(+)}$ in $\overline{D}^{(-)}$ and $\overline{D}^{(+)}$ respectively, which reads

$$\varepsilon \, \frac{\partial U^{(-)}(x,\tau,\varepsilon)}{\partial x} = \varepsilon \, \frac{\partial U^{(+)}(x,\tau,\varepsilon)}{\partial x} \quad \text{for} \quad x = \hat{x}(\tau,\varepsilon). \tag{6.11}$$

For the following it is convenient to use the variable ϱ introduced in (6.4) to rewrite the differential operator

$$L \equiv \varepsilon^2 \, \frac{\partial^2}{\partial x^2} - \varepsilon \, \frac{\partial}{\partial \tau}$$

in the form

$$L \quad \equiv \quad \frac{\partial^2}{\partial \varrho^2} + v(\tau,\varepsilon) \, \frac{\partial}{\partial \varrho} - \varepsilon \, \frac{\partial}{\partial \tau}, \tag{6.12}$$

where $v(\tau,\varepsilon) = \frac{\partial \hat{x}}{\partial \tau}(\tau,\varepsilon)$.

Using (6.12) for $\varepsilon = 0$ we get by the standard representation of $f(u, x, \varepsilon\tau)$ (see, e.g., [1]) for the zero-th order internal layer functions $Q_0^{(-)}$ and $Q_0^{(+)}$ the boundary value problems ($v_0(\tau) = \hat{x}_0'(\tau)$)

$$\begin{aligned} \frac{d^2 Q_0^{(-)}}{d\varrho^2} + v_0(\tau) \frac{dQ_0^{(-)}}{d\varrho} &= f(\varphi_1(\hat{x}_0(\tau),0) + Q_0^{(-)}, \hat{x}_0(\tau), 0), \quad \varrho < 0 \\ Q_0^{(-)}(-\infty,\tau) = 0, \; Q_0^{(-)}(0,\tau) &= \varphi_0(\hat{x}_0(\tau),0) - \varphi_1(\hat{x}_0(\tau),0), \\ \frac{d^2 Q_0^{(+)}}{d\varrho^2} + v_0(\tau) \frac{d^2 Q_0^{(+)}}{d\varrho} &= f(\varphi_2(\hat{x}_0(\tau),0) + Q_0^{(+)}, \hat{x}_0(\tau), 0), \quad \varrho > 0 \\ Q_0^{(+)}(+\infty,\tau) = 0, \; Q_0^{(+)}(0,\tau) &= \varphi_0(\hat{x}_0(\tau),0) - \varphi_2(\hat{x}_0(\tau),0). \end{aligned} \tag{6.13}$$

Using the function

$$\tilde{Q}_0(\varrho,\tau) := \begin{cases} \varphi_1(\hat{x}_0(\tau),0) + Q_0^{(-)}(\varrho,\tau) & \text{for} \quad \varrho \le 0, \\ \varphi_2(\hat{x}_0(\tau),0) + Q_0^{(+)}(\varrho,\tau) & \text{for} \quad \varrho \ge 0, \end{cases}$$

we can write the boundary value problems (6.13) as

$$
\begin{aligned}
&\frac{d^2\tilde{Q}_0}{d\varrho^2} + v_0(\tau)\frac{d\tilde{Q}_0}{d\varrho} = f(\tilde{Q}_0, \hat{x}_0(\tau), 0) \quad \text{for} \quad \varrho \in R, \\
&\tilde{Q}_0(0,\tau) = \varphi_0(\hat{x}_0(\tau), 0), \;\; \tilde{Q}_0(-\infty,\tau) = \varphi_1(\hat{x}_0(\tau), 0), \\
&\tilde{Q}_0(+\infty,\tau) = \varphi_2(\hat{x}_0(\tau), 0).
\end{aligned}
\tag{6.14}
$$

We note that the solution of the boundary value problem (6.14) represents a traveling wave (heteroclinic orbit) connecting the saddles $(\varphi_1, 0)$ and $(\varphi_2, 0)$. It is known (see, e.g., [7]) that (6.14) has such a solution for

$$
v_0(\tau) \equiv \hat{x}_0'(\tau) = \frac{I(\hat{x}_0(\tau), 0)}{\int_{-\infty}^{\infty}\left(\frac{\partial \tilde{Q}_0}{\partial \varrho}\right)^2 d\varrho}. \tag{6.15}
$$

Equation (6.15) represents a differential equation which determines $\hat{x}_0(\tau)$ to a given initial condition $\hat{x}_0(0)$.

The zero-th order C^1-matching condition (6.11) implies

$$
\frac{\partial Q_0^{(-)}}{\partial \varrho}(0,\tau) = \frac{\partial Q_0^{(+)}}{\partial \varrho}(0,\tau).
$$

This condition is satisfied because of the solvability of problem (6.13).

The boundary layer functions Π and R in (6.7) and (6.10), respectively, can be determined by the standard theory (see,e.g.,[1]), and we will not consider their construction here.
Now we determine the first order terms in the asymptotic expansions (6.7) and (6.10). By comparing the first order terms in (6.1) we obtain for $\overline{u}_1^{(-)}$

$$
\overline{u}_1^{(-)}(x,\tau) = -\frac{f_t(\varphi_1(\hat{x}_0(\tau),0), \hat{x}_0(\tau), 0)}{f_u(\varphi_1(\hat{x}_0(\tau),0), \hat{x}_0(\tau), 0)}\,\tau.
$$

If we replace φ_1 by φ_2 in this expression, then we obtain $\overline{u}_1^{(+)}(x,\tau)$.

For the first order internal layer function $Q_1^{(-)}$ we get the boundary value problem

$$
\frac{d^2 Q_1^{(-)}}{d\varrho^2} + v_0(\tau)\frac{dQ_1^{(-)}}{d\varrho} - f_u(\tilde{Q}_0, \hat{x}_0(\tau), 0)Q_1^{(-)} = f_1^{(-)}(\varrho,\tau) \quad \text{for} \quad \varrho < 0, \tag{6.16}
$$

$$
\begin{aligned}
Q_1^{(-)}(-\infty,\tau) \;\; &= \;\; 0, \quad Q_1^{(-)}(0,\tau) = -\overline{u}_1(\hat{x}_0(\tau),\tau) + \\
&+ \;\; \Big[\frac{\partial \varphi_0}{\partial t}(\hat{x}_0(\tau),0) - \varphi_1(\hat{x}_0(\tau),0)\Big]\tau + \\
&+ \;\; \Big[\frac{\partial \varphi_0}{\partial x}(\hat{x}_0(\tau),0) - \frac{\partial \varphi_1}{\partial x}(\hat{x}_0(\tau),0)\Big]\hat{x}_1(\tau) \equiv g_1^{(-)}(\tau),
\end{aligned}
\tag{6.17}
$$

where

$$
\begin{aligned}
f_1^{(-)}(\varrho,\tau) \;\equiv\; & -v_1(\tau)\frac{\partial Q_0}{\partial \varrho}(\varrho,\tau) + \frac{\partial Q_0}{\partial \tau}(\varrho,\tau) + f_u^0 \overline{u}_1^{(-)}(x,\tau) + \\
& +(\hat{x}_1(\tau)+\varrho)\Big[f_u^0 \frac{\partial \varphi_1}{\partial x}(\hat{x}_0(\tau),0) + f_x^0\Big] + f_t^0 \tau.
\end{aligned}
\tag{6.18}
$$

Here, the upper index 0 means that the derivatives f_u^0, f_x^0 and f_t^0 are evaluated at the point $(\tilde{Q}_0(\varrho,\tau), \hat{x}_0(\tau), 0)$, $v_1(\tau)$ is defined by $v_1(\tau) := \hat{x}_1'(\tau)$.
Similarly, $Q_1^{(+)}$ can be defined. The solutions of the boundary value problems for $Q_1^{(-)}$ and $Q_1^{(+)}$ can be represented in the explicit form

$$
\begin{aligned}
Q_1^{(\pm)}(\varrho,\tau) \;=\; & g_1^{(\pm)}(\tau)\frac{\phi(\varrho,\tau)}{\phi(0,\tau)} - \phi(\varrho,\tau)\int_0^{\varrho} \phi^{-2}(\eta,\tau) e^{-v_0(\tau)\eta} \times \\
& \times \int_{\eta}^{\pm\infty} \phi(\sigma,\tau) e^{v_0(\tau)\sigma} f_1^{(\pm)}(\sigma,\tau)\, d\sigma d\eta,
\end{aligned}
\tag{6.19}
$$

where $\phi(\varrho,\tau) = \frac{\partial \tilde{Q}_0}{\partial \varrho}(\varrho,\tau) > 0$.
The first order of C^1-matching condition (6.11) yields

$$
\frac{\partial \varphi_1}{\partial x}(\hat{x}_0(\tau),0) + \frac{\partial Q_1^{(-)}}{\partial \varrho}(0,\tau) = \frac{\partial \varphi_2}{\partial x}(\hat{x}_0(\tau),0) + \frac{\partial Q_1^{(+)}}{\partial \varrho}(0,\tau). \tag{6.20}
$$

Substituting the expression (6.19) into (6.20) and using the formula (6.17), (6.18) we can reduce the conditions (6.20) to a linear algebraic equation for $v_1(\tau)$ (see, e.g., [4]). The construction of the asymptotic expansion can be continued to any order, provided the function f is sufficiently smooth.

6.2 Construction of lower and upper solutions

Our construction of lower and upper solutions which can be used to describe the moving transition layer (front) on a finite interval of τ will follow the scheme presented in the previous section.
We recall that the differential equation (6.15) with a given initial condition $\hat{x}_0(0)$ ($\hat{x}_0(0)$ is any number of the interval $(0,1)$) determines $\hat{x}_0(\tau)$. Under the assumption (H_1) we can conclude from (6.15) that $\hat{x}_0'(\tau)$ is positive as long as $\hat{x}_0(\tau)$ belongs to the interval $[0,1]$. Hence, under our smoothness assumptions, there is a positive number $\hat{\tau}$ such that $\hat{x}_0(\hat{\tau}) > 1$. Let $\tau^* \in (0,\hat{\tau})$ such that $\hat{x}_0(\tau^*) = 1-\delta_1$, where δ_1 is any small positive number. By means of $\hat{x}_0(\tau)$ we define lower and upper solutions to the boundary value problem (6.1), (6.2) for $0 \le \tau \le \tau^*$ as follows

$$
\begin{aligned}
\alpha(x,\tau,\varepsilon) &= \overline{u}_{0\alpha}(x,\tau)+\varepsilon(\overline{u}_1(x,\tau)-\gamma)+Q_{0\alpha}(\varrho_\alpha,\tau) \\
&+ \varepsilon\ Q_{1\alpha}(\varrho_\alpha,\tau)-\varepsilon(e^{-\kappa\varrho_0}+e^{-\kappa\varrho_1}), \\
\beta(x,\tau,\varepsilon) &= \overline{u}_{0\beta}(x,\tau)+\varepsilon(\overline{u}_1(x,\tau)+\gamma)+Q_{0\beta}(\varrho_\beta,\tau), \\
&+ \varepsilon\ Q_{1\beta}(\varrho_\beta,\tau)+\varepsilon(e^{-\kappa\varrho_0}+e^{-\kappa\varrho_1}).
\end{aligned}
\tag{6.21}
$$

Here, the zero-th order terms $\overline{u}_{0\alpha}$ and $\overline{u}_{0\beta}$ are defined by

$$
\overline{u}_{0\alpha} = \begin{cases} \varphi_1(x,0) & \text{for} \quad 0\le x\le x_\alpha(\tau), \\ \varphi_2(x,0) & \text{for} \quad x_\alpha(\tau)\le x\le 1, \end{cases}
$$

$$
\overline{u}_{0\beta} = \begin{cases} \varphi_1(x,0) & \text{for} \quad 0\le x\le x_\beta(\tau), \\ \varphi_2(x,0), & \text{for} \quad x_\beta(\tau)\le x\le 1, \end{cases}
$$

where $x_\alpha(\tau)=\hat{x}_0(\tau)+\tilde{\delta}$, $x_\beta(\tau)=\hat{x}_0(\tau)-\tilde{\delta}$ and $\tilde{\delta}$ is a sufficiently small fixed positive number such that $\hat{x}_0(\tau)+\tilde{\delta}\le 1-\delta_1$ for $0\le\tau\le\tau^*$.

The function $Q_{0\beta}(\varrho_\beta,\tau)$ is defined by boundary value problems which are similar to (6.13):

$$
\begin{aligned}
\frac{d^2Q_{0\beta}^{(-)}}{d\varrho_\beta^2}+v_{0\beta}(\tau)\frac{dQ_{0\beta}^{(-)}}{d\varrho_\beta} &= f(\varphi_1(x_\beta(\tau),0)+Q_{0\beta}^{(-)},x_\beta(\tau),0) \quad \text{for} \quad \varrho_\beta<0, \\
Q_{0\beta}^{(-)}(-\infty,\tau) &= 0,\ Q_{0\beta}^{(-)}(0,x_\beta(\tau))=\varphi_0(x_\beta(\tau),0)-\varphi_1(x_\beta(\tau),0), \\
\frac{d^2Q_{0\beta}^{(+)}}{d\varrho_\beta^2}+v_{0\beta}\frac{dQ_{0\beta}^{(+)}}{d\varrho_\beta} &= f(\varphi_2(x_\beta(\tau),0)+Q_{0\beta}^{(+)},x_\beta(\tau),0) \quad \text{for} \quad \varrho_\beta>0, \\
Q_{0\beta}^{(+)}(\infty,\tau) &= 0,\ Q_{0\beta}^{(+)}(0,\tau)=\varphi_0(x_\beta(\tau),0)-\varphi_2(x_\beta(\tau),0),
\end{aligned}
\tag{6.22}
$$

where

$$
v_{0\beta}(\tau)=\frac{I(x_\beta(\tau),0)}{\int_{-\infty}^{\infty}(\frac{\partial Q_0}{\partial\varrho})^2(\varrho,x_\beta(\tau))d\varrho}-\delta_v,
$$

δ_v is a positive sufficiently small fixed number, $\varrho_\beta=(x-x_\beta(\tau))/\varepsilon$. The function $Q_{0\alpha}(\varrho_\alpha,\tau)$ can be defined analogously if we replace in (6.22) $x_\beta(\tau)$ and ϱ_β by $x_\alpha(\tau)$ and $\varrho_\alpha=(x-x_\alpha(\tau))/\varepsilon$ respectively, and $v_{0\beta}$ by $v_{0\alpha}$, where

$$
v_{0\alpha}(\tau)=\frac{I(x_\alpha(\tau),0)}{\int_{-\infty}^{\infty}(\frac{\partial Q_0}{\partial\varrho})^2(\varrho,x_\alpha(\tau))d\varrho}+\delta_v.
$$

For the first order internal layer function $Q_{1\beta}$ in (6.21) we obtain boundary value problems which are similar to (6.16)–(6.18). Particularly, for the function $Q_{1\beta}^{(-)}$ we get

$$
\frac{d^2Q_{1\beta}^{(-)}}{d\varrho_\beta^2}+v_0(\tau)\frac{dQ_{1\beta}^{(-)}}{d\varrho_\beta}-f_u(\tilde{Q}_{0\beta},x_\beta(\tau),0)Q_{1\beta}^{(-)}=f_1^{(-)}(\varrho_\beta,\tau) \quad \text{for} \quad \varrho_\beta<0,
$$

$$Q_{1\beta}^{(-)}(-\infty,\tau) \;=\; 0,\; Q_1^{(-)}(0,\tau) = -\overline{u}_1(x_\beta(\tau),\tau) + \Big[\frac{\partial\varphi_0}{\partial t}(x_\beta(\tau),0) - \varphi_1(x_\beta(\tau),0)\Big]\tau$$

where $\tilde{Q}_{0\beta} = \tilde{Q}_{0\beta}^{(-)} = Q_{0\beta}^{(-)} + \varphi_1(x_\beta(\tau),0)$ and

$$\begin{aligned} f_1^{(-)}(\varrho_\beta,\tau) = \quad & \varrho_\beta\Big[f_u^0\frac{\partial\varphi_1}{\partial x}(x_\beta(\tau),0) + f_x^0\Big] + \\ & +(f_u^0 - \overline{f}_u)(\overline{u}_1^{(-)} + \gamma) + (f_t^0 - \overline{f}_t)\tau + \frac{\partial Q_0}{\partial\tau}(\varrho,\tau). \end{aligned} \tag{6.23}$$

We note that $f_1^{(-)}(\varrho_\beta,\tau)$ in (6.23) is different from $f_1^{(-)}(\varrho,\tau)$ in (6.18) by setting $\hat{x}_1 = 0$, $v_1 = 0$. The derivatives f_u^0, f_x^0 and f_t^0 here are evaluated at the point $(\tilde{Q}_{0\beta}, x_{0\beta}, 0)$, and the derivatives $\overline{f}_u$, $\overline{f}_x$ and $\overline{f}_t$ are evaluated at the point $(\varphi_1(x_\beta(\tau),0), x_\beta(\tau), 0)$. Analogously, $Q_{1\beta}^{(+)}$ can be defined.

Remark. For the proof of our main result (see Theorem 6.1 below) it is sufficient to use the upper and lower solutions in the form (6.21). If we want to approximation the solution in the transition layer, then we have to construct another expressions for α and β containing $\hat{x}_1(\tau)$ and $v_1(\tau)$.

The function $Q_{1\alpha}$ can be introduced similarly.
Using the expressions for the functions α and β we can verify by means of the standard approach (see, for example, [4], [5]) that they satisfy the following inequalities for $0 < x < 1,\; 0 < \tau < \tau^*$.

$$L\beta := \varepsilon^2\frac{\partial^2\beta}{\partial x^2} - \varepsilon\frac{\partial\beta}{\partial\tau} - f(\beta,x,\tau) \le 0, \quad L\alpha \ge 0.$$

The corresponding inequalities at the boundaries can be satisfied if we take κ in (6.21) sufficiently large (see also (5.7). From the exponential decay of the Q-functions in (6.21) it follows that α and β are ordered, i.e.

$$\alpha(x,\tau,\varepsilon) \le \beta(x,\tau,\varepsilon) \quad \text{for} \quad x\in[0,1],\; \tau\in[0,\tau^*].$$

From the definitioon of α and β in (6.21) it follows that these functions are continuous but not differentiable for $x = x_\alpha(\tau)$ and $x = x_\beta(\tau)$, respectively. The jump of the derivative of β with respect to x at the point $x_\beta(\tau)$ is determined by the expression

$$\frac{\partial\beta}{\partial x}(x_\beta(\tau)+0,\tau,\varepsilon) - \frac{\partial\beta}{\partial x}(x_\beta(\tau)-0,\tau,\varepsilon) = \frac{1}{\varepsilon}\left(\frac{\partial Q_{0\beta}^{(+)}}{\partial\varrho_\beta}(0,\tau) - \frac{\partial Q_{0\beta}^{(-)}}{\partial\varrho_\beta}(0,\tau) + o(1)\right).$$

In order to guarantee that this jump is admissible, that is, it satisfies

$$\frac{\partial\beta}{\partial x}(x_\beta(\tau)+0,\tau,\varepsilon) - \frac{\partial\beta}{\partial x}(x_\beta(\tau)-0,\tau,\varepsilon) < 0, \tag{6.24}$$

we have to ensure

$$\frac{\partial Q_{0\beta}^{(+)}}{\partial\varrho_\beta}(0,\tau) - \frac{\partial Q_{0\beta}^{(-)}}{\partial\varrho_\beta}(0,\tau) < 0. \tag{6.25}$$

To this end, we rewrite the first equation in (6.22) in the following simplified form

$$\frac{d^2u}{d\rho^2} + v\frac{du}{d\rho} = \tilde{f}(u), \tag{6.26}$$

where

$$u = \begin{cases} \tilde{Q}_{0\beta}^{(-)} & \text{for} \quad -\infty < \rho_\beta < 0, \\ \tilde{Q}_{0\beta}^{(+)} & \text{for} \quad 0 < \rho_\beta < +\infty. \end{cases}$$

Equation (6.26) is equivalent to the system

$$\frac{du}{d\rho} = p, \tag{6.27}$$

$$\frac{dp}{d\rho} = -vp + f(u). \tag{6.28}$$

It is easy to verify that in the half plane $p > 0$ all vectors of the vector field defined by (6.27) rotate in mathematically positive sense as v increases. As we mentioned above, system (6.27) has two equilibria $(\varphi_1, 0)$, $\varphi_2, 0)$ which are saddles, where for $v = v_0$ the separatrix $p = \sigma_1(u)$ of the saddle $(\varphi_1, 0)$ and the separatrix $p = \sigma_2(u)$ of the saddle $(\varphi_2, 0)$ form a heteroclinic orbit located in the half plane $p > 0$. From the property that (6.27) is a rotated vector field [10] we obtain that for $v < v_0$

$$\sigma_2(\varphi_0) - \sigma_1(\varphi_0) < 0,$$

and therefore, the jump condition (6.24) is satisfied.

Similarly we can show that the function $\alpha(x, \tau, \varepsilon)$ has an admissible jump of its derivative at the point $x_\alpha(\tau)$:

$$\frac{\partial \alpha}{\partial x}(x_\alpha(\tau) + 0, \tau, \varepsilon) - \frac{\partial \alpha}{\partial x}(x_\alpha(\tau) - 0, \tau, \varepsilon) > 0.$$

Under the additional assumption

(H_2). Suppose $u^0(x, \varepsilon)$ is a step type internal layer function such that

$$\alpha(x, 0, \varepsilon) \leq u^0(x, \varepsilon) \leq \beta(x, 0, \varepsilon)$$

where α and β are defined in (6.21).

We have the following result:

Theorem 6.1 *Suppose the hypotheses* (A_0) - (A_2), (H_1), (H_2) *are valid. Then, for sufficiently small* ε*, the initial boundary value problem* (1.1)–(1.3) *has a unique solution* $u(x, t, \varepsilon)$ *for* $t \in (0, \varepsilon\tau^*)$ *such that*

$$\lim_{\varepsilon \to 0} u(x, \tau, \varepsilon) = \begin{cases} \varphi_1(x, 0), & 0 \leq x < \hat{x}_0(\tau) \\ \varphi_2(x, 0), & \hat{x}_0(\tau) < x \leq 1. \end{cases}$$

6.3 Fast transition to a pure boundary layer solution.

In order to prove that the contrast structure whose existence has been established in Theorem 6.1 changes into a pure boundary layer solution we will use the same scheme as in the proof of Theorem 5.1 in section 5.

First we extend the existence result of the initial-boundary value problem (1.1)–(1.3) to the interval $[0, \tilde{\tau}]$ in which $x_0(\tau)$ crosses the boundary $x = 1$ such that we have $x_0(\tilde{\tau}) > 1$.
In order to show that the solution $u(x, \tau, \varepsilon)$ changes its form by losing its interior layer, we use for that interval the lower solution $\tilde{\alpha}$ in the form

$$\tilde{\alpha}(x, \tau, \varepsilon) = \varphi_1(x, 0) - \gamma\varepsilon - \varepsilon(e^{-\kappa p_0} + e^{-\kappa p_1}).$$

As an upper solution we use the function $\beta(x, \tau, \varepsilon)$, defined in (6.21). By repeating the considerations from section 5 we get the following result.

Theorem 6.2 *Suppose the hypotheses* (A_0)–(A_2), (H_1), (H_2) *are valid. Then there exists a finite time* $\tau = \tilde{\tau}$ *such that at* $t = \tilde{\tau}\varepsilon$ *the solution of the initial-boundary value problem* (1.1)–(1.3) *is in the domain of attraction of the periodic boundary layer solution, and therefore near the moment* $t = \tilde{\tau}\varepsilon$ *the contrast structure solution changes into a pure boundary layer type solution. Moreover we have*

$$\lim_{t\to\infty} \Big[u(x, t, \varepsilon) - u_1(x, t, \varepsilon)\Big] = 0,$$

where $u_1(x, t, \varepsilon)$ *is the periodic boundary layer solution of the problem* (1.1), (1.3).

7 Fast-slow transition from step type contrast structure to a pure boundary layer solution.

We again consider the initial-boundary value problem (1.1)–(1.3) under the conditions (A_0)–(A_3). Furthermore, we suppose that the initial function $u^0(x, \varepsilon))$ is of step-type, where the location of the transition layer is characterized by the point $\tilde{x}_0$. But different from assumption (H_1) introduced in section 5 we suppose that $\tilde{x}_0$ is not near $_0(0)$. In this case, the numerical results (Fig. 2) show a fast motion of the initial step type contrast structure to the neighbourhood of $x_0(0)$, and then a slow motion with $x_0(t)$ to the boundary, where the solution changes from a contrast structure solution into a pure boundary layer solution.
In order to be able to prove an analytic result we introduce the following additional assumption.

(H_3). Suppose for definiteness $\tilde{x}_0 < x_0(0)$ and that $u^0(x, \varepsilon)$ is a step-type internal layer function whose transition point $\tilde{x}_0$ is located near $\hat{x}_0(0)$ and such that

$$\alpha(\hat{x}_0(0), 0, \varepsilon) \le u^0(x, \varepsilon) \le \beta(\hat{x}_0(0), 0, \varepsilon)$$

where α and β are defined by (6.21).

From assumption (A_3) it follows that $I(x,0) > 0$ for $x < x_0(0)$ and $I(x,0) < 0$ for $x > x_0(0)$.
Let δ_1 is a small positive independent on ε number such that $\delta_1 < \delta$ (δ is the number which is used in the construction of lower and upper solutions in (5.5), (5.6)). Under the assumptions above we can apply the results of Theorem 6.1 for the time interval $0 \le t \le \varepsilon\tau_0^*$, where τ_0^* is the time is the stretched scale such that $\hat{x}_0(\tau) = x_0(0) - \delta_1$ ($\hat{x}_0(\tau)$ moves to the point $x_0(0)$ according the results of Theorem 6.1). From the structure of the lower and upper solutions defined by (6.17) it follows that for $t = \varepsilon\tau_0^*$ the solution of problem (1.1) - (1.2) satisfies the condition (A_4) for the time $t = \varepsilon\tau_0^*$. From this time we can apply the results of Theorem 5.1 to describe the motion of step type contrast structure and its transformation into boundary layer solution. Our observations we summarize in the following theorem.

Theorem 7.1 *Assume the hypotheses* (A_1)–(A_3) *and* (H_3) *to be valid. Then for sufficiently small ε there exists a unique solution of problem* (1.1)–(1.3) *which has an interior layer for any $t \in [0, t_0)$, where t_0 is defined by $x_0(t) = 1$. This solution exhibits a phase of fast motion for $0 \le t \le \varepsilon\tau_0^*$ and a phase of slow motion for $\nu \le t < t_0$, where ν is any small number. For any fixed $t > t_0$, $u(x,t,\varepsilon)$ is a pure boundary layer solution. Moreover*

$$\lim_{t\to\infty} \Big[u(x,t,\varepsilon) - u_1(x,t,\varepsilon) \Big] = 0.$$

Remark *For the time interval $[\varepsilon\tau_0^*, \nu]$ there is a transition from the fast to the slow motion which is not described by our approach.*

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