Improved Numerical Methods for the Simulation of Microwave Circuits

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Abstract

The scattering matrix describes monolithic microwave integrated circuits that are connected to transmission lines in terms of their wave modes. Using a finite-volume method the corresponding boundary value problem of Maxwell's equations can be solved by means of a two-step procedure. An eigenvalue problem for non-symmetric matrices yields the wave modes. The eigenfunctions determine the boundary values at the ports of the transmission lines for the calculation of the fields in the three dimensional structure. The electromagnetic fields and the scattering matrix elements are achieved by the solution of large-scale systems of linear equations with indefinite symmetric matrices. Improved numerical solutions for the time and memory consuming problems are treated in this paper. The numerical effort could be reduced considerably.

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1 Introduction

The design of monolithic microwave integrated circuits (MMIC) requires efficient CAD tools in order to avoid costly and time-consuming redesign cycles.

The electromagnetic characteristics of microwave circuits and packages can be described by equivalent circuit in terms of voltages and currents or by the scattering matrix. With growing frequencies the voltage and current definitions become ambiguous and the scattering matrix approach is more appropriate. In order to determine the scattering matrix, the circuit is inserted between transmission lines (see Figure 1). For typical microwave packages one is dealing with transmission lines of the microstrip or coplanar type. The scattering matrix describes the structure in terms of wave modes on these lines.

In this way, a three dimensional boundary-value problem can be formulated using the Maxwellian equations in frequency domain in order to compute the electromagnetic field and, subsequently, the scattering matrix.

The application of a finite-volume method to the boundary value problem for the Maxwellian equations results in the so-called **F**inite-**D**ifference method in **F**requency **D**omain (FDFD) [1], [2], [3]. In contrast to the common finite-difference scheme, this method is based on the integral formulation of Maxwell's equations for each cell. The FDFD allows the computation of the scattering matrix of a given structure for a number of simultaneously excited modes. The typical package structures under investigation are small compared to wavelength and contain inhomogeneous waveguide cross-sections. Thus, approaches in the frequency domain show inherent advantages since their time-domain counterparts require excessive numbers of time-steps and do not provide mode separation.

2 Scattering Matrix

The structure under investigation consists of infinitely long transmission lines and a discontinuity (see Figure 1). The transmission lines are assumed to be longitudinally homogeneous. The discontinuity may have an arbitrary structure. The fields are computed in a rectangular volume, which contains the discontinuity and short parts of the transmission lines. Ports are defined on the transmission lines. The remaining surface of the computational volume is formed by electric or magnetic walls. The incoming modes a_l are

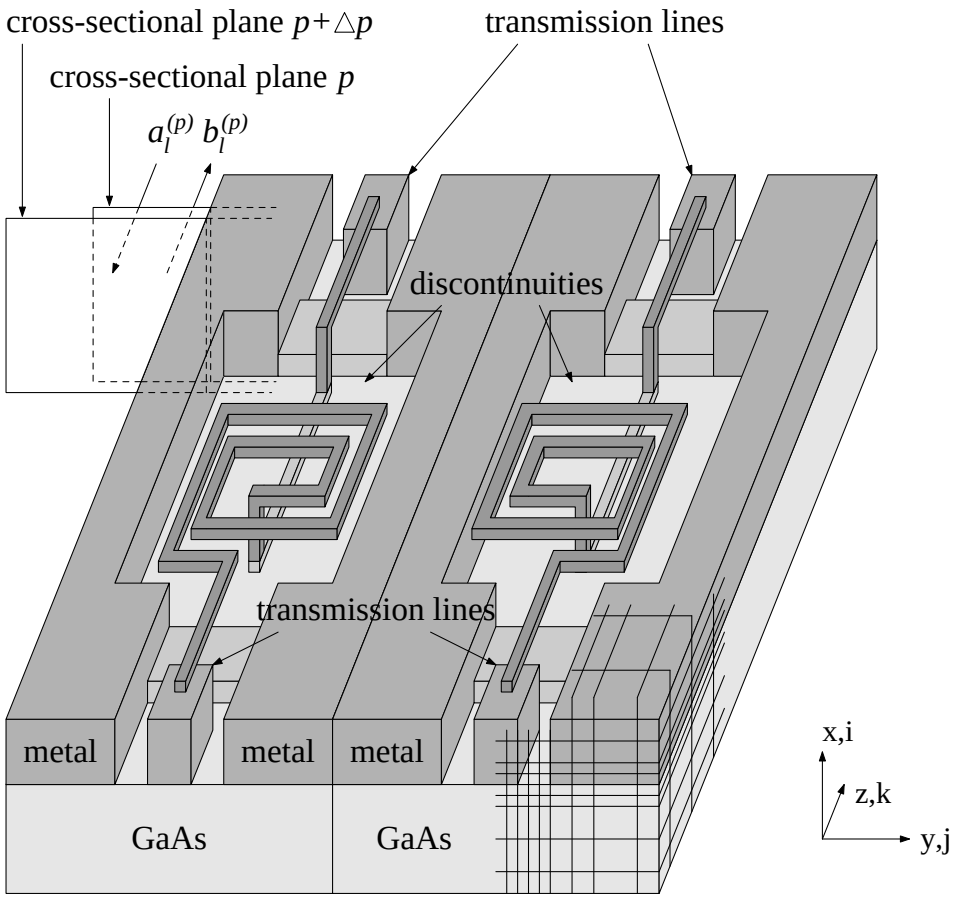

Figure 1: Discretized structure under investigation: coupled spiral inductivities, x direction on a larger scale

changed in the discontinuity. The changed outgoing modes are denoted with b_l. The scattering matrix S describes the energy exchange and phase relation between all outgoing modes $b_l^{(p)}$ and all incoming modes $a_l^{(p)}$ [4]:

$$b_{\rho,\nu} = \sum_{\sigma=1}^{m_s} S_{\rho,\sigma} a_{\sigma,\nu}, \quad S = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1m_s} \\ S_{21} & S_{22} & \cdots & S_{2m_s} \\ \cdots & \cdots & \cdots & \cdots \\ S_{m_s 1} & S_{m_s 2} & \cdots & S_{m_s m_s} \end{pmatrix} = (S_{\rho,\sigma}), \qquad (1)$$

$$\nu, \rho, \sigma = 1(1)m_s, \quad m_s = \sum_{p=1}^{\overline{p}} m^{(p)}. \tag{2}$$

$m^{(p)}$ is the number of modes which have to be taken into account on the port p. $\overline{p}$ is the number of ports. The modes on a port p are numbered with l. Then the indices ρ (and σ) are related to the mode l in the following way

$$\rho = l + \sum_{q=1}^{p-1} m^{(q)}. \tag{3}$$

The scattering matrix can be extracted from the orthogonal decomposition of the electric field at a pair of two neighboring cross-sectional planes p and $p + \Delta p$ (see Figure 1) on each waveguide for a number of linear independent excitations of the transmission lines [4].

3 Boundary Value Problem

We use the integral form of the Maxwellian equations in the frequency domain:

$$\begin{aligned} \oint_{\partial\Omega} \frac{1}{\tilde{\mu}\mu_0}\vec{B} \cdot d\vec{s} &= \int_{\Omega} (\jmath\omega\tilde{\epsilon}\epsilon_0\vec{E}) \cdot d\vec{\Omega}, \quad \oint_{\cup\Omega} (\tilde{\epsilon}\epsilon_0\vec{E}) \cdot d\vec{\Omega} = 0, \\ \oint_{\partial\Omega} \vec{E} \cdot d\vec{s} &= \int_{\Omega} (-\jmath\omega\vec{B}) \cdot d\vec{\Omega}, \quad \oint_{\cup\Omega} \vec{B} \cdot d\vec{\Omega} = 0 \end{aligned} \tag{4}$$

taking into account the constitutive relations

$$\vec{B} = \mu\vec{H}, \;\; \vec{D} = \epsilon\vec{E}, \;\; \text{with } \underline{\epsilon} = \epsilon + \frac{\kappa}{\jmath\omega}, \;\; \mu = \tilde{\mu}\mu_0, \;\; \underline{\epsilon} = \tilde{\epsilon}\epsilon_0. \tag{5}$$

The electric and the magnetic field intensity $\vec{E}$ and $\vec{H}$, and the electric and magnetic flux density $\vec{D}$ and $\vec{B}$, respectively, are complex functions of the spatial coordinates only. ω is the angular frequency and $\jmath^2 = -1$. The permeability μ, the permittivity ϵ, and the conductivity κ are assumed to be scalar functions of the spatial coordinates.

At the port p the transverse electric field $\vec{E}_t(z_p)$ is given by superposing all transmission line modes $\vec{E}_{t,l}(z_p)$ with weighted mode-amplitude sums $w_l(z_p)$:

$$\vec{E}_t(z_p) = \sum_{l=1}^{m^{(p)}} w_l(z_p)\vec{E}_{t,l}(z_p). \tag{6}$$

The transverse electric mode fields $\vec{E}_{t,l}^{(p)} = \vec{E}_{t,l}(z_p)$ are computed using an eigenvalue problem for transmission lines (see section 6).

At all other parts of the enclosure the tangential electric or magnetic field is assumed to be zero:

$$\vec{E}_{tang} = 0 \quad \text{or} \quad \vec{H}_{tang} = 0. \tag{7}$$

The transverse mode fields $\vec{E}_{t,l}^{(p)}$ satisfy an orthogonality relation

$$\int_{\Omega} (\vec{E}_{t,l}^{(p)} \times \vec{H}_{t,m}^{(p)}) \cdot d\vec{\Omega} = \eta_m \delta_{l,m}, \quad \eta_m = 1\text{Watt}. \tag{8}$$

The orthogonality relation (8) is applied at two neighboring cross-sectional planes z_p and $z_p + \Delta z_p$:

$$\begin{aligned} \frac{1}{\eta_m} \int_{\Omega} (\vec{E}_t^{(p)} \times \vec{H}_{t,m}^{(p)}) \cdot d\vec{\Omega} &= a_m^{(p)} + b_m^{(p)} &= w_m^{(p)}, \\ \frac{1}{\eta_m} \int_{\Omega} (\vec{E}_t^{(p+\Delta p)} \times \vec{H}_{t,m}^{(p)}) \cdot d\vec{\Omega} &= a_m^{(p+\Delta p)} + b_m^{(p+\Delta p)} &= w_m^{(p+\Delta p)}. \end{aligned} \tag{9}$$

The weighted mode-amplitude sums $w_l^{(p)}$ are given. Because of

$$a_m^{(p+\Delta p)} = a_m^{(p)} e^{-\jmath k_{z_l}^{(p)} \Delta z_p}, \qquad b_m^{(p+\Delta p)} = b_m^{(p)} e^{+\jmath k_{z_l}^{(p)} \Delta z_p} \tag{10}$$

we can compute the mode amplitudes $a_m^{(p)}$ and $b_m^{(p)}$ from (9), and subsequently, the scattering matrix (1). $k_{z_l}^{(p)}, l = 1(1)m^{(p)}$, are the propagation constants (see section 6) at the port p.

4 Matrix Representation of the Maxwellian Equations

The region is divided into elementary cells (see Figure 1) using a three dimensional nonequidistant orthonormal Cartesian grid. We use staggered grids [5], [6]. The electric field components are located at the centers of the edges of the cell and the magnetic flux density components are normal to the centers of the faces. Thus, the electric field components form a primary grid, and

the magnetic flux density components a dual grid. We use the lowest-order integration formulae

$$\oint_{\partial\Omega} \vec{f} \cdot d\vec{s} \approx \sum(\pm f_i s_i), \quad \int_{\Omega} \vec{f} \cdot d\vec{\Omega} \approx f\Omega \tag{11}$$

in order to approximate Maxwellian equations (4). Thus, we get the matrix representation of (4):

$$\begin{aligned} A^T D_{s/\tilde{\mu}} \vec{b} &= \jmath\omega\epsilon_0\mu_0 D_{A_{\tilde{\epsilon}}} \vec{e}, & B D_{A_{\tilde{\epsilon}}} \vec{e} &= 0, \\ A D_s \vec{e} &= -\jmath\omega D_A \vec{b}, & B^T D_A \vec{b} &= 0. \end{aligned} \tag{12}$$

The vectors $\vec{e}$ and $\vec{b}$ contain the components of the electric field intensity and the components of the magnetic flux density of the elementary cells, respectively. The diagonal matrices $D_{s/\tilde{\mu}}$, $D_{A_{\tilde{\epsilon}}}$, D_s, and D_A contain the information on cell dimension and material for the specified structure and the corresponding mesh. A is defined as the operator of the line integral in the second Maxwellian equation (left formula of the second row of (4)) using the primary grid. B represents the surface integral of the divergence. A and B are sparse, and contain the values 1, -1 and 0 only.

5 System of Linear Algebraic Equations

Eliminating the components of the magnetic flux density from the two equations of the left-hand side of (12) we get the system of linear algebraic equations

$$Q_1 \vec{e} = 0, \quad Q_1 = A^T D_{s/\tilde{\mu}} D_A^{-1} A D_s - k_0^2 D_{A_{\tilde{\epsilon}}}, \quad k_0 = \omega\sqrt{\epsilon_0\mu_0}. \tag{13}$$

The ingoing wave modes at the ports of the structure act as sources for the field inside the discontinuity. Thus, a source term has to be induced by partitioning of the matrix Q_1:

$$Q_1 = Q_{1,A} + Q_{1,r}, \quad Q_{1,A}\vec{e} = -Q_{1,r}\vec{e}, \tag{14}$$

where $Q_{1,r}\vec{e}$ is known. Using $\vec{r} = -Q_{1,r}\vec{e}$ the matrix $Q_{1,A}$ is transformed into the symmetric matrix $\tilde{Q}_{1,A}$, after some manipulations:

$$\tilde{Q}_{1,A}\tilde{\vec{e}} = D_s^{\frac{1}{2}} Q_{1,A} D_s^{-\frac{1}{2}} D_s^{\frac{1}{2}} \vec{e} = -D_s^{\frac{1}{2}} Q_{1,r} \vec{e} = D_s^{\frac{1}{2}} \vec{r} = \tilde{\vec{r}}. \tag{15}$$

Now we derive advantage from the fact, that there is no space charge in our volume, and therefore $\mathrm{div}(\tilde{\epsilon}\epsilon_0\vec{E}) = 0$. The gradient of the electric-field divergence

$$\tilde{\epsilon}\epsilon_0 \nabla(\frac{1}{(\tilde{\epsilon}\epsilon_0)^2}\nabla \cdot \tilde{\epsilon}\epsilon_0\vec{E}) = 0 \tag{16}$$

is equivalent to the matrix equation

$$Q_2\vec{e} = 0 \quad \text{with} \quad Q_2 = D_s^{-1} D_{A_{\tilde{\epsilon}}} B^T D_{V_{\tilde{\epsilon}\tilde{\epsilon}}}^{-1} B D_{A_{\tilde{\epsilon}}}. \tag{17}$$

We carry out a similar partitioning like (14) for Equation (17):

$$Q_2 = Q_{2,A} + Q_{2,r}. \tag{18}$$

Using $Q_{2,r}\vec{e} = 0$ the matrix $Q_{2,A}$ is transformed into the symmetric matrix $\tilde{Q}_{2,A}$, after some manipulations:

$$\tilde{Q}_{2,A}\vec{\tilde{e}} = D_s^{\frac{1}{2}} Q_{2,A} D_s^{-\frac{1}{2}} D_s^{\frac{1}{2}}\vec{e} = -D_s^{\frac{1}{2}} Q_{2,r}\vec{e} = 0, \quad \vec{\tilde{e}} = D_s^{\frac{1}{2}}\vec{e}. \tag{19}$$

Adding Equation (19) to (15) the new system can be solved numerically faster [1] (see also section 8).

6 Eigenvalue Problem

The transverse electric mode fields $\vec{E}_{t,l}^{(p)}$ at the ports have to be computed before we can solve the system of linear equations. Because the transmission lines are longitudinally homogeneous any field can be expanded into a sum of so-called modal fields

$$\vec{E}(x, y, z \pm 2h) = \vec{E}(x, y, z)e^{\mp \jmath k_z 2h}, \tag{20}$$

which vary exponentially in the longitudinal direction. k_z is the propagation constant. $2h$ is the length of an elementary cell in z-direction. We consider the field components in three consecutive elementary cells. The electric field components of the vector $\vec{e}$ (see Equation (13)) $E_{x_{i,j,k+1}}$, $E_{x_{i,j,k-1}}$, $E_{y_{i,j,k+1}}$, $E_{y_{i,j,k-1}}$, $E_{z_{i,j,k-1}}$, $E_{z_{i+1,j,k-1}}$, and $E_{z_{i,j+1,k-1}}$ are expressed by the values of cell k using ansatz (20). The longitudinal electric field components E_z can be eliminated by means of the equation $BD_{A_{\tilde{\epsilon}}}\vec{e} = 0$ (see (12)) [7] . Thus, we

get an eigenvalue problem for the transverse electric field on the transmission line region:

$$C\underline{\vec{e}} = \gamma\underline{\vec{e}}\ , \quad \text{type}(C) = (2n_{xy} - n_b, 2n_{xy} - n_b). \tag{21}$$

$\underline{\vec{e}}$ consists of components $E_{x_{i,j,k}}$ and $E_{y_{i,j,k}}$, $k = const$, of the eigenfunctions. Thus, the problem for the transmission line is reduced to a two dimensional problem. The sparse matrix C is non-symmetric or non-Hermitean in the lossless or the lossy case, respectively. n_{xy} is the number of elementary cells at the port. The size of n_b depends on the boundary conditions at the port. The relations between the eigenvalues γ and the propagation constants k_z are

$$\gamma = e^{-\jmath k_z 2h} + e^{+\jmath k_z 2h} - 2 = -4\sin^2(k_z h) = u + \jmath v, \tag{22}$$

$$k_z = \frac{\jmath}{2h}\ln\left(\frac{\gamma}{2} + 1 + \sqrt{\frac{\gamma}{2}\left(\frac{\gamma}{2} + 2\right)}\right) = \beta - \jmath\alpha. \tag{23}$$

A propagation constant k_z and its corresponding eigenfunction is called a mode. The energy of the complex and evanescent modes decreases exponentially with the distance from the discontinuity. Thus, in technical applications most of the modes can be neglected within the limit of accuracy. Generally speaking, the larger the magnitude of the imaginary part of k_z the stronger the decay. Therefore, to sort the propagation constants according to their importance in our problem, we use the
Criterion: The propagation constants k_z are sorted in ascending order of $\mid \alpha \mid$. In the case if some $\mid \alpha \mid$ have the same value the constants k_z are sorted in descending order of $\mid \beta \mid$.

Computing the wanted propagation constants and the corresponding eigenfunctions the transverse electric fields $\vec{E}_{t,l}^{(p)}$, $l = 1(1)m^{(p)}$, are known at the ports p, and the boundary condition (6) can be build superposing the transmission line modes.

7 Numerical Solution of the Eigenvalue Problem

In an earlier version of the method [3] the complete set of eigenvalues and of corresponding propagation constants was computed and sorted in order to

select the interesting propagation constants. The sparse matrix was stored as a dense matrix.

We avoid the computation of all eigenvalues to find the few required propagation constants using the implicitly restarted Arnoldi method [8], [9]. The sparse storage technique is applied.

The Arnoldi algorithm is called iteratively to solve the standard eigenvalue problem using the inverse mode $C^{-1}x = \frac{1}{\gamma}x$ with the solution of linear algebraic equations. In general the method does not converge using the regular mode for our eigenvalue problem. We use a Gaussian elimination for sparse matrices, to solve the ill-conditioned systems of linear equations.

By means of the Arnoldi iteration we can compute a set of eigenvalues of largest or smallest magnitude, real part or imaginary part, but we can not find in one step the set of eigenvalues according to our criterion. Therefore, we must proceed in two steps.

In a first run we compute a subset $\mathcal{E}$ of eigenvalues γ of smallest magnitude using the Arnoldi method in inverse mode looking for eigenvalues of largest magnitude, and compute the corresponding subset $\bar{\mathcal{E}}$ of propagation constants. However, we have to find a subset $\bar{\mathcal{A}}$ of propagation constants with the smallest magnitude of the imaginary part, but possibly with large real part. In general, we have $\bar{\mathcal{A}} \cap \bar{\mathcal{E}} \neq \emptyset$ but $\bar{\mathcal{A}} \not\subseteq \bar{\mathcal{E}}$.

To search for the corresponding additional eigenvalues, we use a second run of the Arnoldi method with a modified matrix.

The wave number

$$k_f = \omega\sqrt{\underline{\epsilon}\mu} = k_0\sqrt{\tilde{\epsilon}\tilde{\mu}} \tag{24}$$

is an upper bound for the interesting propagation constants of undamped modes in a waveguide. Using the maximum wave number $k^{(max)}$ of the cells we extend the matrix C by a diagonal matrix which consists of the negative elements

$$\gamma_\tau^{(a)} = \gamma^{(max)}(1 + \frac{\tau}{10}), \;\; \tau = 1(1)m_a, \quad \gamma^{(max)} = -4(hk^{(max)})^2. \tag{25}$$

In a second run we compute a subset $\mathcal{E}^l$ of $m_a + m_r$ eigenvalues of smallest real part of the extended matrix C^* using the Arnoldi method in inverse mode. m_r is the number of negative eigenvalues of the subset $\mathcal{E}$ computed in the first run. Separating the new eigenvalues of $\mathcal{E}^l$ and computing the corresponding propagation constants we have found all propagation constants according to our criterion [10].

The modes satisfy the orthogonality relation (8) if $k_{z_l} \neq \mp k_{z_m}$. In the case of multiple eigenvalues the eigenfunctions are orthogonalized according to (8) using the method of Gram-Schmidt.

The computation of all eigenvalues with the QR algorithm and the calculation of the eigenfunctions of the wanted propagation constants by solving systems of linear homogeneous algebraic equations in the original version [1], [3], [11] is very time and memory consuming. Computing the wanted propagation constants only, using of the sparse storage technique, and neglecting relative small elements in the matrix C reduce the computing time from days to minutes for a non-Hermitean matrix C of order $2n_{xy} - n_b \approx 5000$ (see (21)) using a workstation. The total storage requirement is reduced by a factor 18 in the new version for this example.

8 Numerical Solution of the System of Linear Algebraic Equations

The high-dimensional indefinite symmetric system of linear algebraic equations (see (13), (15), (19)) with multiple right hand sides

$$U\vec{\tilde{e}} = \vec{\tilde{r}}, \quad U = \; D_s^{\frac{1}{2}} Q_{1,A} D_s^{-\frac{1}{2}} = \tilde{Q}_{1,A}, \tag{26}$$

$$V\vec{\tilde{e}} = 0, \quad V = \; D_s^{\frac{1}{2}} Q_{2,A} D_s^{-\frac{1}{2}} = \tilde{Q}_{2,A} \tag{27}$$

is solved using iterative methods. The number of right hand sides is m_s (see (2)).

The convergence rate of iterative methods depends on spectral properties of the coefficient matrix U. Thus, we transform the linear system (26) into one that is equivalent in the sense that is has the same solution but more favorable spectral properties. A preconditioner $M = M_1 M_2$ is a matrix that performs such a transformation:

$$M_1^{-1} U M_2^{-1} (M_2 \vec{\tilde{e}}) = M_1^{-1} \vec{\tilde{r}} \tag{28}$$

We use four kinds of preconditioning:

1. The effect of the addition of the two equations (15) and (19) described in section 5 can be interpretated as preconditioning. Using (27) we construct a preconditioner M for the original system (26):

$$M_1^{-1} = I + VU^{-1}, \quad M_1 = (I + VU^{-1})^{-1}, \quad M_2 = I. \tag{29}$$

Substituting (29) in (28) we get

$$(I + VU^{-1})U\vec{\tilde{e}} = (I + VU^{-1})\vec{\tilde{r}} \Rightarrow (U + V)\vec{\tilde{e}} = \vec{\tilde{r}} + V\vec{\tilde{e}} = \vec{\tilde{r}} \tag{30}$$

or

$$\tilde{U}\vec{\tilde{e}} = \vec{\tilde{r}} \quad \text{with} \quad \tilde{U} = U + V. \tag{31}$$

2. A commonly used approach for solving large sparse linear systems is to find sets of unknowns which are independent. A set of such unknowns is called an independent set. Independent set orderings are permutations $\tilde{P}$ to transform the system (31) into the form

$$\tilde{P}\tilde{U}\tilde{P}^T\vec{t} = \begin{pmatrix} \tilde{D} & \tilde{E}^T \\ \tilde{E} & \tilde{H} \end{pmatrix} \begin{pmatrix} \vec{t}_1 \\ \vec{t}_2 \end{pmatrix} = \begin{pmatrix} \vec{s}_1 \\ \vec{s}_2 \end{pmatrix} \tag{32}$$

with

$$\vec{t} = \tilde{P}\vec{\tilde{e}} = \begin{pmatrix} \vec{t}_1 \\ \vec{t}_2 \end{pmatrix}, \qquad \vec{s} = \tilde{P}\vec{\tilde{r}} = \begin{pmatrix} \vec{s}_1 \\ \vec{s}_2 \end{pmatrix}. \tag{33}$$

$\tilde{D}$ is a diagonal matrix, $\tilde{E}$ is a general sparse matrix, and $\tilde{H}$ is a quadratic sparse matrix. The unknowns of the independent set $\tilde{D}$ are eliminated to get the next reduced matrix

$$\hat{U} = \tilde{H} - \tilde{E}\tilde{D}^{-1}\tilde{E}^T, \tag{34}$$

and we have to solve the system of linear equations

$$\hat{U}\vec{t}_2 = (\tilde{H} - \tilde{E}\tilde{D}^{-1}\tilde{E}^T)\vec{t}_2 = \vec{s}_2 - \tilde{E}\tilde{D}^{-1}\vec{s}_1 \tag{35}$$

or

$$\hat{U}\vec{\hat{e}} = \vec{\hat{r}}, \quad \hat{U} = \tilde{H} - \tilde{E}\tilde{D}^{-1}\tilde{E}^T, \ \vec{\hat{e}} = \vec{t}_2, \ \vec{\hat{r}} = \vec{s}_2 - \tilde{E}\tilde{D}^{-1}\vec{s}_1. \tag{36}$$

Thus, we get

$$\vec{t}_1 = \tilde{D}^{-1}(\vec{s}_1 - \tilde{E}^T\vec{t}_2) = \tilde{D}^{-1}(\vec{s}_1 - \tilde{E}^T\vec{\hat{e}}). \tag{37}$$

Then we have to permute the solution vector $\vec{t}$ (see (33)) back to the vector $\vec{\tilde{e}}$ (see (19), (31)).

3. Using a preconditioner $\hat{M} = \hat{M}_1 \hat{M}_2$ Equation (36) can be written as

$$\hat{M}_1^{-1} \hat{U} \hat{M}_2^{-1} (\hat{M}_2 \vec{\hat{e}}) = \hat{M}_1^{-1} \vec{\hat{r}}. \tag{38}$$

Let be

$$\hat{M}_1 = \hat{D}_{\hat{U}}^{\frac{1}{2}}, \quad \hat{M}_2 = \hat{D}_{\hat{U}}^{\frac{1}{2}}, \quad \hat{D}_{\hat{U}} = diag(\hat{U}). \tag{39}$$

Combining (38) and (39) with (36) we obtain

$$\breve{U} \vec{\breve{e}} = \vec{\breve{r}}, \quad \breve{U} = \hat{D}_{\hat{U}}^{-\frac{1}{2}} \hat{U} \hat{D}_{\hat{U}}^{-\frac{1}{2}}, \ \vec{\breve{e}} = \hat{D}_{\hat{U}}^{\frac{1}{2}} \vec{\hat{e}}, \ \vec{\breve{r}} = \hat{D}_{\hat{U}}^{-\frac{1}{2}} \vec{\hat{r}}. \tag{40}$$

4. We construct a SSOR preconditioner for the matrix $\breve{U}$ (see (40)) [12]. If the matrix $\breve{U}$ is decomposed as

$$\breve{U} = I + L + L^T \tag{41}$$

in its diagonal, strict lower, and strict upper triangular part, the SSOR matrices are defined as

$$\breve{M}_1 = (I + \omega L), \quad \breve{M}_2 = (I + \omega L^T) \quad \text{with} \quad 0 < \omega < 2. \tag{42}$$

We have to solve the system of linear algebraic equations

$$\breve{M}_1^{-1} \breve{U} \breve{M}_2^{-1} (\breve{M}_2 \vec{\breve{e}}) = \breve{M}_1^{-1} \vec{\breve{r}}. \tag{43}$$

We use Eisenstat's trick [13]. Because of (see (41))

$$\breve{M}_1^{-1} \breve{U} \breve{M}_2^{-1} = \frac{1}{\omega}((I + \omega L^T)^{-1} + (I + \omega L)^{-1}(I - (2 - \omega)(I + \omega L^T)^{-1}))$$

the matrix vector product $(\breve{M}_1^{-1} \breve{U} \breve{M}_2^{-1}) v$, for any vector v, requires two solves [14] with the triangular matrices $(I + \omega L)$ and $(I + \omega L^T)$ plus a few arithmetical operations.

The Equations (43) are solved with Krylov subspace methods described in [15], [16], [17].

At present, we can handle structures with up to 3 million unknowns on modern workstations with a memory of half a GByte. The computing time for the solution of the linear algebraic equations are reduced compared to the original version [1], [3] by a factor of 10.

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