Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Quasiperiodic regimes in multisection semiconductor lasers

Sergey V. Gonchenko[1], Klaus R. Schneider[2], Dmitry Turaev[3]

submitted: 23 Sep 2004

[1] Institute for Applied Mathematics
and Cybernetics
Uljanova Str. 10
603005 Nizhny Novgorod
Russia
E-mail: gosv100@uic.nnov.ru

[2] current address:
Weierstrass Institute for Applied
Analysis and Stochastics
Mohrenstr. 39
10117 Berlin
Germany
E-mail: schneider@wias-berlin.de

[3] Ben Gurion University of the Negev
Department of Mathematics
P.O. Box 653
Beer-Sheva 84105
Israel
E-mail:turaev@cs.bgu.ac.il

No. 967
Berlin 2004

2000 *Mathematics Subject Classification.* 78A60, 34C29, 34C20, 34C60.

Key words and phrases. multisection semiconductor laser, mode approximation, quasiperiodic regime, normal form, stability.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract. We consider a mode approximation model for the longitudinal dynamics of a multisection semiconductor laser which represents a slow-fast system of ordinary differential equations for the electromagnetic field and the carrier densities. Under the condition that the number of active sections q coincides with the number of critical eigenvalues we introduce a normal form which admits to establish the existence of invariant tori. The case $q = 2$ is investigated in more detail where we also derive conditions for the stability of the quasiperiodic regime.

1. Introduction

Semiconductor lasers play a crucial role in many areas of modern technology. Especially in photonic networks they are used for fast data regeneration. Typically, these devices possess a non-stationary working regime, and their behavior is characterized by a multi-scale dynamics and by occurring of instabilities of higher co-dimension. The construction of semiconductor lasers with several sections allows to control these nonlinear effects.
Under certain physical conditions, the longitudinal dynamics of edge emitting multisection semiconductor lasers can be characterized by the temporal behavior of the electro-magnetic field E and the effective carrier density N within the active zone of the device. The corresponding mathematical model is referred to as traveling wave model (see [5, 6] and references therein). In this model, the time evolution of the state variables is described by the following differential system in some Banach space

$$\begin{aligned} \frac{dE}{dt} &= H(N)E, \\ \frac{dN_j}{dt} &= \varepsilon(f_j(N) - E^T g^j(N)E^*), \quad j = 1, ..., k. \end{aligned} \tag{1.1}$$

Here, E is a complex vector depending on time t and on the one-dimensional space variable z characterizing the longitudinal direction of the laser, E^* represents the complex conjugate of E, and $N = (N_1, ..., N_k)^T$ is a real vector whose components describe the spatially section-wise averaged carrier density in the k active sections. Hence, the variables $N_1, ..., N_k$ depend on time only. $H(N)$ is a first order differential operator with respect to z such that the first subsystem in (1.1) represents a linear hyperbolic system of partial differential equations. $E^T g^j(N)E^*$ is a Hermitian form implying the symmetry of (1.1) with respect to rotation of the complex variable E (phase shift of the electromagnetic field). Thus, system (1.1) couples a linear system of partial differential equations (PDEs) for E with a system of ordinary differential equations (ODEs) for N. The crucial fact that the variables E and N act on different time-scales is expressed by the presence of the small parameter ε which is the ratio between the averaged lifetime of a photon and the averaged lifetime of a carrier (in applications, $\varepsilon \approx 10^{-2}$, or often smaller [2, 4, 5]).

The slow-fast structure of (1.1) permits to derive conditions ensuring the existence of a finite-dimensional invariant manifold such that the PDE-ODE system (1.1) can be reduced to an ODE model. For this purpose we need the following assumption (see [8, 5]):

Assumption: there exist a negative number κ and a simply connected compact set $\mathcal{K} \subset \mathbb{R}^k$ such that for all $N \in \mathcal{K}$ the spectrum of $H(N)$ can be decomposed as

$$\operatorname{spec} H(N) = \sigma_c H(N) \cup \sigma_s H(N),$$

where

$$\operatorname{Re} \sigma_c H(N) = 0, \quad \operatorname{Re} \sigma_s H(N) < \kappa < 0.$$

According to this splitting, to any N in some small neighborhood of $\mathcal{K}$ there exist spectral projections $P_c(N)$ and $P_s(N)$. We assume that $\sigma_c H(N)$ consists of a finite number q of eigenvalues. Let the column vectors of the $q \times q$-matrix $B(N)$ form a basis for the space $ImP_c(N)$, and let E_c be the coordinates in this basis, i.e. $E = B(N)E_c$ for $E \in ImP_c(N)$. In [5] it has been proven that for sufficiently small ε there exists a finite-dimensional, exponentially attracting smooth invariant manifold with the representation $E = \varepsilon\nu(E_c, N, \varepsilon)$, where ν is a smooth bounded function defined for N in some neighborhood of the set $\mathcal{K}$.

On this invariant manifold, system (1.1) takes the form [4, 5, 8]

$$\begin{aligned} \frac{dE_c}{dt} &= \left[H_c(N) - \varepsilon\alpha(N)F(E_c, N) + O(\varepsilon^2)\right] E_c, \\ \frac{dN}{dt} &= \varepsilon F(E_c, N) + O(\varepsilon^2), \end{aligned} \tag{1.2}$$

where $E_c \in \mathbb{C}^q$, $N \in \mathbb{R}^k$, and

$$\begin{aligned} H_c(N) &:= B(N)^{-1} H(N) B(N), \\ \alpha(N) &:= B(N)^{-1} P_c(N) \partial_N B(N), \\ F(E_c, N) &:= \big(f_1(N) - (B(N)E_c)^T g_1(N)(B(N)E_c)^*, ..., \\ &\qquad f_k(N) - (B(N)E_c)^T g_k(N)(B(N)E_c)^*\big). \end{aligned} \tag{1.3}$$

In what follows we use the $q \times q$-matrices $G^j(N)$, which are defined by

$$G^j(N) := B(N)^T g^j(N) B(N)^*, \quad j = 1, ..., q, \tag{1.4}$$

such that we can represent $F(E_c, N)$ in the form

$$F(E_c, N) := \left(f_1(N) - E_c^T G^1(N) E_c^*, ..., f_k(N) - E_c^T G^k(N) E_c^*\right)^T. \tag{1.5}$$

If we drop in (1.2) the $O(\varepsilon^2)$-terms in (1.2), then we get the system

$$\begin{aligned} \frac{dE_c}{dt} &= \left[H_c(N) - \varepsilon\alpha(N)F(E_c, N)\right] E_c, \\ \frac{dN}{dt} &= \varepsilon F(E_c, N), \end{aligned} \tag{1.6}$$

which is called *mode approximation model*. That model is in some sense an implicit system of ordinary differential equations because the functions $H_c(N)$ and $\alpha(N)$ usually are known only implicitly via the solution of the characteristic equation for

H. Mode approximations have been proven to be extremely useful for numerical and analytical investigations of longitudinal effects in multi-section semiconductor lasers since the dimension of system (1.6) is typically low (q is often either 1 or 2; see, e.g.,[1, 7, 3, 4]). In fact, using specific information about the eigenvalues of H_c(N), it is often possible to make further significant simplification of system (1.2).

The goal of this note is to consider the case when for $N \in \mathcal{K}$ the q eigenvalues of $H_c(N)$ are all different. Moreover, we assume that the number of active sections is also q (that is $k = q$) such that the set $\mathcal{K}$ is, typically, a single point in the N-space. In that case we will transform system (1.2) into some normal form which, in particular, permits to establish the existence of invariant tori for system (1.6). The case $q = 2$ will be studied in more detail.

We note that the normal forms we obtain can be viewed as small (of order $O(\sqrt{\varepsilon})$) perturbations of some conservative systems. In the case $q = 2$, for example, the conservative "nucleus" of the normal form (see equation (3.9)) can be written as a Lagrangian system with two degrees of freedom, given by the Lagrangian

$$L = a_1\Big(\frac{du_1}{dt}\Big)^2 + a_2\Big(\frac{du_2}{dt}\Big)^2 + \frac{du_1}{dt}\frac{du_2}{dt} - e^{u_1} - e^{u_2} + b_1u_1 + b_2u_2$$

with certain constants a_1, a_2, b_1, b_2. Clearly, the dynamics of small dissipative perturbations of such systems is not exhausted by invariant tori only. Therefore, further analysis of our normal forms can provide more insight into the dynamics of multi-section lasers.

2. Transformation to a normal form

We consider system (1.2), where we drop the index c, under the following assumptions:

(A_1). There exists $N^0 \in R^q$, $q \geq 2$, such that the $q \times q$ - matrix $H(N^0)$ has q different eigenvalues on the imaginary axis.

(A_2). There is a neighborhood $\mathcal{N}$ of N^0 in R^q such that the matrices H and G^j, and the functions f_j, $j = 1, 2, ..., q$, depend smoothly on N for $N \in \mathcal{N}$.

We denote by $\lambda_j(N) = \mu_j(N) + i\omega_j(N)$ the eigenvalues of $H(N)$ for $N \in \mathcal{N}$. By assumption (A_1) we have

$$\text{(2.1)} \quad \mu_j(N^0) = 0 \quad \text{for} \quad j = 1, \ldots, q, \quad \omega_l(N^0) \neq \omega_m(N^0) \quad \text{for} \quad l \neq m.$$

We also assume

(A_3). For any different pairs (l, s) and (j, p) of indices it holds

$$\omega_l(N^0) - \omega_s(N^0) \neq \omega_j(N^0) - \omega_p(N^0).$$

Using the eigenvectors of $H(N)$ as column vectors forming the matrix $B(N)$ we have the representation $H_c(N) = \mathcal{M}(n) + i\Omega(n)$ with the diagonal matrices

$$\mathcal{M}(n) = \mathrm{diag}(\mu_1(N), \ldots, \mu_q(N)), \quad \Omega(n) = \mathrm{diag}(\omega_1(N), \ldots, \omega_q(N)).$$

Thus, system (1.2) can be rewritten in the form

$$
(2.2)\qquad
\begin{aligned}
\frac{dE}{dt} &= \left[\mathcal{M}(N) + i\Omega(N) - \varepsilon\alpha(N)F(E,N) + O(\varepsilon^2)\right] E,\\
\frac{dN}{dt} &= \varepsilon F(E,N) + O(\varepsilon^2),
\end{aligned}
$$

where the complex vector E has the components $E_1, \dots, E_q$. It can be shown (see [8]) that every bounded orbit of system (2.2) must stay in a region of the phase space, where the variable N is close to N^0, i.e., where $\mu_j(N)$ is small for any j. For the sequel, it turns out to be useful to scale the variable N as $N = N^0 + \sqrt{\varepsilon}n$. Then, system (2.2) takes the form

$$
(2.3)\qquad
\begin{aligned}
\frac{dE}{dt} &= \big[\mathcal{M}(N^0 + \sqrt{\varepsilon}n) + i\Omega(N^0 + \sqrt{\varepsilon}n)\\
&\qquad -\varepsilon\alpha(N^0 + \sqrt{\varepsilon}n)F(E, N^0 + \sqrt{\varepsilon}n) + O(\varepsilon^2)\big] E,\\
\frac{dn}{dt} &= \sqrt{\varepsilon}\, F(E, N^0 + \sqrt{\varepsilon}n) + O(\varepsilon^{3/2}).
\end{aligned}
$$

From the definition of $\alpha(N)$ and $G^j(N)$ in (1.3) and (1.4) respectively, it follows that system (2.3) can be rewritten component-wise as

$$
(2.4)\qquad
\begin{aligned}
\frac{dE_l}{dt} &= \lambda_l(N)E_l\\
&\quad - \varepsilon \sum_{1\le p,j\le q} \alpha^j_{lp}(N)\left(f_j(N) - \sum_{1\le m,s\le q} G^j_{ms}(N)E_m E_s^*\right) E_p\\
&\quad + O(|E|\varepsilon^2), \quad l = 1, \dots, q,\\
\frac{dn_j}{dt} &= \sqrt{\varepsilon}\left(f_j(N) - \sum_{1\le m,s\le q} G^j_{ms}(N)E_m E_s^*\right) + O(\varepsilon^{3/2}),\\
&\quad j = 1, \dots, q.
\end{aligned}
$$

where $|E|$ denotes the Euclidean norm. The following transformation aims to eliminate the terms related to the off-diagonal elements of $G^j_{ms}(N^0 + \sqrt{\varepsilon}n)$ on the right-hand side of dn_j/dt. We apply the transformation

$$
(2.5)\qquad \tilde{n}_j = n_j + \sqrt{\varepsilon}\sum_{m\ne s} E_m E_s^* \frac{G^j_{ms}(N)}{\lambda_m(N) + \lambda_s{}^*(N)},
$$

where $N = N^0 + \sqrt{\varepsilon}n$. The relations $G^j_{ms}(N) = G^j_{sm}(N)^*$ and $\omega_m(N^0) \ne \omega_s(N^0)$ for $m \ne s$ imply that this transformation is real and non-degenerate for sufficiently small ε. From (2.5) and (2.2), (2.4) we obtain

$$
\begin{aligned}
\frac{d\tilde{n}_j}{dt} &= \frac{dn_j}{dt} + \sqrt{\varepsilon}\sum_{m\ne s}\left(\frac{dE_m}{dt}E_s^* + \frac{dE_m^*}{dt}E_s\right)\frac{G^j_{ms}(N)}{\lambda_m(N) + \lambda_s{}^*(N)} + O(\varepsilon^{3/2})\\
&= \sqrt{\varepsilon}\Big(f_j(N) - \sum_{1\le m,s\le q} G^j_{ms}(N)E_m E_s^*\Big)
\end{aligned}
$$

$$+\sqrt{\varepsilon}\sum_{m\neq s}\frac{G^j_{ms}(N)}{\lambda_m(N)+\lambda_s{}^*(N)}\Big(\lambda_m(N)+\lambda_s{}^*(N)\Big)E_mE_s^*+O(\varepsilon^{3/2})$$

$$=\sqrt{\varepsilon}\Big(f_j(N)-\sum_{1\leq m,s\leq q}G^j_{ms}(N)E_mE_s^*\Big)$$

$$+\sqrt{\varepsilon}\sum_{m\neq s}G^j_{ms}(N)E_mE_s^*+O(\varepsilon^{3/2}).$$

Thus, we have

$$\frac{d\tilde{n}_j}{dt} = \sqrt{\varepsilon}\Big(f_j(N)-\sum_{1\leq s\leq q}G^j_{ss}(N)|E_s|^2\Big)+O(\varepsilon^{3/2}). \tag{2.6}$$

We recall that N is defined by $N=N^0+\sqrt{\varepsilon}n$. It can be easily verified that the right hand side of (2.6) keeps its form if we replace N by $N^0+\sqrt{\varepsilon}\,\tilde{n}$.
For the field E the same relations hold as in (2.4), but if we replace N by $N^0+\sqrt{\varepsilon}\tilde{n}$, then we have to consider the first term separately. For this purpose, we rewrite the transformation (2.5) in the form

$$\tilde{n}_j = n_j+\sqrt{\varepsilon}\,h_j(N^0+\sqrt{\varepsilon}n, E, E^*), \tag{2.7}$$

where h_j is defined by

$$h_j(N^0+\sqrt{\varepsilon}\,n, E, E^*) := \sum_{m\neq s}E_mE_s^*\frac{G^j_{ms}(N)}{\lambda_m(N)+\lambda_s{}^*(N)}.$$

Let $h(N^0+\sqrt{\varepsilon}\,n, E, E^*)$ be the column-vector with the components $h_j(N^0+\sqrt{\varepsilon}\,n, E, E^*)$. Then (2.5) can be represented in the form

$$n=\tilde{n}-\sqrt{\varepsilon}\,h(N^0,E,E^*)+O(\varepsilon).$$

By means of that relation we get

$$\lambda_l(N)=\lambda_l(N^0+\sqrt{\varepsilon}\,n)=\lambda_l(N^0+\sqrt{\varepsilon}\,\tilde{n}-\varepsilon h(N^0,E,E^*)+O(\varepsilon^{3/2})),$$

so that we have

$$\lambda_l(N^0+\sqrt{\varepsilon}n)=\lambda_l(N^0+\sqrt{\varepsilon}\,\tilde{n})-\varepsilon M_l(N^0)h(N^0,E,E^*)+O(\varepsilon^{3/2}),$$

where $M_l(N)$ is the row-vector defined by

$$M_l(N)+iW_l(N) := \frac{\partial\lambda_l(N)}{\partial N}. \tag{2.8}$$

Thus, in the new variables, system (2.4) takes the form

$$
\begin{aligned}
\frac{dE_l}{dt} &= \lambda_l(N^0+\sqrt{\varepsilon}\;\tilde n)E_l - \varepsilon M_l(N^0)h(N^0,E,E^*)E_l \\
&\quad -\varepsilon \sum_{1\le j,p\le q} \alpha^j_{lp}(N)\Big(f_j(N) - \sum_{1\le m,s\le q} G^j_{ms}(N)E_mE^*_s\Big)E_p \\
&\quad +O(|E|\varepsilon^2), \quad l=1,\dots,q, \\
\frac{d\tilde n_j}{dt} &= \sqrt{\varepsilon}\left(f_j(N^0+\sqrt{\varepsilon}\;\tilde n) - \sum_{1\le m\le q}|E_m|^2G^j_{mm}(N^0+\sqrt{\varepsilon}\;\tilde n)\right)+O(\varepsilon^{3/2}), \\
&\quad l=1,\dots,q,\; j=1,\dots,q.
\end{aligned} \tag{2.9}
$$

In the following step we eliminate the term

$$\varepsilon M_l(N^0)h(N^0,E,E^*)E_l = \varepsilon\; E_l\sum_{j,m\ne s} M^j_l G^j_{ms}E_mE^*_s$$

in (2.9) by means of the transformation

$$\tilde E_l = E_l + \varepsilon E_l \sum_{j,m\ne s} M^j_l \frac{G^j_{ms}E_mE^*_s}{(\lambda_m+\lambda^*_s)^2}. \tag{2.10}$$

Differentiating (2.10) and taking into account (2.9) we get

$$
\begin{aligned}
\frac{d\tilde E_l}{dt} &= \frac{dE_l}{dt} + \varepsilon\; E_l\sum_{j,m\ne s} M^j_lG^j_{ms}E_mE^*_s\frac{\lambda_m+\lambda^*_s+\lambda_l}{(\lambda_m+\lambda^*_s)^2} \\
&= \lambda_lE_l - \varepsilon\; E_l\sum_{j,m\ne s} M^j_lG^j_{ms}E_mE^*_s - \\
&\quad -\varepsilon \sum_{1\le j,p\le q} \alpha^j_{lp}(N)\Big(f_j(N) - \sum_{1\le m,s\le q} G^j_{ms}(N)E_mE^*_s\Big)E_p \\
&\quad +\;\varepsilon\; E_l\sum_{j,m\ne s} M^j_lG^j_{ms}E_mE^*_s\frac{\lambda_m+\lambda^*_s+\lambda_l}{(\lambda_m+\lambda^*_s)^2} + O(|E|\varepsilon^2) \\
&= \lambda_lE_l + \varepsilon\; E_l\sum_{j,m\ne s} M^j_lG^j_{ms}E_{ms} - \\
&\quad -\varepsilon \sum_{1\le j,p\le q} \alpha^j_{lp}(N)\Big(f_j(N) - \sum_{1\le m,s\le q} G^j_{ms}(N)E_mE^*_s\Big)E_p \\
&\quad + O(|E|\varepsilon^2), \\
&= \lambda_l\tilde E_l - \varepsilon \sum_{1\le j,p\le q} \alpha^j_{lp}(N)\Big(f_j(N) - \sum_{1\le m,s\le q} G^j_{ms}(N)E_mE^*_s\Big)E_p \\
&\quad + O(|E|\varepsilon^2), \quad l=1,\dots,q.
\end{aligned} \tag{2.11}
$$

By means of the next transformation we eliminate all terms on the right-hand side of (2.11) which depend on the optical phase and are of order ε. We apply the following transformation with respect to the $\tilde{E}_l, l = 1, \dots, q$,

$$\overline{E}_l = \tilde{E}_l + \varepsilon \sum_{s\neq l} \sum_{j=1}^{q} \alpha_{ls}^j(N^0) f_j(N^0) \frac{\tilde{E}_s}{\lambda_s(N^0) - \lambda_l(N^0)}$$

$$\text{(2.12)} \qquad - \varepsilon \sum_{j=1}^{q} \sum_{s,m,p} \alpha_{lp}^j(N^0) G_{ms}^j(N^0) \frac{\tilde{E}_m \tilde{E}_s^* \tilde{E}_p}{\lambda_m(N^0) + \lambda_s^*(N^0) + \lambda_p(N^0) - \lambda_l(N^0)},$$

where the last sum is taken over all triples of the indices s, m and p, ranging from 1 to q, excluding those for which $m = l$ and $s = p$ or $p = l$ and $m = s$. By (A_3), the denominator $\lambda_m(N^0) + \lambda_s^*(N^0) + \lambda_p(N^0) - \lambda_l(N^0)$ is non-zero for these indices. Thus, the coordinate transformation (2.12) is well-defined. Differentiating (2.12) we get

$$\frac{d\overline{E}_l}{dt} = \frac{d\tilde{E}_l}{dt} + \varepsilon \sum_{s\neq l} \sum_{j=1}^{q} \alpha_{ls}^j(N^0) f_j(N^0) \frac{E_s \lambda_s(N^0)}{\lambda_s(N^0) - \lambda_l(N^0)} + O(|\tilde{E}|\varepsilon^{3/2})$$

$$-\varepsilon \sum_{j=1}^{q} \sum_{s,m,p} \alpha_{lp}^j(N^0) G_{ms}^j(N^0) \frac{\tilde{E}_m \tilde{E}_s^* \tilde{E}_p(\lambda_m(N^0) + \lambda_s^*(N^0) + \lambda_p(N^0))}{\lambda_m(N^0) + \lambda_s^*(N^0) + \lambda_p(N^0) - \lambda_l(N^0)}.$$

If we substitute for $\dfrac{d\tilde{E}_l}{dt}$ the expression on the right hand side of (2.11), we get

$$\frac{d\overline{E}_l}{dt} = \lambda_l(N)\overline{E}_l - \varepsilon \overline{E}_l \sum_{j} \alpha_{ll}^j(N^0) f_j(N^0) + \varepsilon\, \overline{E}_l \sum_{j,s} \alpha_{ll}^j(N^0) G_{ss}^j(N^0) |\overline{E}_s|^2$$

$$\text{(2.13)} \qquad + \varepsilon\, \overline{E}_l \sum_{j,s\neq l} \alpha_{ls}^j(N^0) G_{ls}^j(N^0) |\overline{E}_s|^2 + O(|\overline{E}|\varepsilon^{3/2}).$$

For the following we denote by R_l and I_l the real and the imaginary parts of the factor of $\varepsilon\overline{E}_l$ in (2.13), respectively:

$$R_l(N^0, \overline{E}, \overline{E}^*) + iI_l(N^0, \overline{E}, \overline{E}^*) := \sum_{j} \alpha_{ll}^j(N^0) f_j(N^0)$$

$$\text{(2.14)} \qquad - \sum_{j,s} \alpha_{ll}^j(N^0) G_{ss}^j(N^0) |\overline{E}_s|^2 - \sum_{j,s\neq l} \alpha_{ls}^j(N^0) G_{ls}^j(N^0) |\overline{E}_s|^2$$

$$:= \sum_{j=1}^{q} \tilde{\alpha}_{lj}(f_j(N^0) - \sum_{s=1}^{q} G_{ss}^j(N^0)|\overline{E}_s|^2) - \sum_{s\neq l} \beta_{ls} |\overline{E}_s|^2,$$

where we use the notation

$$\tilde{\alpha}_{lj} = \alpha_{ll}^j(N^0)$$

$$\beta_{ls} = \textstyle\sum_{j=1}^{q} [\alpha_{ll}^j(N^0) G_{ss}^j(N^0) + \alpha_{ls}^j(N^0) G_{ls}^j(N^0)].$$

From (2.13) and (2.9) we obtain the system

$$
\begin{aligned}
&\frac{d\overline{E}_l}{dt} = \lambda_l(N)\overline{E}_l - \varepsilon\, \overline{E}_l\Big(R_l(N^0,\overline{E},\overline{E}^*) + iI_l(N^0,\overline{E},\overline{E}^*)\Big) + O(|\overline{E}|\varepsilon^{3/2}),\\
(2.15)\qquad &\frac{d\tilde{n}_j}{dt} = \sqrt{\varepsilon}\left(f_j(N^0+\sqrt{\varepsilon}\tilde{n}) - \sum_{1\le m\le q}|\overline{E}_m|^2 G^j_{mm}(N^0+\sqrt{\varepsilon}\tilde{n})\right) + O(\varepsilon^{3/2}),\\
&\quad l=1,\dots,q,\ j=1,\dots,q.
\end{aligned}
$$

Our next goal is to eliminate the term $\varepsilon\ \overline{E}_l R_l(N^0,\overline{E},\overline{E}^*)$ on the right hand side of $d\overline{E}_l/dt$. For this purpose we require:

(A_4). The matrix $M(N^0)$ consisting of the row-vectors defined in (2.8) is invertible.

Under the assumption (A_4) we can introduce the new coordinate $\overline{n}$ implicitly by the system of equations $(l=1,\dots,q)$:

$$
(2.16)\qquad \mu_l(N^0+\sqrt{\varepsilon}\ \overline{n}) = \mu_l(N^0+\sqrt{\varepsilon}\ \tilde{n}) - \varepsilon R_l(N^0,\overline{E},\overline{E}^*).
$$

Let R be the column vector with the components $R_1,\dots,R_q$ and μ be the column vector with the components $\mu_1,\dots,\mu_q$. Under the assumption (A_4) we get from (2.16)

$$
(2.17)\qquad \overline{n} = \tilde{n} + \sqrt{\varepsilon}(M(N^0))^{-1}(M_{00}\tilde{n}\tilde{n} - M_{00}\overline{n}\,\overline{n} - R + O(\sqrt{\varepsilon})),
$$

where M_{00} is the bilinear form defined by

$$
M_{00}NN := \frac{\partial^2\mu}{\partial N^2}(N^0)NN.
$$

Substituting (2.17) into $M_{00}\overline{n}\,\overline{n}$ we get

$$
M_{00}\tilde{n}\,\tilde{n} - M_{00}\overline{n}\,\overline{n} = O(\sqrt{\varepsilon}).
$$

Thus, we have by (2.17)

$$
\overline{n} = \tilde{n} - \sqrt{\varepsilon}M(N^0)^{-1}R + O(\varepsilon),
$$

or

$$
\overline{n}_j = \tilde{n}_j - \sqrt{\varepsilon}\sum_{l=1}^{q} M^{jl}R_l + O(\varepsilon),
$$

where we denote by M^{jl} the entries of the matrix $M(N^0)^{-1}$. Using the above formulas, and taking into account (2.16) we obtain from (2.15)

$$
\begin{aligned}
\frac{d\overline{E}_l}{dt} &= E_l\Big(\mu_l(N^0+\sqrt{\varepsilon}\ \overline{n}) + i\omega_l(N^0+\sqrt{\varepsilon}\ \overline{n}+\varepsilon M(N^0)^{-1}R) - i\varepsilon I_l)\Big) \\
&\quad +O(\varepsilon^{3/2}|\overline{E}|), \\
\frac{d\overline{n}_j}{dt} &= \sqrt{\varepsilon}\Big(f_j(N^0+\sqrt{\varepsilon}\ \overline{n}) - \sum_{1\le m\le q}|\overline{E}_m|^2 G^j_{mm}(N^0+\sqrt{\varepsilon}\ \overline{n}) - \\
&\quad -\sum_{l=1}^{q} M^{jl}\frac{d}{dt}R_l\Big) + O(\varepsilon^{3/2}), \\
&\quad l=1,\dots,q,\ j=1,\dots,q.
\end{aligned} \tag{2.18}
$$

Introducing the notation (see (2.14))

$$
\bar{\alpha}_{lj} := \mathrm{Im}\tilde{\alpha}_{lj} - \sum_{s=1}^{q} W_{ls}\mathrm{Re}\tilde{\alpha}_{sj}, \qquad \bar{\beta}_{ls} := \mathrm{Im}\beta_{ls} - \sum_{p=1}^{q} W_{lp}\mathrm{Re}\beta_{ps},
$$

$$
\bar{G}_{jm}(N^0+\sqrt{\varepsilon}\bar{n}) := G^j_{mm}(N^0+\sqrt{\varepsilon}\bar{n}) - 2\gamma_{jm}\mu_m(N^0+\sqrt{\varepsilon}\bar{n}),
$$

$$
\gamma_{jm} := \sum_{p=1}^{q}\left[\sum_{l=1}^{q} M^{jl}\mathrm{Re}\tilde{\alpha}^p_l G^p_{mm}(N^0) + \sum_{l\neq m} M^{jl}\mathrm{Re}\beta^p_{lm}\right],
$$

where W_{ls} denotes the entries of the matrix $W(N^0)M(N^0)^{-1}$, we may rewrite system (2.18) as follows (note that the corrections $2\gamma_{jm}\mu_m$ to the coefficients G^j_{mm} are of order $\sqrt{\varepsilon}$ because $\mu(N^0)=0$ by assumption):

$$
\begin{aligned}
\frac{d\overline{E}_l}{dt} &= E_l\Big(\mu_l(N^0+\sqrt{\varepsilon}\ \overline{n}) + i\Big[\omega_l(N^0+\sqrt{\varepsilon}\ \overline{n}) - \sqrt{\varepsilon}\sum_{j=1}^{q}\bar{\alpha}_{lj}\frac{d}{dt}\bar{n}_j \\
&\quad -\varepsilon\sum_{s\neq l}\bar{\beta}_{ls}|\overline{E}_s|^2\Big]\Big) + O(\varepsilon^{3/2}|\overline{E}|), \\
\frac{d\overline{n}_j}{dt} &= \sqrt{\varepsilon}\Big(f_j(N^0+\sqrt{\varepsilon}\ \overline{n}) - \sum_{1\le m\le q}|\overline{E}_m|^2\bar{G}_{jm}(N^0+\sqrt{\varepsilon}\ \overline{n})\Big) + O(\varepsilon^{3/2}), \\
&\quad l=1,\dots,q,\ j=1,\dots,q.
\end{aligned} \tag{2.19}
$$

Summarizing our investigations we have the result:

Theorem 1. *Under the assumptions $(A_1)-(A_4)$, to any compact region of the phase space of system (2.4) there is a sufficiently small ε_0 such that for $0<\varepsilon\le\varepsilon_0$ system (2.4) is mapped into system (2.19) by a coordinate transformation, which is $O(\sqrt{\varepsilon})$ close to identity in the given region.*

Remark. System (2.19) is our wanted normal form.

3. The truncated system

If we omit the $\mathrm{O}(\varepsilon^{3/2})$- terms in (2.19) we get the truncated system

$$
\begin{aligned}
\frac{d\overline{E}_l}{dt} &= E_l\Big(\mu_l(N^0+\sqrt{\varepsilon}\,\overline{n}) + i\Big[\omega_l(N^0+\sqrt{\varepsilon}\,\overline{n}) \\
&\quad - \sqrt{\varepsilon}\sum_{j=1}^{q}\bar{\alpha}_{lj}\frac{d}{dt}\bar{n}_j - -\varepsilon\sum_{s\neq l}\bar{\beta}_{ls}|\overline{E}_s|^2\Big]\Big), \\
\frac{d\overline{n}_j}{dt} &= \sqrt{\varepsilon}\left(f_j(N^0+\sqrt{\varepsilon}\,\overline{n}) - \sum_{1\leq s\leq q}|\overline{E}_s|^2\bar{G}_{js}(N^0+\sqrt{\varepsilon}\,\overline{n})\right).
\end{aligned} \tag{3.1}
$$

The following theorem gives an answer to the question about the deviation of the trajectories of system (2.19) from the trajectories of the truncated system (3.1).

Theorem 2. *Let the hypotheses $(A_1)-(A_4)$ to be valid. Then the trajectories of the systems (2.19) and (3.1) starting at the same initial point are uniformly $O(\varepsilon)$-close on a time interval of order $O(\sqrt{1/\varepsilon})$.*

Proof. We write system (3.1) and system (2.19) in the form

$$
\frac{dz}{dt} = \zeta(z,\varepsilon), \quad \frac{dw}{dt} = \zeta(w,\varepsilon) + O(\varepsilon^{3/2}) \tag{3.2}
$$

respectively, where $z, w \in \mathbb{C}^{2q}$. We denote by $(\cdot,\cdot)$ the usual scalar product in $\mathbb{C}^{2q}$ and introduce by $\|v\| = \sqrt{(v,v)}$ a norm in $\mathbb{C}^{2q}$. Let $\mathcal{C}$ be some compact convex region in $\mathbb{C}^{2q}$. We denote by $Z(z,\varepsilon)$ the derivative of $\zeta(z,\varepsilon)$ with respect to z and by $\kappa(\varepsilon)$ the maximal eigenvalue of $\frac{1}{2}(Z(z,\varepsilon)+Z(z,\varepsilon)^{*^{\top}})$ for $z\in\mathcal{C}$. In our case, the relation $\kappa \leq \kappa_0\sqrt{\varepsilon}$ can be easily verified.
Let $z(t,\varepsilon)$ and $w(t,\varepsilon)$ be the solutions of the corresponding systems in (3.2) satisfying $z(0)=w(0)$, and let $\delta(t) = z(t,\varepsilon)-w(t,\varepsilon)$. Under our assumptions we have

$$
\begin{aligned}
\frac{d}{dt}\|\delta(t)\| &\leq \frac{((z(t,\varepsilon),\varepsilon)-(w(t,\varepsilon),\varepsilon)+O(\varepsilon^{3/2}),\delta(t))}{2\|\delta(t)\|} \\
&\quad + \frac{(\delta(t),(z(t,\varepsilon),\varepsilon)-(w(t,\varepsilon),\varepsilon)+O(\varepsilon^{3/2}))}{2\|\delta(t)\|} \\
&= \frac{((Z+Z^{*^{\top}})\delta(t),\delta(t))}{2\|\delta(t)\|} + O(\varepsilon^{3/2}) \\
&\leq \kappa_0\sqrt{\varepsilon}\|\delta(t)\| + O(\varepsilon^{3/2}).
\end{aligned}
$$

Taking into account $\delta(0)=0$ we obtain from this inequality

$$
\|\delta(t)\| \leq O(\varepsilon)e^{\kappa_0\sqrt{\varepsilon}t},
$$

which implies the result claimed in the theorem. □

If we represent $\overline{E}_l(t)$, $l=1,...,q$, in the form

$$
\overline{E}_l(t) = \sqrt{S_l(t)}e^{i\varphi_l(t)}, \tag{3.3}
$$

then we get from (3.1) the system

$$
\begin{aligned}
&\frac{d\varphi_l}{dt} = \omega_l(N^0 + \sqrt{\varepsilon}\ \overline{n}) - \sqrt{\varepsilon}\sum_{j=1}^{q} \bar{\alpha}_{lj}\frac{d}{dt}\bar{n}_j - \varepsilon\sum_{s\neq l}\bar{\beta}_{ls}S_s, \\
(3.4) \qquad &\frac{dS_l}{dt} = 2\mu_l(N^0 + \sqrt{\varepsilon}\overline{n})S_l, \\
&\frac{d\overline{n}_l}{dt} = \sqrt{\varepsilon}\Big(f_j(N^0 + \sqrt{\varepsilon}\overline{n}) - \sum_s \bar{G}_{js}(N^0 + \sqrt{\varepsilon}\overline{n})S_s\Big), \\
&\quad l = 1, ..., q.
\end{aligned}
$$

Let S and f be the column-vectors with the components $S_1, ..., S_q$ and $f_1, ..., f_q$, respectively, let $\bar{G}$ be the matrix with the entries $\bar{G}_{js}, 1 \leq s, j \leq q$. Then the amplitude system to (3.4) can be represented in the form

$$
\begin{aligned}
(3.5) \qquad &\frac{dS_l}{dt} = 2\mu_l(N^0 + \sqrt{\varepsilon}\overline{n})S_l, \\
&\frac{d\overline{n}}{dt} = \sqrt{\varepsilon}\Big(f(N^0 + \sqrt{\varepsilon}\overline{n}) - \bar{G}(N^0 + \sqrt{\varepsilon}\overline{n})S\Big).
\end{aligned}
$$

Using the scaling $\tau = \sqrt{\varepsilon}\, t$, $\tilde{\mu}_l = \mu_l/\sqrt{\varepsilon}$, we get from (3.5)

$$
\begin{aligned}
(3.6) \qquad &\frac{dS_l}{d\tau} = 2\tilde{\mu}_l(N^0 + \sqrt{\varepsilon}\overline{n})S_l, \qquad l = 1, \dots, q \\
&\frac{d\overline{n}}{d\tau} = f(N^0 + \sqrt{\varepsilon}\ \overline{n}) - \bar{G}(N^0 + \sqrt{\varepsilon}\ \overline{n})S.
\end{aligned}
$$

According to $\mu(N^0) = 0$ we obtain $\tilde{\mu}(N^0 + \sqrt{\varepsilon}\overline{n}) = M(N^0)\overline{n} + O(\sqrt{\varepsilon})$. Since $M(N^0)$ is invertible by assumption (A_4), we can implicitly introduce new variables $\eta_1, .., \eta_q$ by $\eta_j = \tilde{\mu}_j(N^0 + \sqrt{\varepsilon}\ \overline{n})$.

Taking into account $\overline{n} = M(N^0)^{-1}\eta + O(\sqrt{\varepsilon})$ we get from (3.6) the system

$$
\begin{aligned}
(3.7) \qquad &\frac{dS_j}{d\tau} = 2\eta_j S_j, \qquad j = 1, \dots, q, \\
&\frac{d\eta}{d\tau} = \hat{f}(N^0 + \sqrt{\varepsilon}\eta) - \hat{G}(N^0 + \sqrt{\varepsilon}\eta)S + O(\varepsilon),
\end{aligned}
$$

where $\hat{f} = M_0 f$, $\hat{G} = M_0\tilde{G}$.

For $\varepsilon = 0$ the amplitude system (3.7) has the form

$$
\begin{aligned}
(3.8) \qquad &\frac{dS_j}{d\tau} = 2\eta_j S_j, \qquad j = 1, \dots, q \\
&\frac{d\eta}{d\tau} = \hat{F}(N^0) - \hat{G}(N^0)S.
\end{aligned}
$$

This system is conservative and reversible: setting $S_j = e^{u_j}$ we get from (3.8)

$$\frac{d^2u}{d\tau^2} = \hat{F}(N^0) - \hat{G}(N^0)\begin{pmatrix} e^{u_1} \\ \vdots \\ e^{u_q} \end{pmatrix}. \tag{3.9}$$

Thus, the amplitude system (3.7) belongs to the class of conservative systems with a small (of order $O(\sqrt{\varepsilon})$) dissipation. In particular, if this system has an exponentially stable equilibrium or a periodic orbit, its Lyapunov exponents must be of order $O(\sqrt{\varepsilon})$ or less, i.e. the stability is rather weak.

4. Existence of invariant tori

By Theorem 2, system (3.4) and therefore also system (3.7) provides a good description of the dynamics of the original system (2.2). For example, the equilibria of (3.7) with non-negative S_j correspond to invariant tori of system (3.1): the dimension of the torus equals q minus the number of zero components of the vector S. Periodic orbits of system (3.7) lying in the region where all S_j are non-negative also correspond to invariant tori of (3.1) with the dimension $(q+1)$ minus the number of identically vanishing S_j. By the $O(\varepsilon)$-closeness of system (3.1) to the original system (2.2) it follows that if the invariant torus is normally-hyperbolic with the transverse Lyapunov exponent of order $O(\sqrt{\varepsilon})$ at least (i.e. if the characteristic exponents of the corresponding equilibrium or the periodic orbit of the amplitude system (3.7) lie on a distance of order at least $O(\sqrt{\varepsilon})$ from the imaginary axis), then this invariant torus persists in the original system for all small ε.

In what follows we investigate the case $q = 2$ in more detail. The amplitude system (3.7) is written here as

$$\begin{aligned} \frac{dS_1}{dt} &= \eta_1 S_1, \\ \frac{dS_2}{dt} &= 2\eta_2 S_2, \\ \frac{d\eta_1}{dt} &= F_1(N^0 + \sqrt{\varepsilon}\eta) - G_{11}(N^0 + \sqrt{\varepsilon}\eta)S_1 - G_{12}(N^0 + \sqrt{\varepsilon}\eta)S_2 + O(\varepsilon), \\ \frac{d\eta_2}{dt} &= F_2(N^0 + \sqrt{\varepsilon}\eta) - G_{21}(N^0 + \sqrt{\varepsilon}\eta)S_1 - G_{22}(N^0 + \sqrt{\varepsilon}\eta)S_2 + O(\varepsilon), \end{aligned} \tag{4.1}$$

where we removed the "hat"-signs from F and G. In the general case, for $\varepsilon = 0$ system (4.1) has a unique equilibrium satisfying $S_1 \neq 0,\ S_2 \neq 0$, namely

$$\begin{aligned} &\eta_1 = \eta_2 = 0, \\ &S_1 = S_1^* = (F_1(N^0)G_{22}(N^0) - F_2(N^0)G_{12}(N^0))/\Delta, \\ &S_2 = S_2^* = (F_2(N^0)G_{11}(N^0) - F_1(N^0)G_{21}(N^0))/\Delta, \end{aligned} \tag{4.2}$$

where

$$\Delta = G_{11}(N^0)G_{22}(N^0) - G_{12}(N^0)G_{21}(N^0).$$

Thus, system (4.2) has an equilibrium with positive S_1 and S_2 if and only if

$$
\begin{aligned}
(F_1(N^0)G_{22}(N^0) - F_2(N^0)G_{12}(N^0))\Delta > 0,\\
(F_2(N^0)G_{11}(N^0) - F_1(N^0)G_{21}(N^0))\Delta > 0.
\end{aligned}
\tag{4.3}
$$

Such equilibrium corresponds to a two-dimensional invariant torus of system (3.4), or, in other words, to a family (parametrized by two initial phases) of two-frequency solutions of (3.4) with frequencies close to $\omega_1(N^0)$ and $\omega_2(N^0)$.

From (1.5) it follows that the original system (2.2) has the following symmetry: if $(E(t), n(t))$ is a solution of (2.2) than also $(E(t)e^{\varphi}, n(t))$ is a solution, where φ is any real number. This symmetry implies that the phase-space can be factorized by identifying all points (E_1, E_2) having the same values of $|E_1|^2$, $|E_2|^2$ and $E_1E_2^*$. The truncated system (3.1) has the same symmetry. In order to prove that the torus, which corresponds to equilibrium (4.2), persists also for small ε, we note that the quasiperiodic solutions which fill it are relative periodic, i.e., they become periodic in the factorized state space.

Since systems (2.2) and (3.4) are close to each other (in the sense of Theorem 2), system (2.2) will have a relative periodic solution close to the relative periodic solution of (3.4), for all sufficiently small ε, provided the latter has no zero multipliers. This condition is equivalent to the requirement that the equilibrium of the amplitude system (4.1) has no zero characteristic root.

The characteristic equation for the equilibrium (4.2) of system(4.1) can be written as

$$
\begin{aligned}
&\lambda^4 - \sqrt{\varepsilon}(p_{11}+p_{22})\lambda^3 + \lambda^2(2S_1^*g_{11} + 2S_2^*g_{22}) +\\
&+\varepsilon(p_{11}p_{22} - p_{21}p_{12})) +\\
&+\sqrt{\varepsilon}\Big(2S_1^*(g_{21}p_{12} + g_{11}p_{22}) + 2S_2^*(g_{12}p_{21} - g_{22}p_{11})\Big)\lambda + 4S_1^*S_2^*\Delta + O(\varepsilon) = 0,
\end{aligned}
\tag{4.4}
$$

where we use the notation $g_{ij} = G_{ij}(N^0)$, $p_{ij} = \partial(F_i - G_{i1}S_1^* - G_{i2}S_2^*)/\partial N_j|_{N=N^0}$. In case $\varepsilon = 0$, where (4.4) can be reduced to the quadratic equation

$$
\varrho^2 - 2(S_1^*g_{11} + S_2^*g_{22})\varrho + 4S_1^*S_2^*\Delta = 0 \tag{4.5}
$$

the condition $S_1^*S_2^*\Delta \neq 0$ implies that no root of (4.5) vanishes, and, therefore, also no root of (4.4) for sufficiently small ε. Thus, we arrive at the following result.

Theorem 3. *For sufficiently small ε, system* (2.2) *with $q = 2$ has a unique two-dimensional invariant torus, that is a family of two-frequency solutions with frequencies close to $\omega_1(N^0)$ and $\omega_2(N^0)$ if and only if the conditions* (4.3) *are fulfilled.*

The corresponding invariant torus will be stable if all roots of the characteristic equation (4.4) are located in the left half plane and have a distance of order larger than $O(\varepsilon)$ to the imaginary axis. As we mentioned above, system (4.1) is $O(\sqrt{\varepsilon})$-close to a conservative system. Therefore, in order to ensure stability, the real parts of the characteristic roots (which tend to zero as $\varepsilon \to 0$) must be of order at least

$O(\sqrt{\varepsilon})$ for non-zero ε. So, we make an ansatz $\lambda = i\nu - \sqrt{\varepsilon}\sigma$ in the characteristic equation (4.4), and get the following equation:

$$\begin{aligned}
&\nu^4 - 2\nu^2(S_1^* g_{11} + S_2^* g_{22}) + 4S_1^* S_2^* \Delta + \\
&\quad + i\nu\sqrt{\varepsilon}[(p_{11} + p_{22} + 4\sigma)\nu^2 - 4\sigma(S_1^* g_{11} + S_2^* g_{22}) + \\
&\qquad + (S_1^*(g_{21}p_{12} - g_{11}p_{22}) + S_2^*(g_{12}p_{21} - g_{22}p_{11}))] + O(\varepsilon) = 0.
\end{aligned}$$

In the limit $\varepsilon = 0$, this yields

$$\nu^4 - 2\nu^2(S_1^* g_{11} + S_2^* g_{22}) + 4S_1^* S_2^* \Delta = 0,$$

$$(p_{11} + p_{22} + 4\sigma)\nu^2 = 4\sigma(S_1^* g_{11} + S_2^* g_{22}) + 2S_1^*(g_{11}p_{22} - g_{21}p_{12}) + 2S_2^*(g_{22}p_{11} - g_{12}p_{21}).$$

Thus, the stability condition for small ε requires that all solutions ν^2 and σ of these equations must be real and positive. A routine computation shows that this requirement is equivalent to the following set of inequalities:

$$\begin{aligned}
&S_1^* g_{11} + S_2^* g_{22} > 2\sqrt{S_1^* S_2^* \Delta}, \qquad p_{11} + p_{22} < 0, \\
&|S_1^*(g_{11}p_{22} - g_{21}p_{12}) + S_2^*(g_{22}p_{11} - g_{12}p_{21}) + |p_{11} + p_{22}|(S_1^* g_{11} + S_2^* g_{22})| < \\
&\quad < |p_{11} + p_{22}|\sqrt{(S_1^* g_{11} + S_2^* g_{22})^2 - 4S_1^* S_2^* \Delta}.
\end{aligned} \tag{4.6}$$

Hence, we have the following result.

Theorem 4. *The two-dimensional invariant torus established in Theorem 3 is asymptotically stable for sufficiently small ε provided the inequalities* (4.6) *hold.*

Acknowledgements

The second author was partially supported by the the grants CRDF No.Ru-M1-2583-MO-04, RFFI No. 02-01-00273, “Universitety Rossii” No. 1905.

References

[1] U. Bandelow, M. Radziunas, V. Tronciu, H.-J. Wünsche, F. Henneberger, *Tailoring the dynamics of diode lasers by dispersive reflectors*, in Proceedings of SPIE, vol. 3944, 2000, pp. 536–545.

[2] T. Erneux, G. Kozyreff, *Nearly vertical Hopf bifurcation for a passively Q-switched microchip laser*, J.Stat.Phys. 101, 2000, pp 543–552.

[3] M. Radziunas, H.-J. Wünsche, B. Sartorius, O. Brox, D. Hoffmann, K. Schneider, D. Marcenac, *Modeling Self-Pulsating DFB Lasers with Integrated Phase Tuning Section*, IEEE J. of Quantum Electronics 36, No.9, pp.1026-1034, 2000.

[4] J. Sieber, *Numerical bifurcation analysis for multi-section semiconductor lasers*, SIAM J. of Appl. Dyn. Sys. 1, 2002, pp 248–270.

[5] J. Sieber, *Longtime behavior of the coupled traveling-wave model for semiconductor lasers*, submitted to J. of Differ. Eqs.

[6] B. Tromborg, H.E. Lassen, H. Olesen, *Travelling Wave Analysis of Semiconductor Lasers*, IEEE J. of Quant. El. 30, No. 5, pp. 939-956, 1994.

[7] V.Z. Tronciu, H.-J. Wünsche, J. Sieber, K. Schneider, F. Henneberger, *Dynamics of single mode semiconductor lasers with passive dispersive reflectors*, Opt. Comm.182, 2000, pp. 221–228.

[8] D. TURAEV, *Fundamental obstacles to selfpulsations in low-intensity lasers*. WIAS Preprint 629, 2001