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On a Class of Compactly Epi-Lipschitzian Sets

Abderrahim Jourani[1]

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[1] Université de Bourgogne
Département de Mathématiques
Analyse Appliquée et Optimisation
BP 47870
21078 Dijon Cedex
France
E-Mail: jourani@u-bourgogne.fr

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Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
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D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
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Abstract

The paper is devoted to the study of the so-called compactly epi-Lipschitzian sets. These sets are needed for many aspects of generalized differentiation, particulary for necessary optimality conditions, stability of mathematical programming problems and calculus rules for subdifferentials and normal cones. We present general conditions under which sets defined by general constraints are compactly epi-Lipschitzian. This allows us to show how the compact epi-Lipschitzness properties behave under set intersections.

1 Introduction.

In 1978, Rockafellar introduced the concept of epi-Lipschitzian sets in order to get a property of interior tangent vectors in finite dimension. He showed that the boundary of the set must be Lipschitzian around any boundary point where the Clarke's tangent cone to the set at this point has an interior. This result has been extended to the infinite dimensional situation by Borwein and Strojwas [2] by introducing the class of compactly epi-Lipschitzian (CEL) sets. A subset C in some Banach space X is said [2] to be compactly epi-Lipschitzian at $\bar{x} \in C$ if there exist $\gamma > 0$, a neighbourhood V of $\bar{x}$ and a compact set $H \subset X$ such that

$$C \cap V + t\gamma B_X \subset C - tH, \quad \text{for all} \quad t \in]0, \gamma[.$$

This class includes all finite dimensional and all epi-Lipschitzian sets. It is worth to note that in infinite dimension, there is no relationship between compact sets and CEL sets (because these later are never compact in infinite dimensional spaces). This also implies that the CEL sets are useful only in infinite dimensional spaces. Borwein and Strojwas [2-4] and Borwein [1] obtained strong results related to these sets. They investigate the relationship between the Clarke's tangent cone and the limit inferior of contingent cones at a neighbouring point, which allows them to generalize the results by Penot [20], Cornet [5] and Treiman [24]. In their paper, Borwein and Strojwas [3] showed that a Banach space is reflexive iff for every closed set in this space the Clarke's tangent cone is contained in the limit inferior of the convex hull of the weak-contingent cones. In [10], the author proved that in Asplund spaces and for CEL sets the previous inclusion holds as equality. In joint papers with Thibault [14-15], the author gave necessary conditions for CEL sets and used these sets to obtain new necessary optimality conditions to vector optimization problems and new chain rules for the Ioffe's approximate subdifferential. Other applications to the marginal function and the intersection formulae are given in the paper [11]. In [9], Ioffe established complete characterization of CEL sets in terms of his approximate subdifferential.

In [21], Rockafellar gave a sufficient condition for a set to be epi-Lipschitzian. He proved that if in a Banach space X, a set C has an inequality representation, that is,

$$C = \{x \in X : \ f(x) \leq 0\}$$

and if $0 \notin \partial_c f(\bar{x})$, for $\bar{x} \in f^{-1}(0)$, then C is epi-Lipschitzian at $\bar{x}$. Here ∂_c denotes the Clarke's subdifferential. In [6], Cornet and Czarnecki showed that in finite dimension, any epi-Lipschitzian set C at some point $\bar{x}$ can be represented as a nondegenerate inequality at this point, i.e., there exists a locally Lipschitzian function f at $\bar{x}$ and a neigbourhood U of $\bar{x}$ such that

$$f(\bar{x}) = 0, \quad 0 \notin \partial_c f(\bar{x}), \quad U \cap C = \{x \in X : \ f(x) \leq 0\}.$$

The function that they used is the following :

$$f(x) = \operatorname{dist}(C, x) - \operatorname{dist}(C^c, x).$$

Natural questions are obvious:

Can we get the same result using the approximate subdifferential?

Can we get a nondegenerate inequality representation of CEL sets?

What about the intersection of CEL sets?

It is well-known that for any locally Lipschitzian function f around $\bar{x}$ we have

$$\partial_c f(\bar{x}) = \overline{\text{co}} \partial_A f(\bar{x})$$

where ∂_A denotes the Ioffe's approximate subdifferential. Paradoxically, if f is only lower semicontinuous the approximate subdifferential may contain (strictly) the Clarke's subdifferential. Thus, can we expect the set

$$C = \{x \in X : \ f(x) \leq 0\}$$

to be epi-Lipschitzian under the weaker condition $0 \notin \partial_A f(\bar{x})$? As it is shown in the following example, the answer is negative. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a function defined by $f(x, y) = |x| - |y|$. One can check that $0 \notin \partial_A f(0)$ while the set $C = \{(x, y) : \ |x| \leq |y|\}$ fails to be epi-Lipschitzian at 0.

In this paper we will show that the weaker condition $0 \notin \partial_A f(\bar{x})$, for a lower semicontonuous function f near $\bar{x} \in f^{-1}(0)$, with CEL epigraph, implies that the set C is CEL at $\bar{x}$. In fact we will establish a more general result on sets defined by generalized inequalities, i.e., sets of the form

$$C = \{x \in X : \ g(x) \in A, \, x \in B\}$$

where g is a mapping between two given Banach spaces and A and B are closed sets in these spaces. We will give sufficient conditions under which C is CEL.

Another question that we land here is about the intersection of CEL sets. As it will be stated later a simple example shows that the intersection of CEL sets fails to be CEL. We will give verifiable conditions under which this intersection remains CEL.

2 Approximate subdifferentials and preliminaries

Throughout we shall assume that X and Y are Banach spaces, X^* and Y^* are their topological duals and $\langle \cdot, \cdot \rangle$ is the pairing between the spaces. We denote by B_X, $B_{X^*}, \cdots$ the closed unit balls of X, $X^*, \cdots$. By $d(\cdot, S)$ we denote the usual distance function to the set S

$$d(x, S) = \inf_{u \in S} \|x - u\|.$$

We write $x \longrightarrow^f x_0$ and $x \rightarrow^S x_0$ to express $x \rightarrow x_0$ with $f(x) \rightarrow f(x_0)$ and $x \rightarrow x_0$ with $x \in S$, respectively.

If f is an extended-real-valued function on X, we write for any subset S of X

$$f_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ +\infty & \text{othewise.} \end{cases}$$

The function

$$d^- f(x, h) = \liminf_{\substack{u \rightarrow h \\ t \downarrow 0}} t^{-1}(f(x + tu) - f(x))$$

is the lower Dini directional derivative of f at x and the Dini ε-subdifferential of f at x is the set

$$\partial_\varepsilon^- f(x) = \{x^* \in X^* : \langle x^*, h \rangle \leq d^- f(x; h) + \varepsilon \|h\|, \forall h \in X\}$$

for $x \in Domf$ and $\partial_\varepsilon^- f(x) = \emptyset$ if $x \notin Domf$, where $Domf$ denotes the effective domain of f. For $\varepsilon = 0$ we write $\partial^- f(x)$.

By $\mathcal{F}(X)$ we denote the collection of finite dimensional subspaces of X. The approximate subdifferentials of f at $x_0 \in Domf$ is defined by the following expressions (see Ioffe [7-8])

$$\partial_A f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \longrightarrow^f x_0} \partial^- f_{x+L}(x) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \longrightarrow^f x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f_{x+L}(x)$$

where $\limsup_{x \xrightarrow{f} x_o} \partial^- f_{x+L}(x) = \{x^* \in X^* : x^* = w^* - lim\, x_i^*, x_i^* \in \partial^- f_{x_i+L}(x_i), x_i \xrightarrow{f} x_0\}$, that is, the set of w^*-limits of all such nets.

It is easily seen that the multivalued function

$$x \rightarrow \partial_A f(x)$$

is upper semicontinuous in the following sense

$$\partial_A f(x_0) = \limsup_{x \xrightarrow{f} x_0} \partial_A f(x)$$

and in [8] Ioffe has shown that when S is a closed subset of X and $x_0 \in S$

$$\partial_A d(x_0, S) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{S} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- d_{x+L}(x, S).$$

The following sum rule has been established by Ioffe in [8] for a more general situation. For the purpose of our discusion, a semi-Lipschitz case suffices.

Theorem 2.1 *[8]. Let $f : X \to \mathbb{R}$ be a function which is lower semicontinuous near x_0 and $g : X \to \mathbb{R}$ be a function which is Lipschitz around x_0. Then*

$$\partial_A(f+g)(x_0) \subset \partial_A f(x_0) + \partial_A g(x_0).$$

In the sequel we shall need the following class of mappings between Banach spaces.

Definition 2.1 *[23] [15]. A mapping $g : X \mapsto Y$ is said to be strongly compactly Lipschitzian (s.c.L.) at a point x_0 if there exist a multivalued function $R : X \mapsto 2^{Comp(Y)}$, where $Comp(Y)$ denotes the set of all norm compact subsets of Y, and a function $r : X \times X \to \mathbb{R}_+$ satisfying*

(i) $\lim_{\substack{x \to x_0 \\ h \to 0}} r(x, h) = 0,$

(ii) there exists $\alpha > 0$ such that

$$t^{-1}[g(x+th) - g(x)] \in R(h) + \|h\| r(x, th) B_Y$$

for all $x \in x_0 + \alpha B_X, h \in \alpha B_X$ and $t \in]0, \alpha[$,

(iii) $R(0) = \{0\}$ and R is upper semicontinuous.

It can be shown [23] that every s.c.L. mapping is locally Lipschitzian. In finite dimensions the concepts coincide.

Recently we have developped in [13] a chain rule for this class of mappings. Let us note that this chain rule has been obtained before by Ioffe in [8] for maps with compact prederivatives.

Theorem 2.2 *[13]. Let $g : X \to Y$ be s.c.L. at x_0 and let $f : Y \to \mathbb{R}$ be locally Lipschitz at $g(x_0)$. Then $f \circ g$ is locally Lipschitz at x_0 and*

$$\partial_A(f \circ g)(x_0) \subset \bigcup_{y^* \in \partial_A f(g(x_0))} \partial_A(y^* \circ g)(x_0).$$

To complete this section we note the following property of s.c.L. mappings which is a direct consequence of Proposition 2.3 in [13].

Proposition 2.1 *Let $g : X \to Y$ be s.c.L. at x_0 and let (y_i^*) any bounded net of Y^* which w^*-converges to zero in Y^* and let (x_i) be a net norm-converging to x_0 in X. If $x_i^* \in \partial_A(y_i^* \circ g)(x_i)$, then (x_i^*) w^*-converges to zero in X^*.*

Before stating the following theorem which will be one of the main tool of section 5 let us recall the following notion by Borwein and Strojwas [2]. A set $S \subset X$ is said to be *compactly epi-Lipschitzian* (CEL for short) at $x_0 \in S$ if there exist $\gamma > 0$ and a norm compact set $H \subset X$ such that

$$S \cap (x_0 + \gamma B_X) + t\gamma B_X \subset S - tH, \quad \text{for all} \quad t \in]0, \gamma[.$$

We close this section by recalling the following results from [15].

Theorem 2.3 *Let $A \subset Y$ and $B \subset X$ be two closed subsets and $g : X \to Y$ be s.c.L. at $\bar{x} \in B \cap g^{-1}(A)$. Suppose that D is compactly epi-Lipschitz at $g(x_0)$. Suppose also that the following regularity condition holds at $\bar{x}$*

$$[y^* \in \partial_A d(g(\bar{x}), A) \quad \textit{and} \quad 0 \in \partial_A(y^* \circ g + d(\cdot, B))(\bar{x})] \Longrightarrow y^* = 0.$$

Then for some real numbers $a \geq 0$ and $r > 0$

$$d(x, B \cap g^{-1}(A - y)) \leq a d(g(x) + y, A)$$

for all $x \in B \cap (\bar{x} + rB_X)$ and $y \in rB_Y$.

3 Characterization of CEL sets.

We begin this section by recalling that a set $K^* \subset X^*$ is (weak-star) locally compact if every point of K^* lies in a weak-star neighbourhood V^* such that $\mathrm{cl}^*(V^*) \cap K^*$ is weak-star compact. The first important property of these cones has been established by Loewen in [16] in a reflexive Banach space (but the proof works in any Banach space). He showed that if (x_i^*) is a net in a locally compact cone K^* then

$$(x_i^*) \text{ weak-star converges to 0 iff it converges in norm to 0.}$$

In the same paper, Loewen showed that if H is a norm-compact subset of X, the following set is locally compact

$$c(H) = \{x^* \in X^* : \quad \|x^*\| \leq \max_{h \in H} |\langle x^*, h \rangle|\rangle\}.$$

In [10], the author showed that a weak-star closed cone K^* is locally compact iff there exist $h_1, \cdots, h_n \in X$ such that $K^* \subset c(\{h_1, \cdots, h_n\})$.

The following result is not new. It is a consequence of the results by the author in [10], by the author in a joint work with Thibault [15] and by Ioffe [9].

Theorem 3.1 *Let C be a closed set in X containing $\bar{x}$. Then the following assertions are equivalent :*

i) C is CEL at $\bar{x}$.

ii) There exist a weak-star closed locally compact cone K_1^* and $\varepsilon_1 > 0$ such that

$$\partial_A d(C, x) \subset K_1^*, \quad \forall x \in C \cap (\bar{x} + \varepsilon_1 B_X).$$

iii) There exist a weak-star closed locally compact cone K_2^* and $\varepsilon_2 > 0$ such that

$$\partial_\varepsilon^- \xi(C, x) \subset K_2^* + \varepsilon B_{X^*}, \quad \forall x \in C \cap (\bar{x} + \varepsilon_2 B_X) \text{ and } \varepsilon \in]0, \varepsilon_2[.$$

If in addition X is Asplund then the above assertions are equivalent to the following ones:

iv) There exist a weak-star closed locally compact cone K_3^* and $\varepsilon_3 > 0$ such that

$$\partial_F d(C, x) \subset K_3^*, \quad \forall x \in C \cap (\bar{x} + \varepsilon_3 B_X).$$

v) There exist a weak-star closed locally compact cone K_4^* and $\varepsilon_4 > 0$ such that

$$\partial^- \xi(C, x) \subset K_4^*, \quad \forall x \in C \cap (\bar{x} + \varepsilon_4 B_X).$$

Moreover assertions ii) $-$ v) hold with the same locally compact cone which can be taken equal to $c(\{h_1, \cdots, h_n\})$, for some $h_1, \cdots, h_n$ in X.

Here ∂_F denotes the limiting Fréchet subdifferential (see [17]) and $\xi(C, \cdot)$ denotes the indicator function of C.

Proof. *i)* $\Longrightarrow$ *ii)*: See Jourani and Thibault [15].

ii) $\Longrightarrow$ *i)*: See Ioffe [9].

See for the other equivalences the paper by Jourani [10].$\diamondsuit$

4 Main results.

As we said in the introduction the approximate subdifferential of lower semicontinuous functions may be bigger than the Clarke's one (∂_c) and is always contained in it for any locally Lipschitzian function. In [10], the author showed that for a function f whose epigraph is CEL at $(\bar{x}, f(\bar{x}))$ one has

$$\partial_A f(\bar{x}) \subset \partial_c f(\bar{x})$$

and for any CEL set S at $\bar{x}$

$$N_A(S, \bar{x}) = \mathbb{R}_+ \partial_A d(S, \bar{x}) \tag{1}$$

We shall consider sets defined by generalized inequalities. Let $g : X \mapsto Y$ be a mapping and let A and B be closed sets of X and Y respectively. We set

$$C = \{x \in X : \ g(x) \in A, \, x \in B\}.$$

Our first result gives sufficient conditions for C to be CEL.

Theorem 4.1 *Suppose that*

i) g is strongly compactly Lipschitzian at $\bar{x}$.

ii) A and B are CEL at $g(\bar{x})$ and $\bar{x}$ respectively.

Then we have either

a) There exists $y^* \in \partial_A d(A, g(\bar{x}))$, with $y^* \neq 0$, such that

$$0 \in \partial_A (y^* \circ g)(\bar{x}) + \partial_A d(B, \bar{x}).$$

or

b) For all $y^* \in \partial_A d(A, g(\bar{x}))$, with $y^* \neq 0$, we have

$$0 \notin \partial_A (y^* \circ g)(\bar{x}) + \partial_A d(B, \bar{x}).$$

In the later case C is CEL at $\bar{x}$ with

$$N_A(C, \bar{x}) \subset \bigcup_{y^* \in N_A(A, g(\bar{x}))} \partial_A (y^* \circ g)(\bar{x}) + N_A(B, \bar{x}).$$

Remark 1. In fact we may prove (in case b)) that, under the assumptions of the theorem, the multivalued function $F : Y \mapsto X$ defined by

$$F(y) = \{x \in X : \ g(x) + y \in A, \, x \in B\}$$

is uniformly compactly Lipschitzian in the sense of Jourani and Thibault [14] (which implies that the set C is CEL at $\bar{x}$). By Theorem 4.7 in [14] and Theorem 2.3, we obtain that the graph GrF of F is CEL at $(0, \bar{x})$.

Remark 2. We may consider, instead of C, any multivalued mapping. But as our objective here is to study only CEL sets, we will consider the general situation in another paper.

In connection with the work by Cornet and Czarnecki [6], the following question arises:

Can we characterize CEL sets in terms of their associate distance?

We mean that if a set C is CEL at $\bar{x}$, does $0 \notin \partial_A \Delta_C(\bar{x})$? Where $\Delta_C(x) = d(C,x) - d(C^c,x)$. Unfortunately the answer is negative. Take for example, in infinite dimensional space, a CEL set C at $\bar{x}$ with empty interior. Then we have $\Delta_C(x) = d(C,x)$ and $0 \in \partial_A \Delta_C(\bar{x})$. Note that CEL sets with empty interior already exist. Take $X = l^\infty(\mathbb{N})$, $f(x) = \liminf |x_n|$ and $C = \{x \in X : \ f(x) \leq 0\}$. Then C is CEL at 0 and has no interior. Moreover both $\partial_A f(0)$ and $\partial_A \Delta_C(0)$ contain 0.

In [19], Mordukhovich and Wang established results on the intersection of the so-called sequentially normally compact sets in Asplund spaces by using the limiting Fréchet subdifférential. Note that every CEL set is sequentially normally compact. The proof proposed in [19] is based on an extremal principle. Here we use the approximate subdifferential which is more suitable to CEL sets and works in any Banach space. The proof proposed here is different from that in [19].

It is easy to show that the union and the product of CEL sets are CEL. But, what about the intersection of CEL sets? The answer is unfortunately negative. To see this take, in infinite dimensional spaces, a closed convex pointed cone K with interior. Set $A = K$ and $B = -K$. It is clear that A and B are CEL at 0, but the intersection $A \cap B = \{0\}$ fails to be CEL. So we need other assumptions to ensure the above property. The following result is a direct application of our Theorem 4.1.

Theorem 4.2 *Let $A \subset X$ and $B \subset X$ be nonempty closed CEL sets at $\bar{x}$. Then we have either*

i) $\partial_A d(A,\bar{x}) \cap (-\partial_A d(B,\bar{x}))) \neq \{0\}$

or

ii) $\partial_A d(A,\bar{x}) \cap (-\partial_A d(B,\bar{x}))) = \{0\}$. *In this case $A \cap B$ is CEL at $\bar{x}$ with*

$$N_A(A \cap B, \bar{x}) \subset N_A(A,\bar{x}) + N_A(B,\bar{x}).$$

Remark 3. Note that the inclusions in Theorems 4.1 and 4.2 hold with only A CEL (and not both A and B) and with N_G instead of N_A, where $N_G(D,\bar{x}) = \mathbb{R}_+ \partial_A d(D,\bar{x})$.

Next let us present an important corollary of Theorem 4.2 that establishes the CEL property for constraint sets defined by inequalities with lower semicontinuous functions.

Corollary 4.1 *Let $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function around $\bar{x}$ such that*

i) $f(\bar{x}) = 0$, *and* $0 \notin \partial_A f(\bar{x})$;

ii) The epigraph of f is CEL at $(\bar{x}, 0)$.

Then the set $C = \{x \in X : \ f(x) \leq 0\}$ *is CEL at* $\bar{x}$ *with*

$$N_A(C, \bar{x}) \subset \mathbb{R}_+(\partial_A f(\bar{x}) \cup \partial_A^\infty f(\bar{x})).$$

Where $\partial_A^\infty f(\bar{x})$ *denotes the singular approximate subdifferential of* f *at* $\bar{x}$ *(see [7]).*

Proof. Introduce the sets $A = \text{epi } f$, the epigraph of f, and $B = X \times \mathbb{R}_-$. Then A and B are CEL at $(\bar{x}, 0)$ and since $0 \notin \partial_A f(\bar{x})$ we get

$$\partial_A d(A, (\bar{x}, 0)) \cap (-\partial_A d(B, (\bar{x}, 0))) = \{0\}.$$

So, by Theorem 4.2, $A \cap B$ is CEL at $(\bar{x}, 0)$ and hence C is also CEL at $\bar{x}$. The inclusion follows from a simple computation.$\diamondsuit$

Remark 4. Taking into account Remark 1, the qualification condition *i)* ensures that

$$\text{epi} f \text{ is CEL at } (\bar{x}, 0) \text{ iff } C \text{ is CEL at } \bar{x}.$$

Because of the Remark 3, the inclusion in the corollary holds for any lower semicontinuous function f (whose epigraph is not necessarily CEL) satisfying

$$f(\bar{x}) = 0, \quad \text{and} \quad 0 \notin \partial_A f(\bar{x}).$$

Similar inclusion was obtained by Mordukhovich and Wang [18] for the limiting Fréchet subdifferential. We recall that the limiting Fréchet subdifferential of f at $\bar{x}$, with $f(\bar{x}) < +\infty$, is given by

$$\partial_F f(\bar{x}) = w^* - seq - \limsup_{\substack{x \to \bar{x} \\ f(x) \to f(\bar{x}) \\ \varepsilon \to 0^+}} \partial_\varepsilon f(x)$$

where

$$\partial_\varepsilon f(x) = \{x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\| h \|} \geq -\varepsilon\}$$

is the $\varepsilon-$Fréchet subdifferential of f at x. The limiting Fréchet normal cone $N_F(C, \bar{x})$ to a closed set $C \subset X$ at a point $\bar{x} \in C$ is defined by

$$N_F(C, \bar{x}) = \partial_F \xi(C, \bar{x}).$$

This subdifferential has chain rules in Asplund Banach spaces, i. e., Banach spaces on which every continuous convex function is Fréchet differentiable at a dense set of points. It is shown in [10, Therem 4.1] that if X is a weakly compactly generated (WCG) Asplund space (say a reflexive Banach space) and $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function with CEL epigraph at $(\bar{x}, f(\bar{x}))$ then

$$\partial_A f(\bar{x}) = \partial_F f(\bar{x}).$$

So we can give the following result.

Theorem 4.3 *Suppose that* X *and* Y *are WCG Asplund spaces. Then Theorems 4.1 and 4.2 and Corollary 4.1 remain valid if we replace there* ∂_A *and* N_A *by* ∂_F *and* N_F.

5 Proof of the main results.

We prove only Theorem 4.1. First we establish the following technical lemma.

Lemma 5.1 *Let $t, \varepsilon \in]0,1[$ and let Q be a compact subset of X containing 0, C a closed set in X, with $c_1, \bar{x} \in C$ and $b \in \varepsilon B_X$ such that*

$$\|c_1 - \bar{x}\| \leq \varepsilon \text{ and } (c_1 + t(b+Q)) \cap C = \emptyset \tag{2}$$

Then there exist $\bar{q} \in Q$ and $\bar{c} \in C$ such that

$$\|\bar{c} - \bar{x}\| \leq r(t, \varepsilon), \quad \|\bar{q}\| < 1$$

and

$$\Psi(\bar{q}, \bar{c}) \leq \Psi(q, c) + \sqrt{\varepsilon} t \|(\bar{q}, \bar{c}) - (q, c)\|, \quad \forall (q, c) \in Q \times C$$

where $\Psi(q, c) = \|c_1 + t(b+q) - c\| + \sqrt{\varepsilon} t \|q\|$ and $r(t, \varepsilon) = \sqrt{\varepsilon} t + \varepsilon t^2 + \varepsilon + 2t(\varepsilon + diam Q)$.

Proof. We follow the proof in [9]. Consider the real number $\gamma = \sqrt{\varepsilon} t$ and the function ϕ defined by

$$\phi(q) = d(c_1 + t(b+q), C) + \gamma \|q\|.$$

We have by (2) and the compactness property of Q that there exists $q_1 \in Q$ such that

$$0 < \phi(q_1) = \min_{q \in Q} \phi(q) \leq \phi(0) \leq t\varepsilon \tag{3}$$

Set $\beta = d(c_1 + t(b + q_1), C)$ and choose $0 < \lambda < \min(\gamma, 1 - \sqrt{\varepsilon})$ and $c_2 \in C$ such that

$$\|c_1 + t(b + q_1) - c_2\| \leq \beta + \lambda^2 \tag{4}$$

Consider the function Ψ defined by

$$\Psi(q, c) = \|c_1 + t(b+q) - c\| + \gamma \|q\|.$$

Then

$$\Psi(q_1, c_2) \leq d(c_1 + t(b + q_1), C) + \gamma \|q_1\| + \lambda^2 \leq \inf_{(q,c) \in Q \times C} \Psi(q, c) + \lambda^2.$$

(This later is due to relation (3)). By Ekeland variational principle, there exists $(\bar{q}, \bar{c}) \in Q \times C$ such that

$$\|q_1 - \bar{q}\| \leq \lambda, \quad \|c_2 - \bar{c}\| \leq \lambda \tag{5}$$

$$\Psi(\bar{q}, \bar{c}) \leq \Psi(q, c) + \lambda \|(\bar{q}, \bar{c}) - (q, c)\|, \quad \forall (q, c) \in Q \times C.$$

By relations (2)-(5), we get

$$\|\bar{c}-\bar{x}\| \leq r(t,\varepsilon) \quad \text{and} \quad \|\bar{q}\| < 1. \qquad \diamondsuit$$

Proof of Theorem 4.1. Consider the collection of all pairs $\alpha = (L,\varepsilon)$, with L being a finite dimensional space of X and $\varepsilon \in]0,1[$ and endow this collection with the order $\alpha' = (L',\varepsilon') \prec (L,\varepsilon) = \alpha$ iff $L' \subset L$ and $\varepsilon \leq \varepsilon'$. Set

$$Q_\alpha = L_\alpha \cap B_X$$

where L_α and ε_α are the component of the pair α.

Suppose that $a)$ of the theorem does not hold. We will prove that C is CEL at $\bar{x}$. So suppose the contrary. Then for any α there are $c_\alpha^1 \in C$, with $\|c_\alpha^1 - \bar{x}\| \leq \varepsilon_\alpha$, $b_\alpha \in \varepsilon_\alpha B$ and $t_\alpha \in]0,\varepsilon_\alpha[$ such that

$$(c_\alpha^1 + t_\alpha(b_\alpha + Q_\alpha)) \cap C = \emptyset.$$

By Lemma 5.1 there are $\bar{q}_\alpha \in Q_\alpha$ and $\bar{c}_\alpha \in C$ such that

$$\|\bar{c}_\alpha - \bar{x}\| \leq r(t_\alpha,\varepsilon_\alpha), \quad \|\bar{q}_\alpha\| < 1$$

and

$$\Psi(\bar{q}_\alpha,\bar{c}_\alpha) \leq \Psi(q,c) + \sqrt{\varepsilon_\alpha}t_\alpha\|(\bar{q}_\alpha,\bar{c}_\alpha)-(q,c)\|, \quad \forall (q,c) \in Q_\alpha \times C.$$

From Theorem 2.3, there exists a constant $a>0$ (not depending on α for $\alpha_0 \prec \alpha$) such that $(\bar{q}_\alpha,\bar{c}_\alpha)$ is a local solution of the function

$$\Psi(q,c)+\sqrt{\varepsilon_\alpha}t_\alpha\|(\bar{q}_\alpha,\bar{c}_\alpha)-(q,c)\|+(1+t_\alpha+\sqrt{\varepsilon_\alpha}t_\alpha)(a(d(A,g(x))+d(B,x))+d(Q_\alpha,q)).$$

By Theorem 2.2 there exist $y_\alpha^* \in a(1+t_\alpha+\sqrt{\varepsilon_\alpha}t_\alpha)\partial_A d(A,g(\bar{c}_\alpha))$ and $x_\alpha^* \in \partial_A(y_\alpha^* \circ g)(\bar{c}_\alpha) + a(1+t_\alpha+\sqrt{\varepsilon_\alpha}t_\alpha)\partial_A d(B,\bar{c}_\alpha)$ such that

$$(0,-x_\alpha^*) \in \partial_A\Psi(\bar{q}_\alpha,\bar{c}_\alpha) + \sqrt{\varepsilon_\alpha}t_\alpha(B_{X^*}\times B_{X^*}) + (1+t_\alpha+\sqrt{\varepsilon_\alpha}t_\alpha)\partial_A d(Q_\alpha,\bar{q}_\alpha)\times\{0\}.$$

Since $\|\bar{q}_\alpha\|<1$ and $c_\alpha^1 + t_\alpha(b_\alpha+\bar{q}_\alpha) - \bar{c}_\alpha \neq 0$ we get

$$\partial_A d(Q_\alpha,\bar{q}_\alpha) = L_\alpha^\perp \cap B_{X^*} \quad \text{and} \quad \partial_A\Psi(\bar{q}_\alpha,\bar{c}_\alpha) \subset \{(t_\alpha q_\alpha^*, -q_\alpha^*) : \ \|q_\alpha^*\| = 1\}.$$

Thus there exists $q_\alpha^* \in L_\alpha^\perp + \sqrt{\varepsilon_\alpha}B_{X^*}$, with $\|q_\alpha^*\|=1$, such that

$$\|x_\alpha^* + q_\alpha^*\| \leq \sqrt{\varepsilon_\alpha}t_\alpha \tag{6}$$

Now by the definition of x_α^* there exist $u_\alpha^* \in \partial_A(y_\alpha^*\circ g)(\bar{c}_\alpha)$ and $v_\alpha^* \in a(1+t_\alpha+\sqrt{\varepsilon_\alpha}t_\alpha)\partial_A d(B,\bar{c}_\alpha)$ such that

$$x_\alpha^* = u_\alpha^* + v_\alpha^*.$$

Note that by (6) we have

$$1-\sqrt{\varepsilon_\alpha}t_\alpha \leq \|u_\alpha^*\|+\|v_\alpha^*\|, \quad \|v_\alpha^*\| \leq a(1+t_\alpha+\sqrt{\varepsilon_\alpha}t_\alpha), \quad \|u_\alpha^*\| \leq (1+t_\alpha+\sqrt{\varepsilon_\alpha}t_\alpha)(1+a).$$

Extracting subnets if necessary we may assume that (u_α^*) and (v_α^*) weak-star converge respectively to u^* and v^*, with $v^* \in a\partial_A d(B,\bar{x})$, and $(\|u_\alpha^*\|)$ and $(\|v_\alpha^*\|)$ converge to u and v with $u+v \geq 1$. We have :

Case 1: $u \neq 0$. Now since $u_\alpha^* \in \partial_A(y_\alpha^* \circ g)(\bar{c}_\alpha)$ and

$$\|u_\alpha^*\| \leq K_g\|y_\alpha^*\| \leq K_g a(1+t_\alpha+\sqrt{\varepsilon_\alpha}t_\alpha)$$

where K_g is a Lipschitz constant of g at $\bar{x}$, then extracting subnet we may assume that (y_α^*) weak-star converges to $y^* \in a\partial_A d(A, g(\bar{x}))$ such that (Proposition 2.1)

$$u^* \in \partial_A(y^* \circ g)(\bar{x})$$

and since A is CEL at $g(\bar{x})$, we get $y^* \neq 0$. On the other hand, as the sets $L_\alpha^\perp + \sqrt{\varepsilon_\alpha}B_{X^*}$ form a basis for the weak-star topology in X^*, it follows from (6) that both (x_α^*) and (q_α^*) weak-star converge to 0. So that

$$0 \neq y^* \in a\partial_A d(A, g(\bar{x})) \text{ and } 0 \in \partial_A(y^* \circ g)(\bar{x}) + a\partial_A d(B,\bar{x})$$

and this contradicts our assumption 'a) does not hold'.

Case 2: $u = 0$. In this case $v \neq 0$ and then $v^* \neq 0$ because C is CEL at $\bar{x}$ and as above we get $y^* \in a\partial_A d(A, g(\bar{x}))$ such that

$$-v^* \in \partial_A(y^* \circ g)(\bar{x})$$

and this contradicts again our assumption 'a) does not hold'.

The proof of the inclusion follows by a simple computation. ♢

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