INVERSE PROBLEM OF DIFFRACTIVE OPTICS: CONDITIONAL STABILITY

[1] Gottfried Bruckner, [2] Jin Cheng and [3] Masahiro Yamamoto

[1] Weierstrass Institut für Angewandte Analysis und Stochastik
Mohrenstrasse 39 Berlin D-10117 Germany
e-mail: bruckner@wias-berlin.de
[2] Department of Mathematics, Fudan University
Shanghai 200433 China
e-mail: jcheng@fudan.edu.cn
[3] Department of Mathematical Sciences, the University of Tokyo
Komaba 3-8-1 Meguro Tokyo 153-8914 Japan
e-mail: myama@ms.u-tokyo.ac.jp

Abstract. In this paper, we prove conditional stability for an inverse problem in diffractive optics of determining a periodic curve from far field observations on a segment, in the case of perfect reflection. Our proof is based on a Carleman estimate for the Laplace operator.

§1. Introduction.

We consider the scattering by the perfectly reflecting periodic structure and we discuss the two dimensional modelling. According to Bao [3], Bao, Dobson and Cox [4], Hettlich and Kirsch [14], Petit [19], we can formulate the problem as follows. Let $f \in C^2(\mathbb{R})$ be 2π-periodic, $f(x) < 0$ for $x \in \mathbb{R}$. We set

$$\Omega_f = \{(x,y); y > f(x),\, x \in \mathbb{R}\}. \tag{1.1}$$

Then we regard $\partial\Omega_f = \{(x,y); y = f(x),\, x \in \mathbb{R}\}$ as a periodic interface which we should determine by scattering data. For this, we introduce an incident field $u^I(x,y;k)$ given by

$$u^I(x,y;k) = \exp\{ik(x\sin\theta - y\cos\theta)\}. \tag{1.2}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Here $i = \sqrt{-1}$ and $k > 0$ is a wave number. Throughout this paper, we assume

$$0 < |\theta| < \frac{\pi}{2} \tag{1.3}$$

and

$$0 < k < \frac{1}{2\pi}. \tag{1.4}$$

Then the resulting scattering field $u^S(x, y; k)$ satisfies the Helmholtz equation with the perfect reflection boundary condition:

$$\Delta u^S + k^2 u^S = 0 \qquad in \quad \Omega_f. \tag{1.5}$$

$$u^S + u^I = 0 \qquad on \quad \partial\Omega_f. \tag{1.6}$$

$$u^S \text{ is bounded as } y \longrightarrow \infty. \tag{1.7}$$

Moreover throughout this paper, we pose the $(k \sin\theta)$-quasi-periodicity condition for u^S:

$$u^S(x + 2\pi, y; k) = \exp(2\pi i k \sin\theta) u^S(x, y; k) \tag{1.8}$$

for all $(x, y) \in \mathbb{R}^2$ (see e.g., [3], [4], [14]). For the unique existence of u^S satisfying (1.5) - (1.8), see Kirsch [16], [17], Wilcox [21], for example.

We can state our inverse problem.

Inverse Problem of Diffractive Optics. Determine $y = f(x)$, $x \in \mathbb{R}$ from the measurements

$$u^S(x, 0; k), \quad x \in (0, 2\pi),$$

where u^S satisfies (1.5) - (1.8).

For this inverse problem, the uniqueness is proved for a lossy medium (i.e. $\mathrm{Im}k > 0$) by Bao [3], and for the case of $k \in \mathbb{R}$ by Hettlich and Kirsch [14]. We further refer to Ammari [2]. See Bruckner, Cheng and Yamamoto [7] for the uniqueness in our inverse problem with discrete observations $u(t_j, 0; k)$ where $\{t_j\}_{j\in\mathbb{N}} \subset (0, 2\pi)$. Moreover Bao and Friedman [5] proved local stability around a fixed f_0. To the authors' knowledge, however, there are no global stability results.

By the $(k\sin\theta)$-quasi-periodicity, setting

$$u = u(x, y; k) = u^I(x, y; k) + u^S(x, y; k), \tag{1.9}$$

we can rewrite (1.5) - (1.8) in terms of the total field u:

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega_f \tag{1.10}$$

$$u = 0 \qquad \text{on } \partial\Omega_f \tag{1.11}$$

$$u(x + 2\pi, y; k) = \exp(2\pi i k \sin\theta) u(x, y; k). \tag{1.12}$$

$$u - u^I \text{ is bounded as } y \longrightarrow \infty. \tag{1.13}$$

Since k is fixed such that (1.4) is true, we simply write $u(x, y)$ in place of $u(x, y; k)$. Then our inverse problem is equivalent to: *determine $y = f(x)$, $x \in \mathbb{R}$ from the measurements*

$$u(x, 0), \qquad x \in (0, 2\pi), \tag{1.14}$$

where u satisfies (1.10) - (1.13).

The purpose of this paper is to establish the conditional stability, which implies conditional well-posedness by combining with the uniqueness result by Bao [3] for the inverse diffractive optics problem in the case of (1.4).

Our method originates from other inverse problem of determining a part of a boundary (Bukhgeim, Cheng and Yamamoto [9], [10]) where the Cauchy problem for the Laplace equation is used for determining a part and conditional stability is proved. The point of the method in [9], [10], is that one needs not assume the boundary condition on the whole boundary of the domain under consideration. This is essential for the application to the present inverse problem. As for determination of parts of boundaries, we refer also to Alessandrini and Rondi [1], Beretta and Vessella [6], Rondi [20], where more general elliptic equations are considered but the boundary conditions must be assumed on the whole boundary, so that their method is not applicable to our present inverse problem.

On the other hand, the method in Bukhgeim, Cheng and Yamamoto [9], [10] relies on the maximum principle, so that discussions for the general Helmholtz equation are difficult. Therefore, in this paper, for the stability, we have to assume (1.4), which admits us to estimate interior values of a solution to the Helmholtz equation by the boundary values.

Furthermore the conditional stability is very helpful for convergence rates of Tikhonov's regularized solutions (see, e.g., Cheng and Yamamoto [12]), and in a succeeding paper, we will apply the conditional stability to the Tikhonov regularization.

This paper is organized as follows:

Section 2. Main result for conditional stability

Section 3. A generalized maximum principle

Section 4. Bounds of solutions to the forward problem

Section 5. First part of the proof

Section 6. Second part of the proof

Section 7. Third part of the proof

Section 8. Concluding remarks.

§2. Main result.

For fixed positive constants M_0, M, κ, and a_0, a such that $0 < M \le a_0 \le a$ and $0 < \kappa < 1$, we set

$$\begin{aligned} \mathcal{F} = \Big\{ & f \in C^{3+\kappa}(\mathbb{R}); \|f\|_{C^{3+\kappa}[0,2\pi]} \le M_0, \quad f \text{ is } (2\pi)\text{-periodic}, \\ & \frac{d^j f}{dx^j}(0) = \frac{d^j f}{dx^j}(2\pi), \quad j = 0,1,2,3, \\ & f(0) = f(2\pi) = -a_0, \, -a \le f(x) \le -M, \, 0 \le x \le 2\pi \Big\} \end{aligned} \tag{2.1}$$

as an admissible set of unknown surfaces. Here and henceforth let

$$\|f\|_{C^{3+\kappa}[0,2\pi]} = \sum_{j=0}^{3} \left\| \frac{d^j f}{dx^j} \right\|_{C[0,2\pi]} + \sup_{0 \le x, x' \le 2\pi, x \ne x'} \frac{\left| \frac{d^3 f}{dx^3}(x) - \frac{d^3 f}{dx^3}(x') \right|}{|x - x'|^\kappa}.$$

Let us set

$$\Omega_f = \{(x,y); y > f(x), \quad x \in \mathbb{R}\} \tag{2.2}$$

for $f \in \mathcal{F}$.

For $f_j \in \mathcal{F}$, $j = 1,2$, let us consider

$$\Delta u_j + k^2 u_j = 0 \quad \text{in } \Omega_{f_j} \tag{2.3}$$

$$u_j = 0 \quad \text{on } \partial\Omega_{f_j} \tag{2.4}$$

u_j is $(k \sin\theta)$-quasi-periodic, that is,

$$u_j(x + 2\pi, y) = \exp(2\pi i k \sin\theta) u_j(x,y). \tag{2.5}$$

We further assume that

$$u_j - u^I \text{ is bounded as } y \longrightarrow \infty. \tag{2.6}$$

We are ready to state our main result on the conditional stability in determining $f_1, f_2 \in \mathcal{F}$:

Theorem 2.1. *We assume (1.4). Then there exists a constant $C = C(k, \theta, \mathcal{F}) > 0$ such that*

$$\|f_1 - f_2\|_{C[0,2\pi]} \le \frac{C}{\left|\log\log \frac{1}{\|(u_1-u_2)(\cdot,0)\|_{H^1(0,2\pi)}}\right|} \tag{2.7}$$

provided that $f_1, f_2 \in \mathcal{F}$.

§3. A generalized maximum principle for the Helmholtz equation.

For the Helmholtz equation

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{3.1}$$

the maximum principle does not hold in general. However in the case where $k > 0$ is small, we prove

Lemma 3.1. *Let $D \subset [0, 2\pi] \times \mathbb{R}$ be a domain and let*

$$0 \le k < \frac{1}{2\pi}. \tag{3.2}$$

Then for a solution $v \in C^2(D) \cap C(\overline{D})$ to (3.1), we have

$$\|v\|_{C(\overline{D})} \le (1 - 4\pi^2 k^2)^{-\frac{1}{2}} \|v\|_{C(\partial D)}. \tag{3.3}$$

Proof. The proof is based on the argument for the proof of a lemma in Zhou [22]. Let

$$\omega(x, y) = 2\pi^2 - \frac{1}{2}x^2, \qquad (x, y) \in D$$

and let $0 \le k < \frac{1}{2\pi}$. Then

$$\begin{aligned}&\Delta(v^2 - 2k^2\|v\|^2_{C(\overline{D})}\omega(x, y))\\ =&2v\Delta v + 2|\nabla v|^2 + 2k^2\|v\|^2_{C(\overline{D})} = -2k^2v^2 + 2|\nabla v|^2 + 2k^2\|v\|^2_{C(\overline{D})} \ge 0 \quad \text{on } \overline{D}.\end{aligned}$$

Therefore the maximum principle (e.g. [13]) yields

$$
\begin{aligned}
& v(x,y)^2 - 2k^2\|v\|^2_{C(\overline{D})}\omega(x,y) \\
\le & \sup_{(x,y)\in\partial D}(v(x,y)^2 - 2k^2\|v\|^2_{C(\overline{D})}\omega(x,y)) \le (\sup_{(x,y)\in\partial D}|v(x,y)|)^2.
\end{aligned}
$$

Here we note that $\omega(x,y)\ge 0$, $(x,y)\in\partial D$, because $x\in[0,2\pi]$. Consequently we obtain

$$
(1-4\pi^2k^2)\|v\|^2_{C(\overline{D})} \le (1-2k^2\max_{(x,y)\in\overline{D}}|\omega(x,y)|)\|v\|^2_{C(\overline{D})} \le \|v\|^2_{C(\partial D)},
$$

which is (3.3). Thus the proof of Lemma 3.1 is complete.

§4. Bounds of solutions to the forward problem.

In this section, we will prove upper and lower bounds of solutions to the Helmholtz equation, which are uniform in $f\in\mathcal{F}$.

Lemma 4.1. *Let $u=u(x,y)$ satisfy*

$$
\Delta u + k^2 u = 0 \qquad \textit{in } \Omega_f \tag{4.1}
$$

$$
u = 0 \qquad \textit{on } \partial\Omega_f \tag{4.2}
$$

$$
u \textit{ is } (k\sin\theta)\textit{-quasi-periodic} \tag{4.3}
$$

$$
u - u^I \textit{ is bounded as } y\longrightarrow\infty. \tag{4.4}
$$

Moreover we choose $M_1>0$ such that

$$
0 < M_1 < \frac{M^2}{a}. \tag{4.5}
$$

Then there exist constants $M_2 > 0$ *and* $m > 0$ *dependent only on* $\mathcal{F}$ *and* M_1 *such that*

$$\|u\|_{C^2(\overline{\Omega_f})} \le M_2 \tag{4.6}$$

and

$$\|u\|_{C([0,2\pi]\times[-M_1,0])} \ge m > 0 \tag{4.7}$$

for all $f \in \mathcal{F}$.

For proving this lemma, we need some results about the forward problem.

Lemma 4.2. *Suppose that* $0 < \tau < 1$ *is constant and a* (2π)*-periodic function* $f \in C^{3+\tau}(\mathbb{R})$ *satisfies*

$$\begin{cases} \dfrac{d^j f}{dx^j}(0) = \dfrac{d^j f}{dx^j}(2\pi),\ j = 0,1,2,3, \quad f(0) = f(2\pi) = -a_0, \\ -a \le f(x) \le -M,\ 0 \le x \le 2\pi. \end{cases}$$

Let $w = w(x,y)$ *satisfy*

$$\begin{cases} \Delta w + k^2 w = 0 \qquad \text{in } \Omega_f, \\ w = 0 \qquad \text{on } \partial\Omega_f, \\ w \text{ is } (k\sin\theta)\text{-quasi-periodic}, \\ w - u^I \text{ is bounded as } y \longrightarrow \infty. \end{cases}$$

Then there exist constants $M_3 = M_3(f, a_0, a, M) > 0$ *and* $m_1 = m_1(f, a_0, a, m) > 0$ *such that*

$$\|w\|_{C^2(\overline{\Omega_f})} \le M_3 \tag{4.8}$$

and

$$\|w\|_{C([0,2\pi]\times[-M_1,0])} \ge m_1 > 0. \tag{4.9}$$

Proof. We will use the integral equation method which is outlined in [16].

Since $0 < k < \frac{1}{2\pi}$ and $0 < |\theta| < \frac{\pi}{2}$, we have that $|n + k\cos\theta| \neq k$ for all $n \in \mathbb{Z}$.

Therefore the free space quasi-periodic Green function can be defined as

$$G(x,y) = \frac{i}{4\pi}\sum_{n\in\mathbb{Z}} \frac{1}{\beta_n}\exp[i\alpha_n(x_1 - y_1) + i\beta_n|x_2 - y_2|], \qquad x \neq y$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, $\alpha_n = n + k\cos\theta$ and

$$\beta_n = \begin{cases} \sqrt{k^2 - \alpha_n^2}, & |\alpha_n| \leq k \\ i\sqrt{\alpha_n^2 - k^2}, & |\alpha_n| > k. \end{cases}$$

By Theorems 4 and 5 in [16], we can express $w^S \equiv w - u^I$ as

$$w^S(x) = \int_0^{2\pi} \left(\frac{\partial}{\partial\nu(y)} - i\right) G(x,y)|_{y_2 = f(y_1)} \sqrt{1 + f'(y_1)^2}\varphi(y_1, f(y_1))dy_1, \tag{4.10}$$

and $\widehat{\varphi}(x_1) = \exp(-ik(\cos\theta)x_1)\varphi(x_1, f(x_1))$ satisfies

$$\widehat{\varphi}(x_1) + \int_0^{2\pi} K_f(x_1, y_1)\widehat{\varphi}(y_1)dy_1 = -e^{-ik(\sin\theta)f(x_1)}. \tag{4.11}$$

Here

$$\begin{aligned} &K_f(x_1, y_1) \\ =&e^{ik\cos\theta(y_1 - x_1)} \left(\frac{\partial}{\partial\nu(y)} + i\right) G(x_1, f(x_1), y_1, y_2)|_{y_2 = f(y_1)} \sqrt{1 + f'(y_1)^2}. \end{aligned} \tag{4.12}$$

It can be shown that the Green function $G(x,y)$ has the same singularity as the fundamental solution $\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x - y|)$ of the two dimensional Helmholtz equation and $\Phi - G$ is analytic in $[(0, 2\pi) \times \mathbb{R}] \times [(0, 2\pi) \times \mathbb{R}]$ ([16]). Here $H_0^{(1)}(\cdot)$ is the Hankel function of the first kind and of order zero (e.g., Kress [18]).

It is easy to verify that the integral operator in (4.11) is a compact operator from $C^{2+\tau}[0, 2\pi]$ to $C^{2+\tau}[0, 2\pi]$. Since the integral equation (4.11) is uniquely solvable ([16]), we know that there exists a constant $M_3' > 0$ such that

$$||\widehat{\varphi}||_{C^{2+\tau}[0,2\pi]} \leq M_3'.$$

Coming back to (4.10), we can have that there exists a constant $M_3 > 0$, which may depend on f, such that $\|w\|_{C^2(\overline{\Omega_f})} \le M_3$. Thus the proof of (4.8) is complete.

Next we will prove (4.9). If it is not true, then we have $w(x_1, x_2) = 0$ for $(x_1, x_2) \in [0, 2\pi] \times [-M_1, 0]$. By the unique continuation for the Helmholtz equation, $w(x_1, x_2) = 0$ for $(x_1, x_2) \in \Omega_f$. This is a contradiction. The proof of Lemma 4.2 is complete.

Proof of Lemma 4.1. We will prove (4.6). Assume that (4.6) is not true. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ such that

$$\|u_n\|_{C^2(\overline{\Omega_{f_n}})} \longrightarrow \infty, \qquad as \quad n \to \infty, \tag{4.13}$$

where u_n is the solution to the problem (4.1) - (4.4) in the domain Ω_{f_n}.

On the other hand, since $f_n \in \mathcal{F}$, there exists a subsequence of $\{f_n\}_{n\in\mathbb{N}}$, which we still denote by $\{f_n\}_{n\in\mathbb{N}}$, such that

$$f_n \longrightarrow g \qquad \text{in} \quad C^{3+\frac{\kappa}{2}}[0, 2\pi]$$

as $n \longrightarrow \infty$. By the arguments in Lemma 4.2, we know that there exists a unique solution $\widehat{\varphi}_n \in C^{2+\frac{\kappa}{2}}[0, 2\pi]$ to the integral equation (4.11) with $f = f_n$. Moreover it can be directly verified that

$$\|\widehat{\varphi}_n - \widehat{\psi}\|_{C^{2+\frac{k}{2}}[0,2\pi]} \longrightarrow 0, \qquad n \to \infty,$$

where $\widehat{\psi} \in C^{2+\frac{\kappa}{2}}[0, 2\pi]$ is the unique solution to the integral equation (4.11) with $f = g$.

Then we have that, for $n \in \mathbb{N}$,

$$\|u_n\|_{C^2(\overline{\Omega_{f_n}})} \le C \|\widehat{\varphi_n}\|_{C^{2+\frac{\kappa}{2}}[0,2\pi]} \le M_2',$$

where $C > 0$ and $M_2' > 0$ are constants which are dependent on $\mathcal{F}$ but independent of n. This is a contradiction to (4.13). We complete the proof of (4.6).

The proof of (4.7) is similar to the proof of (4.6). Assume that (4.7) is not true. Then there exists $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ such that

$$\|u_n\|_{C([0,2\pi]\times[-M_1,0])} \longrightarrow 0$$

as $n \to \infty$.

Since $f_n \in \mathcal{F}$, there exists a subsequence of $\{f_n\}_{n\in\mathbb{N}}$, which we still denote by the same notation, such that

$$f_n \to g \qquad \text{in} \quad C^{2+\frac{\kappa}{2}}[0,2\pi]$$

as $n \longrightarrow \infty$.

It can be easily verified that

$$\|u_n - v\|_{C([0,2\pi]\times[-M_1,0])} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty,$$

where v is the solution to (4.1) - (4.4) in Ω_g. Then we have

$$v(x_1, x_2) = 0, \qquad (x_1, x_2) \in [0, 2\pi] \times [-M_1, 0].$$

By the unique continuation for the Helmholtz equation, we can obtain that

$$v(x_1, x_2) = 0, \qquad (x_1, x_2) \in \Omega_g.$$

This is a contradiction. The proof of (4.7) is complete.

§5. First part of the proof; Cauchy problem for the Helmholtz equation.

Let

(5.1) $$\gamma = \{(x, y); y = f(x), \quad 0 \le x \le 2\pi\},$$

where we set

$$f(x) = \max\{f_1(x), f_2(x)\},$$

and let Ω denote the domain bounded by $[0, 2\pi] \times \{0\}$, γ, $x = 0$ and $x = 2\pi$.

We can prove that Ω satisfies the uniform interior cone condition.

In fact, it is sufficient to prove that at every intersection point of f_1 and f_2, the cone property holds. Let $f_1(x^*) = f_2(x^*)$ with some $x^* \in [0, 2\pi]$. If $\frac{df_1}{dx}(x^*) \neq \frac{df_2}{dx}(x^*)$, then the cone property is straightforward. Therefore we have to consider the case of $\frac{df_1}{dx}(x^*) = \frac{df_2}{dx}(x^*)$. We will prove that $f = \max_{j=1,2} f_j$ is C^1 at x^*. In fact, let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence converging to x^*. Then

$$\lim_{n\to\infty} \frac{f(x_n) - f(x^*)}{x_n - x^*} = \frac{df_1}{dx}(x^*) = \frac{df_2}{dx}(x^*). \tag{5.2}$$

Let $\{x'_n\}$ be any subsequence of $\{x_n\}$. Then we can choose a subsequence $\{x''_n\}$ of $\{x'_n\}$ satisfying, say, $f(x''_n) = f_2(x''_n)$ for all n''. Hence

$$\lim_{n\to\infty} \frac{f(x''_n) - f(x^*)}{x''_n - x^*} = \lim_{n\to\infty} \frac{f_2(x''_n) - f_2(x^*)}{x''_n - x^*} = \frac{df_2}{dx}(x^*).$$

Since we can extract a subsequence converging to the unique limit from any subsequence, the limit (5.2) is true. Therefore we see that f is C^1 at x^*. Thus Ω satisfies the cone condition.

We set

$$\varepsilon_1 = \|(u_1 - u_2)(\cdot, 0)\|_{H^1(0,2\pi)} + \left\| \left(\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial y} \right)(\cdot, 0) \right\|_{L^2(0,2\pi)}.$$

Set $v = u_1 - u_2$. Then in Ω, it follows from (2.3) that $\Delta v + k^2 v = 0$. We will estimate v on γ.

Since Ω satisfies the uniform interior cone condition, we can apply Theorem 6.2

in Cheng and Yamamoto [11] in view of Lemma 4.1, so that we can prove:

There exist constants $C > 0$ and $\mu \in (0,1)$ dependent on $\mathcal{F}$, k, θ, such that

$$\|u_1 - u_2\|_{C(\overline{\gamma})} \le \frac{C}{\left|\log \frac{1}{\varepsilon_1}\right|^{\mu}}. \tag{5.3}$$

Here and henceforth $C > 0$ denotes a generic constant dependent on m, M, k, θ, a, but independent of choices f_j. On the basis of (5.3), we can prove

Lemma 5.1. *There exist constants $C > 0$ and $\mu \in (0,1)$ dependent on $\mathcal{F}$, k, θ, such that*

$$\|u_1 - u_2\|_{C(\overline{\gamma})} \le \frac{C}{\left|\log \frac{1}{\|(u_1-u_2)(\cdot,0)\|_{H^1(0,2\pi)}}\right|^{\mu}}. \tag{5.4}$$

Proof. We will estimate

$$\left\|\left(\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial y}\right)(\cdot,0)\right\|_{L^2(0,2\pi)}$$

by $\|(u_1-u_2)(\cdot,0)\|_{H^1(0,2\pi)}$. Noting that $0 > \max_{j=1,2}\{f_j(x); x \in \mathbb{R}\}$, we apply Theorem 2.1 in Bao [3], so that we see

$$\left(\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial y}\right)(\cdot,0) = B(u_1-u_2)(\cdot,0),$$

where B is a pseudodifferential operator of order one, and

$$\|Bg\|_{L^2(0,2\pi)} \le C\|g\|_{H^1(0,2\pi)}.$$

Therefore

$$\left\|\left(\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial y}\right)(\cdot,0)\right\|_{L^2(0,2\pi)} \le C\|(u_1-u_2)(\cdot,0)\|_{H^1(0,2\pi)}. \tag{5.5}$$

Combination of (5.3) with (5.5) yields (5.4). Thus the proof of Lemma 5.1 is complete.

§6. Second part of the proof; construction of a family of subdomains.

In this section, we construct domains Ω_0, Ω_0' (see (6.4), (6.8)) and a family of domains $\eta_{\ell,x_0}^{-1}(\Omega_0)$, and establish a lower bound of derivatives of the Green function for Δ in $\eta_{\ell,x_0}^{-1}(\Omega_0)$ with the homogeneous Dirichlet boundary condition. Such a family is essential in completing the proof of Theorem 2.1 in Section 7 where we estimate the solutions u_1 and u_2 and establish estimation of the distance between the two curves $y = f_j(x)$, $j = 1, 2$.

We set

$$\begin{aligned}\mathcal{F}_0 = \Big\{ & f \in C^{3+\kappa}(\mathbb{R}); \|f\|_{C^{3+\kappa}[0,2\pi]} \le M_0, \quad f \text{ is } (2\pi)\text{-periodic}, \\ & \frac{d^j f}{dx^j}(0) = \frac{d^j f}{dx^j}(2\pi), \quad j = 0,1,2,3, \\ & -a \le f(x) \le -M,\ 0 \le x \le 2\pi \Big\} \end{aligned} \tag{6.1}$$

We note that $\mathcal{F} = \mathcal{F}_0 \cap \{f; f(0) = f(2\pi) = -a_0\}$.

We call $K \subset \mathbb{R}^2$ *a finite cone with vertex* $(0,0)$ if $K = B_1 \cap \{(\lambda x, \lambda y) \in \mathbb{R}^2; \lambda > 0,\ (x,y) \in B_2\}$ where $B_1, B_2 \subset \mathbb{R}^2$ are open balls, $(0,0) \notin B_2$ and B_1 is centred at $(0,0)$. We set $K(x_0,y_0) = K + (x_0,y_0) = \{(x+x_0, y+y_0); (x,y) \in K\}$. Then $K(x_0,y_0)$ is a finite cone with vertex (x_0,y_0). Noting that $f \in \mathcal{F}_0$ satisfies the uniform cone condition, we can take $\theta_0 \in (0, \frac{\pi}{4}]$ such that for every $f \in \mathcal{F}_0$ and for every $x \in [0, 2\pi]$, a finite cone $K(x, f(x))$ whose angle is $2\theta_0$ and whose centre line is parallel to the y-axis, is contained in $\{(x,y); y > f(x)\}$. Therefore

$$-a + x\cot\theta_0 > f(x) \quad \text{for } 0 < x < a\tan\theta_0 \tag{6.2}$$

if $f \in \mathcal{F}_0$ and $f(0) = -a$.

Here we recall (4.5), that is, $0 < M_1 < \frac{M^2}{a}$. Set

$$\widetilde{f}_0(x) = \min\left\{ -a + x\cot\theta_0, \frac{M^2 - M_1 a}{3\pi a} x - M \right\} \tag{6.3}$$

for $0 \le x \le \frac{3\pi a M}{M^2 - M_1 a}$. Then

$$\widetilde{f}_0(x) \ge f(x) \quad \text{for} \quad 0 \le x \le \frac{3\pi a M}{M^2 - M_1 a}$$

if $f \in \mathcal{F}_0$ and $f(0) = -a$.

Let

$$\Omega_0 = \left\{ (x, y);\ 0 < x < \frac{3\pi a M}{M^2 - M_1 a},\ \widetilde{f}_0(x) < y < 0 \right\}. \tag{6.4}$$

Moreover we set

$$\ell\Omega_0 = \{(\ell x, \ell y); (x, y) \in \Omega_0\} \quad \text{for } \frac{M}{a} \le \ell \le 1. \tag{6.5}$$

Then we can prove

$$\ell\Omega_0 \subset \{(x, y); y > f(x)\}, \quad \text{if } f \in \mathcal{F}_0 \text{ and } f(0) = -\ell a. \tag{6.6}$$

In fact,

$$\ell\Omega_0 \cap \{(x, y); y \ge -M\} \subset \{(x, y); y > f(x)\}$$

by $f \in \mathcal{F}_0$. Let $(x, y) \in \ell\Omega_0 \cap \{(x, y); y < -M\}$. Then

$$\ell\Omega_0 \cap \{(x, y); y < -M\} \subset K(0, -\ell a)$$

by (6.3). Therefore, by the choice of θ_0, it follows that $y > f(x)$. Thus (6.6) is seen.

We define a transform $\eta = \eta_{\ell, x_0} : (x, y) \longrightarrow (\xi_1, \xi_2)$ by

$$\begin{cases} \xi_1 = \dfrac{1}{\ell}(x - x_0) \\ \xi_2 = \dfrac{1}{\ell} y \end{cases}$$

for $\frac{M}{a} \le \ell \le 1$. Then

$$\eta_{\ell, x_0}^{-1}(\xi_1, \xi_2) = (\ell\xi_1 + x_0, \ell\xi_2).$$

It follows from (6.6) that

$$\eta^{-1}_{\ell,x_0}(\Omega_0) \subset \{(x,y); f(x) < y < 0\}, \quad \text{if } f \in \mathcal{F}_0 \text{ and } f(x_0) = -\ell a. \tag{6.7}$$

In fact, let $f \in \mathcal{F}_0$ and $f(x_0) = -\ell a$. we set $\widetilde{f}(\xi_1) = f(\xi_1 + x_0)$. Then $\widetilde{f} \in \mathcal{F}_0$ and $\widetilde{f}(0) = -\ell a$. Hence by (6.6), we obtain $\ell\Omega_0 \subset \{(\xi_1,\xi_2); \xi_2 > \widetilde{f}(\xi_1)\}$, that is,

$$\begin{aligned}\eta^{-1}_{\ell,x_0}(\Omega_0) = \ell\Omega_0 + (x_0, 0) &\subset \{(\xi_1 + x_0, \xi_2); \xi_2 > \widetilde{f}(\xi_1)\} \\ =&\{(\xi_1,\xi_2); \xi_2 > \widetilde{f}(\xi_1 - x_0) = f(\xi_1)\},\end{aligned}$$

which is (6.7).

Moreover we will take a subdomain Ω_0' such that

$$\overline{\Omega_0'} \subset \Omega_0, \qquad \|u\|_{L^2(\Omega_0')} \ge c_0 m \tag{6.8}$$

if u satisfies

$$\|u\|_{C^2([0,2\pi]\times[-M_1,0])} \le M_2, \tag{6.9}$$

$$\|u\|_{C([0,2\pi]\times[-M_1,0])} \ge m > 0 \tag{6.10}$$

and

$$u(x+2\pi, y) = \exp(2\pi i k \sin\theta) u(x,y), \quad 0 \le x \le 2\pi, \ -M \le y \le 0. \tag{6.11}$$

Here we recall (4.5). Here $c_0 > 0$ is a constant which depends only on M_0, M_1, M_2, m, $\mathcal{F}$.

We take sufficiently small $d > 0$ so that

$$\begin{cases} d < \dfrac{2M_1 a}{M}, \quad d < \dfrac{m}{4M_2}, \quad d < \dfrac{\pi a}{3M}, \\ d < \dfrac{M_1 a}{M} - M_1, \quad \dfrac{3\pi a}{M} - d < \dfrac{3\pi a M}{M^2 - M_1 a}. \end{cases} \tag{6.12}$$

We set

$$\Omega_0' = \left(\frac{\pi a}{M} - 3d, \frac{3\pi a}{M} - d\right) \times \left(-\frac{M_1 a}{M}, -\frac{d}{2}\right). \tag{6.13}$$

By (4.5), (6.3), (6.4) and (6.12), we can directly verify that $\overline{\Omega_0'} \subset \Omega_0$.

Let u satisfy (6.9) - (6.11). First let

$$\|u(\cdot, h)\|_{C[0,2\pi]} \ge m > 0$$

for $-\frac{d}{2} \le h \le 0$. Then we set

$$B(\widetilde{x}, h; d) = \{(x, y); |x - \widetilde{x}|^2 + (y - h)^2 < d^2\}.$$

Then we see that

$$\Omega_0' \supset \bigcup \left\{ B(\widetilde{x}, h; d) \cap \{y < h - \frac{d}{2}\};\ \frac{\pi a}{M} - 2d < \widetilde{x} < \frac{3\pi a}{M} - 2d \right\}. \tag{6.14}$$

In fact, the set at the right hand side is included in $\left(\frac{\pi a}{M} - 3d, \frac{3\pi a}{M} - d\right) \times \left(h - d, h - \frac{d}{2}\right)$. Moreover by $d < \frac{2M_1 a}{M}$ in (6.12) and $-\frac{d}{2} \le h \le 0$, we have $-\frac{M_1 a}{M} < h - d < h - \frac{d}{2} < -\frac{d}{2}$, and (6.14) is seen.

Since

$$\left| \left[\frac{\pi a}{M} - 2d, \frac{3\pi a}{M} - 2d\right] \right| = \frac{2\pi a}{M} \ge 2\pi,$$

there exists

$$x_u^* \in \left[\frac{\pi a}{M} - 2d, \frac{3\pi a}{M} - 2d\right] \quad \text{such that } |u(x_u^*, h)| \ge m > 0, \tag{6.15}$$

in view of (6.11). Then by (6.9) and $d < \frac{m}{4M_2}$ in (6.12), we apply the mean value theorem to have

$$|u(x, y)| \ge \frac{m}{2}, \quad (x, y) \in B(x_u^*, h; d) \cap \left\{ y < h - \frac{d}{2} \right\},$$

that is,

$$\|u\|_{L^2(\Omega_0')} \geq \left(\int_{B(x_u^*,h;d)\cap\{y<h-\frac{d}{2}\}} |u(x,y)|^2 dxdy\right)^{\frac{1}{2}}$$

$$\geq \frac{m}{2}\left|B(x_u^*,h;d)\cap\left\{y<h-\frac{d}{2}\right\}\right|^{\frac{1}{2}} = \frac{dm}{2}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)^{\frac{1}{2}}. \tag{6.16}$$

Next we assume

$$\|u(\cdot,h)\|_{C[0,2\pi]} \geq m > 0$$

for $-M_1 \leq h \leq -\frac{d}{2}$. Then, since $-\frac{M_1}{M}a \leq -M_1$, we have $(x,h)\in\overline{\Omega_0'}$ for $\frac{\pi a}{M}-3d \leq x \leq \frac{3\pi a}{M}-d$. Therefore, if x_u^* satisfies (6.15), then by $d<\frac{M_1 a}{M}-M_1$ in (6.12), we have $B(x_u^*,h;d)\cap\{y<h\}\subset\Omega_0'$, so that

$$|B(x_u^*,h;d)\cap\Omega_0'| \geq \frac{1}{2}|B(x_u^*,h;d)|. \tag{6.17}$$

Moreover by (6.9) and $d<\frac{m}{4M_2}$ in (6.12), we see that

$$|u(x,y)| \geq \frac{m}{2} > 0, \quad (x,y)\in B(x_u^*,h;d)\cap\Omega_0'. \tag{6.18}$$

Therefore, by (6.17) and (6.18), we have

$$\|u\|_{L^2(\Omega_0')} \geq \left(\int_{B(x_u^*,h;d)\cap\Omega_0'} |u(x,y)|^2 dxdy\right)^{\frac{1}{2}}$$

$$\geq \frac{m}{2}|B(x_u^*,h;d)\cap\Omega_0'|^{\frac{1}{2}} \geq \frac{m}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} d. \tag{6.19}$$

Therefore taking

$$c_0 = \min\left\{\frac{d}{2}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)^{\frac{1}{2}}, \quad \frac{d}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\right\},$$

we see from (6.16) and (6.19) that (6.8) holds true.

By $\frac{M}{a} \le \ell \le 1$, we have

$$
\begin{aligned}
&\eta^{-1}_{\ell,x_0}(\Omega_0') = \{(\ell\xi_1 + x_0, \ell\xi_2); (\xi_1,\xi_2) \in \Omega_0'\} \\
&= \left\{(\ell\xi_1 + x_0, \ell\xi_2); \frac{\pi a}{M} - 3d < \xi_1 < \frac{3a\pi}{M} - d, -\frac{M_1}{M}a < \xi_2 < -\frac{d}{2}\right\} \\
&= \left\{(\xi_1,\xi_2); x_0 + \frac{\pi a\ell}{M} - 3\ell d < \xi_1 < x_0 + \frac{3\pi a\ell}{M} - \ell d, -\frac{M_1 a\ell}{M} < \xi_2 < -\frac{\ell d}{2}\right\} \\
&\supset \left\{(\xi_1,\xi_2); x_0 + \frac{\pi a\ell}{M} - 3\ell d < \xi_1 < x_0 + \frac{3\pi a\ell}{M} - \ell d, -M_1 < \xi_2 < -\frac{\ell d}{2}\right\}.
\end{aligned}
\tag{6.20}
$$

Consequently by (4.5) we obtain

$$
\eta^{-1}_{\ell,x_0}(\Omega_0') \subset \{(x,y); y > -M\}. \tag{6.21}
$$

Moreover

$$
\int_{\eta^{-1}_{\ell,x_0}(\Omega_0')} |u(x,y)|^2 dxdy = \ell^2 \int_{\Omega_0'} |u(\ell\xi_1 + x_0, \ell\xi_2)|^2 d\xi_1 d\xi_2.
$$

Therefore if we further choose small $d > 0$, then, noting that the length of $\eta^{-1}_{\ell,x_0}(\Omega_0')$ in the x-direction is not shorter than 2π and taking (6.11) into consideration, similarly to (6.8), we see that

$$
\|u\|_{L^2(\eta^{-1}_{\ell,x_0}(\Omega_0'))} \ge c_0\ell m \ge \frac{c_0 M}{a} m, \tag{6.21}
$$

if u satisfies (6.9) - (6.11).

On the other hand, since $\overline{\Omega_0'} \subset \Omega_0$ and η^{-1}_{ℓ,x_0} is an isomorphism, we see that

$$
\overline{\eta^{-1}_{\ell,x_0}(\Omega_0')} \subset \eta^{-1}_{\ell,x_0}(\overline{\Omega_0'}) \subset \eta^{-1}_{\ell,x_0}(\Omega_0). \tag{6.22}
$$

Since $\partial\Omega_0$ is approximated by C^3-curves in the interior of Ω_0, we see that there exists the Green function with the same property as in the case of C^3-domains (e.g. Section 16 of Chapter 3 in Itô [15]). Let $G_0(x,y,\xi_1,\xi_2)$ and $G_{\ell,x_0}(x,y,\xi_1,\xi_2)$ be the Green functions for Δ in Ω_0 and $\eta_{\ell,x_0}^{-1}(\Omega_0)$ respectively with the homogeneous Dirichlet boundary condition. Then we directly see that

$$
\begin{aligned}
&G_{\ell,x_0}(x,y,\xi_1,\xi_2) = \ell^2 G_0(\eta_{\ell,x_0}(x,y), \eta_{\ell,x_0}(\xi_1,\xi_2)),\\
&\qquad (x,y),(\xi_1,\xi_2) \in \eta_{\ell,x_0}^{-1}(\Omega_0).
\end{aligned}
\tag{6.23}
$$

We consider a Dirichlet problem in $\eta_{\ell,x_0}^{-1}(\Omega_0)$:

$$
\begin{cases}
\Delta\Psi = 0 & \text{in } \eta_{\ell,x_0}^{-1}(\Omega_0),\\
\Psi_{|\partial(\eta_{\ell,x_0}^{-1}(\Omega_0))} = \psi.
\end{cases}
\tag{6.24}
$$

Then, since the domain $\eta_{\ell,x_0}^{-1}(\Omega_0)$ is parametrized by means of a linear function in $\ell \in \left[\frac{M}{a}, 1\right]$ and x_0, we see that there exists a constant $C = C(m, M_0, M_1, M, k, \theta, a) > 0$, independent of $\ell \in \left[\frac{M}{a}, 1\right]$ and x_0 such that

$$
\|\partial_\nu \Psi\|_{L^2(\partial(\eta_{\ell,x_0}^{-1}(\Omega_0)))} \le C\|\psi\|_{H^1(\partial(\eta_{\ell,x_0}^{-1}(\Omega_0)))}.
\tag{6.25}
$$

Here ∂_ν denotes the normal derivative on $\partial(\eta_{\ell,x_0}^{-1}(\Omega_0))$.

Moreover, in view of (6.22), (6.23) and the positivity of $-\frac{\partial G_{\ell,x_0}}{\partial\nu}$ (e.g. Theorem 18.2 in Chapter 4 in Itô [15]), since $\overline{\Omega_0'} \subset \Omega_0$ and $\frac{M}{a} \le \ell \le 1$, we see

$$
\begin{aligned}
&\min\left\{-\frac{\partial G_{\ell,x_0}}{\partial x}(x_0,y,\xi_1,\xi_2);\ -\ell a \le y \le -M, (\xi_1,\xi_2) \in \eta_{\ell,x_0}^{-1}(\Omega_0')\right\}\\
\ge&\ell \min\left\{-\frac{\partial G_0}{\partial x}(0,y,\xi_1,\xi_2);\ -a \le y \le -\frac{M}{\ell}, (\xi_1,\xi_2) \in \Omega_0'\right\}\\
\ge&\frac{M}{a}\min\left\{-\frac{\partial G_0}{\partial x}(0,y,\xi_1,\xi_2);\ -a \le y \le -M, (\xi_1,\xi_2) \in \Omega_0'\right\}\\
=&\mu_0 > 0.
\end{aligned}
\tag{6.26}
$$

Here $\mu_0 > 0$ depends only on a, M_0, M_1, M and m, θ, k, and independent of $\ell \in \left[\frac{M}{a}, 1\right]$ and x_0.

§7. Third part of the proof; estimation of distance between two curves.

Let $|(f_1 - f_2)(x)|$ take the maximum at a point $x_0 \in [0, 2\pi]$. Without loss of generality, we may assume that $f_2(x_0) > f_1(x_0)$ and therefore

$$\|f_1 - f_2\|_{C[0,2\pi]} = f_2(x_0) - f_1(x_0).$$

We recall that the curve γ is defined by (5.1).

In this section, we will prove

Lemma 7.1. *There exists a constant $C > 0$ such that*

$$|f_1(x_0) - f_2(x_0)| \le \frac{C}{\log \frac{1}{\delta}}$$

if $\|u_1 - u_2\|_{C(\overline{\gamma})} \le \delta$.

Once Lemma 7.1 is proved, we can directly complete the proof of Theorem 2.1 by Lemma 5.1.

Henceforth we define the distance between a point (p, q) and the curve $y = f(x)$ by

$$\text{dist}\,((p,q), f) = \inf_{t\in\mathbb{R}} (|t-p|^2 + |f(t) - q|^2)^{\frac{1}{2}}. \tag{7.1}$$

Then we prove

Lemma 7.2. *There exists a constant $m_0 = m_0(M, a) > 0$ such that*

$$dist\,((p,q), f) \ge m_0 |f(p) - q|$$

for all $f \in \mathcal{F}$ and $(p, q) \in [0, 2\pi] \times [-a, 0]$.

Proof of Lemma 7.2. We set $f'(t) = \frac{df}{dt}(t)$. We note that for a parameter s, the point $(t + sf'(t), f(t) - s)$ is on the line normal to the curve $y = f(x)$ at $(t, f(t))$

and that if for $(p,q) \in [0,2\pi] \times [-a,0]$, in (7.1), the minimum is attained at $t_0 \in \mathbb{R}$, then $(p-t_0, q-f(t_0))$ is orthogonal to the tangential vector $(1, f'(t_0))$. Therefore

$$\mathrm{dist}\,((p,q),f) = \inf\{|s|(|f'(t)|^2+1)^{\frac{1}{2}}; p = t + sf'(t), q = f(t) - s, t, s \in \mathbb{R}\}.$$

We choose $\Lambda > 0$ such that

$$\Lambda = \max\{M_0 + a + 2\pi, \quad 2\pi + (a + M_0)M_0\}.$$

Then for any $(p,q) \in [0,2\pi] \times [-a,0]$, if $|t| > \Lambda$, then

$$\mathrm{dist}\,((p,q),f) < ((t-p)^2 + (f(t)-q)^2)^{\frac{1}{2}}. \tag{7.2}$$

In fact,

$$\mathrm{dist}\,((p,q),f) = \inf_{t\in\mathbb{R}}((t-p)^2 + (f(t)-q)^2)^{\frac{1}{2}} \le |f(p) - q| \le M_0 + a$$

by $f \in \mathcal{F}$. On the other hand, $|t| > \Lambda$ yields that $|t-p| \ge \Lambda - 2\pi$ for $0 \le p \le 2\pi$. By the definition of Λ, we have

$$M_0 + a < \Lambda - 2\pi \le |t-p| \le ((t-p)^2 + (f(t)-q)^2)^{\frac{1}{2}}.$$

Therefore (7.2) is seen.

Hence we obtain

$$\begin{aligned} &\mathrm{dist}\,((p,q),f) = \min\{|s|(|f'(t)|^2+1)^{\frac{1}{2}}; p = t + sf'(t), q = f(t) - s, \\ &|t| \le \Lambda,\ s \in \mathbb{R}\}. \end{aligned} \tag{7.3}$$

Moreover we note that

$$\begin{aligned} &\{(t,s) \in [-\Lambda,\Lambda] \times \mathbb{R}; p = t + sf'(t), q = f(t) - s\} \\ &\text{is not empty for } (p,q) \in [0,2\pi] \times [-a,0]. \end{aligned} \tag{7.4}$$

In fact, we define a mapping $G_f : (t,s) \longrightarrow (x,y)$ by

$$\begin{cases} x = t + sf'(t), \\ \\ y = f(t) - s. \end{cases}$$

Then we have to prove

$$G_f((-\Lambda,\Lambda)\times(-a,-M+a)) \supset [0,2\pi]\times[-a,0]$$

for $f \in \mathcal{F}$.

Let $(p,q) \in [0,2\pi]\times[-a,0]$ be arbitrary. Then we have to prove the existence of $(t,s) \in [-\Lambda,\Lambda]\times[-a,-M+a]$ such that $p = t + sf'(t)$ and $q = f(t) - s$. For the proof of (7.4), it is sufficient to verify the existence of a root t of

$$t = p + (q - f(t))f'(t) \equiv H(t).$$

By the definition of Λ, for fixed $f \in \mathcal{F}$ and $(p,q) \in [0,2\pi]\times[-a,0]$, we can prove that H maps $[-\Lambda,\Lambda]$ into $[-\Lambda,\Lambda]$. In fact,

$$|H(t)| \le |p| + |q - f(t)||f'(t)| \le 2\pi + (a + M_0)M_0 \le \Lambda.$$

Therefore the Brouwer fixed point theorem yields the existence of a fixed point $t = H(t)$ (e.g. Theorem 10.1 in Gilbarg and Trudinger [13]). On the other hand, since $s = f(t) - y$, $-a \le f(t) \le -M$ and $0 \le -y \le a$, we have $-a \le s \le -M + a$. Hence (7.4) follows.

Note that if $p = t + sf'(t)$ and $q = f(t) - s$ for some $t, s \in \mathbb{R}$, then $|f(p) - q| = |f(t+sf'(t)) - (f(t)-s)|$. In view of (7.3), for the proof of the lemma, it is sufficient to verify

$$\sup\Bigg\{ \frac{|f(t+sf'(t)) - (f(t)-s)|}{|s|(|f'(t)|^2+1)^{\frac{1}{2}}}; \\ 0 \le p \le 2\pi,\ -a \le q \le 0,\ p = t + sf'(t),\ q = f(t) - s, \\ -\Lambda \le t \le \Lambda,\ -a \le s \le -M + a,\ f \in \mathcal{F}\Bigg\} < \infty.$$

By the mean value theorem, we can choose $\chi \in (0,1)$ to obtain

$$|f(t+sf'(t)) - (f(t)-s)| = |f(t) + f'(t+\chi sf'(t))sf'(t) - f(t) + s|$$

$$\leq |s|(1 + |f'(t+\chi sf'(t))||f'(t)|) \leq |s|(1+M_0^2)$$

for $f \in \mathcal{F}$. Hence

$$\begin{aligned}
&\sup\Bigg\{ \frac{|f(t+sf'(t)) - (f(t)-s)|}{|s|(|f'(t)|^2+1)^{\frac{1}{2}}}; \\
&0 \leq p \leq 2\pi,\ -a \leq q \leq 0,\ p = t+sf'(t),\ q = f(t)-s, \\
&-\Lambda \leq t \leq \Lambda,\ -a \leq s \leq -M+a,\ f \in \mathcal{F} \Bigg\} \\
\leq & \frac{1+M_0^2}{(|f'(t)|^2+1)^{\frac{1}{2}}} \leq 1+M_0^2.
\end{aligned}$$

Thus the proof of Lemma 7.2 is complete.

We set

$$\Omega(f) = \{(x,y); 0 < x < 2\pi,\ f(x) < y < 0\}.$$

Let D be the connected component of $\Omega(f_1) \setminus \Omega(f_2)$ which includes the segment $x = x_0$. By $f_1, f_2 \in \mathcal{F}$, the graphs of f_1 and f_2 intersect and so D is bounded by the graphs of $y = f_1(x)$ and $y = f_2(x)$, $0 \leq x \leq 2\pi$. We set

$$D_h = \{(x,y);\ \mathrm{dist}\,((x,y), \partial D) \geq h\} \tag{7.5}$$

for $h > 0$.

If

$$|(f_1 - f_2)(x_0)| \leq \frac{1}{\log\frac{1}{\delta}}, \tag{7.6}$$

then we have already proved Lemma 7.1 with $C = 1$. Therefore we assume that

$$|(f_1 - f_2)(x_0)| \geq \frac{1}{\log\frac{1}{\delta}}. \tag{7.7}$$

We set

$$A = (x_0, f_2(x_0)), \quad B = (x_0, f_1(x_0)). \tag{7.8}$$

We recall that we are assuming that $f_2(x_0) > f_1(x_0)$, without loss of generality. We further set

$$h = \frac{1}{4} \frac{1}{\log \frac{1}{\delta}} \tag{7.9}$$

and

$$\begin{cases} P = \left(x_0, f_2(x_0) - \dfrac{f_2(x_0) - f_1(x_0)}{4} \right), \\ Q = \left(x_0, f_1(x_0) + \dfrac{f_2(x_0) - f_1(x_0)}{4} \right). \end{cases} \tag{7.10}$$

We note that $P, Q \in D$. Furthermore by Lemma 7.2, we see that

$$\overline{PQ} \subset D_{m_0 h}. \tag{7.11}$$

(Here and henceforth, $\overline{PQ}$ denotes the segment connecting P and Q which includes P and Q.) In fact, let $(p, q) \in \overline{PQ}$. Then, by (7.7), we see that $|f_1(p) - q|$, $|f_2(p) - q| \geq h$, so that Lemma 7.2 implies

$$\operatorname{dist}((p, q), f_1), \quad \operatorname{dist}((p, q), f_2) \geq m_0 h,$$

that is, $(p, q) \in D_{m_0 h}$.

By $-a \leq f_1(x_0) \leq -M$, we can take $\ell \in \left[\frac{M}{a}, 1\right]$ such that $f_1(x_0) = -\ell a$. Noting that $f_1 \in \mathcal{F}$, we apply (6.7) to obtain

$$\eta^{-1}_{\ell, x_0}(\Omega_0) \subset \{(x, y); y > f_1(x)\}. \tag{7.12}$$

For simplicity, we set

$$E = \eta^{-1}_{\ell, x_0}(\Omega_0) \tag{7.13}$$

and

$$E' = \eta_{\ell,x_0}^{-1}(\Omega_0'). \tag{7.14}$$

Then by (6.4) and (6.21), we have

$$\{(x_0, y); f_1(x_0) < y < 0\} \subset \partial E, \quad \overline{PQ} \subset \partial E \tag{7.15}$$

and

$$\|u_1\|_{L^2(E')} \ge c_1 m. \tag{7.16}$$

Moreover by (6.26), we obtain

$$-\frac{\partial G_E}{\partial x}(x_0, y, \xi_1, \xi_2) \ge \mu_0 > 0 \quad \text{if } (x_0, y) \in \overline{PQ} \text{ and } (\xi_1, \xi_2) \in E', \tag{7.17}$$

where G_E is the Green function of Δ with the homogeneous Dirichlet boundary condition in E.

We will consider a function u_1 in the domain D. We have

$$\Delta u_1 + k^2 u_1 = 0 \qquad \text{in } D. \tag{7.18}$$

Setting $\gamma_j = \partial D \cap \partial\Omega_{f_j}$, $j = 1, 2$, we see from the remark before (7.5) that $\partial D = \gamma_1 \cup \gamma_2$. On γ_2, the condition (2.4) yields $u_2 = 0$. Consequently

$$|u_1| = |u_2 + u_1 - u_2| = |u_1 - u_2| \le \delta \quad \text{on } \gamma_2. \tag{7.19}$$

Therefore, since $u_1 = 0$ on γ_1, we obtain $|u_1| \le \delta$ on ∂D. By Lemma 3.1,

$$|u_1| \le (1 - 4\pi^2 k^2)^{-1/2}\delta \qquad \text{in } D. \tag{7.20}$$

Hence, by (7.20) and the Schauder interior estimate (e.g. Theorem 6.2 in Gilbarg and Trudinger [13]), we have

$$\begin{aligned} &\left|\frac{\partial^2 u_1}{\partial x^2}\right|, \left|\frac{\partial^2 u_1}{\partial x \partial y}\right|, \left|\frac{\partial^2 u_1}{\partial y^2}\right| \le \frac{C\delta}{m_0^2 h^2}, \\ &|\nabla u_1| \le \frac{C\delta}{m_0 h} \quad \text{in } D_{m_0 h}, \end{aligned} \tag{7.21}$$

where $C > 0$ depends only on k.

We choose

$$\kappa \in \left(\frac{1}{2}, \frac{2}{3}\right) \tag{7.22}$$

such that

$$\frac{1}{2}\frac{1}{\log\frac{1}{\delta}}, \qquad \left(\frac{1}{\log\frac{1}{\delta}}\right)^2 \ge \delta^\kappa, \tag{7.23}$$

because $\delta > 0$ can be assumed to be sufficiently small. Then, by (7.7) and (7.21), we obtain

$$|\overline{PQ}| = \frac{f_2(x_0) - f_1(x_0)}{2} \ge \delta^\kappa \tag{7.24}$$

and

$$\begin{aligned} &\left|\frac{\partial^2 u_1}{\partial x^2}\right|^2, \left|\frac{\partial^2 u_1}{\partial x \partial y}\right|^2, \left|\frac{\partial^2 u_1}{\partial y^2}\right|^2, \\ &|u_1|^2, |\nabla u_1|^2 \le C\delta^{2(1-\kappa)} \quad \text{in } D_{m_0 h}. \end{aligned} \tag{7.25}$$

Furthermore let us take a sub-segment in $\overline{PQ}$ with the starting point P' and the end point Q' such that

$$|\overline{PP'}| = |\overline{QQ'}| = \frac{1}{4}|\overline{PQ}|. \tag{7.26}$$

In E, we will use the following Carleman estimate with non-homogeneous boundary value (Lemma 2.4 in Bukhgeim [8]):

$$\begin{aligned} &\int_E \left(\Delta\psi|u_1|^2 + (\Delta\psi - 1)|\nabla u_1|^2\right) e^\psi \, dxdy \\ \le &\int_E |\Delta u_1|^2 e^\psi \, dxdy + \int_{\partial E} \left(\partial_\nu\psi(|u_1|^2 + |\nabla u_1|^2) + 8|\partial_{\nu^\perp}\nabla u_1||\nabla u_1|\right) e^\psi \, d\sigma \end{aligned} \tag{7.27}$$

for real-valued $\psi \in C^2(\overline{E})$. Here and henceforth $\partial_{\nu^\perp}$ and ∂_ν denote the tangential derivative and the normal derivative respectively.

Let us choose the weight function ψ in the form:

$$\psi = \psi_1 + s\Psi. \tag{7.28}$$

Here $s \ge 0$ is a parameter,

$$\begin{cases} \Delta\psi_1 = |k|^4 + 1 \qquad \text{in } E, \\ \\ \psi_{1|\partial E} = 0, \end{cases} \tag{7.29}$$

$$\begin{cases} \Delta\Psi = 0 \qquad \text{in } E, \\ \\ \Psi_{|\partial E} = \psi_0, \end{cases} \tag{7.30}$$

where

$$\begin{cases} \psi_0 \in C^\infty(\partial E), \quad 0 \le \psi_0 \le 1, \\ \\ \psi_0(\xi_1, \xi_2) = 0, \quad (\xi_1, \xi_2) \notin \overline{PQ}, \qquad \psi_0(\xi_1, \xi_2) = 1, \quad (\xi_1, \xi_2) \in \overline{P'Q'}. \end{cases} \tag{7.31}$$

Here we note that the segment $\overline{P'Q'}$ is strictly included in $\overline{PQ}$, and we can take such ψ_0. By (7.9), (7.10) and (7.23), we have

$$|\nabla\psi_0| \le C\delta^{-\kappa} \quad \text{on } \partial E \tag{7.32}$$

as well as (7.31). Therefore (6.25) yields

$$\|\partial_\nu \Psi\|_{L^2(\partial E)} \le C\delta^{-\kappa} \tag{7.33}$$

in view of (7.32), where the constant $C > 0$ is independent of δ.

Let

$$\Psi(\xi_1, \xi_2) = \min_{(x,y)\in E'} \Psi(x, y). \tag{7.34}$$

Then, by (7.31), (7.17), (7.10) and (7.26), we obtain

$$\begin{aligned} \Psi(\xi_1, \xi_2) &= \int_{\partial E} -\frac{\partial G_E}{\partial \nu}(x, y, \xi_1, \xi_2)\psi_0(x, y)d\sigma \\ &= \int_{\overline{PQ}} -\frac{\partial G_E}{\partial \nu}(x_0, y, \xi_1, \xi_2)\psi_0(x, y)d\sigma \ge \int_{\overline{P'Q'}} -\frac{\partial G_E}{\partial \nu}(x_0, y, \xi_1, \xi_2)d\sigma \\ &\ge \min_{(x_0,y)\in\overline{P'Q'}} \left(-\frac{\partial G_E}{\partial \nu}(x_0, y, \xi_1, \xi_2)\right) |\overline{P'Q'}| \\ &\ge \mu_0|\overline{P'Q'}| = \frac{1}{4}\mu_0(f_2(x_0) - f_1(x_0)). \end{aligned} \tag{7.35}$$

Now we return to (7.27). Since $\Delta u_1 = -k^2 u_1$ in E, from (7.27), we obtain

$$
\begin{aligned}
&\min\{|k|^4, 1\} \int_E (|u_1|^2 + |\nabla u_1|^2) e^{\psi} \, dx dy \\
&\le \int_{\partial E} \{(\partial_\nu \psi_1 + s\partial_\nu \Psi)(|u_1|^2 + |\nabla u_1|^2) + 8|\partial_{\nu^\perp} \nabla u_1||\nabla u_1|\} e^{s\psi_0} d\sigma.
\end{aligned} \tag{7.36}
$$

By (7.35) and (7.16), the left-hand side of (7.36) can estimated from below as follows:

$$
\begin{aligned}
&\min\{|k|^4, 1\} \int_E (|u_1|^2 + |\nabla u_1|^2) e^{\psi} \, dx dy \\
\ge &\min\{|k|^4, 1\} \int_{E'} (|u_1|^2 + |\nabla u_1|^2) e^{\psi} \, dx dy \\
\ge &c_1^2 m^2 \min\{|k|^4, 1\} \exp(-\|\psi_1\|_{L^\infty(E')}) \exp(s\Psi(\xi_1, \xi_2)) \\
\ge &C_2 \exp\left(\frac{1}{4}\mu_0 s(f_2(x_0) - f_1(x_0))\right)
\end{aligned} \tag{7.37}
$$

with some $C_2 > 0$ independent of δ.

Next we will estimate the right-hand side of the inequality (7.36) from above. Let us decompose $\partial E = \overline{PQ} \cup (\partial E \setminus \overline{PQ})$. On $\overline{PQ}$, we have that, according to (7.11), (7.25) and (7.33),

$$
\begin{aligned}
&\int_{\overline{PQ}} \{(\partial_\nu \psi_1 + s\partial_\nu \Psi)(|u_1|^2 + |\nabla u_1|^2) + 8|\partial_{\nu^\perp} \nabla u_1||\nabla u_1|) e^{s\psi_0} d\sigma \\
\le &Ce^s \{\delta^{2(1-\kappa)}(1+s)\delta^{-\kappa} + \delta^{2(1-\kappa)}\} \\
\le &Ce^s \{\delta^{2-3\kappa}(1+s) + \delta^{2-2\kappa}\} \le Ce^{2s}\delta^{2-3\kappa}.
\end{aligned} \tag{7.38}
$$

Here we have used $1 + s \le e^s$ for $s > 0$.

By the maximum principle, we have $0 < \Psi < 1$ in E. Hence, by $\Psi_{\partial E \setminus \overline{PQ}} = 0$, the strong maximum principle yields $\partial_\nu \Psi \le 0$ on $\partial E \setminus \overline{PQ}$. Therefore, by (7.25), we find that

$$\begin{aligned}&\int_{\partial E\setminus\overline{PQ}}\{(\partial_\nu\psi_1+s\partial_\nu\Psi)(|u_1|^2+|\nabla u_1|^2)+8|\partial_{\nu^\perp}\nabla u_1||\nabla u_1|)e^{s\psi_0}d\sigma\\ &\le\int_{\partial E\setminus\overline{PQ}}\{\partial_\nu\psi_1(|u_1|^2+|\nabla u_1|^2)+8|\partial_{\nu^\perp}\nabla u_1||\nabla u_1|\}d\sigma\le C(M_0).\end{aligned} \tag{7.39}$$

Here we also used (4.5). Applying the estimates (7.37) - (7.39) into the inequality (7.36), we obtain

$$\exp\left(\frac{1}{4}\mu_0 s(f_2(x_0)-f_1(x_0))\right)\le C(1+e^{2s}\delta^{2-3\kappa}). \tag{7.40}$$

Now we can choose $s>0$ such that

$$e^{2s}\delta^{2-3\kappa}=1,$$

that is,

$$s=\frac{2-3\kappa}{2}\log\frac{1}{\delta}. \tag{7.41}$$

Since we can assume $0<\delta<1$ and $\frac{1}{2}<\kappa<\frac{2}{3}$, we have that $\log\frac{1}{\delta}>0$ and $s>0$. Now taking the logarithm of the both sides of (7.40) with s given by (7.41), we obtain

$$f_2(x_0)-f_1(x_0)\le\frac{C}{\log\frac{1}{\delta}}.$$

Thus the proof of Lemma 7.1 is complete.

Acknowlegements. This work has been done within the framework of the research project for diffractive optics organised by Dr. Volker Schulz at Weierstrass Institut für Angewandte Analysis und Stochastik (WIAS), Berlin. The second named author is partly supported by NSF of China (No. 19971016). The paper was finished during stays of the second and third named authors at WIAS in 2000. They thank WIAS for the kind invitation.

References

1. G. Alessandrini and L. Rondi, *Stable determination of a crack in a planar inhomogeneous conductor*, SIAM J. Math. Anal. **30** (1999), 326–340.
2. H. Ammari, *Uniqueness theorems for an inverse problem in a doubly periodic structure*, Inverse Problems **11** (1995), 823–833.
3. G. Bao, *A uniqueness theorem for an inverse problem in periodic diffractive optics*, Inverse Problems **10** (1994), 335–340.
4. G. Bao, D.C. Dobson and J.A. Cox, *Mathematical studies in rigorous grating theory*, J. Opt. Soc. Amer. **12** (1995), 1029–1042.
5. G. Bao and A. Friedman, *Inverse problems for scattering by periodic structures*, Arch. Rat. Mech. Anal. **132** (1995), 49–72.
6. E. Beretta and S. Vessella, *Stable determination of boundaries from Cauchy data*, SIAM J. Math. Anal. **30** (1999), 220–232.
7. G. Bruckner, J. Cheng and M. Yamamoto, *Uniqueness of determining a periodic structure from discrete far field observations*, WIAS Preprint No.605, Weierstrass Institut für Angewandte Analysis und Stochastik (WIAS), Berlin, 2000.
8. A.L. Bukhgeim, *Extension of solutions of elliptic equations from discrete sets*, J. Inverse and Ill-posed Problems **1** (1993), 17–32.
9. A.L. Bukhgeim, J. Cheng and M. Yamamoto, *Stability for an inverse boundary problem of determining a part of a boundary*, Inverse Problems **15** (1999), 1021–1032.
10. A.L. Bukhgeim, J. Cheng and M. Yamamoto, *Conditional stability in an inverse problem of determining a non-smooth boundary*, J. Math. Anal. Appl. **242** (2000), 57–74.
11. J. Cheng and M. Yamamoto, *Unique continuation on a line for harmonic functions*, Inverse Problems **14** (1998), 869–882.
12. J. Cheng and M. Yamamoto, *One new strategy for a priori choice of regularizing parameters in Tikhonov's regularization*, Inverse Problems **16** (2000), L31–L38.
13. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1977.
14. F. Hettlich and A. Kirsch, *Schiffer's theorem in inverse scattering theory for periodic structures*, Inverse Problems **13** (1997), 351–361.
15. S. Itô, *Diffusion Equations*, American Mathematical Society, Providence, Rhode Island, 1992.
16. A. Kirsch, *Diffraction by periodic structures*, "Inverse Problems in Mathematical Physics", Lecture Notes in Physics **422** (1993), 87–102.
17. A. Kirsch, *Uniqueness theorems in inverse scattering theory for periodic structures*, Inverse Problems **10** (1994), 145–152.
18. R. Kress, *Linear Integral Equations*, Springer-Verlag, Berlin, 1989.
19. R. Petit, *Electromagnetic Theory of Gratings*, Springer-Verlag, Berlin, 1980.
20. L. Rondi, *Optimal stability estimates for the determination of defects by electrostatic measurements*, Inverse Problems **15** (1999), 1193–1212.
21. C.H. Wilcox, *Scattering Theory for Diffraction Gratings*, Springer-Verlag, Berlin, 1984.
22. C. Zhou, *The equivalence of the uniqueness of the Dirichlet problem and the maximum principle for elliptic systems*, J. Math. Anal. Appl. **183** (1994), 208–215.