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# FINITE ELEMENT APPROXIMATION OF TRANSPORT OF REACTIVE SOLUTES IN POROUS MEDIA*

## Part 2 Error Estimates for Equilibrium Adsorption Processes

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**ABSTRACT** In this paper we analyse a fully practical piecewise linear finite element approximation; involving numerical integration, backward Euler time discretisation and possibly regularization and relaxation; of the following degenerate parabolic equation arising in a model of reactive solute transport in porous media: Find $u(x,t)$ such that

$$\partial_t u + \partial_t[\varphi(u)] - \Delta u = f \quad \text{in} \quad \Omega \times (0,T]$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T] \qquad u(\cdot,0) = g(\cdot) \quad \text{in } \Omega$$

for known data $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, $f$, $g$ and a monotonically increasing $\varphi \in C^0(\mathbb{R}) \cap C^1(-\infty,0] \cup (0,\infty)$ satisfying $\varphi(0) = 0$, which is only locally Hölder continuous, with exponent $p \in (0,1)$, at the origin; e.g. $\varphi(s) \equiv [s]_+^p$. This lack of Lipschitz continuity at the origin limits the regularity of the unique solution $u$ and leads to difficulties in the finite element error analysis.

## 1. INTRODUCTION

This is the second of two papers, in which we study finite element approximations of degenerate parabolic systems and equations, as they arise in the modelling of reactive solute transport in porous media. Here we concentrate on a quasistationarily described equilibrium adsorption reaction, leading to

$$\partial_t(\Theta u) + \rho\partial_t v - \nabla.(\Theta\underline{\underline{D}}\nabla u - \underline{q}u) = f \quad \text{in } Q_T \tag{1.1a}$$

$$v = \varphi(u) \quad \text{in } Q_T , \tag{1.1b}$$

supplemented by initial and boundary conditions for the unknown function u, the dissolved concentration. Here $\Omega$ is a bounded domain in $\mathbb{R}^d$, $1 \le d \le 3$, $[0,T]$ a fixed time interval and $Q_t \equiv \Omega \times (0,t]$, for $t \in (0,T]$. For more remarks on the model we refer to the introduction of part 1 and for a complete account to the literature cited there. The parameter functions $\Theta$, $\underline{q}$, $\underline{\underline{D}}$, $\rho$ are given and fulfill:

$$\partial_t\Theta + \nabla.\underline{q} = 0 \ , \quad \Theta(x,t) \ge \Theta_0 > 0, \quad \rho(x) \ge \rho_0 > 0 \quad \text{in } Q_T \tag{1.2}$$

and further conditions such that the linear part of (1.1) defines a uniformly parabolic operator. The nonlinearity $\varphi$ - the adsorption isotherm - is monotone non-decreasing, but there are typical examples, which are not Lipschitz continuous at $u = 0$ such as is the Freundlich isotherm

$$\varphi(u) \equiv \alpha u^p \quad \text{for } u \ge 0, \text{ where } \alpha \in \mathbb{R}^+ \text{ and } p \in (0,1). \tag{1.3}$$

Thus in general equation (1.1) is degenerate, exhibiting a finite speed of propagation property, such that a front given by the boundary of the support of u, is preserved. In fact, there is a close relation between equation (1.1) and the well-investigated (generalised) "porous medium equation" (see e.g. Aronson (1986)) which reads

$$\partial_t[\phi(u)] - \Delta u = f \quad \text{in } Q_T \tag{1.4a}$$

with $\phi(u) \equiv \text{sgn}(u)|u|^{1/m}$ for some $m > 1$; i.e. (a model problem of) (1.1) is of the form (1.4a), and (1.1) and (1.4a) are equivalent if we assume that for some $\alpha > 0$

$$\phi'(u) \geq \alpha \quad \text{for all } u \in \mathbb{R},\ u \neq 0. \tag{1.4b}$$

A sufficient condition for the finite speed of propagation property is

$$1/\phi \in L^1(0,\delta) \quad \text{for some } \delta > 0, \tag{1.5}$$

see Watanabe (1988), which has also been proven to be necessary in the one-dimensional case, see Peletier (1974). This condition is satisfied by (1.3), see also section 2. A common desription of chemical non-equlibrium has the form of relaxation kinetics, i.e.

$$\partial_t v = k(\varphi(u)-v) \tag{1.6}$$

with a rate parameter $k > 0$. Equations (1.1a), (1.6) in general form a degenerate system with the aforementioned property. In part 1 we gave a fairly complete order of convergence analysis in energy norms for Galerkin finite element approximations of the system (1.1a), (1.6); based on a technique which is at least applicable for time independent and smooth $\Theta$, $\underline{q}$, $\underline{\underline{D}}$. However, the fact that we analysed the Galerkin procedure assumes in addition that the system is not convection-dominated, where we would encounter all the well-known difficulties. This analysis has been presented for a model problem, to which we will restrict ourselves later on.

One may expect that for $k \to \infty$ $(P_k) \equiv$ (1.1a), (1.6) approximates $(P) \equiv$ (1.1). This may be called a kinetic approximation and will be made rigorous in section 2. The aim of this paper is to exploit the kinetic approximation as a proof technique device (and possibly also as an algorithmic device) to study order of convergence estimates for problem (P) on the basis of the results of part 1 for the relaxed problem $(P_k)$. There it turned out to be advantageous to introduce a regularized system $(P_{k,\varepsilon})$ obtained by substituting $\varphi$ by a Lipschitz continuous $\varphi_\varepsilon$, differing only near $u = 0$. The relaxation and regularization is a proof device insofar as the order of convergence estimates, established for the finite element approximation of $(P_{k,\varepsilon})$ for appropriate $k = O(h^{-\gamma})$, $\varepsilon = O(h^{\mu})$, where h is the mesh parameter, then carry over to the corresponding finite element approximations of $(P_\varepsilon)$, the regularized version of (P), and (P). The situation is different, if we improve on the estimates by taking a non-degeneracy condition into account. Then we cannot dispense with the regularization. The non-degeneracy condition describes the minimal growth of u away from the front. In the one-dimensional case the following result has been established by Aronson et al. (1983). We will assume later on, that $\varphi$ is Hölder continuous near $u = 0$ with exponent $p \in (0,1]$. If in addition the exponent is sharp, i.e.

$$\varphi(u) \geq \alpha u^p \quad \text{for } u \in [0,\delta_0] \text{ and for some } \alpha,\ \delta_0 > 0 \tag{1.7}$$

then:

**(N.D.)**
$$A_\varepsilon(t) \leq C\varepsilon, \tag{1.8a}$$

where

$$A_\varepsilon(t) \equiv \int_0^t \underline{m}(\Omega_\varepsilon(s))\, ds, \tag{1.8b}$$

$$\Omega_\varepsilon(t) \equiv \{\ x \in \Omega : u(x,t) \in (0,\varepsilon^{1/(1-p)})\ \}, \tag{1.8c}$$

and $\underline{m}$ is the Lebesgue measure.

For ease of exposition we will develop our results for the following model problem, which keeps the specific difficulty of the non-Lipschitz nonlinearity, but reduces the handling of standard terms:

**(P)** Find $u(x,t)$ such that

$$\partial_t u + \partial_t[\varphi(u)] - \Delta u = f \quad \text{in } Q_T$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T] \qquad u(\cdot,0) = g(\cdot) \quad \text{in } \Omega,$$

where we make the following assumptions on the given data:

Assumptions **(D1)**: $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, with either $\Omega$ convex polyhedral or $\partial\Omega \in C^{1,1}$, $f \in L^\infty(Q_T)$, $g \in L^\infty(\Omega) \cap H_0^1(\Omega)$ and $\varphi \in C^0(\mathbb{R})$ is such that

(i) $\quad \varphi(0) = 0$, $\varphi(s) > 0\ \forall\ s > 0$ and $\varphi$ is monotonically increasing $\quad$ (1.9a)

(ii) $\quad \varphi \in C^1(-\infty,0] \cup (0,\infty)$ $\quad$ (1.9b)

(iii) there exist $L \in \mathbb{R}^+$ and $\varepsilon_0$, $p \in (0,1]$ such that

$$|\varphi(a)-\varphi(b)| \le L|a-b|^p \quad \text{for all } a,b \in [0,\varepsilon_0]. \tag{1.9c}$$

The layout of this paper is as follows. In the next section we establish the existence and uniqueness of a solution to (P) by firstly establishling these results for a regularised relaxed version ($P_{k,\varepsilon}$). In section 3 we consider a continuous in time continuous piecewise linear finite element approximation in space to (P). In section 4 we consider a more practical approximation employing numerical integration on the nonlinear term. Finally in section 5 we consider a fully practical approximation involving discretisation in time using the backward Euler method.

The most complete order of convergence analysis until now for the finite element approximation of the porous medium equation, involving time discretisation and numerical integration has been given by Nochetto & Verdi (1988). Contrary to our approach they consider this approximation directly, taking regularization but not relaxation of the problem into account. A proviso in the comparison lies in the fact that in some places we require the mesh to be (weakly) acute, whereas they do not. In their main error bound, (3.4), the term $\varepsilon^{-2}h^4/\tau$ appears, where $\tau$ is the time step size. This gives the unnatural feature of an error bound deteriorating for fixed h (and $\varepsilon$) as $\tau \to 0$. Our approach, firstly leads to their resulting error bounds with a less severe time step constraint; that is, $\tau \le Ch$ as opposed to their restrictions $\tau = Ch^{1+p}$ and $\tau = Ch^{4/(3-p)}$ on not assuming and assuming (N.D.), respectively. Furthermore, under some additional assumptions we can improve on their error bounds in some cases. More details about these comparisons are given at the end of section 5.

Finally, we note that one could employ alternative forms of relaxation, not considered here. The description of a physically caused non-equilibrium may lead to

$$\partial_t v = k(u-\varphi^{-1}(v)). \tag{1.10}$$

For a nonlinearity of the type (1.3), $\varphi^{-1}$ is Lipschitz continuous, i.e. (1.1a), (1.10) is a regular system. This type of relaxation was used by Verdi & Visintin (1988) for the Stefan problem.

Throughout the paper we adopt the standard notation for Sobolev spaces. We note that the seminorm $|\cdot|_{H^1(\Omega)}$ and norm $\|\cdot\|_{H^1(\Omega)}$ are equivalent on $H^1_0(\Omega)$. The standard $L^2$ inner product over $\Omega$ is denoted by $(\cdot,\cdot)$. Throughout C denotes a generic positive constant independent of $\varepsilon$ the regularisation parameter, k the relaxation parameter and h the mesh spacing.

## 2. THE CONTINUOUS PROBLEM

In this section we establish existence and uniqueness of a solution to (P). In doing so we will develop various bounds that will be useful in analysing the error in the finite element approximation of (P). Firstly we introduce a regularized version of $(P_\varepsilon)$:

$(P_\varepsilon)$ Find $u_\varepsilon(x,t)$ such that

$$\partial_t u_\varepsilon + \partial_t[\varphi_\varepsilon(u_\varepsilon)] - \Delta u_\varepsilon = f \quad \text{in } Q_T$$

$$u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \times (0,T] \qquad u_\varepsilon(\cdot,0) = g(\cdot) \quad \text{in } \Omega,$$

where $\varphi_\varepsilon \in C^{0,1}_{loc}(\mathbb{R})$ is such that

(i) $\qquad \varphi_\varepsilon(s) \equiv \varphi(s)$ for $s \notin (0,\varepsilon^{1/(1-p)})$ $\qquad$ (2.1a)

(ii) $\qquad \varphi_\varepsilon(s)$ is strictly monotonically increasing on $[0,\varepsilon^{1/(1-p)}]$ $\qquad$ (2.1b)

(iii) $\qquad$ for $m \in \mathbb{N}$ there exists a $M(m) \in \mathbb{R}^+$:

$$\varphi_\varepsilon(b)-\varphi_\varepsilon(a) \le M(m)\varepsilon^{-1}(b-a) \qquad \text{for } -m \le a \le b \le m. \tag{2.1c}$$

Note that M can be chosen independently of m, if $\varphi'$ is bounded in $\mathbb{R}\backslash(0,\delta)$ for some $\delta > 0$. In addition we set

$$\Phi_\varepsilon(s) \equiv \int_0^s \varphi_\varepsilon(\sigma)\, d\sigma\ . \tag{2.2}$$

It is a simple matter to deduce from the conditions (2.1) that for all $|a|$, $|b| \le m$

$$[M(m)]^{-1}\varepsilon|\varphi_\varepsilon(a)-\varphi_\varepsilon(b)|^2 \le [\varphi_\varepsilon(a)-\varphi_\varepsilon(b)](a-b) \le M(m)\varepsilon^{-1}|a-b|^2\ , \tag{2.3a}$$

and

$$\varphi_\varepsilon(\varepsilon^{1/(1-p)}) = \varphi(\varepsilon^{1/(1-p)}) \le L\varepsilon^{p/(1-p)}. \tag{2.3b}$$

with L as in (1.9c). The simplest choice for $\varphi_\varepsilon$ is the linear regularization

$$\varphi_\varepsilon(s) \equiv \varepsilon^{-1/(1-p)}\varphi(\varepsilon^{1/(1-p)})s \qquad \text{for } s \in (0,\varepsilon^{1/(1-p)}). \tag{2.4}$$

In addition it is useful to consider the following problem in which the reaction process is relaxed in time with $k > 0$ being the given relaxation parameter.

$(P_{k,\varepsilon})$ Find $\{u_{k,\varepsilon}(x,t), v_{k,\varepsilon}(x,t)\}$ such that

$$\partial_t u_{k,\varepsilon} + \partial_t v_{k,\varepsilon} - \Delta u_{k,\varepsilon} = f \quad \text{in } Q_T \qquad u_{k,\varepsilon} = 0 \quad \text{on } \partial\Omega$$

$$\partial_t v_{k,\varepsilon} = k(\varphi_\varepsilon(u_{k,\varepsilon}) - v_{k,\varepsilon}) \quad \text{in } Q_T$$

$$u_{k,\varepsilon}(\cdot,0) = g(\cdot) \qquad v_{k,\varepsilon}(\cdot,0) = \varphi_\varepsilon(g(\cdot)) \quad \text{in } \Omega.$$

The above problem has been studied in part 1. We adopt the notion of weak solution defined there and below we recall some of the results.

**Theorem 2.1**

Let the Assumptions (D1) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$ and $k > 0$ there exists a unique weak solution $\{u_{k,\varepsilon}, v_{k,\varepsilon}\}$ to $(P_{k,\varepsilon})$ such that

$$\underline{u} \le u_{k,\varepsilon} \le \bar{u} \quad \text{and} \quad \underline{v} \le v_{k,\varepsilon} \le \bar{v} \quad \text{in } Q_T \tag{2.5a}$$

$$|\nabla u_{k,\varepsilon}|^2_{L^2(Q_T)} + |\partial_t u_{k,\varepsilon}|^2_{L^2(Q_T)} + |\partial_t v_{k,\varepsilon}|^2_{L^\infty(Q_T)} \le C(k), \tag{2.5b}$$

where $\underline{u}$, $\bar{u}$, $\underline{v}$, $\bar{v} \in C(\bar{\Omega})$ are all independent of $\varepsilon$ and $k$. Furthermore, if $g$ and $f \ge 0$ one can take $\underline{u} = \underline{v} = 0$.

Proof: This result with $\underline{u}$, $\bar{u}$, $\underline{v}$, $\bar{v} \in C[0,T]$, all independent of $\varepsilon$ and uniformly bounded in $k$, is proved in Theorem 2.1 of part 1 in the case $v_{k,\varepsilon}(\cdot,0) = \tilde{g}(\cdot) \in L^\infty(\Omega)$ for all $k > 0$ and $\varepsilon \in (0,\varepsilon_0]$. That proof is easily adapted to the present case. Furthermore, noting Remark 2.1 of part 1 yields the above choice of $\underline{u}$, $\bar{u}$, $\underline{v}$, $\bar{v}$. We note for later purposes that $\underline{u}$ and $\bar{u}$ depend only on $\Omega$, $|f|_{L^\infty(Q_T)}$ and $|g|_{L^\infty(\Omega)}$. □

**Lemma 2.1**

Under Assumptions (D1) we have for all $\varepsilon \in (0,\varepsilon_0]$, $k > 0$ and $t \in (0,T]$ that

$$\varepsilon|\nabla\varphi_\varepsilon(u_{k,\varepsilon})|^2_{L^2(Q_T)} + (\Phi_\varepsilon(u_{k,\varepsilon}(\cdot,t)),1) + k|\varphi_\varepsilon(u_{k,\varepsilon}) - v_{k,\varepsilon}|^2_{L^2(Q_T)} + \\ + |v_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + k^{-1}|\partial_t v_{k,\varepsilon}|^2_{L^2(Q_T)} \le C. \tag{2.6}$$

Proof: This result is proved in Lemma 2.2 of part 1 in the case $v_{k,\varepsilon}(\cdot,0) = \tilde{g}(\cdot) \in L^\infty(\Omega)$ for all $k > 0$ and $\varepsilon \in (0,\varepsilon_0]$. That proof is easily adapted to the present case. □

For $k \in \mathbb{R}^+_e \equiv \mathbb{R}^+ \cup \{\infty\}$ and for sufficiently smooth $w$ we set

$$\|w\|^2_{E_1(k,t)} \equiv |w|^2_{L^2(Q_t)} + \tfrac{1}{2} k^{-1}|w(\cdot,t)|^2_{L^2(\Omega)}$$

and

$$\|w\|^2_{E_2(k,t)} \equiv \|w\|^2_{E_1(k,t)} + \tfrac{1}{2}\Big|\nabla\int_0^t w(\cdot,s)ds\Big|^2_{L^2(\Omega)} + k^{-1}|\nabla w|^2_{L^2(Q_t)}.$$

**Lemma 2.2**

Let the Assumptions (D1) hold. For $0 < k_2 < k_1$ and for all $\varepsilon \in (0,\varepsilon_0]$ let $\{u_{k_i}, v_{k_i}\}$ be the unique weak solution to $(P_{k_i,\varepsilon})$, $i = 1, 2$. Then for all $t \in (0,T]$ we have that

$$\|u_{k_1,\varepsilon}-u_{k_2,\varepsilon}\|^2_{E_2(k_1,t)} + \varepsilon|\varphi_\varepsilon(u_{k_1,\varepsilon})-\varphi_\varepsilon(u_{k_2,\varepsilon})|^2_{L^2(Q_t)} + \varepsilon\|v_{k_1,\varepsilon}-v_{k_2,\varepsilon}\|^2_{E_1(k_1,t)}$$
$$\le Ck_2^{-2}|\partial_t v_{k_2,\varepsilon}|^2_{L^2(Q_T)} \le Ck_2^{-1}. \qquad (2.7)$$

Proof: Let $e^u_{k,\varepsilon} \equiv u_{k_1,\varepsilon}-u_{k_2,\varepsilon}$ and $e^v_{k,\varepsilon} \equiv v_{k_1,\varepsilon}-v_{k_2,\varepsilon}$. Then subtracting the first equation in $(P_{k_2,\varepsilon})$ from that in $(P_{k_1,\varepsilon})$, multiplying by $\int_s^t e^u_{k,\varepsilon}(\cdot,\sigma)d\sigma$, integrating over $Q_t$, where $s$ is the integration variable in time, and performing integration by parts yields that

$$|e^u_{k,\varepsilon}|^2_{L^2(Q_t)} + \tfrac{1}{2}\Big|\nabla\int_0^t e^u_{k,\varepsilon}(\cdot,s)ds\Big|^2_{L^2(\Omega)} = -\int_0^t(e^v_{k,\varepsilon}(\cdot,s)),e^u_{k,\varepsilon}(\cdot,s))ds. \qquad (2.8)$$

Repeating the above, but multiplying by $e^u_{k,\varepsilon}(\cdot,s)$ in place of $\int_s^t e^u_{k,\varepsilon}(\cdot,\sigma)d\sigma$ yields that

$$\tfrac{1}{2}|e^u_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + |\nabla e^u_{k,\varepsilon}|^2_{L^2(Q_t)} = -\int_0^t(\partial_s e^v_{k,\varepsilon}(\cdot,s)),e^u_{k,\varepsilon}(\cdot,s))ds. \qquad (2.9)$$

Combining (2.8) and (2.9), and noting (2.3a) yields that

$$\|e^u_{k,\varepsilon}\|^2_{E_2(k_1,t)} + [M(m)]^{-1}\varepsilon|\varphi_\varepsilon(u_{k_1,\varepsilon})-\varphi_\varepsilon(u_{k_2,\varepsilon})|^2_{L^2(Q_t)}$$

$$\le \|e^u_{k,\varepsilon}\|^2_{E_2(k_1,t)} + \int_0^t(\varphi_\varepsilon(u_{k_1,\varepsilon}(\cdot,s))-\varphi_\varepsilon(u_{k_2,\varepsilon}(\cdot,s)),e^u_{k,\varepsilon}(\cdot,s))ds$$

$$= (k_1^{-1}-k_2^{-1}) \int_0^t(\partial_s v_{k_2,\varepsilon}(\cdot,s),e^u_{k,\varepsilon}(\cdot,s))ds \le Ck_2^{-2}|\partial_t v_{k_2,\varepsilon}|^2_{L^2(Q_T)}, \qquad (2.10)$$

where $[\inf \underline{u}, \sup \bar{u}] \subseteq [-m,m]$, see Theorem 2.1.

Finally, subtracting the second equation in $(P_{k_2,\varepsilon})$ from that in $(P_{k_1,\varepsilon})$, multiplying by $e^v_{k,\varepsilon}$ and integrating over $Q_t$ yields that

$$\|e^v_{k,\varepsilon}\|^2_{E_1(k_1,t)} = \int_0^t(\varphi_\varepsilon(u_{k_1,\varepsilon}(\cdot,s))-\varphi_\varepsilon(u_{k_2,\varepsilon}(\cdot,s)),e^v_{k,\varepsilon}(\cdot,s))ds +$$

$$+ (1-k_1^{-1}k_2) \int_0^t(\varphi_\varepsilon(u_{k_2,\varepsilon}(\cdot,s))-v_{k_2,\varepsilon}(\cdot,s),e^v_{k,\varepsilon}(\cdot,s))ds$$

$$\le C\varepsilon^{-1}k_2^{-2}|\partial_t v_{k_2,\varepsilon}|^2_{L^2(Q_T)}. \qquad (2.11)$$

Combining (2.10) and (2.11) yields the first inequality in (2.7). The second inequality then follows from the bound (2.6). □

Assumptions (D2): In addition to the Assumptions (D1) we assume that $f \in H^1(0,T;L^2(\Omega))$, $g \in H^2(\Omega)$ and that $k \ge k_0$.

Lemma 2.3

Under Assumptions (D2) we have for all $\varepsilon \in (0,\varepsilon_0]$ and $t \in (0,T]$ that

$$|\nabla u_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + |\partial_t u_{k,\varepsilon}|^2_{L^2(Q_T)} + \varepsilon|\partial_t v_{k,\varepsilon}|^2_{L^2(Q_T)} + \varepsilon|\partial_t[\varphi_\varepsilon(u_{k,\varepsilon})]|^2_{L^2(Q_T)} +$$

$$+ k^{-1}\left[|\partial_t u_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\partial_t v_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + |\nabla(\partial_t u_{k,\varepsilon})|^2_{L^2(Q_T)}\right] \le C. \qquad (2.12)$$

Proof: The result (2.12) is proved in Lemma 2.3 in part 1. □

We will prove existence of solutions of problems $(P_\varepsilon)$ or $(P)$ in the following sense.

Definition: $u_\varepsilon$ is a weak solution to $(P_\varepsilon)$ if $u_\varepsilon \in L^2(0,T;H^1_0(\Omega))$ is such that $\varphi_\varepsilon(u_\varepsilon) \in L^2(Q_T)$ and for all test functions $\eta \in L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega))$ with $\eta(\cdot,T) = 0$ in $\Omega$

$$\int_{Q_T} \{ - [u_\varepsilon + \varphi_\varepsilon(u_\varepsilon)]\partial_t \eta + \nabla u_\varepsilon . \nabla\eta - f\eta \} \, dxdt - ([g(\cdot) + \varphi_\varepsilon(g(\cdot))], \eta(\cdot,0)) = 0.$$

A similar definition holds for (P) with $u_\varepsilon$ and $\varphi_\varepsilon(u_\varepsilon)$ replace by $u$ and $\varphi(u)$.

**Remark 2.1**

For problem $(P_\varepsilon)$ we will also use the stronger notion defined in part 1: $u_\varepsilon \in L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega))$ is such that $\varphi_\varepsilon(u_\varepsilon) \in H^1(0,T;L^2(\Omega))$ and for all $\eta \in L^2(0,T;H^1_0(\Omega))$

$$\int_{Q_T} \{ \partial_t [u_\varepsilon + \varphi_\varepsilon(u_\varepsilon)]\eta + \nabla u_\varepsilon . \nabla\eta - f\eta \} \, dxdt = 0, \qquad u_\varepsilon(\cdot,0) = g(\cdot) \text{ on } \Omega.$$

**Lemma 2.4**

Let the Assumptions (D1) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$ if there exists a weak solution $u_\varepsilon$ of $(P_\varepsilon)$, then it is unique. Furthermore, if it is a solution in the sense of Remark 2.1, then

$$|\nabla u_\varepsilon|^2_{L^2(Q_T)} \le C[\,|f|^2_{L^2(Q_T)} + |g|^2_{L^2(\Omega)} + |\varphi_\varepsilon(g)|^2_{L^2(\Omega)}]. \tag{2.13}$$

Proof: Let $e \equiv u^1_\varepsilon - u^2_\varepsilon$, where $u^1_\varepsilon$, $u^2_\varepsilon$ are two weak solutions of $(P_\varepsilon)$. Then subtracting the two defining equations and choosing $\eta(\cdot,t) \equiv \int_t^T e(\cdot,s)ds$ yields

$$|e|^2_{L^2(Q_T)} + \tfrac{1}{2}|\nabla \int_0^T e(\cdot,s)ds|^2_{L^2(\Omega)} \le 0$$

and hence uniqueness.

If $u_\varepsilon$ is a solution in the sense of Remark 2.1, then we can choose $\eta \equiv u_\varepsilon$ in Remark 2.1 and obtain

$$\tfrac{1}{2}|u_\varepsilon(\cdot,T)|^2_{L^2(\Omega)} + |\nabla u_\varepsilon|^2_{L^2(Q_T)} + \int_{Q_T} \partial_t[\varphi_\varepsilon(u_\varepsilon)]u_\varepsilon = \tfrac{1}{2}|g|^2_{L^2(\Omega)} + \int_{Q_T} f u_\varepsilon.$$

Noting that

$$\int_{Q_T} \partial_t[\varphi_\varepsilon(u_\varepsilon)]u_\varepsilon = [(\varphi_\varepsilon(u_\varepsilon(\cdot,T)),u_\varepsilon(\cdot,T)) - (\varphi_\varepsilon(g),g)] -$$

$$- (\Phi_\varepsilon(u_\varepsilon(\cdot,T)) - \Phi_\varepsilon(g(\cdot)),1) \ge -\tfrac{1}{2}\, [\,|\varphi_\varepsilon(g)|^2_{L^2(\Omega)} + |g|^2_{L^2(\Omega)}],$$

since $0 \le \Phi_\varepsilon(s) \le \varphi_\varepsilon(s)s$ for all $s \in \mathbb{R}$; the desired result (2.13) then follows from a Gronwall inequality. □

**Theorem 2.2**

Let the Assumptions (D1) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$ there exists a unique weak solution $u_\varepsilon$ to $(P_\varepsilon)$ and for all $k > 0$ and $t \in (0,T]$ we have that

$$\|u_\varepsilon - u_{k,\varepsilon}\|^2_{E_2(\infty,t)} + \varepsilon|\varphi_\varepsilon(u_\varepsilon)-\varphi_\varepsilon(u_{k,\varepsilon})|^2_{L^2(Q_t)} + \varepsilon|\varphi_\varepsilon(u_\varepsilon)-v_{k,\varepsilon}|^2_{L^2(Q_t)}$$

$$\le Ck^{-2}|\partial_t v_{k,\varepsilon}|^2_{L^2(Q_T)} \le Ck^{-1}. \tag{2.14a}$$

In addition

$$|\nabla u_\varepsilon|_{L^2(Q_T)} \le C \tag{2.14b}$$

and

$$\underline{u} \le u_\varepsilon \le \bar{u} \quad \text{in } Q_T, \tag{2.14c}$$

where $\underline{u}$, $\bar{u} \in C^0(\bar{\Omega})$ are independent of $\varepsilon$. Moreover, if $g$ and $f \ge 0$ then $u_\varepsilon \ge 0$ in $Q_T$.

Proof: Firstly, we establish the existence of a weak solution to $(P_\varepsilon)$. Let $k_n \to \infty$ as $n \to \infty$ and let $\{u_{k_n,\varepsilon}, v_{k_n,\varepsilon}\}$ be the unique weak solution to $(P_{k_n,\varepsilon})$. It follows from (2.7) that $\{u_{k_n,\varepsilon}, v_{k_n,\varepsilon}\}$ is Cauchy in $L^2(Q_T) \times L^2(Q_T)$ and therefore $\{u_{k_n,\varepsilon}, v_{k_n,\varepsilon}\} \to \{u_\varepsilon, v_\varepsilon\}$ in $L^2(Q_T) \times L^2(Q_T)$ as $n \to \infty$. In particular the bounds (2.5a) hold true for the limit.

We next restrict ourselves to Assumptions (D2) and show the existence of a solution in the sense of Remark 2.1. Due to Lemma 2.3 $\{u_{k_n,\varepsilon}\}$ is bounded in $L^2(0,T;H^1_0(\Omega))$ and in $H^1(0,T;L^2(\Omega))\cap\{\eta:\eta(\cdot,0)=g(\cdot)\}$ and $\{v_{k_n,\varepsilon}\}$ is bounded in

$H^1(0,T;L^2(\Omega))\cap\{\eta:\eta(\cdot,0)=\varphi_\varepsilon(g(\cdot))\}$. Therefore $\{u_{k_n,\varepsilon}, v_{k_n,\varepsilon}\}$ converges weakly in the corresponding spaces to $\{u_\varepsilon, v_\varepsilon\}$. Thus we have shown all the properties in Remark 2.1, if we can verify that $v_\varepsilon \equiv \varphi_\varepsilon(u_\varepsilon)$. Because of (2.1c) and (2.5a) it follows from $u_{k_n,\varepsilon} \to u_\varepsilon$ in $L^2(Q_T)$ that $\varphi_\varepsilon(u_{k_n,\varepsilon}) \to \varphi_\varepsilon(u_\varepsilon)$ in $L^2(Q_T)$. Furthermore, $(\partial_t v_{k_n,\varepsilon})/k_n = \varphi_\varepsilon(u_{k_n,\varepsilon})-v_{k_n,\varepsilon}$ and the left hand side converges to zero in $L^2(Q_T)$ due to (2.6). Hence passing to the limit yields $v_\varepsilon \equiv \varphi_\varepsilon(u_\varepsilon)$. Therefore under Asssumptions (D2) there exists a solution $u_\varepsilon$ to $(P_\varepsilon)$, in the sense of Remark 2.1, satisfying (2.14c). Uniqueness follows from Lemma 2.4. (2.14b) follows from (2.13) and (2.14a) follows directly from (2.7) by letting $k_1 \to \infty$.

We now weaken the assumptions to (D1) and approximate the data with $g^j \in H^2(\Omega)$ and $f^j \in H^1(0,T;L^2(\Omega))$ such that $g^j$ and $f^j$ are uniformly bounded independently of j and

$$g^j \to g,\ \varphi_\varepsilon(g^j) \to \varphi_\varepsilon(g) \quad \text{in } L^2(\Omega) \quad \text{and} \quad f^j \to f \text{ in } L^2(Q_T). \tag{2.15}$$

Then there exist corresponding solutions $u^j_\varepsilon$ of problem $(P_\varepsilon)$, in the sense of Remark 2.1. Let $e^u \equiv u^j_\varepsilon-u^i_\varepsilon$ and $e^v \equiv v^j_\varepsilon-v^i_\varepsilon$. Subtracting the corresponding solutions and using the test function $\eta(\cdot,t) \equiv \int_t^T e^u(\cdot,s)ds$ and performing integration by parts yields

$$|e^u|^2_{L^2(Q_T)} + \tfrac{1}{2}\Big|\nabla\int_0^T e^u(\cdot,t)dt\Big|^2_{L^2(\Omega)} + \varepsilon|\varphi_\varepsilon(u^j_\varepsilon)-\varphi_\varepsilon(u^i_\varepsilon)|^2_{L^2(Q_T)}$$
$$\le C|[g^j+\varphi_\varepsilon(g^j)] - [g^i+\varphi_\varepsilon(g^i)]|^2_{L^2(\Omega)} + C|f^j-f^i|^2_{L^2(Q_T)}, \tag{2.16}$$

where we have noted (2.3a). Therefore $\{u^j_\varepsilon\}$ is Cauchy in $L^2(Q_T)$ and $\{u^j_\varepsilon\} \to \{u_\varepsilon\}$ in $L^2(Q_T)$. Next we note that the bounds (2.5a) hold for $\{u^j_\varepsilon\}$ with $\underline{u}$ and $\bar{u}$ independent of j (and $\varepsilon$), see (2.15) and the proof of Theorem 2.1. Therefore we conclude, as above, that $\varphi_\varepsilon(u^j_\varepsilon) \to \varphi_\varepsilon(u_\varepsilon)$ in $L^2(Q_T)$. In addition, it follows from (2.13) that $\{u^j_\varepsilon\}$ is bounded in $L^2(0,T;H^1_0(\Omega))$ and therefore weakly convergent in this space. This enables us to pass to the limit $j \to \infty$ in the defining equation for a weak solution and hence conclude the existence proof and that (2.14b) holds. Uniqueness and the bound (2.14c) follow as above.

Finally, we need to prove the bound (2.14a). Firstly, we approximate the data as in (2.15) for problem $(P_{k,\varepsilon})$ and establish (2.14a) for $\{u_\varepsilon, u_{k,\varepsilon}, v_{k,\varepsilon}\}$ replaced by $\{u^j_\varepsilon, u^j_{k,\varepsilon}, v^j_{k,\varepsilon}\}$. With a similar proof as for (2.16) we conclude the analogous stability result for $(P_{k,\varepsilon})$. Furthermore, noting that (2.6) holds independently of (D2), we can pass to the limit $j \to \infty$ in (2.14a) and hence obtain the desired result. □

**Lemma 2.5**

Under Assumptions (D2) we have for all $\varepsilon \in (0,\varepsilon_0]$ that

$$\|u_\varepsilon - u_{k,\varepsilon}\|^2_{E_2(\infty,t)} + \varepsilon|\varphi_\varepsilon(u_\varepsilon)-\varphi_\varepsilon(u_{k,\varepsilon})|^2_{L^2(Q_t)} + \varepsilon|\varphi_\varepsilon(u_\varepsilon)-v_{k,\varepsilon}|^2_{L^2(Q_t)} \le C\varepsilon^{-1}k^{-2}. \qquad (2.17)$$

Proof: The result follows from the first inequality in (2.14a) and from (2.12). □

**Theorem 2.3**

Let the Assumptions (D1) hold. Then there exists a unique weak solution u to (P) and for all $\varepsilon \in (0,\varepsilon_0]$ and $t \in (0,T]$ we have that

$$\|u-u_\varepsilon\|^2_{E_2(\infty,t)} + \varepsilon|\varphi(u)-\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} \le C\ A_\varepsilon(t)\ \varepsilon^{(1+p)/(1-p)} \qquad (2.18)$$

In addition, the bounds (2.14c) hold for u and if g and f $\ge 0$ then u $\ge 0$ in $Q_T$.

Proof: To prove existence of a weak solution to (P) we let $\varepsilon \to 0$ in $(P_\varepsilon)$. Due to (2.14a) the unique weak solution $u_\varepsilon$ to $(P_\varepsilon)$ is such that $\{u_\varepsilon, \varphi_\varepsilon(u_\varepsilon)\}$ is the limit of a sequence $\{u_{k_n,\varepsilon}, \varphi_\varepsilon(u_{k_n,\varepsilon})\}$ with respect to $\|\cdot\|_{E_2(\infty,t)}$ and

$|\cdot|_{L^2(Q_t)}$ for $k_n \to \infty$, where $u_{k_n,\varepsilon}$ is the unique weak solution to $(P_{k_n,\varepsilon})$. We apply Lemma 2.1 of part 1 to $u_{k_n,\varepsilon_i} - u_{k_n,\varepsilon_j}$ and let $k_n \to \infty$ to conclude for $\varepsilon_0 \geq \varepsilon_j \geq \varepsilon_i > 0$ that

$$\|u_{\varepsilon_i} - u_{\varepsilon_j}\|^2_{E_2(\infty,t)} + \varepsilon_j|\varphi_{\varepsilon_i}(u_{\varepsilon_i}) - \varphi_{\varepsilon_j}(u_{\varepsilon_j})|^2_{L^2(Q_t)} \leq C\varepsilon_j^{(1+p)/(1-p)}.$$

Thus $\{u_{\varepsilon_i}\}$ is Cauchy in $L^2(Q_T)$, i.e. $u_\varepsilon \to u$ in $L^2(Q_T)$. The bounds (2.14c) also hold true for the limit and due to (2.14b) we have also weak convergence in $L^2(0,T;H^1_0(\Omega))$. Therefore to pass to the limit in the weak formulation we only have to show that $\varphi_{\varepsilon_i}(u_{\varepsilon_i}) \to \varphi(u)$ in $L^2(Q_T)$. This is done as in the proof of Theorem 2.2 of part 1: We have from (1.9), (2.1) and (2.3b) that

$$\begin{aligned}|\varphi(u) - \varphi_{\varepsilon_i}(u_{\varepsilon_i})|_{L^2(Q_T)} &\leq |\varphi(u) - \varphi(u_{\varepsilon_i})|_{L^2(Q_T)} + |\varphi(u_{\varepsilon_i}) - \varphi_{\varepsilon_i}(u_{\varepsilon_i})|_{L^2(Q_T)} \\ &\leq C|u - u_{\varepsilon_i}|^p_{L^2(Q_T)} + C\varepsilon_i^{p/(1-p)}\end{aligned}$$

and hence the desired result. Uniqueness follows as for $(P_\varepsilon)$, see Lemma 2.4, with $\varphi_\varepsilon$ replaced by $\varphi$.

Finally to show (2.18), for convenience we repeat a simplified version of the proof of Lemma 2.1 of part 1: Let $e^u \equiv u - u_\varepsilon$ and $e^v \equiv \varphi(u) - \varphi_\varepsilon(u_\varepsilon)$. Using once again the primitive of $e^u$ as a test function yields that

$$\|e^u\|^2_{E_2(\infty,t)} = -\int_0^t (e^v(\cdot,s), e^u(\cdot,s))ds$$

and therefore

$$\begin{aligned}\|e^u\|^2_{E_2(\infty,t)} &+ [M(m)]^{-1}\varepsilon|\varphi(u) - \varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} \\ &\leq \int_0^t (\varphi(u(\cdot,s)) - \varphi_\varepsilon(u_\varepsilon(\cdot,s)), (\zeta - u)(\cdot,s))ds;\end{aligned}$$

where $\zeta \equiv \varphi_\varepsilon^{-1}(\varphi(u))$ if $\varphi(u) \in (0, \varphi(\varepsilon^{p/(1-p)}))$ and $\zeta \equiv u$ otherwise, and $[\inf \underline{u}\ ,\ \sup \bar{u}] \subseteq [-m,m]$. Hence the desired result (2.18) follows from noting (1.8) and (2.3). □

**Lemma 2.6**

Under Assumptions (D2) we have for all $t \in (0,T]$ that the unique weak solutions $u$ and $u_{k,\varepsilon}$ of (P) and $(P_{k,\varepsilon})$ are such that

(i) On choosing $\varepsilon = Ck^{-(1-p)} \le \varepsilon_0$

$$\|u-u_{k,\varepsilon}\|_{E_2(\infty,t)} \le Ck^{-(1+p)/2}, \qquad |\varphi(u)-\varphi_\varepsilon(u_{k,\varepsilon})|_{L^2(Q_t)} \le Ck^{-p}. \tag{2.19a}$$

The above also holds true with $u_{k,\varepsilon}$ and $\varphi_\varepsilon(u_{k,\varepsilon})$ replaced by $u_k$ and $\varphi(u_k)$, respectively; where $u_k$ is the unique weak solution of $(P_k)$.

(ii) On assuming (N.D.) and choosing $\varepsilon = Ck^{-2(1-p)/(3-p)} \le \varepsilon_0$

$$\|u-u_{k,\varepsilon}\|_{E_2(\infty,t)} \le Ck^{-2/(3-p)}, \qquad |\varphi(u)-\varphi_\varepsilon(u_{k,\varepsilon})|_{L^2(Q_t)} \le Ck^{-(1+p)/(3-p)}. \tag{2.19b}$$

Proof: The desired results follow immediately from (2.17), (2.18) and Theorem 2.2 of part 1. □

Problem (P) is strongly related to a degenerate problem, which has been investigated intensively, the (generalised) porous medium equation

$$\partial_t w - \Delta[\beta(w)] = f \quad \text{in } Q_T, \tag{2.20}$$

where $\beta$: $\mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, and without loss of generality $\beta(0) = 0$. The (classical) porous medium equation is given by $\beta(w) \equiv \mathrm{sgn}(w)|w|^m$ for some $m > 1$. A change of variables yields (1.4a) with $\phi \equiv \beta^{-1}$. Obviously (P) is of the form (1.4a). On the other hand, (1.4a) can be written in the form of (P), if we assume that $\phi$ satisfies (1.9) and, as Nochetto & Verdi (1988), for some $\alpha > 0$

$$\phi'(s) \ge \alpha \quad \text{for all } s \in \mathbb{R}, \tag{2.21}$$

where we allow for $\phi'(0) = \infty$; as we can substitute $\partial_t u$ by $\alpha\partial_t u$ in the definition of (P), which amounts to substituting $\varphi$ by $\varphi/\alpha$ and scaling $t$ by $1/\alpha$. Actually, we can even cast problem (1.4a) in the form of (P) if we only assume for every $m > 0$

$$\phi'(s) \ge \alpha(m) > 0 \quad \text{for all } s \in [-m,m]. \tag{2.22}$$

This condition is satisfied, if e.g. $\beta \in C^1(\mathbb{R})$ and $\beta'(s) > 0$ for $s \neq 0$. The above statement can be seen as follows . As already noted, we can substitute $\partial_t u$ by $\alpha\partial_t u$ in the definition of problems $(P_{k,\varepsilon})$, $(P_\varepsilon)$ and (P) without affecting the developed theory. In particular the bounds $\underline{u}$, $\bar{u}$ for the u-component are independent of $\alpha$ and $\varphi$. Choose $m = \max\{\|\underline{u}\|_{L^\infty(\Omega)}, \|\bar{u}\|_{L^\infty(\Omega)}\}$ and $\alpha = \alpha(m)$ according to (2.23). This $\alpha$ we take in the definition of (P) and $\varphi \equiv \varphi^{(\alpha)}$ defined by

$$\varphi^{(\alpha)}(s) \equiv \begin{cases} \phi(-m) + \alpha m & s \le -m \\ \phi(s) - \alpha s & |s| \le m \\ \phi(m) - \alpha m & s \ge m \end{cases} . \tag{2.23}$$

Then $\varphi^{(\alpha)}$ satisfies (1.9), if $\phi$ does so, and as the solution of (P) fulfills $\|u\|_{L^\infty(Q_T)} \le m$, we have that $\alpha u + \varphi^{(\alpha)}(u) \equiv \phi(u)$, i.e. the solution of (1.4a) is the solution (P).

The existence result for (P) in Theorem 2.3 is not new. It is quite comparable to the basic results for the generalised porous medium equation, (compare e.g. Sacks (1983)). What is of importance for the following, is the precise information about its approximation by $(P_{k,\varepsilon})$.

## 3. A CONTINUOUS IN TIME FINITE ELEMENT APPROXIMATION

We now consider the continuous piecewise linear finite element approximation to $(P_{k,\varepsilon})$. We make the following assumptions on the data and triangulation:

Assumptions **(D3)**: In addition to the assumptions (D2) we assume that the constant M in (2.1c) can be chosen uniformly for all $s \in \mathbb{R}$. (In view of the bounds (2.14c) for u, see Theorem 2.3, this is always achievable by changing $\varphi(s)$ for $|s| \geq m = \max\{-\underline{u}, \bar{u}\}$) Let $\Omega^h$ be a polyhedral approximation to $\Omega$ defined by $\bar{\Omega}^h \equiv \cup_{\kappa \in T^h} \bar{\kappa}$ with $\text{dist}(\partial\Omega, \partial\Omega^h) \leq Ch^2$; where $T^h$ is a partitioning consisting of regular simplices $\kappa$ with $h_\kappa \equiv \text{diam}(\kappa)$ and $h \equiv \max_{\kappa \in T^h} h_\kappa$. For ease of exposition we assume that $\Omega^h \subseteq \Omega$.

We introduce

$$S^h \equiv \{ \chi \in C(\bar{\Omega}^h) : \chi|_\kappa \text{ is linear for all } \kappa \in T^h \}$$

and
$$S^h_0 \equiv \{ \chi \in S^h : \chi = 0 \text{ on } \partial\Omega^h \}.$$

In the analysis that follows we extend $\chi \in S^h$ from $\bar{\Omega}^h$ to $\bar{\Omega}^h \backslash \bar{\Omega}$ by zero. Let $\pi_h : C^0(\bar{\Omega}) \to S^h$ denote the interpolation operator such that for any $w \in C^0(\bar{\Omega})$, $\pi_h w \in S^h$ satisfies

$$(\pi_h w)(x_i) = w(x_i) \quad \text{for all nodes } x_i \text{ of the partition } T^h.$$

Let $P^0_h : L^2(\Omega) \to S^h$ denote the $L^2$ projection such that for any $w \in L^2(\Omega)$, $P^0_h w \in S^h$ satisfies

$$(w - P^0_h w, \chi) = 0 \quad \forall\, \chi \in S^h.$$

Let $P^1_h : H^1_0(\Omega) \to S^h_0$ denote the $H^1$ semi-norm projection such that for any $w \in H^1_0(\Omega)$, $P^1_h w \in S^h_0$ satisfies

$$(\nabla(w - P^1_h w), \nabla\chi) = 0 \quad \forall\, \chi \in S^h_0.$$

We recall the standard approximation results, for all $\kappa \in T^h$

$$|w - \pi_h w|_{W^{m,q}(\kappa)} \leq Ch_\kappa^{2-m} |w|_{W^{2,q}(\kappa)} \quad \text{for } m = 0 \text{ and } 1 \text{ and}$$
$$\forall\, q \in [1,\infty] \text{ if } d \leq 2 \text{ and } \forall\, q \in (3/2,\infty] \text{ if } d = 3 \qquad (3.1a)$$

$$|w-P^0_h w|_{L^2(\Omega)} \le Ch^m|w|_{H^m(\Omega)} \quad \text{for } m = 0,\ 1 \text{ and } 2 \tag{3.1b}$$

and

$$|w-P^1_h w|_{L^2(\Omega)} + h|w-P^1_h w|_{H^1(\Omega)} \le Ch^m|w|_{H^m(\Omega)} \quad \text{for } m = 1 \text{ and } 2; \tag{3.1c}$$

where in (3.1a) we note the imbedding $W^{2,1}(\kappa) \subset C^0(\bar{\kappa})$ in the case d = 2, see for example p300 in Kufner et al. (1977).

Another result that will be useful later is that

$$|(I-\pi_h)\varphi_\varepsilon(\chi)|_{L^2(\Omega)} \le h|\nabla\pi_h[\varphi_\varepsilon(\chi)]|_{L^2(\Omega)} \quad \forall\ \chi \in S^h_0 . \tag{3.2}$$

This result is proved in Elliott (1987), p68, with h replaced by Ch on the righthand side of (3.2). However, it is easy to see from this proof that C can be taken as 1.

The approximation to $(P_{k,\varepsilon})$ we wish to consider first is:

$(P^h_{k,\varepsilon})$ Find $u^h_{k,\varepsilon} \in H^1(0,T;S^h_0)$ and $v^h_{k,\varepsilon} \in H^1(0,T;S^h)$ such that

$$(\partial_t u^h_{k,\varepsilon} + \partial_t v^h_{k,\varepsilon},\chi) + (\nabla u^h_{k,\varepsilon},\nabla\chi) = (f,\chi) \quad \forall\ \chi \in S^h_0$$

$$(\partial_t v^h_{k,\varepsilon},\chi) = k(\varphi_\varepsilon(u^h_{k,\varepsilon})-v^h_{k,\varepsilon},\chi) \quad \forall\ \chi \in S^h$$

$$u^h_{k,\varepsilon}(\cdot,0) = P^1_h g(\cdot) \qquad v^h_{k,\varepsilon}(\cdot,0) = P^0_h[\varphi_\varepsilon(g(\cdot))].$$

**Theorem 3.1**

Let the Assumptions (D3) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$ and $h > 0$ there exists a unique solution $\{u^h_{k,\varepsilon}, v^h_{k,\varepsilon}\}$ to $(P^h_{k,\varepsilon})$ and

$$\|u^h_{k,\varepsilon}\|_{L^\infty(Q_T)},\ \|v^h_{k,\varepsilon}\|_{L^\infty(Q_T)} \le C(k,h). \tag{3.3}$$

Proof: See the proof of Theorem 3.1 in part 1. □

Firstly, we have the following analogue of Lemma 2.3.

**Lemma 3.1**

Under Assumptions (D3) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$|\nabla u^h_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + |\partial_t u^h_{k,\varepsilon}|^2_{L^2(Q_T)} + \varepsilon|\partial_t v^h_{k,\varepsilon}|^2_{L^2(Q_T)} + \varepsilon|\partial_t[\varphi_\varepsilon(u^h_{k,\varepsilon})]|^2_{L^2(Q_T)} +$$
$$+ k^{-1}\Big[|\partial_t u^h_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\partial_t v^h_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + |\nabla(\partial_t u^h_{k,\varepsilon})|^2_{L^2(Q_T)}\Big]$$
$$\le C(1+k\varepsilon^{-2}h^4). \tag{3.4}$$

Proof: See the proof of Lemma 3.1 in part 1. □

In order to analyse the approximation $(P^h_{k,\varepsilon})$ it is convenient to introduce the associated linear problem :

$(P^{h,*}_{k,\varepsilon})$ Find $u^{h,*}_{k,\varepsilon} \in H^1(0,T;S^h_0)$ and $v^{h,*}_{k,\varepsilon} \in H^1(0,T;S^h)$ such that

$$(\partial_t u^{h,*}_{k,\varepsilon} + \partial_t v^{h,*}_{k,\varepsilon},\chi) + (\nabla u^{h,*}_{k,\varepsilon},\nabla\chi) = (f,\chi) \quad \forall\ \chi \in S^h_0$$
$$(\partial_t v^{h,*}_{k,\varepsilon},\chi) = k(\varphi_\varepsilon(u_{k,\varepsilon})-v^{h,*}_{k,\varepsilon},\chi) \quad \forall\ \chi \in S^h$$
$$u^{h,*}_{k,\varepsilon}(\cdot,0) = P^1_h g(\cdot) \qquad v^{h,*}_{k,\varepsilon}(\cdot,0) = P^0_h[\varphi_\varepsilon(g(\cdot))].$$

The existence and uniqueness of $\{u^{h,*}_{k,\varepsilon}, v^{h,*}_{k,\varepsilon}\}$ solving $(P^{h,*}_{k,\varepsilon})$ for all $\varepsilon \in (0,\varepsilon_0]$ and $h > 0$ is easily established and we have the following result.

**Lemma 3.2**

Under Assumptions (D3) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$\|u^{h,*}_{k,\varepsilon}-u^h_{k,\varepsilon}\|^2_{E_2(k,t)} + k^{-2}\varepsilon|\nabla(u^{h,*}_{k,\varepsilon}-u^h_{k,\varepsilon})(\cdot,t)|^2_{L^2(\Omega)} +$$
$$+ \varepsilon|\varphi_\varepsilon(u_{k,\varepsilon})-\varphi_\varepsilon(u^h_{k,\varepsilon})|^2_{L^2(Q_t)} + \varepsilon\|v^{h,*}_{k,\varepsilon}-v^h_{k,\varepsilon}\|^2_{E_1(k,t)}$$
$$\le C\varepsilon^{-1}|u_{k,\varepsilon}-u^{h,*}_{k,\varepsilon}|^2_{L^2(Q_t)}. \tag{3.5}$$

Proof: See the proof of Lemma 3.2 in part 1. □

**Lemma 3.3**

Under Assumptions (D3) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$|u_{k,\varepsilon}-u^{h,*}_{k,\varepsilon}|^2_{L^2(Q_t)} + h^2|\nabla\int_0^t(u_{k,\varepsilon}-u^{h,*}_{k,\varepsilon})(\cdot,s)ds|^2_{L^2(\Omega)}$$
$$\le Ch^4\Big[|u_{k,\varepsilon}|^2_{L^2(0,t;H^2(\Omega))} + |g|^2_{H^2(\Omega)}\Big] \le C\varepsilon^{-1}h^4 \tag{3.6a}$$

and

$$|\nabla(u_{k,\varepsilon}-u^{h,*}_{k,\varepsilon})(\cdot,t)|^2_{L^2(\Omega)}$$
$$\le Ch^2\Big[|u_{k,\varepsilon}|^2_{H^1(0,t;H^1(\Omega))} + |u_{k,\varepsilon}|^2_{L^\infty(0,t;H^2(\Omega))}\Big] \le C\varepsilon^{-1}kh^2. \tag{3.6b}$$

Proof: The first set of inequalities in (3.6a&b) are proved in Lemma 3.3 of part 1. The second inequalities in (3.6a&b) follow from noting (2.12) and under the stated assumptions on $\Omega$ that for $r \in [1,\infty]$

$$|u_{k,\varepsilon}|_{L^r(0,T;H^2(\Omega))}$$
$$\le C\Big[|\partial_t u_{k,\varepsilon}|_{L^r(0,T;L^2(\Omega))} + |\partial_t v_{k,\varepsilon}|_{L^r(0,T;L^2(\Omega))} + |f|_{L^r(0,T;L^2(\Omega))}\Big]$$

and hence from (2.12) that

$$|u_{k,\varepsilon}|^2_{L^2(0,T;H^2(\Omega))} \le C\varepsilon^{-1} \quad \text{and} \quad |u_{k,\varepsilon}|^2_{L^\infty(0,T;H^2(\Omega))} \le C\varepsilon^{-1}k. \tag{3.7}$$

□

One could approximate directly the problem $(P_\varepsilon)$ without relaxing the reaction process by introducing

$(P^h_\varepsilon)$ Find $u^h_\varepsilon \in H^1(0,T;S^h_0)$ such that

$$(\partial_t u^h_\varepsilon + \partial_t[\varphi_\varepsilon(u^h_\varepsilon)],\chi) + (\nabla u^h_\varepsilon,\nabla\chi) = (f,\chi) \quad \forall\, \chi \in S^h_0$$
$$u^h_\varepsilon(\cdot,0) = P^1_h g(\cdot)$$

In addition one could approximate (P) without relaxing the reaction process or regularizing by introducing the problem $(P^h)$, the same as $(P^h_\varepsilon)$ with $\varphi_\varepsilon$ replaced by $\varphi$. We have the following result.

**Theorem 3.2**

Let the Assumptions (D3) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$ and $h > 0$ there exist unique solutions $u^h_\varepsilon$ to $(P^h_\varepsilon)$ and $u^h$ to $(P^h)$. In addition for all $t \in (0,T]$ we have that

$$\|u^h_\varepsilon - u^h_{k,\varepsilon}\|^2_{E_2(\infty,t)} + \varepsilon|\varphi_\varepsilon(u^h_\varepsilon)-\varphi_\varepsilon(u^h_{k,\varepsilon})|^2_{L^2(Q_t)} + \varepsilon|\varphi_\varepsilon(u^h_\varepsilon)-v^h_{k,\varepsilon}|^2_{L^2(Q_t)}$$
$$\le Ck^{-2}|\partial_t v^h_{k,\varepsilon}|^2_{L^2(Q_T)} \le C\varepsilon^{-1}k^{-2}(1+k\varepsilon^{-2}h^4) \qquad (3.8a)$$

and

$$\|u^h - u^h_\varepsilon\|^2_{E_2(\infty,t)} + \varepsilon|\varphi(u^h)-\varphi_\varepsilon(u^h_\varepsilon)|^2_{L^2(Q_t)} \le C\varepsilon^{(1+p)/(1-p)}. \qquad (3.8b)$$

Proof: Existence and uniqueness of solutions to $(P^h_\varepsilon)$ and $(P^h)$ follow from discrete analogues of Theorems 2.2 and 2.3. The first inequality in (3.8a) and (3.8b) are discrete analogues of the first inequality in (2.14a) and (2.18), respectively, and are proved in similar ways. The second inequality in (3.8a) follows from (3.4). □

**Theorem 3.3**

Under Assumptions (D3) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$|u-u^h_{k,\varepsilon}|^2_{L^2(Q_t)} + \varepsilon|\varphi(u)-\varphi_\varepsilon(u^h_{k,\varepsilon})|^2_{L^2(Q_t)}$$
$$\le C[A_\varepsilon(t)\ \varepsilon^{(1+p)/(1-p)} + \varepsilon^{-1}k^{-2} + \varepsilon^{-2}h^4] \qquad (3.9a)$$
$$\varepsilon|\varphi(u)-v^h_{k,\varepsilon}|^2_{L^2(Q_t)}$$
$$\le C[A_\varepsilon(t)\ \varepsilon^{(1+p)/(1-p)} + \varepsilon^{-1}k^{-2} + \varepsilon^{-3}k^{-1}h^4 + \varepsilon^{-2}h^4]. \qquad (3.9b)$$

Proof: The result (3.9a) follows directly from (2.18), (2.17), (3.6) and (3.5). The result (3.9b) follows directly from (3.9a) and (3.8a). □

**Corollary 3.3**

Let the Assumptions (D3) hold, then for all $h > 0$ and $t \in (0,T]$:

(i) Under no assumptions on non-degeneracy, we have on choosing $\varepsilon = Ch^{4(1-p)/(3-p)} \leq \varepsilon_0$ and $k = Ch^{-4/(3-p)}$ that

$$|u-u^h_{k,\varepsilon}|_{L^2(Q_T)} \leq Ch^{2(1+p)/(3-p)}, \tag{3.10a}$$

$$|\int_0^t (u-u^h_{k,\varepsilon})(\cdot,s)ds|_{H^1(\Omega)} \leq Ch^{(1+p)/(3-p)} \tag{3.10b}$$

and

$$|\varphi(u)-\varphi_\varepsilon(u^h_{k,\varepsilon})|_{L^2(Q_T)} + |\varphi(u)-v^h_{k,\varepsilon}|_{L^2(Q_T)} \leq Ch^{4p/(3-p)}. \tag{3.10c}$$

(ii) Assuming (N.D.) and choosing $\varepsilon = Ch^{2(1-p)/(2-p)} \leq \varepsilon_0$ and $k = Ch^{(p-3)/(2-p)}$ we have that

$$|u-u^h_{k,\varepsilon}|_{L^2(Q_T)} \leq Ch^{2/(2-p)}, \tag{3.11a}$$

$$|\int_0^t (u-u^h_{k,\varepsilon})(\cdot,s)ds|_{H^1(\Omega)} \leq Ch^{1/(2-p)} \tag{3.11b}$$

and

$$|\varphi(u)-\varphi_\varepsilon(u^h_{k,\varepsilon})|_{L^2(Q_T)} + |\varphi(u)-v^h_{k,\varepsilon}|_{L^2(Q_T)} \leq Ch^{(1+p)/(2-p)}. \tag{3.11c}$$

Proof: Noting the non-degeneracy condition (N.D.)≡(1.8a) in the case of (3.11); (3.10a&c) and (3.11a&c) follow directly from (3.9a&b). (3.10b) and (3.11b) follow from (2.18), (2.17), (3.6) and (3.5). □

**Theorem 3.4**

Let the Assumptions (D3) hold. Then for all $h > 0$ and $t \in (0,T]$ the unique solutions $u^h_\varepsilon$ to $(P^h_\varepsilon)$ and $u^h$ to $(P^h)$ satisfy the following error bounds

(i) (3.10) for $\{u^h_{k,\varepsilon}, v^h_{k,\varepsilon}, \varphi_\varepsilon(u^h_{k,\varepsilon})\}$ replaced by (a) $\{u^h_\varepsilon, \varphi_\varepsilon(u^h_\varepsilon), \varphi_\varepsilon(u^h_\varepsilon)\}$ with $\varepsilon = Ch^{4(1-p)/(3-p)} \leq \varepsilon_0$ and (b) $\{u^h, \varphi(u^h), \varphi(u^h)\}$.

and (ii) (3.11), assuming (N.D.) holds, for $\{u^h_{k,\varepsilon}, v^h_{k,\varepsilon}, \varphi_\varepsilon(u^h_{k,\varepsilon})\}$ replaced by $\{u^h_\varepsilon, \varphi_\varepsilon(u^h_\varepsilon), \varphi_\varepsilon(u^h_\varepsilon)\}$ with $\varepsilon = Ch^{2(1-p)/(2-p)} \leq \varepsilon_0$.

Proof: The above error bounds follow by combining (3.8) with (3.10) and (3.11). □

**Remark 3.1**

If we know that u satisfies the non-degeneracy condition (N.D.), then from the error estimates above it is better to approximate (P) by $(P^h_\varepsilon)$, with the appropriate choice of $\varepsilon$, rather than $(P^h)$. □

## 4. A MORE PRACTICAL CONTINUOUS IN TIME FINITE ELEMENT APPROXIMATION

The standard Galerkin approximation analysed above is not practical as it requires the term $(\varphi_\varepsilon(u^h_{k,\varepsilon}),\chi)$ to be integrated exactly. This is obviously difficult in practice and it is computationally more convenient to consider a scheme where numerical integration is applied to all the terms and the initial data is interpolated as opposed to being projected. Below we introduce and analyse such a scheme.

For all $w_1,\ w_2 \in C^0(\bar{\Omega}^h)$ we set

$$(w_1,w_2)^h \equiv \int_{\Omega^h} \pi_h(w_1w_2)$$

as an approximation to $(w_1,w_2)$. On setting

$$|w|_h \equiv [(w,w)^h]^{½} \quad \text{for } w \in C^0(\bar{\Omega}^h),$$

we recall the well-known results

$$|\chi|_{L^2(\Omega^h)} \le |\chi|_h \le C_1|\chi|_{L^2(\Omega^h)} \quad \forall\ \chi \in S^h, \tag{4.1a}$$

and for $m = 0$ or $1$

$$\left|\int_{\Omega^h} \chi_1\chi_2 - (\chi_1,\chi_2)^h\right| \le C_2h^{1+m}\|\chi_1\|_{H^1(\Omega^h)}\|\chi_2\|_{H^m(\Omega^h)} \quad \forall\ \chi_1,\ \chi_2 \in S^h. \tag{4.1b}$$

We make the following assumptions on the data.

Assumptions **(D4)**: In addition to the Assumptions (D3) we assume that

$$f \in H^1(0,T;C^0(\bar{\Omega}))\cap L^2(0,T;H^2(\Omega)).$$

A more practical approximation to $(P_{k,\varepsilon})$ than $(P^h_{k,\varepsilon})$ is then :

$(\hat{P}^h_{k,\varepsilon})$ Find $\hat{u}^h_{k,\varepsilon} \in H^1(0,T;S^h_0)$ and $\hat{v}^h_{k,\varepsilon} \in H^1(0,T;S^h)$ such that

$$(\partial_t\hat{u}^h_{k,\varepsilon} + \partial_t\hat{v}^h_{k,\varepsilon},\chi)^h + (\nabla\hat{u}^h_{k,\varepsilon},\nabla\chi) = (f,\chi)^h \ \forall\ \chi \in S^h_0$$

$$(\partial_t\hat{v}^h_{k,\varepsilon},\chi)^h = k(\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})-\hat{v}^h_{k,\varepsilon},\chi)^h \ \forall\ \chi \in S^h$$

$$\hat{u}^h_{k,\varepsilon}(\cdot,0) = \pi_hg(\cdot) \qquad \hat{v}^h_{k,\varepsilon}(\cdot,0) = \pi_h[\varphi_\varepsilon(g(\cdot))].$$

We have the following analogues of Theorem 3.1 and Lemmas 3.1.

**Theorem 4.1**

Let the Assumptions (D4) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ there exists a unique solution $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}\}$ to $(\hat{P}^h_{k,\varepsilon})$ and $\|\hat{u}^h_{k,\varepsilon}\|_{L^\infty(Q_T)}$, $\|\hat{v}^h_{k,\varepsilon}\|_{L^\infty(Q_T)}$ $\le C(k,h)$.

Proof: See the proof of Theorem 4.1 in part 1. $\square$

**Lemma 4.1**

Under Assumptions (D4) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$
\begin{aligned}
&|\nabla \hat{u}^h_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + |\partial_t \hat{u}^h_{k,\varepsilon}|^2_{L^2(Q_T)} + \\
&\quad + \varepsilon|\partial_t \hat{v}^h_{k,\varepsilon}|^2_{L^2(Q_T)} + \varepsilon|\partial_t \pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|^2_{L^2(Q_T)} + \\
&\quad + k^{-1}\Big[|\partial_t \hat{u}^h_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\partial_t \hat{v}^h_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + |\nabla(\partial_t \hat{u}^h_{k,\varepsilon})|^2_{L^2(Q_T)}\Big] \le C.
\end{aligned}
\tag{4.2}
$$

Proof: See the proof of Lemma 4.1 in part 1. $\square$

Assumptions (D5): In addition to the Assumptions (D4) we assume that

The triangulation $T^h$ is such that (i) for $d = 2$ it is weakly acute; that is, for any pair of adjacent triangles the sum of opposite angles relative to the common side does not exceed $\pi$; and (ii) for $d = 3$ the angle between the vectors normal to any two faces of the same tetrahedron must not exceed $\pi/2$, see Kerkhoven & Jerome (1990).

From the above it is easy to deduce that the stiffness matrix $\{(\nabla\chi_i,\nabla\chi_j)\}^I_{i,j=1}$; where $\{x_i\}^I_{i=1}$ are the internal nodes of the partitioning and $\chi_j \in S^h_0$ is such that $\chi_j(x_i) = \delta_{ij}$, $i,j = 1 \to I$; is an M-matrix. From this property one can deduce that

$$
M^{-1}\varepsilon|\nabla\pi_h[\varphi_\varepsilon(\chi)]|^2_{L^2(\Omega)} \le (\nabla\chi, \nabla\pi_h[\varphi_\varepsilon(\chi)]) \quad \forall\ \chi \in S^h_0, \tag{4.3}
$$

see §2.4.2 of Nochetto (1991).

**Corollary 4.1**

Let the Assumptions (D5) hold. Then the unique solution $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}\}$ to $(\hat{P}^h_{k,\varepsilon})$, $\varepsilon \in (0,\varepsilon_0]$ and $h > 0$, satisfies the bounds (2.5a). In particular, if g and f $\geq 0$ then $\hat{u}^h_{k,\varepsilon}$, $\hat{v}^h_{k,\varepsilon} \geq 0$ in $Q_T$.

Proof: See the proof of Corollary 4.1 in part 1. □

We now have the analogue of Lemma 2.1.

**Lemma 4.2**

Under Assumptions (D5) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$\varepsilon|\nabla\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|^2_{L^2(Q_T)} + (\Phi_\varepsilon(\hat{u}^h_{k,\varepsilon}(\cdot,t)),1)^h + k|\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]-\hat{v}^h_{k,\varepsilon}|^2_{L^2(Q_T)} + \\ + |\hat{v}^h_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + k^{-1}|\partial_t\hat{v}^h_{k,\varepsilon}|^2_{L^2(Q_T)} \leq C. \tag{4.4}$$

Proof: See the proof of Lemma 4.2 in part 1.

We now prove the analogue of Lemma 4.2 for the solution $\{u^h_{k,\varepsilon}, v^h_{k,\varepsilon}\}$ of $(P^h_{k,\varepsilon})$.

**Lemma 4.3**

Under Assumptions (D5) we have for all $\varepsilon \in (0,\varepsilon_0]$, and for all $h > 0$, provided $M\varepsilon^{-1}kh^2 \leq 1$, and $t \in (0,T]$ that

$$\varepsilon|\nabla[\pi_h\varphi_\varepsilon(u^h_{k,\varepsilon})]|^2_{L^2(Q_T)} + (\Phi_\varepsilon(u^h_{k,\varepsilon}(\cdot,t)),1) + k|\varphi_\varepsilon(u^h_{k,\varepsilon})-v^h_{k,\varepsilon}|^2_{L^2(Q_T)} + \\ + |v^h_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + k^{-1}|\partial_t v^h_{k,\varepsilon}|^2_{L^2(Q_T)} \leq C. \tag{4.5}$$

Proof: See the proof of Lemma 4.3 in part 1. □

**Lemma 4.4**

Under Assumptions (D5) we have for all $\varepsilon \in (0,\varepsilon_0]$, and for all $h > 0$, provided $\varepsilon^{-1}h^2 \le C$, and $t \in (0,T]$ that

$$\|u^h_{k,\varepsilon}-\hat{u}^h_{k,\varepsilon}\|^2_{E_2(k,t)} + \varepsilon|\varphi_\varepsilon(u^h_{k,\varepsilon})-\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})|^2_{L^2(Q_t)} + \varepsilon\|v^h_{k,\varepsilon}-\hat{v}^h_{k,\varepsilon}\|^2_{E_1(k,t)}$$
$$\le C[\varepsilon^{-1}+\|\pi_h\varphi_\varepsilon(g)\|^2_{H^1(\Omega)}]h^2 + C[h^4+|(I-\pi_h)\varphi_\varepsilon(g)|^2_{L^2(\Omega)}]$$
$$\le C\varepsilon^{-1}h^2. \tag{4.6}$$

Proof: The first inequality in (4.6) is proved in Lemma 4.4 of part 1. From (3.2), (3.1a) and (2.1c) we have that

$$|(I-\pi_h)\varphi_\varepsilon(g)|^2_{L^2(\Omega)} = \le 2[|(I-\pi_h)\varphi_\varepsilon(\pi_h g)|^2_{L^2(\Omega)} + |\varphi_\varepsilon(g)-\varphi_\varepsilon(\pi_h g)|^2_{L^2(\Omega)}]$$
$$\le 2h^2|\nabla[\pi_h\varphi_\varepsilon(\pi_h g)]|^2_{L^2(\Omega)} + C\varepsilon^{-2}h^4 \le C\varepsilon^{-1}h^2, \tag{4.7a}$$

since from (4.3) it follows using a Young's inequality that

$$M^{-1}\varepsilon|\nabla[\pi_h\varphi_\varepsilon(\pi_h g)]|^2_{L^2(\Omega)} \le (\nabla(\pi_h g),\nabla[\pi_h\varphi_\varepsilon(\pi_h g)])$$
$$= -(\Delta g,\pi_h\varphi_\varepsilon(\pi_h g)) - (\nabla(g-\pi_h g),\nabla[\pi_h\varphi_\varepsilon(\pi_h g)])$$
$$\le C(1+\varepsilon^{-1}h^2). \tag{4.7b}$$

Noting these bounds yields the second inequality in (4.6). □

We now improve on the bound (4.6) in the physically interesting case of given data $g$ and $f \ge 0$ yielding $u$, $\hat{u}^h_{k,\varepsilon} \ge 0$ in $Q_T$.

Assumptions (D6): In addition to the Assumptions (D5) we assume that (i) $\Omega \subset \mathbb{R}^d$, $d = 1$ or $2$, and $T^h$ is a quasi-uniform partition if $d=2$; (ii) $g$ and $f \ge 0$ and (iii) $\varphi \in C^2(0,\infty)$ such that $\varphi''(s) \le 0$ for all $s > 0$ and there exist an $s_0$ such that $\varphi(s) \ge s\varphi'(s)$ for all $s \in (0,s_0)$.

We set $\varphi_\varepsilon$ to be the following quadratic regularization of $\varphi$

$$\varphi_\varepsilon(s) \equiv \begin{cases} \varphi(s) & \text{for } s \geq \delta \\ as^2 + bs & \text{for } s \in [0,\delta]\ ; \\ bs & \text{for } s \leq 0 \end{cases} \tag{4.8}$$

where $a \equiv \delta^{-1}\varphi'(\delta) - \delta^{-2}\varphi(\delta)$, $b \equiv -\varphi'(\delta) + 2\delta^{-1}\varphi(\delta)$ and $\delta \equiv \varepsilon^{1/(1-p)}$ so that $\varphi_\varepsilon \in C^1(\mathbb{R})$.

As (ii) $\Rightarrow u \geq 0$ in $Q_T$, see Theorem 2.3, we can choose $\varphi(s)$ for $s < 0$ as we please. As (iii) holds it follows for $\varepsilon$ sufficiently small that $0 < b \leq C_1\varepsilon^{-1}$ and $-C_2\varepsilon^{(p-2)/(1-p)} \leq a \leq 0$, see (2.3b), and hence $\varphi_\varepsilon$ satisfies the conditions (2.1b&c). Extending $\varphi$ so that $\varphi(s) \equiv \varphi_\varepsilon(s)$ for $s \leq 0$, we have that (1.9) holds and $\varphi_\varepsilon$ satisfies (2.1a). Therefore all the results proved so far in this paper hold under the Assumptions (D6). We note for example that $\varphi(s) \equiv s^p$ for $s \geq 0$ with $p \in (0,1)$ satisfies (1.9) and (iii) above.

From (i) we have the discrete Sobolev imbedding result

$$|\chi|_{L^\infty(\Omega)} \leq C[\ln(1/h)]^r|\chi|_{H^1(\Omega)} \leq C[\ln(1/h)]^r|\nabla\chi|_{L^2(\Omega)} \quad \forall\ \chi \in S^h_0,$$

where $r = 0$ if $d = 1$ and $r = \frac{1}{2}$ if $d = 2$; see for example p67 in Thomée (1984). As noted in part 1, the quasi-uniformity restriction is not really restrictive in practice.

Lemma 4.5

Under Assumptions (D6) there exists an $\tilde{\varepsilon}_0 \leq \varepsilon_0$ such that we have for all $\varepsilon \in (0,\tilde{\varepsilon}_0]$ and for all h, provided $M\varepsilon^{-1}kh^2 \leq 1$, and $t \in (0,T]$ that

$$\|u^h_{k,\varepsilon}-\hat{u}^h_{k,\varepsilon}\|^2_{E_2(k,t)} + \varepsilon|\pi_h[\varphi_\varepsilon(u^h_{k,\varepsilon})-\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|^2_{L^2(Q_t)} \leq$$

$$Ck[\ln(1/h)]^{2r}\varepsilon^{-2}h^4[\,|u_{k,\varepsilon}|^2_{L^2(0,t;H^2(\Omega))}+\varepsilon^{-2/(1-p)}|\nabla(u_{k,\varepsilon}-u^h_{k,\varepsilon})|^4_{L^4(0,t;L^2(\Omega))}] +$$

$$+ Ck[\varepsilon^{-1}+\|\pi_h\varphi_\varepsilon(g)\|^2_{H^1(\Omega)}]h^4 + C[\ln(1/h)]^{2r}|(I-\pi_h)\varphi_\varepsilon(g)|^2_{L^1(\Omega)}$$

$$\leq Ck[\ln(1/h)]^{2r}\varepsilon^{-3}h^4\{1+\varepsilon^{-(4-2p)/(1-p)}kh^4[1+\varepsilon^{-2}kh^2]\} \tag{4.9a}$$

and

$$\varepsilon\|v^h_{k,\varepsilon}-\hat{v}^h_{k,\varepsilon}\|^2_{E_1(k,t)}$$
$$\leq C[h^2+\varepsilon|\pi_h[\varphi_\varepsilon(u^h_{k,\varepsilon})-\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|^2_{L^2(Q_t)}+\varepsilon k^{-1}|(I-\pi_h)\varphi_\varepsilon(g)|^2_{L^2(\Omega)}]$$
$$\leq C[k^{-1}h^2+\varepsilon|\pi_h[\varphi_\varepsilon(u^h_{k,\varepsilon})-\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|^2_{L^2(Q_t)}], \tag{4.9b}$$

where $r = 0$ if $d = 1$ and $r = ½$ if $d = 2$.

Proof: The first inequalities in (4.9a) and (4.9b) are proved in Lemma 4.5 in part 1. We note, in a similar manner to (4.25) in part 1, that

$$\begin{aligned}|(I-\pi_h)\varphi_\varepsilon(g)|^2_{L^1(\Omega)} &= \leq 2[|(I-\pi_h)\varphi_\varepsilon(\pi_h g)|^2_{L^1(\Omega)} + |\varphi_\varepsilon(g)-\varphi_\varepsilon(\pi_h g)|^2_{L^1(\Omega)}]\\ &\leq Ch^4\Big[\sum_{K\in T^h}|\varphi_\varepsilon(\pi_h g)|_{W^{2,1}(K)}\Big]^2 + C\varepsilon^{-2}h^4\\ &\leq C\varepsilon^{-2}h^4[1+\varepsilon^{-2/(1-p)}h^4],\end{aligned} \tag{4.10a}$$

since from (4.8) we have that $-\varphi''_\varepsilon(\cdot) \leq C\varepsilon^{(p-2)/(1-p)}$ and hence from (2.1c) and (3.1a) it follows using a Young's inequality that

$$\begin{aligned}\sum_{K\in T^h}|\varphi_\varepsilon(\pi_h g)|_{W^{2,1}(K)} &\leq |(\varphi''_\varepsilon(\pi_h g)\nabla(\pi_h g),\nabla(\pi_h g))| = |(\nabla[\varphi'_\varepsilon(\pi_h g)],\nabla(\pi_h g))|\\ &\leq |(\varphi'_\varepsilon(\pi_h g),\Delta g)| + |\int_{\partial\Omega}\varphi'_\varepsilon(\pi_h g)\nabla g.\underline{n}| + |(\varphi''_\varepsilon(\pi_h g)\nabla(\pi_h g),\nabla(g-\pi_h g))|\\ &\leq C\varepsilon^{-1} + C\varepsilon^{(p-2)/(1-p)}h^2,\end{aligned} \tag{4.10b}$$

where $\underline{n}$ is the outward unit normal to $\partial\Omega$. From (3.1c), (3.7), (3.5) and (3.6a) and an inverse inequality we have that

$$\begin{aligned}|\nabla(u_{k,\varepsilon}-u^h_{k,\varepsilon})|^2_{L^2(Q_T)} &\leq 2[|\nabla(u_{k,\varepsilon}-P^1_h u_{k,\varepsilon})|^2_{L^2(Q_T)} + |\nabla(P^1_h u_{k,\varepsilon}-u^h_{k,\varepsilon})|^2_{L^2(Q_T)}]\\ &\leq C\varepsilon^{-1}h^2 + Ch^{-2}|P^1_h u_{k,\varepsilon}-u^h_{k,\varepsilon}|^2_{L^2(Q_T)}\\ &\leq C\varepsilon^{-1}h^2 + Ch^{-2}[|u_{k,\varepsilon}-u^{h,*}_{k,\varepsilon}|^2_{L^2(Q_T)} + |u^{h,*}_{k,\varepsilon}-u^h_{k,\varepsilon}|^2_{L^2(Q_T)}]\\ &\leq C\varepsilon^{-2}h^2\end{aligned} \tag{4.11a}$$

and from (3.5) and (3.6b)

$$\begin{aligned}&|\nabla(u_{k,\varepsilon}-u^h_{k,\varepsilon})|^2_{L^\infty(0,t;L^2(\Omega))}\\ &\quad\leq 2[|\nabla(u_{k,\varepsilon}-u^{h,*}_{k,\varepsilon})|^2_{L^\infty(0,t;L^2(\Omega))} + |\nabla(u^{h,*}_{k,\varepsilon}-u^h_{k,\varepsilon})|^2_{L^\infty(0,t;L^2(\Omega))}]\\ &\quad\leq C\varepsilon^{-1}kh^2[1+\varepsilon^{-2}kh^2].\end{aligned} \tag{4.11b}$$

Hence it follows that

$$|\nabla(u_{k,\varepsilon}-u^h_{k,\varepsilon})|^4_{L^4(0,t;L^2(\Omega))} \le C\varepsilon^{-3}kh^4[1+\varepsilon^{-2}kh^2]. \tag{4.12}$$

The second inequality in (4.9a) then follows from the bounds (4.10a), (4.12), (4.7b) and (3.7). The second inequality in (4.9b) follows from (4.7a). □

One can also consider the corresponding direct approximations without relaxing the reaction:

$(\hat{P}^h_\varepsilon)$ Find $\hat{u}^h_\varepsilon \in H^1(0,T;S^h_0)$ such that

$$(\partial_t \hat{u}^h_\varepsilon + \partial_t[\varphi_\varepsilon(\hat{u}^h_\varepsilon)],\chi)^h + (\nabla\hat{u}^h_\varepsilon,\nabla\chi) = (f,\chi)^h \quad \forall\ \chi \in S^h_0$$

$$\hat{u}^h_\varepsilon(\cdot,0) = \pi_h g(\cdot).$$

and $(\hat{P}^h)$, the same as $(\hat{P}^h_\varepsilon)$ with $\varphi_\varepsilon$ replaced by $\varphi$. We have the following results.

**Theorem 4.2**

Let the Assumptions (D4) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$ and $h > 0$ there exist unique solutions $\hat{u}^h_\varepsilon$ to $(\hat{P}^h_\varepsilon)$ and $\hat{u}^h$ to $(\hat{P}^h)$. Moreover, for all $t \in (0,T]$ we have that

$$\|\hat{u}^h_\varepsilon-\hat{u}^h_{k,\varepsilon}\|^2_{E_2(\infty,t)} + \varepsilon|\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)-\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|^2_{L^2(Q_t)} + \varepsilon|\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]-\hat{v}^h_{k,\varepsilon}|^2_{L^2(Q_t)}$$

$$\le Ck^{-2}|\partial_t\hat{v}^h_{k,\varepsilon}|^2_{L^2(Q_T)} \le C\varepsilon^{-1}k^{-2} \tag{4.13a}$$

and

$$\|\hat{u}^h-\hat{u}^h_\varepsilon\|^2_{E_2(\infty,t)} + \varepsilon|\pi_h[\varphi(\hat{u}^h)-\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|^2_{L^2(Q_t)} \le C\varepsilon^{(1+p)/(1-p)}. \tag{4.13b}$$

In addition under the Assumptions (D5) $\hat{u}^h_\varepsilon$ and $\hat{u}^h$ satisfy the first bound in (2.5a). In particular, if g and f ≥ 0 then $\hat{u}^h_\varepsilon$ and $\hat{u}^h \ge 0$ in $Q_T$.

Proof: Existence and uniqueness of solutions to $(\hat{P}^h_\varepsilon)$ and $(\hat{P}^h)$ follow from discrete analogues of Theorems 2.2 and 2.3. The first inequality in (4.13a) and (4.13b) are discrete analogues of the first inequality in (2.14a) and (2.18), respectively, and are proved in similar ways. The second inequality in (4.13a) follows from (4.2). The first bound in (2.5a) follows from (4.13), the equivalence of norms on $S^h$ and Corollary 4.1. □

**Theorem 4.3**

Under Assumptions (D5) we have for all $\varepsilon \in (0,\varepsilon_0]$ and for all $h > 0$, provided $\varepsilon^{-1}h^2 \le C$, and $t \in (0,T]$ that

$$|u-\hat{u}^h_{k,\varepsilon}|^2_{L^2(Q_t)} + \varepsilon|\varphi(u)-\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})|^2_{L^2(Q_t)} + \varepsilon|\varphi(u)-\hat{v}^h_{k,\varepsilon}|^2_{L^2(Q_t)}$$
$$\le C[A_\varepsilon(t)\ \varepsilon^{(1+p)/(1-p)} + \varepsilon^{-1}(k^{-2}+h^2)]. \tag{4.14}$$

Under Assumptions (D6) there exists an $\tilde{\varepsilon}_0 \le \varepsilon_0$ such that we have for all $\varepsilon \in (0,\tilde{\varepsilon}_0]$, and for all h, provided $M\varepsilon^{-1}kh^2 \le 1$, and $t \in (0,T]$ that

$$|u-\hat{u}^h_{k,\varepsilon}|^2_{L^2(Q_t)} \le C[A_\varepsilon(t)\ \varepsilon^{(1+p)/(1-p)} + \varepsilon^{-1}k^{-2}] +$$
$$+ Ck[\ln(1/h)]^{2r}\varepsilon^{-3}h^4\{1+\varepsilon^{-(4-2p)/(1-p)}kh^4[1+\varepsilon^{-2}kh^2]\} \tag{4.15a}$$

and

$$\varepsilon|\varphi(u)-\hat{v}^h_{k,\varepsilon}|^2_{L^2(Q_t)} + \varepsilon|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|^2_{L^2(Q_t)}$$
$$\le C[A_\varepsilon(t)\ \varepsilon^{(1+p)/(1-p)} + \varepsilon^{-1}k^{-2} + k^{-1}h^2 + h^2] +$$
$$+ Ck[\ln(1/h)]^{2r}\varepsilon^{-3}h^4\{1+\varepsilon^{-(4-2p)/(1-p)}kh^4[1+\varepsilon^{-2}kh^2]\}, \tag{4.15b}$$

where $r = 0$ if $d = 1$ and $r = ½$ if $d = 2$.

Proof: The results (4.14) and (4.15a) follow immediately from combining (3.9) with (4.6) and (4.9a), respectively. (4.15b) follows similarly from (3.9) and (4.9) after noting (3.2), (4.5) and (4.13a). □

**Corollary 4.3a**

Let Assumptions (D5) hold, then for all $h > 0$ and $t \in (0,T]$:

(i) Under no assumptions on non-degeneracy and on choosing $\varepsilon = Ch^{1-p} \le \varepsilon_0$ and $k = Ch^{-1}$, we have that

$$|u-\hat{u}^h_{k,\varepsilon}|_{L^2(Q_T)} + |\int_0^t (u-\hat{u}^h_{k,\varepsilon})(\cdot,s)ds|_{H^1(\Omega)} \le Ch^{(1+p)/2} \tag{4.16a}$$

and

$$|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|_{L^2(Q_T)} + |\varphi(u)-\hat{v}^h_{k,\varepsilon}|_{L^2(Q_T)} \le Ch^p. \tag{4.16b}$$

(ii) On assuming (N.D.) and choosing $\varepsilon = Ch^{2(1-p)/(3-p)} \le \varepsilon_0$ and $k = Ch^{-1}$ we have that

$$|u-\hat{u}^h_{k,\varepsilon}|_{L^2(Q_T)} + |\int_0^t (u-\hat{u}^h_{k,\varepsilon})(\cdot,s)ds|_{H^1(\Omega)} \le Ch^{2/(3-p)} \tag{4.17a}$$

and

$$|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|_{L^2(Q_T)} + |\varphi(u)-\hat{v}^h_{k,\varepsilon}|_{L^2(Q_T)} \le Ch^{(1+p)/(3-p)}. \tag{4.17b}$$

Proof: The results follow directly from (4.14), (4.6), (2.18), (2.17), (3.6), (3.5), (3.2) and (4.4). □

**Corollary 4.3b**

Under Assumptions (D6) we have for all $t \in (0,T]$

(i) Under no assumptions on non-degeneracy and on choosing $\varepsilon = C\{h^2[\ln(1/h)]^r\}^{2(1-p)/(5-2p)} \le \tilde{\varepsilon}_0$ and $k = C\{h^2[\ln(1/h)]^r\}^{-2/(5-2p)}$ we have for all $p \in (½,1]$ and $h \le h_0$

$$|u-\hat{u}^h_{k,\varepsilon}|_{L^2(Q_T)} \le C\{h^2[\ln(1/h)]^r\}^{(1+p)/(5-2p)}, \tag{4.18a}$$

$$|\int_0^t (u-\hat{u}^h_{k,\varepsilon})(\cdot,s)ds|_{H^1(\Omega)} \le C\{h^2[\ln(1/h)]^r\}^{3/[2(5-2p)]} \tag{4.18b}$$

and

$$|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|_{L^2(Q_T)} + |\varphi(u)-\hat{v}^h_{k,\varepsilon}|_{L^2(Q_T)} \le C\{h^2[\ln(1/h)]^r\}^{q/(5-2p)}, \tag{4.18c}$$

where $q = \min\{2p, 3/2\}$.

(ii) Assuming (N.D.) and on choosing $\varepsilon = C\{h^2[\ln(1/h)]^r\}^{4(1-p)/(13-7p)} \le \tilde{\varepsilon}_0$ and $k = C\{h^2[\ln(1/h)]^r\}^{-2(3-p)/(13-7p)}$, we have for all $p \in (1/3,1]$ and $h \le h_0$

$$|u-\hat{u}^h_{k,\varepsilon}|_{L^2(Q_T)} \le C\{h^2[\ln(1/h)]^r\}^{4/(13-7p)} \tag{4.19a}$$

$$|\int_0^t (u-\hat{u}^h_{k,\varepsilon})(\cdot,s)ds|_{H^1(\Omega)} \le C\{h^2[\ln(1/h)]^r\}^{3(3-p)/[2(13-7p)]} \tag{4.19b}$$

and

$$|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|_{L^2(Q_T)} + |\varphi(u)-\hat{v}^h_{k,\varepsilon}|_{L^2(Q_T)} \le C\{h^2[\ln(1/h)]^r\}^{q/(13-7p)}, \tag{4.19c}$$

where $q = \min\{2(p+1),3(3-p)/2\}$.

Proof: The results follow directly from (4.15), (4.9), (2.18), (2.17), (3.6) and (3.5). □

We note that (4.18a&c) improve on (4.16a&b), and (4.19a&c) improve on (4.17a&b).

**Theorem 4.4**

Let the Assumptions (D5) hold. Then for all $h > 0$ and $t \in (0,T]$ the unique solutions $\hat{u}^h_\varepsilon$ to $(\hat{P}^h_\varepsilon)$ and $\hat{u}^h$ to $(\hat{P}^h)$ satisfy the following error bounds

(i) (4.16) for $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})\}$ replaced by (a) $\{\hat{u}^h_\varepsilon, \pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)], \varphi_\varepsilon(\hat{u}^h_\varepsilon)\}$ with $\varepsilon = Ch^{1-p} \le \varepsilon_0$ and (b) $\{\hat{u}^h, \pi_h[\varphi(\hat{u}^h)], \varphi(\hat{u}^h)\}$.

(ii) (4.17), assuming (N.D.) holds, for $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})\}$ replaced by $\{\hat{u}^h_\varepsilon, \pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)], \varphi_\varepsilon(\hat{u}^h_\varepsilon)\}$ with $\varepsilon = Ch^{2(1-p)/(3-p)} \le \varepsilon_0$.

Finally, under Assumptions (D6) for all $h \le h_0$ the following error bounds hold:

(i) (4.18) with $p \in (½,1]$ for $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})\}$ replaced by (a) $\{\hat{u}^h_\varepsilon, \pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)], \varphi_\varepsilon(\hat{u}^h_\varepsilon)\}$ with $\varepsilon = C\{h^2[\ln(1/h)]^r\}^{2(1-p)/(5-2p)} \le \tilde{\varepsilon}_0$ and (b) $\{\hat{u}^h, \pi_h[\varphi(\hat{u}^h)], \varphi(\hat{u}^h)\}$.

(ii) (4.19), assuming (N.D.) holds, with $p \in (1/3,1]$ for $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})\}$ replaced by $\{\hat{u}^h_\varepsilon, \pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)], \varphi_\varepsilon(\hat{u}^h_\varepsilon)\}$ with $\varepsilon = C\{h^2[\ln(1/h)]^r\}^{4(1-p)/(13-7p)} \le \tilde{\varepsilon}_0$.

Proof: The above error bounds follow by combining (4.13) with (4.16)-(4.19).

□

## 5. A FULLY DISCRETE AND PRACTICAL FINITE ELEMENT APPROXIMATION

In this section we analyse the following fully discrete practical approximation to $(P_{k,\varepsilon})$ with timestep $\tau = T/N$ :

$(\hat{P}^{h,\tau}_{k,\varepsilon})$ For $n = 1 \to N$ find $\hat{u}^{h,n}_{k,\varepsilon} \in S^h_0$ and $\hat{v}^{h,n}_{k,\varepsilon} \in S^h$ such that

$$\tau^{-1}((\hat{u}^{h,n}_{k,\varepsilon} - \hat{u}^{h,n-1}_{k,\varepsilon}) + (\hat{v}^{h,n}_{k,\varepsilon} - \hat{v}^{h,n-1}_{k,\varepsilon}),\chi)^h + (\nabla\hat{u}^{h,n}_{k,\varepsilon},\nabla\chi) = (f^n,\chi)^h \;\forall\; \chi \in S^h_0$$

$$\tau^{-1}(\hat{v}^{h,n}_{k,\varepsilon} - \hat{v}^{h,n-1}_{k,\varepsilon},\chi)^h = k\;(\varphi_\varepsilon(\hat{u}^{h,n}_{k,\varepsilon}) - \hat{v}^{h,n}_{k,\varepsilon},\chi)^h \;\forall\; \chi \in S^h$$

$$\hat{u}^{h,0}_{k,\varepsilon}(\cdot) = \pi_h g(\cdot) \qquad \hat{v}^{h,0}_{k,\varepsilon}(\cdot) = \pi[\varphi_\varepsilon g(\cdot)],$$

where $f^n(\cdot) \equiv f(\cdot,n\tau)$.

Let $\hat{U}_{k,\varepsilon} \in L^\infty(0,T;S^h_0)$ and $\hat{V}_{k,\varepsilon} \in L^\infty(0,T;S^h)$ be such that for $n = 1 \to N$

$$\hat{U}_{k,\varepsilon}(\cdot,t) \equiv \hat{u}^{h,n}_{k,\varepsilon}(\cdot) \quad\text{and}\quad \hat{V}_{k,\varepsilon}(\cdot,t) \equiv \hat{v}^{h,n}_{k,\varepsilon}(\cdot) \quad \text{if } t \in ((n-1)\tau,n\tau];$$

and $\hat{U}^L_{k,\varepsilon} \in C^0([0,T];S^h_0)$ and $\hat{V}^L_{k,\varepsilon} \in C^0([0,T];S^h)$ be such that for $n = 1 \to N$

$$\hat{U}^L_{k,\varepsilon}(\cdot,t) \equiv [(t-(n-1)\tau)\hat{u}^{h,n}_{k,\varepsilon}(\cdot) + (n\tau-t)\hat{u}^{h,n-1}_{k,\varepsilon}(\cdot)]/\tau \quad \text{if } t \in [(n-1)\tau,n\tau]$$

and

$$\hat{V}^L_{k,\varepsilon}(\cdot,t) \equiv [(t-(n-1)\tau)\hat{v}^{h,n}_{k,\varepsilon}(\cdot) + (n\tau-t)\hat{v}^{h,n-1}_{k,\varepsilon}(\cdot)]/\tau \quad \text{if } t \in [(n-1)\tau,n\tau].$$

Then $(\hat{P}^{h,\tau}_{k,\varepsilon})$ can be restated: for almost every $t \in (0,T]$

$$(\partial_t\hat{U}^L_{k,\varepsilon} + \partial_t\hat{V}^L_{k,\varepsilon},\chi)^h + (\nabla\hat{U}_{k,\varepsilon},\nabla\chi) = (\hat{f},\chi)^h \;\forall\; \chi \in S^h_0$$

$$(\partial_t\hat{V}^L_{k,\varepsilon},\chi)^h = k\;(\varphi_\varepsilon(\hat{U}_{k,\varepsilon}) - \hat{V}_{k,\varepsilon},\chi)^h \;\forall\; \chi \in S^h$$

$$\hat{U}^L_{k,\varepsilon}(\cdot,0) = \pi_h g(\cdot) \qquad \hat{V}^L_{k,\varepsilon}(\cdot,0) = \pi[\varphi_\varepsilon g(\cdot)],$$

where $\hat{f}(\cdot,t) \equiv f^n(\cdot) \equiv f(\cdot,n\tau)$ if $t \in ((n-1)\tau,n\tau]$, $n = 1 \to N$.

**Theorem 5.1**

Let the Assumptions (D4) hold. Then for $\varepsilon \in (0,\varepsilon_0]$, $h$ and $\tau > 0$ there exists a unique solution $\{\hat{U}_{k,\varepsilon},\hat{V}_{k,\varepsilon}\}$ to $(\hat{P}^{h,\tau}_{k,\varepsilon})$. Moreover, if the Assumptions (D5) hold then

$$\underline{U} \le \hat{U}_{k,\varepsilon} \le \bar{U} \quad\text{and}\quad \underline{V} \le \hat{V}_{k,\varepsilon} \le \bar{V} \quad \text{in } Q_T, \tag{5.1}$$

where $\underline{U}$, $\bar{U}$, $\underline{V}$ and $\bar{V} \in \mathbb{R}$ are independent of $h$, $\tau$, $\varepsilon$ and $k$. In particular, if $g$ and $f \ge 0$ then one can take $\underline{U} = \underline{V} = 0$.

Proof: See the proof of Theorem 5.1 in part 1. □

**Lemma 5.1**

Under the Assumptions (D4) we have for all $\varepsilon \in (0,\varepsilon_0]$, h, $\tau > 0$ and $m = 0 \to N$ that

$$\|\hat{u}^h_{k,\varepsilon}-\hat{U}_{k,\varepsilon}\|^2_{E_2(k,m\tau)} + \varepsilon|\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})-\varphi_\varepsilon(\hat{U}_{k,\varepsilon})]|^2_{L^2(Q_{m\tau})} + \varepsilon\|\hat{v}^h_{k,\varepsilon}-\hat{V}_{k,\varepsilon}\|^2_{E_1(k,m\tau)}$$
$$\le C\tau^2\Big\{ |\partial_t\hat{u}^h_{k,\varepsilon}|^2_{L^2(Q_T)} + (\tau+k^{-1})^{-1}|\nabla(\partial_t\hat{u}^h_{k,\varepsilon})|^2_{L^2(Q_T)} +$$
$$+|\partial_t\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})]|^2_{L^2(Q_T)} + |\partial_t\hat{v}^h_{k,\varepsilon}|^2_{L^2(Q_T)} + |\partial_t[\pi_h f]|^2_{L^2(Q_T)} \Big\}.$$

Proof: See the proof of Lemma 5.1 in part 1. □

**Corollary 5.1**

Under the Assumptions (D4) we have for all $\varepsilon \in (0,\varepsilon_0]$, h, $\tau > 0$ and $m = 0 \to N$ that

$$\|\hat{u}^h_{k,\varepsilon}-\hat{U}_{k,\varepsilon}\|^2_{E_2(k,m\tau)} + \varepsilon|\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})-\varphi_\varepsilon(\hat{U}_{k,\varepsilon})]|^2_{L^2(Q_{m\tau})} + \varepsilon\|\hat{v}^h_{k,\varepsilon}-\hat{V}_{k,\varepsilon}\|^2_{E_1(k,m\tau)}$$
$$\le C[\varepsilon^{-1}+(\tau+k^{-1})^{-1}k]\tau^2. \qquad (5.2)$$

Proof: The result (5.2) follows immediately from Lemma 5.1 and the bound (4.2). □

Below we will present an alternative bound to (5.2). Firstly, we prove appropriate analogues of Lemmas 4.1 and 4.2.

Lemma 5.2

Under Assumptions (D4) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h$, $\tau > 0$ and $t \in (0,T]$ that

$$|\nabla \hat{U}_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + |\partial_t \hat{U}^L_{k,\varepsilon}|^2_{L^2(Q_T)} + \varepsilon|\partial_t \hat{V}^L_{k,\varepsilon}|^2_{L^2(Q_T)} +$$
$$+ \varepsilon\,\tau \sum_{n=1}^{N} |\pi_h[\varphi_\varepsilon(\hat{U}^L_{k,\varepsilon}(\cdot,n\tau))-\varphi_\varepsilon(\hat{U}^L_{k,\varepsilon}(\cdot,(n-1)\tau))]/\tau|^2_{L^2(\Omega)} +$$
$$+ k^{-1}\left[|\partial_t \hat{U}^L_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\partial_t \hat{V}^L_{k,\varepsilon}(\cdot,t)|^2_{L^2(\Omega)}\right] \le C[1+(k\tau)^{-1}]. \tag{5.3}$$

Proof: We adapt the proof given for Lemma 2.3 in part 1. We adopt the difference notation $D^+_t \hat{u}^{h,n}_{k,\varepsilon} \equiv (\hat{u}^{h,n+1}_{k,\varepsilon}-\hat{u}^{h,n}_{k,\varepsilon})/\tau$, $D^-_t \hat{u}^{h,n}_{k,\varepsilon} \equiv (\hat{u}^{h,n}_{k,\varepsilon}-\hat{u}^{h,n-1}_{k,\varepsilon})/\tau$ and $\delta^2_t \equiv D^+_t D^-_t$ and note that

$$\sum_{n=1}^{m} [(a^n-a^{n-1})a^n] \equiv ½\,[(a^m)^2 - (a^0)^2 + \sum_{n=1}^{m} (a^n-a^{n-1})^2]. \tag{5.4}$$

Subtracting successive equations in $(P^{h,\tau}_{k,\varepsilon})$ yields for $n = 1 \to N-1$

$$(\delta^2_t[\hat{u}^{h,n}_{k,\varepsilon} + \hat{v}^{h,n}_{k,\varepsilon}],\chi)^h + (\nabla D^+_t \hat{u}^{h,n}_{k,\varepsilon},\nabla\chi) = (D^+_t f^n,\chi)^h \qquad \forall\, \chi \in S^h_0 \tag{5.5a}$$
$$(\delta^2_t \hat{v}^{h,n}_{k,\varepsilon},\chi)^h = k\,(D^+_t[\varphi_\varepsilon(\hat{u}^{h,n}_{k,\varepsilon}) - \hat{v}^{h,n}_{k,\varepsilon}],\chi)^h \qquad \forall\, \chi \in S^h, \tag{5.5b}$$

and hence

$$(k^{-1}\delta^2_t \hat{u}^{h,n}_{k,\varepsilon} + D^+_t[\hat{u}^{h,n}_{k,\varepsilon} + \varphi_\varepsilon(\hat{u}^{h,n}_{k,\varepsilon})],\chi)^h + (k^{-1}\nabla D^+_t \hat{u}^{h,n}_{k,\varepsilon} + \nabla \hat{u}^{h,n+1}_{k,\varepsilon},\nabla\chi)$$
$$= (f^{n+1} + k^{-1}D^+_t f^n,\chi)^h \qquad \forall\, \chi \in S^h_0. \tag{5.6}$$

Choosing $\chi \equiv D^+_t \hat{u}^{h,n}_{k,\varepsilon}$ in (5.6) and summing from $n = 1 \to m$ and noting (5.4) yields for $m = 1 \to N-1$ that

$$½\,k^{-1}[\,|D^+_t \hat{u}^{h,m}_{k,\varepsilon}|^2_h + \tau^2 \sum_{n=1}^{m} |\delta^2_t \hat{u}^{h,n}_{k,\varepsilon}|^2_h - |D^+_t \hat{u}^{h,0}_{k,\varepsilon}|^2_h] +$$
$$+ ½\,[\,|\nabla \hat{u}^{h,m+1}_{k,\varepsilon}|^2_{L^2(\Omega)} + (\tau+2k^{-1})\tau \sum_{n=1}^{m} |\nabla(D^+_t \hat{u}^{h,n}_{k,\varepsilon})|^2_{L^2(\Omega)} - |\nabla \hat{u}^{h,1}_{k,\varepsilon}|^2_{L^2(\Omega)}] +$$
$$+ \tau \sum_{n=1}^{m} (D^+_t[\hat{u}^{h,n}_{k,\varepsilon} + \varphi_\varepsilon(\hat{u}^{h,n}_{k,\varepsilon})],D^+_t \hat{u}^{h,n}_{k,\varepsilon})^h = \tau \sum_{n=1}^{m} (f^{n+1} + k^{-1}D^+_t f^n,D^+_t \hat{u}^{h,n}_{k,\varepsilon})^h$$

and hence

$$k^{-1}|D_t^+\hat{u}_{k,\varepsilon}^{h,m}|_h^2 + \tau\sum_{n=0}^{m}|D_t^+\hat{u}_{k,\varepsilon}^{h,n}|_h^2 + |\nabla\hat{u}_{k,\varepsilon}^{h,m+1}|_{L^2(\Omega)}^2 + 2\tau\sum_{n=0}^{m}(D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,n})],D_t^+\hat{u}_{k,\varepsilon}^{h,n})^h$$
$$\leq (\tau+k^{-1})|D_t^+\hat{u}_{k,\varepsilon}^{h,0}|_h^2 + 2\tau(D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,0})],D_t^+\hat{u}_{k,\varepsilon}^{h,0})^h +$$
$$+ |\nabla\hat{u}_{k,\varepsilon}^{h,1}|_{L^2(\Omega)}^2 + \tau\sum_{n=1}^{m}|f^{n+1} + k^{-1}D_t^+f^n|_h^2 . \tag{5.7}$$

Next we note from the first equations in $(\hat{P}_{k,\varepsilon}^{h,\tau})$ and the initial conditions that

$$(1+k\tau)(D_t^+\hat{v}_{k,\varepsilon}^{h,0},\chi)^h = k\tau(D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,0})],\chi)^h \qquad \forall\ \chi \in S^h \tag{5.8a}$$

and hence

$$(D_t^+\hat{u}_{k,\varepsilon}^{h,0},\chi)^h + [k\tau/(1+k\tau)](D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,0})],\chi)^h + (\nabla\hat{u}_{k,\varepsilon}^{h,1},\nabla\chi) = (f^1,\chi)^h \ \forall\ \chi \in S_0^h. \tag{5.8b}$$

In addition it follows from (2.3a) that for $n = 0 \to N-1$

$$M^{-1}\varepsilon|D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,n})]|_h^2 \leq (D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,n})],D_t^+\hat{u}_{k,\varepsilon}^{h,n})^h. \tag{5.9}$$

Choosing $\chi \equiv D_t^+\hat{u}_{k,\varepsilon}^{h,0}$ in (5.8b) and noting (5.9) and (3.1a) yields that

$$\tau|D_t^+\hat{u}_{k,\varepsilon}^{h,0}|_h^2 + 2\tau[k\tau/(1+k\tau)](D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,0})],D_t^+\hat{u}_{k,\varepsilon}^{h,0})^h + |\nabla\hat{u}_{k,\varepsilon}^{h,1}|_{L^2(\Omega)}^2$$
$$\leq |\nabla\hat{u}_{k,\varepsilon}^{h,0}|_{L^2(\Omega)}^2 + \tau|f^1|_h^2 \leq C. \tag{5.10}$$

From (5.7), (5.9), (5.10) and the assumptions on f it follows that

$$k^{-1}|D_t^+\hat{u}_{k,\varepsilon}^{h,m}|_h^2 + \tau\sum_{n=0}^{m}|D_t^+\hat{u}_{k,\varepsilon}^{h,n}|_h^2 + |\nabla\hat{u}_{k,\varepsilon}^{h,m+1}|_{L^2(\Omega)}^2 + \varepsilon\tau\sum_{n=0}^{m}|D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,n})]|_h^2$$
$$\leq C[1+(k\tau)^{-1} + \tau\sum_{n=1}^{N}|f^{n+1} + k^{-1}D_t^+f^n|_h^2] \leq C[1+(k\tau)^{-1}]. \tag{5.11}$$

Choosing $\chi \equiv D_t^+\hat{v}_{k,\varepsilon}^{h,n}$ in (5.5b) and summing from $n = 1 \to m$ and noting (5.4) yields for $m = 1 \to N-1$ that

$$k^{-1}|D_t^+\hat{v}_{k,\varepsilon}^{h,m}|_h^2 + \tau\sum_{n=1}^{m}|D_t^+\hat{v}_{k,\varepsilon}^{h,n}|_h^2 \leq k^{-1}|D_t^+\hat{v}_{k,\varepsilon}^{h,0}|_h^2 + \tau\sum_{n=1}^{m}|D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,n})]|_h^2.$$

Hence noting (5.8a) we obtain for $m = 1 \to N-1$ that

$$k^{-1}|D_t^+\hat{v}_{k,\varepsilon}^{h,m}|_h^2 + \tau\sum_{n=0}^{m}|D_t^+\hat{v}_{k,\varepsilon}^{h,n}|_h^2 \leq \tau\sum_{n=0}^{m}|D_t^+[\varphi_\varepsilon(\hat{u}_{k,\varepsilon}^{h,n})]|_h^2. \tag{5.12}$$

Combining (5.11) and (5.12) and noting (4.1a) yields the desired result (5.3). □

**Lemma 5.3**

Under the Assumptions (D5) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h$, $\tau > 0$ and for $m = 0 \to N$ that

$$\|\hat{u}^h_{k,\varepsilon}-\hat{U}_{k,\varepsilon}\|^2_{E_2(k,m\tau)} + \varepsilon|\pi_h[\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})-\varphi_\varepsilon(\hat{U}_{k,\varepsilon})]|^2_{L^2(Q_{m\tau})} + \varepsilon\|\hat{v}^h_{k,\varepsilon}-\hat{V}_{k,\varepsilon}\|^2_{E_1(k,m\tau)}$$
$$\le C(\varepsilon^{-1}\tau^2+\varepsilon^{-1}k^{-1}\tau+k^{-2}). \tag{5.13}$$

Proof: Let $E_u \equiv \hat{u}^h_{k,\varepsilon}-\hat{U}_{k,\varepsilon}$, $E^L_u \equiv \hat{u}^h_{k,\varepsilon}-\hat{U}^L_{k,\varepsilon}$, $E_v \equiv \hat{v}^h_{k,\varepsilon}-\hat{V}_{k,\varepsilon}$, $E^L_v \equiv \hat{v}^h_{k,\varepsilon}-\hat{V}^L_{k,\varepsilon}$ and $E_f \equiv f-\hat{f}$. Firstly, we note that

$$|\hat{U}^L_{k,\varepsilon}-\hat{U}_{k,\varepsilon}|^2_{L^2(Q_T)} \equiv \sum_{n=1}^{N}\int_{(n-1)\tau}^{n\tau}(n\tau-t)^2|\partial_t\hat{U}^L_{k,\varepsilon}|^2\,dt \le \tau^2|\partial_t\hat{U}^L_{k,\varepsilon}|^2_{L^2(Q_T)}. \tag{5.14a}$$

and the equivalent result with U replaced by V. Hence it follows from (5.3) that

$$|\hat{U}^L_{k,\varepsilon}-\hat{U}_{k,\varepsilon}|^2_{L^2(Q_T)} + \varepsilon|\hat{V}^L_{k,\varepsilon}-\hat{V}_{k,\varepsilon}|^2_{L^2(Q_T)} \le C[\tau^2+k^{-1}\tau]. \tag{5.14b}$$

Similarly, we have that $|E_f|^2_{L^2(Q_T)} \le C\tau^2$.

It follows from $(\hat{P}^h_{k,\varepsilon})$ and $(\hat{P}^{h,\tau}_{k,\varepsilon})$ that $E^L_u(\cdot,0) = 0$, $E^L_v(\cdot,0) = 0$ and for almost every $t \in (0,T]$

$$(\partial_t E^L_u + \partial_t E^L_v,\chi)^h + (\nabla E_u,\nabla\chi) = (E_f,\chi)^h \qquad \forall\ \chi \in S^h_0 \tag{5.15a}$$
$$(\partial_t E^L_v,\chi)^h = k\,([\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})-\varphi_\varepsilon(\hat{U}_{k,\varepsilon})] - E_v,\chi)^h \qquad \forall\ \chi \in S^h. \tag{5.15b}$$

Choosing $\chi \equiv \int_s^t E_u(\cdot,\sigma)d\sigma$ in (5.15a), integrating over $(0,t)$ in time, where s is the integration variable in time yields that

$$\int_0^t|E_u(\cdot,s)|^2_h\,ds + \tfrac{1}{2}|\nabla\int_0^t E_u(\cdot,s)ds|^2_{L^2(\Omega)} =$$
$$\int_0^t([\hat{U}^L_{k,\varepsilon}-\hat{U}_{k,\varepsilon}-E^L_v](\cdot,s) + \int_0^s E_f(\cdot,\sigma)d\sigma, E_u(\cdot,s))^h ds. \tag{5.16}$$

Similarly choosing $\chi \equiv E_u$ in (5.15a) yields that

$$\tfrac{1}{2}|E^L_u(\cdot,t)|^2_h + \int_0^t|\nabla E_u(\cdot,s)|^2_{L^2(\Omega)}ds =$$
$$\int_0^t[([\hat{U}_{k,\varepsilon}-\hat{U}^L_{k,\varepsilon}](\cdot,s),\partial_s E^L_u(\cdot,s))^h + ([E_f-\partial_s E^L_v](\cdot,s),E_u(\cdot,s))^h]ds. \tag{5.17}$$

From (5.16), (5.17), (5.15b), (4.2), (5.3) and (5.14b) it follows that

$$\int_0^t |E_u(\cdot,s)|_h^2\, ds + \tfrac{1}{2}\Big|\nabla\int_0^t E_u(\cdot,s)ds\Big|_{L^2(\Omega)}^2 + k^{-1}\Big[\tfrac{1}{2}|E_u^L(\cdot,t)|_h^2 + \int_0^t |\nabla E_u(\cdot,s)|_{L^2(\Omega)}^2 ds\Big]$$

$$+ \int_0^t (\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon}(\cdot,s)) - \varphi_\varepsilon(\hat{U}_{k,\varepsilon}(\cdot,s)), E_u(\cdot,s))^h ds$$

$$= \int_0^t ([(\hat{U}^L_{k,\varepsilon} - \hat{U}_{k,\varepsilon}) + (\hat{V}^L_{k,\varepsilon} - \hat{V}_{k,\varepsilon})](\cdot,s), E_u(\cdot,s))^h\, ds +$$

$$+ k^{-1}\int_0^t ([\hat{U}_{k,\varepsilon} - \hat{U}^L_{k,\varepsilon}](\cdot,s), \partial_s E_u^L(\cdot,s))^h\, ds +$$

$$+ \int_0^t (k^{-1}E_f(\cdot,s) + \int_0^s E_f(\cdot,\sigma)d\sigma, E_u(\cdot,s))^h ds$$

$$\le C[\varepsilon^{-1} + (k\tau)^{-1}][\tau^2 + k^{-1}\tau] \le C(\varepsilon^{-1}\tau^2 + \varepsilon^{-1}k^{-1}\tau + k^{-2}). \qquad (5.18)$$

The desired result for u in (5.13) then follows from (5.18), (4.1a) and (2.3a).

Similarly choosing $\chi \equiv E_v$ in (5.15b) yields that

$$\tfrac{1}{2}\, k^{-1}|E_v^L(\cdot,t)|_h^2 + \int_0^t |E_v(\cdot,s)|_h^2\, ds =$$

$$\int_0^t (\varphi_\varepsilon(\hat{u}^h_{k,\varepsilon}(\cdot,s)) - \varphi_\varepsilon(\hat{U}_{k,\varepsilon}(\cdot,s)), E_v(\cdot,s))^h ds +$$

$$+ k^{-1}\int_0^t ([\hat{V}_{k,\varepsilon} - \hat{V}^L_{k,\varepsilon}](\cdot,s), \partial_s E_v^L(\cdot,s))^h\, ds. \qquad (5.19)$$

The desired result for v in (5.13) then follows from (5.19), the result for u in (5.13), (5.14b), (5.3), (4.2) and (4.1a). □

**Theorem 5.2**

(a) Let the Assumptions (D5) hold. Then for the stated choices of $\varepsilon$ and k, we have that the error bounds (4.16) and (4.17) hold for $t = m\tau$, $m = 0 \to N$, with $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})\}$ replaced by $\{\hat{U}_{k,\varepsilon}, \hat{V}_{k,\varepsilon}, \varphi_\varepsilon(\hat{U}_{k,\varepsilon})\}$ with $\tau = Ck^{-1} \le Ch$.

(b) Let Assumptions (D6) hold. Then for the stated choices of $\varepsilon$ and k, we have that the error bounds (4.18) with $\tau = Ck^{-1} \le C\{h^2[\ln(1/h)]^r\}^{2/(5-2p)}$ and $p \in (\tfrac{1}{2}, 1]$ and (4.19) with $\tau = Ck^{-1} \le C\{h^2[\ln(1/h)]^r\}^{2(3-p)/(13-7p)}$ and $p \in (1/3, 1]$ hold for $h \le h_0$ and $t = m\tau$, $m = 0 \to N$, with $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})\}$ replaced by $\{\hat{U}_{k,\varepsilon}, \hat{V}_{k,\varepsilon}, \varphi_\varepsilon(\hat{U}_{k,\varepsilon})\}$.

Proof: These results follow from balancing the terms (5.13), (4.14), (4.15a), (4.9a), (4.6), (3.5), (3.6a), (2.17) and (2.18). We note that using (5.2) in place of (5.13) leads to a more restrictive bound on $\tau$. $\square$

Finally we extend the above results to the problems:

$(\hat{P}^{h,\tau}_{\varepsilon})$ For $n = 1 \to N$ find $\hat{u}^{h,n}_{\varepsilon} \in S^h_0$ such that

$$\tau^{-1}((\hat{u}^{h,n}_{\varepsilon} - \hat{u}^{h,n-1}_{\varepsilon}) + [\varphi_{\varepsilon}(\hat{u}^{h,n}_{\varepsilon}) - \varphi_{\varepsilon}(\hat{u}^{h,n-1}_{\varepsilon})],\chi)^h + (\nabla\hat{u}^{h,n}_{\varepsilon},\nabla\chi) = (f^n,\chi)^h \qquad \forall\ \chi \in S^h_0$$

$$\hat{u}^{h,0}_{\varepsilon}(\cdot) = \pi_h g(\cdot).$$

and $(\hat{P}^{h,\tau})$, the same as problem $(\hat{P}^{h,\tau}_{\varepsilon})$ with $\varphi_{\varepsilon}$ replaced by $\varphi$.

**Theorem 5.3**

Let the Assumptions (D4) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$, h and $\tau > 0$ there exist unique solutions $\hat{U}_{\varepsilon}$ to $(\hat{P}^{h,\tau}_{\varepsilon})$ and $\hat{U}$ to $(\hat{P}^{h,\tau})$. Moreover, for $m = 0 \to N$ we have that

$$\|\hat{U}_{\varepsilon}-\hat{U}_{k,\varepsilon}\|^2_{E_2(\infty,m\tau)} + \varepsilon|\pi_h[\varphi_{\varepsilon}(\hat{U}_{\varepsilon})-\varphi_{\varepsilon}(\hat{U}_{k,\varepsilon})]|^2_{L^2(Q_{m\tau})} + \varepsilon|\pi_h[\varphi_{\varepsilon}(\hat{U}_{\varepsilon})]-\hat{V}_{k,\varepsilon}|^2_{L^2(Q_{m\tau})}$$
$$\le Ck^{-2}|\partial_t\hat{V}_{k,\varepsilon}|^2_{L^2(Q_T)} \le C\varepsilon^{-1}k^{-2}[1+(k\tau)^{-1}] \tag{5.20a}$$

and

$$\|\hat{U}-\hat{U}_{\varepsilon}\|^2_{E_2(\infty,m\tau)} + \varepsilon|\pi_h[\varphi(\hat{U})-\varphi_{\varepsilon}(\hat{U}_{\varepsilon})]|^2_{L^2(Q_{m\tau})} \le C\varepsilon^{(1+p)/(1-p)}. \tag{5.20b}$$

Moreover, under the Assumptions (D5) we have (i) the first bound in (5.1) holds true for $\hat{U}_{\varepsilon}$ and $\hat{U}$. In particular, if g and f $\ge 0$ then $\hat{U}_{\varepsilon}$ and $\hat{U} \ge 0$ in $Q_T$; (ii) on choosing $\tau \le Ch$, the following error bounds hold for $t = m\tau$, $m = 0 \to N$, and the stated choices of $\varepsilon$ (a) (4.16) and (4.17) with $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_{\varepsilon}(\hat{u}^h_{k,\varepsilon})\}$ replaced by $\{\hat{U}_{\varepsilon}, \pi_h[\varphi_{\varepsilon}(\hat{U}_{\varepsilon})], \varphi_{\varepsilon}(\hat{U}_{\varepsilon})\}$ and (b) (4.16) with $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_{\varepsilon}(\hat{u}^h_{k,\varepsilon})\}$ replaced by $\{\hat{U}, \pi_h[\varphi(\hat{U})], \varphi(\hat{U})\}$.

In addition, under the Assumptions (D6) we have that the following error bounds hold for $t = m\tau$, $m = 0 \to N$, and the stated choices of $\varepsilon$ (i) (4.18) for $p \in (\frac{1}{2},1]$ and $\tau \le C\{h^2[\ln(1/h)]^r\}^{2/(5-2p)}$ with $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})\}$ replaced by (a) $\{\hat{U}_\varepsilon, \pi_h[\varphi_\varepsilon(\hat{U}_\varepsilon)], \varphi_\varepsilon(\hat{U}_\varepsilon)\}$ and (b) $\{\hat{U}, \pi_h[\varphi(\hat{U})], \varphi(\hat{U})\}$; (ii) (4.19) for $p \in (1/3,1]$, $\tau \le C\{h^2[\ln(1/h)]^r\}^{2(3-p)/(13-7p)}$ and $h \le h_0$ with $\{\hat{u}^h_{k,\varepsilon}, \hat{v}^h_{k,\varepsilon}, \varphi_\varepsilon(\hat{u}^h_{k,\varepsilon})\}$ replaced by $\{\hat{U}_\varepsilon, \pi_h[\varphi_\varepsilon(\hat{U}_\varepsilon)], \varphi_\varepsilon(\hat{U}_\varepsilon)\}$.

Proof: Existence and uniqueness of solutions to $(\hat{P}^{h,\tau}_\varepsilon)$ and $\hat{U}$ to $(\hat{P}^{h,\tau})$ follow as in the proof of Theorem 5.1 of part 1. The first inequality in (5.20a) and (5.20b) are discrete analogues of the first inequality in (2.14a) and (2.18), respectively, and are proved in similar ways. The second inequality in (5.20a) follows from (5.3). The first bound in (5.1) follows from (5.20) and the equivalence of norms on $S^h$. The above error bounds follow by combining (5.20) with Theorem 5.2. □

As stated in sections 1 and 2, problem (P) is equivalent to the generalised porous medium equation, whose finite element approximation by $(\hat{P}^{h,\tau}_\varepsilon)$ is analysed in Nochetto & Verdi (1988). There the error bounds (4.16a) and (4.17a) for $\hat{u}^h_{k,\varepsilon}$ replaced by $\hat{U}_\varepsilon$ are proved under the same choices of $\varepsilon$, but with $\tau = Ch^{1+p}$ and $\tau = Ch^{4/(3-p)}$, respectively. Therefore Theorem 5.3 above improves on these results as we require only $\tau \le Ch$. As stated previously we have assumed that the mesh is (weakly) acute, whereas they do not. Furthermore, under additional assumptions we have the improved error bounds (4.18) and (4.19).

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