Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

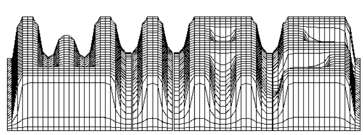
Correct voltage distribution for axisymmetric sinusoidal modeling of induction heating with prescribed current, voltage, or power

Olaf Klein, Peter Philip

submitted: August 14th 2001

Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstraße 39
D – 10117 Berlin
Germany
E-Mail: klein@wias-berlin.de
E-Mail: philip@wias-berlin.de

Preprint No. 669
Berlin 2001



2000 *Mathematics Subject Classification.* 78A25, 65Z05.

Key words and phrases. Sinusoidal induction heating, axisymmetric modeling, voltage distribution, numerical simulation.

1998 PACS numbers. 84.32.Hh, 02.60.Cb.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider the problem of determining the voltage in coil rings, which arise as an axisymmetric approximation of a single connected induction coil during modeling of induction heating. Assuming axisymmetric electromagnetic fields with sinusoidal time dependence, the voltages are computed from the condition that the total current must be equal in each ring. Depending on which of the quantities *total current*, *total voltage*, or *total power* is to be prescribed, the ring voltages are given by different linear systems of complex equations. In two sets of numerical simulations, varying the number of coil rings, we compare results using the correct voltage distribution to the corresponding results using the simple homogeneous voltage distribution.

1 Introduction

A model for the heating of workpieces by induction is presented in [CRS94] and [RS96]. The considered setup consists of a cylindrically symmetric workpiece placed inside an induction coil. It is assumed that a sinusoidal alternating voltage is imposed in the coil, resulting in an alternating current that generates a rapidly oscillating magnetic field, inducing eddy currents in the conducting parts of the workpiece. The goal is to compute the distribution of heat sources in the workpiece caused by the eddy currents due to the Joule effect.

To be able to consider the problem in an axisymmetric setting, the induction coil is replaced by N cylindrical rings. Assuming the involved quantities Q to be cylindrically symmetric in space and to have a sinusoidal time dependence, for each Q there is a representation

$$Q(r,z,t) = Q_0(r,z)\sin(\omega t + \phi(r,z)) = \mathrm{Im}(e^{i\omega t}\mathbf{Q}(r,z)), \tag{1.1}$$

where (r,z) denote cylindrical coordinates, t denotes time, ω is the angular frequency, ϕ is a phase shift, and $\mathbf{Q}$ is the complex representation of Q. In the sequel bold face will always denote the complex representation of a quantity.

The power density per volume of the heat sources can be computed from the current density $\mathbf{j}$:

$$\mu(r,z) = \frac{|\mathbf{j}(r,z)|^2}{2\,\sigma(r,z)}, \tag{1.2}$$

where σ denotes the electrical conductivity.

It is shown in [CRS94] and [RS96] that given the total voltage in each coil ring $\mathbf{v}_k$, $k = 1, \ldots, N$, there is a complex-valued magnetic scalar potential ϕ such that (cf. [RS96, Eq. (28)])

$$\mathbf{j} = \begin{cases} -i\omega\,\sigma\,\phi + \frac{\sigma\,\mathbf{v}_k}{2\pi r} & \text{in each induction coil ring,} \\ -i\omega\,\sigma\,\phi & \text{in all other conducting materials.} \end{cases} \tag{1.3}$$

The potential ϕ is determined from the system of elliptic partial differential equations [RS96, (22), (29), and (30)] with the boundary conditions $\phi = 0$ on the symmetry axis $r = 0$ and [RS96, Eq. (19)] at infinity. Since the boundary condition does not play an essential part in our considerations, we replaced the outer boundary condition [RS96, Eq. (19)] by the simpler hypothesis $\phi = 0$ sufficiently far from the coil.

Thus the complete system for ϕ is given by (1.4), where we have rewritten [RS96, (22), (29), (30)] and $\phi = 0$ in terms of the quantity $\psi := r \cdot \phi$, which we found to be more suitable for our numerical approach via a finite volume discretization:

$$-\nu\,\mathrm{div}\,\frac{\mathrm{grad}\,\psi}{r^2} = 0 \quad \text{in insulators,} \tag{1.4a}$$

$$-\nu\,\mathrm{div}\,\frac{\mathrm{grad}\,\psi}{r^2} + \frac{i\,\omega\sigma\psi}{r^2} = \frac{\sigma\,\mathbf{v}_k}{2\pi r^2} \quad \text{in the } k\text{-th coil ring,} \tag{1.4b}$$

$$-\nu\,\mathrm{div}\,\frac{\mathrm{grad}\,\psi}{r^2} + \frac{i\,\omega\sigma\psi}{r^2} = 0 \quad \text{in other conducting materials,} \tag{1.4c}$$

completed by the interface conditions

$$\left(\frac{\nu\restriction_{\text{Material}_1}}{r^2}\,\mathrm{grad}\,\psi\restriction_{\text{Material}_1}\right) \bullet \vec{n}_{\text{Material}_1} = \left(\frac{\nu\restriction_{\text{Material}_2}}{r^2}\,\mathrm{grad}\,\psi\restriction_{\text{Material}_2}\right) \bullet \vec{n}_{\text{Material}_1} \tag{1.4d}$$

on interfaces between Material_1 and Material_2, and the outer boundary condition

$$\psi\restriction_{\text{outer boundary}} = 0. \tag{1.4e}$$

ν denotes the reciprocal of the magnetic permeability, $\restriction$ denotes the restriction to the respective material, and $\vec{n}_{\text{Material}_1}$ denotes the outer unit normal vector to Material_1.

Interface condition (1.4d) is valid under the assumption that there are no surface currents. It is also assumed that ϕ (and thus ψ) is continuous throughout the whole domain.

In [CRS94] and [RS96] the coil ring voltages $\mathbf{v}_k$ are treated as given. If the coil rings constitute the two-dimensional approximation of a single, connected coil, the total current in each coil ring must be the same. This is not reflected in the simple approximation by a homogeneous distribution of a total imposed voltage V to the rings, letting $\mathbf{v}_k = V/N$, which we used to perform numerical simulations of induction heating in [KP01]. To ensure identical total currents in the different rings, the $\mathbf{v}_k$ can be determined from a linear system as described in Sec. 2. To the authors' knowledge an account of this approach is lacking from the literature. It is the objective of the second section of this article to provide the determining equations of the $\mathbf{v}_k$ for the three important cases of a prescribed total current J, a prescribed total voltage V, and a prescribed total power P.

In Sec. 3 we present numerical simulations of heat source distributions to assess the improvement gained when using the correct distribution of Sec. 2 instead of the homogeneous voltage distribution.

2 Correct Voltage Distribution to the Coil Rings

2.1 Formulation of Equations

For each solution ψ of (1.4), the corresponding total current in the k-th coil ring is given by

$$\mathbf{j}_k(\mathbf{v}_k, \psi) = \frac{\mathbf{v}_k}{2\pi} \int_{\Omega_k} \frac{\sigma}{r} \, \mathrm{d}r\mathrm{d}z \; - i\omega \int_{\Omega_k} \frac{\sigma\psi}{r} \, \mathrm{d}r\mathrm{d}z \,, \tag{2.1}$$

Ω_k denoting the two-dimensional domain of the k-th coil ring. One can now set up a joint determining system for ψ and the $\mathbf{v}_k$ by combining (2.3) or (2.5) with (1.4), depending on whether the total current or the total voltage is to be prescribed. Scaling of the solution to (1.4) and (2.5) allows to prescribe the total power.

Prescription of Total Current: In each coil ring the total current must be equal to the prescribed current

$$j(t) = \mathrm{Im}(e^{i\omega t}\, J). \tag{2.2}$$

Therefore, ψ and the $\mathbf{v}_k$ must satisfy the system

$$J = \mathbf{j}_k(\mathbf{v}_k, \psi), \quad k \in \{1, \ldots, N\}. \tag{2.3}$$

Prescription of Total Voltage: If the total voltage

$$v(t) = \mathrm{Im}(e^{i\omega t}\, V) \tag{2.4}$$

is to be prescribed, then ψ and the $\mathbf{v}_k$ must satisfy

$$\mathbf{j}_k(\mathbf{v}_k, \psi) = \mathbf{j}_{k+1}(\mathbf{v}_{k+1}, \psi), \quad k \in \{1, \ldots, N-1\}, \tag{2.5a}$$

$$V = \sum_{k=1}^{N} \mathbf{v}_k, \tag{2.5b}$$

since the total current must be equal in each ring ((2.5a)) and the total voltage is the sum of the ring voltages ((2.5b)). It is noted that instead of solving (1.4) and (2.5) one can also solve (1.4) and (2.3) for a reference current, and then scale the solution to the prescribed voltage.

Prescription of Total Power. Let $(\psi^{(1)}, \mathbf{v}_1^{(1)}, \ldots, \mathbf{v}_N^{(1)})$ denote the solution to (1.4) and (2.5). Then for each $\lambda \in \mathbb{R}$ it is $(\lambda\psi^{(1)}, \lambda\mathbf{v}_1^{(1)}, \ldots, \lambda\mathbf{v}_N^{(1)})$ a solution of (1.4) and

(2.5) with V replaced by λV. This scaled solution corresponds to the average total power

$$P^{(\lambda)} = \frac{\lambda^2 V^2 \,\mathrm{Re}\left(\mathbf{j}_1(\mathbf{v}_1^{(1)}, \psi^{(1)})\right)}{2}. \tag{2.6}$$

Thus, to prescribe the average total power P, set

$$\lambda := \frac{1}{V}\sqrt{\frac{2P}{\mathrm{Re}\left(\mathbf{j}_1(\mathbf{v}_1^{(1)}, \psi^{(1)})\right)}}. \tag{2.7}$$

2.2 Decomposition for Numerical Solution

As the system (1.4) is linear, (1.4) plus either (2.3) or (2.5) can be decomposed into $N+1$ (numerically) simpler problems: N reference problems of the form (1.4) and one problem of the form (2.3) or (2.5).

To that end choose an arbitrary reference voltage $V_{\mathrm{ref}} \neq 0$. Then the l-th reference problem $\mathrm{P}_{l,\mathrm{ref}}$ consists of (1.4) with

$$\mathbf{v}_k = \mathbf{v}_{l,k,\mathrm{ref}} := \begin{cases} V_{\mathrm{ref}} & \text{for } k = l, \\ 0 & \text{for } k \neq l. \end{cases} \tag{2.8}$$

It is noted that for each $P_{l,\mathrm{ref}}$ the matrix M of the linear discrete problem arising from a fixed spatial discretization is the same. Hence the numerically costly procedure of inverting M (rank(M)>10 000 not being unusual in applications) has to be performed only once.

Let $\psi_{l,\mathrm{ref}}$ be the solution to $\mathrm{P}_{l,\mathrm{ref}}$ and

$$\mathbf{j}_{l,k,\mathrm{ref}} := \mathbf{j}_k(\mathbf{v}_{l,k,\mathrm{ref}}, \psi_{l,\mathrm{ref}}) = \begin{cases} \frac{V_{\mathrm{ref}}}{2\pi}\int_{\Omega_k}\frac{\sigma}{r}\,\mathrm{d}r\mathrm{d}z \; - i\omega\int_{\Omega_k}\frac{\sigma\psi_{l,\mathrm{ref}}}{r}\,\mathrm{d}r\mathrm{d}z & \text{for } k = l, \\ \qquad\qquad\qquad\quad - i\omega\int_{\Omega_k}\frac{\sigma\psi_{l,\mathrm{ref}}}{r}\,\mathrm{d}r\mathrm{d}z & \text{for } k \neq l. \end{cases} \tag{2.9}$$

Then for arbitrary complex numbers $\mathbf{c}_l$ the function ψ

$$\psi := \sum_{l=1}^{N} \mathbf{c}_l \cdot \psi_{l,\mathrm{ref}} \tag{2.10}$$

is a solution to (1.4) with $\mathbf{v}_k = \mathbf{c}_k V_{\mathrm{ref}}$ and the corresponding total current in the k-th ring is given by

$$\mathbf{j}_k(\mathbf{c}_k V_{\mathrm{ref}}, \psi) = \sum_{l=1}^{N} \mathbf{c}_l \cdot \mathbf{j}_{l,k,\mathrm{ref}}. \tag{2.11}$$

It remains to determine the numbers $\mathbf{c}_l$ such that $(\psi, \mathbf{v}_1, \ldots, \mathbf{v}_N)$ is a solution of (2.3) or (2.5), respectively.

Prescription of Total Current: The $\mathbf{c}_l$ must satisfy the linear system

$$J = \sum_{l=1}^{N} \mathbf{c}_l \cdot \mathbf{j}_{l,k,\mathrm{ref}}, \quad k \in \{1, \ldots, N\}. \tag{2.12}$$

Prescription of Total Voltage: The $\mathbf{c}_l$ must satisfy the linear system

$$\sum_{l=1}^{N} \mathbf{c}_l \cdot \mathbf{j}_{l,k,\mathrm{ref}} = \sum_{l=1}^{N} \mathbf{c}_l \cdot \mathbf{j}_{l,k+1,\mathrm{ref}}, \quad k \in \{1, \ldots, N-1\}, \tag{2.13a}$$

$$V = \sum_{l=1}^{N} \mathbf{c}_l \cdot V_{\mathrm{ref}}. \tag{2.13b}$$

Prescription of Total Power: Let $\mathbf{c}_1^{(1)}, \ldots, \mathbf{c}_N^{(1)}$ denote the solution to (2.13) with $V = V_{\mathrm{ref}}$. To prescribe the total power P one has to scale with

$$\lambda := \frac{1}{V_{\mathrm{ref}}} \sqrt{\frac{2P}{\mathrm{Re}\left(\sum_{l=1}^{N} \mathbf{c}_l^{(1)} \cdot \mathbf{j}_{l,1,\mathrm{ref}}\right)}}. \tag{2.14}$$

3 Numerical Simulations

To assess the error made when distributing the total voltage homogeneously to the coil rings as compared with determining the ring voltages according to Sec. 2, we performed numerical calculations of heat source distributions, comparing the results of both methods.

The simulations were carried out for an induction heated apparatus used in sublimation growth of SiC (cf. [PAC+99, Fig. 2], s. Fig. 1 for geometric proportions). The conducting part of the apparatus consists of graphite ($\sigma = 6.8 \cdot 10^5 \frac{1}{\Omega\mathrm{m}}$, $\rho = 1750 \frac{\mathrm{kg}}{\mathrm{m}^3}$). The induction coil consists of N hollow rings made of copper ($\sigma = 5.9 \cdot 10^7 \frac{1}{\Omega\mathrm{m}}$, $\rho = 8930 \frac{\mathrm{kg}}{\mathrm{m}^3}$). For the simulation set (a) we have set $N = 5$, only using the upper five rings of Fig. 1, whereas for set (b) $N = 9$ (all rings of Fig. 1). In each material the reciprocal of the magnetic permeability is assumed to be $\nu = \mu_0^{-1} = \frac{10^7}{4\pi} \cdot \frac{\mathrm{Am}}{\mathrm{Vs}}$. As we are using the outer boundary condition $\psi = 0$, the domain where (1.4) is solved has been chosen much larger then the apparatus itself: $\psi = 0$ was set at $z = -0.8$ m, $z = 1.0$ m, $r = 1.2$ m, and $r = 1.0$ mm (to permit the evaluation of terms with r in the denominator). For a more detailed discussion of the dependence of the solution on the domain size see [KP01].

We set the angular frequency $\omega = 2\pi f$, $f = 10$ kHz. The total voltage is prescribed according to (2.4), letting $V = V_{\mathrm{eff}}\sqrt{2}$, $V_{\mathrm{eff}} = 230$ V. For the simulations $(\mathrm{a})_{\mathrm{cor}}$ and $(\mathrm{b})_{\mathrm{cor}}$ the $\mathbf{c}_l$ were determined from (2.13) ($V_{\mathrm{ref}} = 1$ V), and (1.4) was solved with $\mathbf{v}_k = \mathbf{c}_k V_{\mathrm{ref}}$, while for the simululations $(\mathrm{a})_{\mathrm{hom}}$ and $(\mathrm{b})_{\mathrm{hom}}$, (1.4) was solved with $\mathbf{v}_k = V/N$.

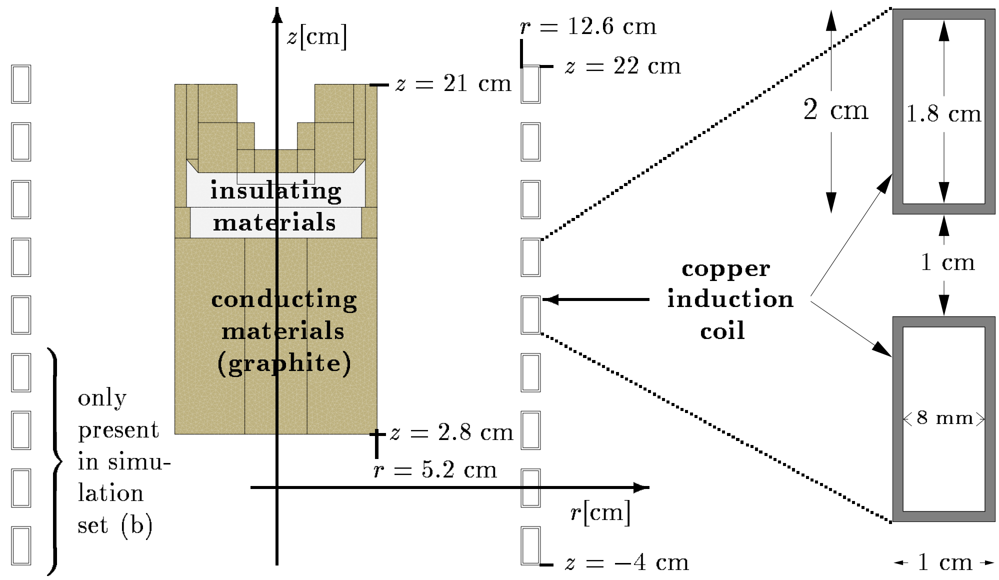


Figure 1: Geometric setup used in simulations. Shape of apparatus is taken from [PAC+99, Fig. 2].

A finite volume method based on [Fuh97] was used for the space discretization of (1.4). The discrete scheme has been implemented using the *pdelib* program package, being developed at the *Weierstrass Institute of Applied Analysis and Stochastics (WIAS)*, Berlin (cf. [FKL98]).

For each simulation $(a)_{cor}$, $(a)_{hom}$, $(b)_{cor}$, and $(b)_{hom}$, the currents $\mathbf{j}_k$ are computed numerically according to (2.11) and (2.9), where the integrals are approximated by sums over finite volume elements. For $(a)_{hom}$ and $(b)_{hom}$ the results in Tab. 1 show that the current in the outer rings is more than twice the current in the center ring. For $(a)_{cor}$ and $(b)_{cor}$ the total currents were calculated to $\mathbf{j}_k = (41.0 - 937i)$ A and $\mathbf{j}_k = (18.2 - 417i)$ A in each ring, respectively.

k	1	2	3	4	5	6	7	8	9
$(a)_{hom}$: $\mathrm{Re}(\mathbf{j}_k)$[A]	63.6	26.0	27.8	26.4	65.3				
$(a)_{hom}$: $\mathrm{Im}(\mathbf{j}_k)$[A]	-1330	-712	-674	-720	-1360				
$(b)_{hom}$: $\mathrm{Re}(\mathbf{j}_k)$[A]	33.0	13.4	14.5	14.9	15.0	14.8	14.1	12.5	32.2
$(b)_{hom}$: $\mathrm{Im}(\mathbf{j}_k)$[A]	-722	-379	-346	-336	-333	-333	-339	-369	-711

Table 1: Total currents in coil rings (numbered from top to bottom) computed during simulations $(a)_{hom}$ and $(b)_{hom}$, respectively.

Table 2 shows that in $(a)_{cor}$ the voltages in the outermost rings deviate from the homogeneous value of 65.1 V by some 10 percent, whereas the corresponding deviation in $(b)_{cor}$ is some 20 percent (the homogeneous value being 36.1 V). For the inner rings the correct voltages differ up to 10 percent from the homogeneous voltages.

Figure 2 depicts the heat sources established during $(a)_{cor}$, $(a)_{hom}$, $(b)_{cor}$, and $(b)_{hom}$,

k	1	2	3	4	5	6	7	8	9
$(a)_{cor}$: $\mathrm{Re}(\mathbf{v}_k)[V]$	60.1	67.9	70.2	67.6	59.5				
$(a)_{cor}$: $\mathrm{Im}(\mathbf{v}_k)[V]$	0.238	0.012	-0.099	-0.121	-0.031				
$(b)_{cor}$: $\mathrm{Re}(\mathbf{v}_k)[V]$	30.4	34.9	37.7	39.1	39.7	39.3	38.0	35.4	30.8
$(b)_{cor}$: $\mathrm{Im}(\mathbf{v}_k)[V]$	-0.001	-0.134	-0.206	-0.216	-0.167	-0.060	0.095	0.265	0.423

Table 2: Voltages in coil rings (numbered from top to bottom) computed during simulations $(a)_{cor}$ and $(b)_{cor}$, respectively.

together with the respective absolute and relative errors. In (a) the maximum relative error of about 10 percent occurs in regions of small heat sources. In (b), however, relative errors of 10 percent are found in the lower part of the workpiece where the heat sources are not too far from the maximum. Moreover, in (b) the relative errors reach up to 20 percent in regions of smaller heat sources.

Finally, Tab. 3 contains the average total powers established during the simulations, where the average electrical powers P_{el} are computed directly from the complex representations of the current and the voltage by numerical integration over the coil rings, whereas the average powers of heat sources $P_{heat}^{[wp]}$ and $P_{heat}^{[coil]}$ are calculated by numerical integration of the heat sources in the workpiece and in the coil, respectively. It is seen that both methods give the same powers up to three digits accuracy.

	$(a)_{cor}$	$(a)_{hom}$	$(b)_{cor}$	$(b)_{hom}$
P_{el} [kW]	6.67	6.74	2.95	2.89
$P_{heat}^{[wp]}$ [kW]	5.10	5.16	2.35	2.27
$P_{heat}^{[coil]}$ [kW]	1.57	1.58	0.60	0.62

Table 3: Average total electrical power P_{el} and average total powers of heat sources in the workpiece $P_{heat}^{[wp]}$ and in the coil $P_{heat}^{[coil]}$ as computed during simulations $(a)_{cor}$, $(a)_{hom}$, $(b)_{cor}$, and $(b)_{hom}$, respectively.

4 Conclusions

It was shown how one can ensure that the total currents in each coil ring are identical during axisymmetrical simulations of induction heating by determining the ring voltages from a linear system of equations. In two sets of numerical simulations we found that one can gain between 10 and 20 percent accuracy in the resulting heat sources by using the presented method as compared to using an equal voltage in each coil ring. The relative errors were found larger in the simulation with 9 rings than in the simulation with 5 rings. The numerical expenditure is higher for the determination of the correct voltage distribution, since the system of partial differential equations needs to be solved once for each ring instead of just once. However, for the number of rings we considered, the numerical cost is not significantly higher

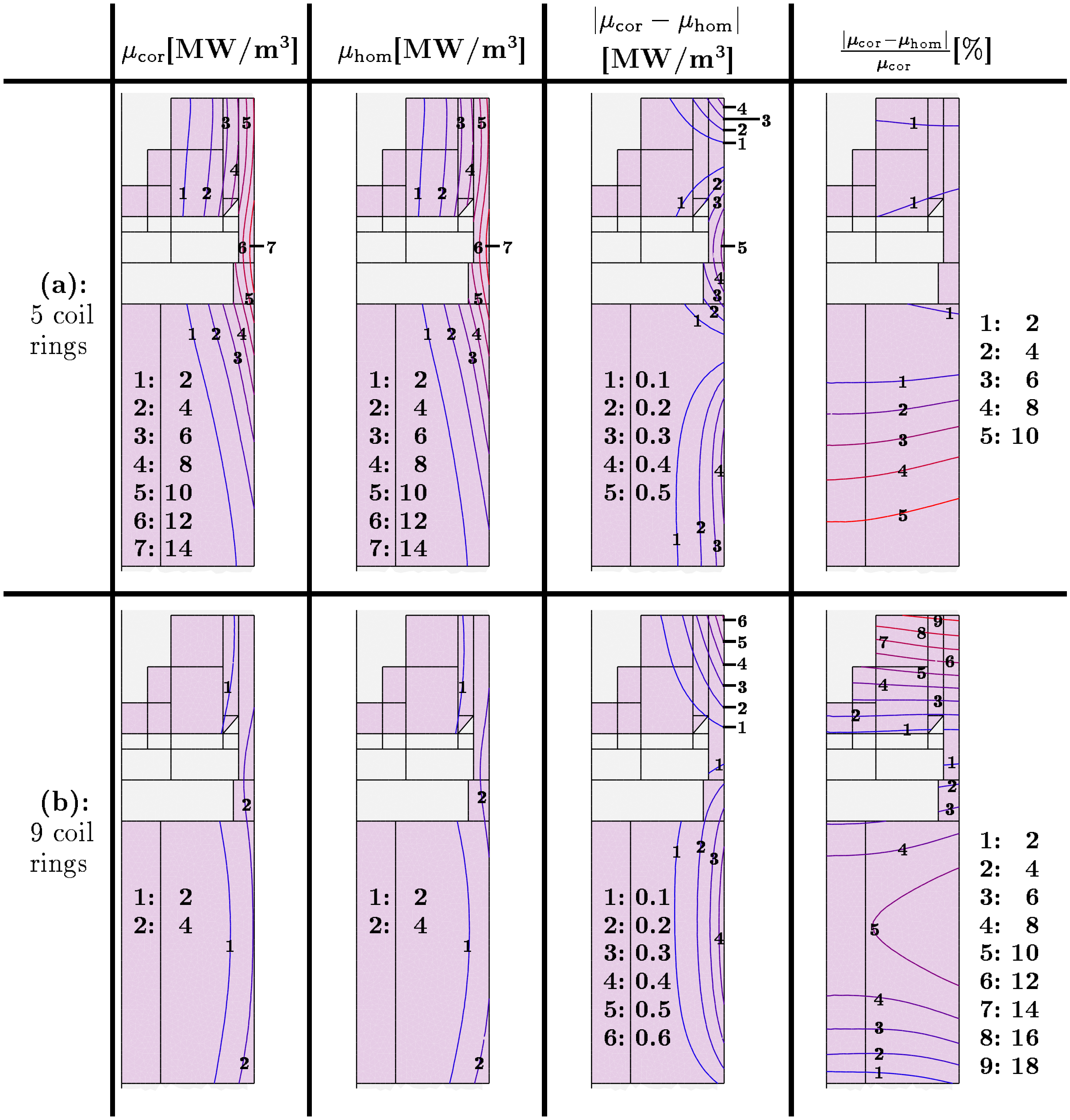


Figure 2: Results of heat sources μ_{cor} computed in simulations (a)$_{\mathrm{cor}}$ and (b)$_{\mathrm{cor}}$, μ_{hom} computed in (a)$_{\mathrm{hom}}$ and (b)$_{\mathrm{hom}}$, of the respective absolute errors, and of the respective relative errors.

due to fact that the matrix of the discrete linear system is identical in each of the N problems, and thus it needs to be inverted only once.

Acknowledgments

We thank Klaus Gärtner, Jürgen Sprekels, and Wolf Weiß of the WIAS Berlin for helpful discussions and advice.

We gratefully acknowledge financial support by the *Bundesministerium für Bildung, Wissenschaft, Forschung und Technologie*[1] *(BMBF)* within the program *Mathematische Verfahren zur Lösung von Problemstellungen in Industrie und Wirtschaft*[2] # 03SP7FV16.

References

[col96] *Modelling and optimization of distributed parameter systems (Warsaw, 1995)*. New York, Chapman & Hall, 1996.

[col97] *Modeling and Computation in Environmental Sciences*. Notes on Numerical Fluid Mechanics, vol. 59, Braunschweig, Germany, Vieweg, 1997.

[CRS94] S. Clain, J. Rappaz, and M. Swierkosz. *Coupling between nonlinear Maxwell and heat equations for an induction heating problem: Modeling and numerical methods*, Finite element methods. 50 years of the Courant element. (M. Krizek et al., ed.). Lect. Notes Pure Appl. Math., vol. 164, Marcel Dekker, Inc., New York, NY, 1994, pp. 163–171.

[FKL98] J. Fuhrmann, Th. Koprucki, and H. Langmach. *pdelib: An open modular tool box for the numerical solution of partial differential equations. Design patterns*, Proceedings of the 14th GAMM Seminar on Concepts of Numerical Software, Kiel. Notes on Numerical Fluid Mechanics, Vieweg, Braunschweig, Germany, 1998, in press.

[Fuh97] J. Fuhrmann. *On numerical solution methods for nonlinear parabolic problems*, in [col97], pp. 170–180.

[KP01] O. Klein and P. Philip. *Transient numerical investigation of induction heating during sublimation growth of silicon carbide single crystals.* Preprint No. 659, Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin, 2001.

[PAC+99] M. Pons, M. Anikin, K. Chourou, J.M. Dedulle, R. Madar, E. Blanquet, A. Pisch, C. Bernard, P. Grosse, C. Faure, G. Basset, and Y. Grange. *State of the art in the modelling of SiC sublimation growth.* Mater. Sci. Eng. B **61-62** (1999), 18–28.

[1] German Ministry for Education, Science, Research, and Technology

[2] Mathematical Methods Solving Problems in Industry and Business

[RS96] J. Rappaz and M. Swierkosz. *Modelling in numerical simulation of electromagnetic heating*, in [col96], pp. 313–320.