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A collocation method for the solution of the first boundary value problem of elasticity in a polygonal domain in $\mathbb{R}^2$

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Abstract

We present a spline collocation method for the numerical solution of a system of integral equations on a polygon in $\mathbb{R}^2$. This integral equation arises if one solves the first boundary value problem for the Lamé equation with a double layer potential. The derivation and the analysis of the integral equation is given in detail. The optimal order of the spline collocation method is proved for sufficiently graded meshes.

1 Introduction

In this paper we consider a collocation method for the approximate solution of a boundary integral equation for the first boundary value problem for the Lamé equation in $\Omega \subset \mathbb{R}^2$ (see [15]). We assume that the domain Ω has a polygonal boundary.

To derive the integral equation of the second kind we use a double layer potential and a generalized stress tensor (see [14],[11]). The resulting integral operator is not compact in the case of a polygonal boundary so the standard theory for collocation methods does not apply here. To the authors knowledge this special boundary integral equation is used here for the first time to approximate the solution of the Lamé equation in polygonal domains. In order to prove that the boundary integral equation has always a solution we first have to show some properties of the double layer potential and here we imitate the proofs of Costabel [4].

To analyze the integral equation we first localize the integral operator around each corner and show the Fredholm property and the existence of the inverse for these localized operators. Related results were obtained in the papers [18], [19], and [16]. With this result we can prove the unique solvability of the integral equation on the polygon. We also prove a regularity result for the solutions of the integral equations and this shows that the use of higher order splines makes sense.

We use continuous splines of any order and graded meshes to get the optimal order of convergence. In order to show the stability of our method we have to modify the spline space in the vicinity of each corner. This technique is well known. In [5] it is used for the solution of integral equations of the second kind with noncompact integral operators and in [3] and in [7] this technique is used for the solution of the Laplace equation in polygonal domains. The proof of stability relies on the stability of the finite section method for systems of Wiener–Hopf operators [9].

The outline of the paper is as follows: In section 2 we derive the integral equation and prove some results for the double layer potential and some uniqueness results for weak solutions of the Lamé equation.

In section 3 we first localize the integral operator around each corner and then we study the localized operators. We put this results together to prove that the integral equation on the boundary of Ω has a solution for every right hand side in $L^2(\partial\Omega)$. If the right hand side has a higher regularity then the solution becomes more regular, i.e. belongs to certain weighted Sobolev spaces.

In section 4 we define the meshes and the spline spaces which we use. Then we prove the stability of our method if the the meshes fulfill some simple condition and if the spline space is suitably modified. A further approximation result then shows the order of convergence of our method.

2 The boundary value problem and the corresponding boundary integral equation

In this section we define the boundary value problem, which we will study, and we introduce the generalized stress operator (see [11]). We extend the trace operator and the generalized stress operator to a sufficiently large function space. We prove the uniqueness of the interior Dirichlet problem and the exterior boundary value problem, where the generalized stress is prescribed at the boundary. The mapping properties of the single and double layer operator are studied and at the end of the section we derive the boundary integral operator, which we study in the following sections. We follow closely the article [4] of M. Costabel.
Let $\Omega \subset R^2$ be an open bounded domain with polygonal boundary Γ. We denote by Ω^c the complement of $\overline{\Omega}$, $\Omega^c := \mathbb{R}^2 \setminus \overline{\Omega}$, and we assume that Ω is contained in some sufficiently large ball $B_{R_0}(0)$, $R_0 > 0$.
For functions $\vec{u} = (u_1, u_2)^T \in \left(H^2(\Omega)\right)^2$ the Lamé operator P is defined by

$$P\vec{u} \;:=\; -\mu\Delta\vec{u} - (\lambda + \mu)\mathrm{grad}(\mathrm{div}\vec{u}), \qquad \mu > 0,\, \lambda \geq 0. \tag{2.1}$$

It is the aim of the sections two and three to study the existence and the properties of the solution of the equation

$$\left.\begin{array}{lcl} (P\vec{u})(x) & = & 0, \quad x \in \Omega \\ \vec{u}|_\Gamma & = & \vec{f}, \quad \vec{f} \in \left(H^{\frac{1}{2}}(\Gamma)\right)^2 \end{array}\right\} \tag{2.2}$$

with the help of a corresponding boundary integral equation.
The operator P can be written in another way with the help of the following definitions (see [15] for the physical meaning of the terms)

$$\left.\begin{array}{lcl} \varepsilon_{i,j}(\vec{u}) & := & \frac{1}{2}(\partial_i u_j + \partial_j u_i) \\ \sigma_{i,j}(\vec{u}) & := & \lambda(\varepsilon_{1,1}(\vec{u}) + \varepsilon_{2,2}(\vec{u}))\delta_{i,j} + 2\mu\varepsilon_{i,j}(\vec{u}) \end{array}\right\} \quad i,j = 1,2 \tag{2.3}$$

The i th component of $P\vec{u}$ can be written as

$$(P\vec{u})_i \;=\; -\sum_{j=1}^{2} \partial_j(\sigma_{i,j}(\vec{u})). \tag{2.4}$$

We further introduce the following notations

$$V^i \;:=\; \left(H^i(\Omega)\right)^2,\; i = 0,1,2, \text{ and } \; V_0^1 \;:=\; \left(H_0^1(\Omega)\right)^2. \tag{2.5}$$

Formula (2.4) and a partial integration gives us the following relation (first Green formula) for functions $\vec{u} \in V^2$ and $\vec{v} \in V^1$:

$$\int_\Omega (P\vec{u})\cdot\vec{v}dx \;=\; \Phi_\Omega(\vec{u},\vec{v}) - \int_\Gamma (\mathcal{T}_\mu(n_y)\vec{u})\cdot\vec{v}\,ds_y, \tag{2.6}$$

where n_y denotes the outer normal at the point y on Γ. The symmetric bilinear form $\Phi_\Omega(\cdot,\cdot)$ on V^1 is given by

$$\begin{aligned}\Phi_\Omega(\vec{u},\vec{v}) \;:=\; & \int_\Omega \lambda\left(\sum_{j=1}^{2}\varepsilon_{jj}(\vec{u})\right)\left(\sum_{j=1}^{2}\varepsilon_{jj}(\vec{v})\right) + \\ & 2\mu\sum_{i,j=1}^{2}\varepsilon_{i,j}(\vec{u})\varepsilon_{i,j}(\vec{v})dx.\end{aligned} \tag{2.7}$$

The generalized stress operator $\mathcal{T}_\mu$ (see [11]) is defined in the following way

$$\begin{aligned}\mathcal{T}_\kappa(n)\vec{u} \;:=\; & \mu\begin{pmatrix}(\nabla u_1)\cdot n\\(\nabla u_2)\cdot n\end{pmatrix} + (\lambda+\mu)\mathrm{div}(\vec{u})n \\ & +\kappa\begin{pmatrix}\partial_1 u_2 n_2 - \partial_2 u_2 n_1\\ \partial_2 u_1 n_1 - \partial_1 u_1 n_2\end{pmatrix},\end{aligned} \tag{2.8}$$

with $\kappa\in\mathbb{R}$, $n=(n_1,n_2)^T\in\mathbb{R}^2$, $\vec{u}\in V^1$.
Because of the symmetry of $\Phi_\Omega(\cdot,\cdot)$ we get the second Green formula for $\vec{u},\vec{v}\in V^2$.

$$\begin{aligned}\int_\Omega (P\vec{u})\cdot\vec{v}-(P\vec{v})\cdot\vec{u}dx \;=\; & \int_\Gamma (\mathcal{T}_\mu(n_y)\vec{v})\cdot\vec{u}-(\mathcal{T}_\mu(n_y)\vec{u})\cdot\vec{v}ds_y \\ =\; & \int_\Gamma (\mathcal{T}_{\mu+\omega}(n_y)\vec{v})\cdot\vec{u}-(\mathcal{T}_{\mu+\omega}(n_y)\vec{u})\cdot\vec{v}\,ds_y + \omega\, A,\end{aligned}$$

where A is given by

$$\begin{aligned}A \;=\; & \int_\Gamma\Big(\left(-\partial_2 u_2 v_1 + \partial_2 u_1 v_2 + \partial_2 v_2 u_1 - \partial_2 v_1 u_2\right)n_1(y) \\ & \left(\partial_1 u_2 v_1 - \partial_1 u_1 v_2 - \partial_1 v_2 u_1 + \partial_1 v_1 u_2\right)n_2(y)\Big)ds_y.\end{aligned}$$

A short calculation shows

$$\begin{aligned}A \;=\; & \int_\Omega\Big(-\partial_{12}^2 u_2 v_1 - \partial_2 u_2\partial_1 v_1 + \partial_{12}^2 u_1 v_2 + \partial_2 u_1\partial_1 v_2 \\ & +\partial_{12}^2 v_2 u_1 + \partial_2 v_2\partial_1 u_1 - \partial_{12}^2 v_1 u_2 - \partial_2 v_1\partial_1 u_2 \\ & +\partial_{12}^2 u_2 v_1 + \partial_1 u_2\partial_2 v_1 - \partial_{12}^2 u_1 v_2 - \partial_1 u_1\partial_2 v_2 \\ & -\partial_{12}^2 v_2 u_1 - \partial_1 v_2\partial_2 u_1 + \partial_{12}^2 v_1 u_2 + \partial_1 v_1\partial_2 u_2\Big)dx \\ =\; & \int_\Omega 0\,dx \\ =\; & 0.\end{aligned}$$

by Gauss' formula. So we finally get the second Green formula in the following form

$$\int_\Omega (P\vec{u}) \cdot \vec{v} - (P\vec{v}) \cdot \vec{u}\, dx \;=\; \int_\Gamma \mathcal{T}_\kappa(n_y)\vec{v} \cdot \vec{u} - \mathcal{T}_\kappa(n_y)\vec{u} \cdot \vec{v}\, ds_y \tag{2.9}$$

for $\vec{u}, \vec{v} \in V^2$, $\kappa \in \mathbb{R}$.
Let $\vec{f} \in V^0$ be given. The function $\vec{u} \in V_0^1$ is the weak solution of

$$\left.\begin{aligned} P\vec{u} &= \vec{f} \\ u|_\Gamma &= 0 \end{aligned}\right\} \tag{2.10}$$

if and only if

$$\Phi_\Omega(\vec{u}, \vec{\phi}) \;=\; \int_\Omega \vec{f} \cdot \vec{\phi}\, dx,\ \forall \vec{\phi} \in V_0^1. \tag{2.11}$$

Korn's inequality (see [8]) says, that there are constants $c_1, c_2 > 0$, which depend only on Ω, for which

$$c_1 \|\vec{u}\|^2_{V_0^1} \;\leq\; \Phi_\Omega(\vec{u}, \vec{u}) \;\leq\; c_2 \|\vec{u}\|^2_{V_0^1},\ \forall \vec{u} \in V_0^1. \tag{2.12}$$

Equation (2.12) together with the Lax–Milgram Lemma gives us the following result.

Corollary 2.1. *The equation (2.10) has always a uniquely determined weak solution.*

In the following we denote by γ_0 the trace operator

$$\gamma_0 \vec{u} \;:=\; \vec{u}|_\Gamma. \tag{2.13}$$

Gagliardo's Trace Lemma (see [4]) implies

$$\gamma_0 \;:\; H^s_{loc}(\mathbb{R}^2) \;\to\; H^{s-\frac{1}{2}}(\Gamma),\ s \in (\frac{1}{2}, 1], \text{is continuous} \tag{2.14}$$

and has a continuous right inverse γ_0^-

$$\gamma_0^- \;:\; H^{s-\frac{1}{2}}(\Gamma) \;\to\; H^s_{loc}(\mathbb{R}^2). \tag{2.15}$$

We now get the following existence theorem.

Lemma 2.2. *For every $\vec{v} \in \left(H^{\frac{1}{2}}(\Gamma)\right)^2$ there exists a unique solution $T\vec{v}$ of the equation*

$$\left.\begin{aligned} P\vec{u} &= 0 \\ \gamma_0\vec{u} &= \vec{v} \end{aligned}\right\} \tag{2.16}$$

The mapping $\vec{v} \to T\vec{v}$ is linear and continuous, i.e. there exists a constant $c_T > 0$, such that

$$\|T\vec{v}\|_{V^1} \leq c_T \|\vec{v}\|_{\left(H^{\frac{1}{2}}(\Gamma)\right)^2}. \tag{2.17}$$

Proof: We first note that the mapping

$$\vec{\phi} \to S\vec{\phi} := \Phi_\Omega(-\gamma_0^- \vec{v}, \vec{\phi}),\ \vec{\phi} \in V_0^1,$$

is continuous, $\|S\| \leq \widetilde{C}_1 \|\vec{v}\|_{\left(H^{\frac{1}{2}}(\Gamma)\right)^2}$. The Lax–Milgram Lemma and Korn's inequality (2.12) show that there exists exactly one $\vec{u} \in V_0^1$ with

$$\Phi_\Omega(\vec{u}, \vec{\phi}) = \Phi_\Omega(-\gamma_0^- \vec{v}, \vec{\phi}),\ \forall \vec{\phi} \in V_0^1,$$

and

$$\|\vec{u}\|_{V_0^1} \leq \widetilde{C}_2 \|S\| \leq \widetilde{C}_1 \widetilde{C}_2 \|\vec{v}\|_{\left(H^{\frac{1}{2}}(\Gamma)\right)^2}.$$

We define

$$T\vec{v} := \vec{u} + \gamma_0^- \vec{v},$$

and get

$$\Phi_\Omega(T\vec{v}, \vec{\phi}) = 0,\ \forall \vec{\phi} \in V_0^1.$$

So $T\vec{v}$ has the following properties

a. $P(T\vec{v}) = 0$ (see (2.10) and (2.11))

b. $\|T\vec{v}\|_{V^1} \leq (\widetilde{C}_1 \widetilde{C}_2 + \widetilde{C}_1) \|\vec{v}\|_{\left(H^{\frac{1}{2}}(\Gamma)\right)^2}$

c. $\gamma_0(T\vec{v}) = \gamma_0 \vec{u} + \gamma_0 \gamma_0^- \vec{v} = \vec{v}$

So T is a continuous mapping and $T\vec{v}$ solves (2.16). We only have to show that $T\vec{v}$ is the only solution to (2.16). If $T_1\vec{v}$ is a second solution for (2.16), then we get

$$T\vec{v} - T_1\vec{v} \in V_0^1$$

and

$$\Phi_\Omega(T\vec{v} - T_1\vec{v}, \vec{\phi}) = 0,\ \forall \vec{\phi} \in V_0^1.$$

By (2.12) this implies $T\vec{v} - T_1\vec{v} = 0$ and therefore $T\vec{v}$ is uniquely determined. $\square$

We denote by $G(x,y)$ the fundamental solution for the operator P:

$$P_y G(y,x) \;:=\; \delta(x-y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =: \delta(x-y) I_{2\times 2}, \tag{2.18}$$

where the index y denotes the differentiation with respect to y. The function G is given by (see [1])

$$G(y,x) \;:=\; \frac{1}{4\pi\mu(\lambda+2\mu)}\Big(-(\lambda+3\mu)\ln(r) I_{2\times 2} + (\lambda+\mu)\frac{(x-y)(x-y)^T}{r^2}\Big), \tag{2.19}$$

$x, y \in \mathbb{R}^2$, $r := \|x-y\|^2$, $x \neq y$. G is the kernel for the Green operator for P. We will denote also the Green operator by G. If we substitute $G(x,y)$ for $\vec{u}(y)$ in the Green formula (2.9) then we get

$$\begin{aligned} \vec{v}(x) \;&=\; \int_\Omega G(y,x)^T (P\vec{v})(y)dy \\ &\quad + \int_\Gamma G(y,x)^T \Big(\mathcal{T}_{\kappa,y}(n_y)\vec{v}(y)\Big) - \Big(\mathcal{T}_{\kappa,y}(n_y)G(y,x)\Big)^T \vec{v}(y)\, ds_y, \\ &\qquad x \in \Omega,\ \vec{v} \in V^2. \end{aligned} \tag{2.20}$$

By V_P^1 we denote the set of all functions $\vec{u} \in V^1$, for which the distributional derivative $P\vec{u}$ belongs to V^0. The norm on V_P^1 is given by

$$\|\vec{u}\|_{V_P^1}^2 \;:=\; \|\vec{u}\|_{V^1}^2 + \|P\vec{u}\|_{V^0}^2. \tag{2.21}$$

Now we extend the generalized stress operator to functions in V_P^1. First we recall the following lemma from [10, p. 113].

Lemma 2.3. *V^2 is dense in V_P^1.*

The next lemma is an easy consequence of our definitions.

Lemma 2.4. *Let $\vec{u} \in V_P^1$. The mapping*

$$\begin{aligned} \vec{\phi} \;&\to\; <\gamma_1^{(\mu)}\vec{u}, \vec{\phi}> \\ &:=\; \Phi_\Omega(\vec{u}, \gamma_0^- \vec{\phi}) - \int_\Omega (P\vec{u})\cdot(\gamma_0^- \vec{\phi})dx \end{aligned} \tag{2.22}$$

is a continuous linear functional $\gamma_1^{(\mu)}\vec{u}$ on $\left(H^{\frac{1}{2}}(\Gamma)\right)^2$, which coincides for $\vec{u} \in V^2$ with

$$\vec{\phi} \;\rightarrow\; \int_\Gamma \left(\mathcal{T}_{\mu,y}(n_y)\vec{u}\right) \cdot \vec{\phi}\, ds_y. \tag{2.23}$$

The mapping

$$\gamma_1^{(\mu)} \;:\; V_P^1 \;\rightarrow\; \left(H^{-\frac{1}{2}}(\Gamma)\right)^2 \tag{2.24}$$

is continuous.
Proof:

a. The continuity of $\gamma_1^{(\mu)}$ follows from the continuity of γ_0^- and from $P\vec{u} \in V^0$.

b. Formula (2.23) follows from the first Green formula (2.6).

c. The continuity in (2.24) follows from the definition of the norm in (2.21) and the continuity of Φ_Ω. □

Remark: That $\gamma_1^{(\mu)}$ coincides with $\mathcal{T}_\mu(n_y)$ for functions in V^2 and the density result in Lemma 2.3 show that the definition of $\gamma_1^{(\mu)}$ is independent of the chosen operator γ_0^-. The operator γ_0^- is not unique.

As a next step we define $\mathcal{T}_0$ and then $\mathcal{T}_\kappa$, $\kappa \in [0,\mu]$, for functions in V_P^1. The starting point is formula (2.1) for P. For $\vec{u} \in V^2$ and $\vec{v} \in V^1$ we get by partial integration

$$\begin{aligned}
\int_\Omega (P\vec{u}) \cdot \vec{v} dx \;&=\; -\int_\Omega \mu\left(\partial_{11}^2 u_1 + \partial_{22}^2 u_1\right) v_1 + (\lambda+\mu)\left(\partial_{11}^2 u_1 + \partial_{12}^2 u_2\right) v_1 \\
&\qquad + \mu\left(\partial_{11}^2 u_2 + \partial_{22}^2 u_2\right) v_2 + (\lambda+\mu)\left(\partial_{12}^2 u_1 + \partial_{22}^2 u_2\right) v_2\, dx \\
&= \int_\Omega \mu\left(\partial_1 u_1 \partial_1 v_1 + \partial_2 u_1 \partial_2 v_1 + \partial_1 u_2 \partial_1 v_2 + \partial_2 u_2 \partial_2 v_2\right) + \\
&\qquad (\lambda+\mu)\left(\partial_1 u_1 \partial_1 v_1 + \partial_2 u_2 \partial_1 v_1 + \partial_1 u_1 \partial_2 v_2 + \partial_2 u_2 \partial_2 v_2\right) dx \\
&\quad - \int_\Gamma \mu\left(\partial_1 u_1 n_1 + \partial_2 u_1 n_2\right) v_1 + (\lambda+\mu)\left(\partial_1 u_1 n_1 + \partial_2 u_2 n_1\right) v_1 \\
&\qquad + \mu\left(\partial_1 u_1 n_1 + \partial_2 u_2 n_2\right) v_2 + (\lambda+\mu)\left(\partial_1 u_1 n_2 + \partial_2 u_2 n_2\right) v_2\, ds_y \\
&= \int_\Omega \mu\left(\sum_{i,j=1}^{2} \partial_i u_j \partial_i v_j\right) + (\lambda+\mu)\mathrm{div}(\vec{u})\mathrm{div}(\vec{v}) dx \\
&\quad + \int_\Gamma \left(\mathcal{T}_0(n_y)\vec{u}\right) \cdot \vec{v} ds_y \\
&=: \widetilde{\Phi}_\Omega(\vec{u},\vec{v}) - \int_\Gamma \left(\mathcal{T}_0(n_y)\vec{u}\right) \cdot \vec{v} ds_y.
\end{aligned} \tag{2.25}$$

For $\widetilde{\Phi}_\Omega$ we get the following properties

$$\left.\begin{array}{rcl}\widetilde{\Phi}_\Omega(\vec{u},\vec{v}) & = & \widetilde{\Phi}_\Omega(\vec{v},\vec{u}), \quad \vec{u},\vec{v}\in V^1 \\ \displaystyle\sum_{i,j=1}^{2}\int_\Omega |\partial_j u_i|^2\,dx & \leq & \widetilde{\Phi}_\Omega(\vec{u},\vec{u}), \quad \vec{u}\in V^1\end{array}\right\} \tag{2.26}$$

Now we extend the operator $\mathcal{T}_0$ to functions in V^1_P. By convex combination we then define $\mathcal{T}_\kappa$, $\kappa\in[0,\mu]$, for functions in V^1_P.

Lemma 2.5. *a. Let $\vec{u}\in V^1_P$. The mapping*

$$\begin{array}{rcl}\vec{\phi} & \rightarrow & <\gamma_1^{(0)}\vec{v},\vec{\phi}> \\ & & := \widetilde{\Phi}_\Omega(\vec{u},\gamma_0^-\vec{\phi}) - \displaystyle\int_\Omega (P\vec{u})\cdot(\gamma_0^-\vec{\phi})dx\end{array} \tag{2.27}$$

is a continuous functional $\mathcal{T}_0$ on $\left(H^{\frac{1}{2}}(\Gamma)\right)^2$, which coincides for $\vec{u}\in V^2$ with the mapping

$$\vec{\phi} \rightarrow \int_\Gamma (\mathcal{T}_0(n_y)\vec{u})\cdot\vec{\phi}\,ds_y.$$

The mapping

$$\gamma_1^{(0)} : V^1_P \rightarrow \left(H^{-\frac{1}{2}}(\Gamma)\right)^2$$

is continuous.

b. For $\kappa\in[0,\mu]$, $\kappa=\widetilde{\lambda}\mu$, $\widetilde{\lambda}\in[0,1]$, we define

$$\gamma_1^{(\kappa)} := \widetilde{\lambda}\gamma_1^{(\mu)} + (1-\widetilde{\lambda})\gamma_1^{(0)}. \tag{2.28}$$

The mapping $\gamma_1^{(\kappa)}$ is contiuous from V^1_P into $\left(H^{-\frac{1}{2}}(\Gamma)\right)^2$. On V^2, $\gamma_1^{(\kappa)}$ coincides with $\mathcal{T}_\kappa(n_y)$. We further get

$$<\gamma_1^{(\kappa)}\vec{u},\vec{\phi}> = \Phi_\Omega^{(\kappa)}(\vec{u},\gamma_0^-\vec{\phi}) - \int_\Omega (P\vec{u})\cdot\gamma_0^-\vec{\phi}ds_y, \tag{2.29}$$

where

$$\Phi_\Omega^{(\kappa)}(\cdot,\cdot) := \widetilde{\lambda}\Phi_\Omega^{(\kappa)}(\cdot,\cdot) + (1-\widetilde{\lambda})\Phi_\Omega^{(\kappa)}(\cdot,\cdot). \tag{2.30}$$

Because of (2.26) the inequality of Korn (2.12) also holds for $\Phi_\Omega^{(\kappa)}$, $\kappa\in[0,\mu]$.

For a function $\vec{u} \in (L^2(\mathbb{R}^2))^2$ with $\vec{u}|_\Omega \in V^1$ and $\vec{u}|_{\Omega^c} \in (H^1_{loc}(\Omega^c))^2$ the traces $\gamma_0(\vec{u}|_\Omega)$ and $\gamma_0(\vec{u}|_{\Omega^c})$ are well defined. Let

$$[\gamma_0 \vec{u}] \;:=\; \gamma_0(\vec{u}|_\Omega) - \gamma_0(\vec{u}|_{\Omega^c}) \in \left(H^{\frac{1}{2}}(\Gamma)\right)^2. \tag{2.31}$$

For a function $\vec{u} \in (H^1_{loc}(\Omega^c))^2$ and $P\vec{u} \in (L^2_{loc}(\mathbb{R}^2))^2$ the operator $\gamma^{(\kappa)}_{1,\Omega^c}$, $\kappa \in [0,\mu]$, is given by (2.28) and (2.22), where Ω has to be replaced by Ω^c. Here we will assume that $\mathrm{supp}(\gamma_0^- \vec{v}) \subset B_{2R_0}(0)$, $\forall \vec{v} \in \left(H^{\frac{1}{2}}(\Gamma)\right)^2$. We will denote the set of all functions $\vec{u} \in (H^1_{loc}(\Omega^c))^2$ with $P\vec{u} \in (L^2_{loc}(\mathbb{R}^2))^2$ by $V^1_P(\Omega^c)$.
If $\vec{u} \in (L^2(\mathbb{R}^2))^2$ with $\vec{u}|_\Omega \in V^1_P$ and $\vec{u}|_{\Omega^c} \in V^1_P(\Omega^c)$ then we define

$$[\gamma_1^{(\kappa)} \vec{u}] \;:=\; \gamma_1^{(\kappa)}(\vec{u}|_\Omega) - \gamma^{(\kappa)}_{1,\Omega^c}(\vec{u}|_{\Omega^c}). \tag{2.32}$$

Now we can formulate the following lemma.

Lemma 2.6. *Let $\kappa \in [0,\mu]$.*

a. *For $\vec{u} \in V^1_P$ and $\vec{v} \in V^1$ the first Green formula holds:*

$$\int_\Omega (P\vec{u}) \cdot \vec{v}\, dx \;=\; \Phi^{(\kappa)}_\Omega(\vec{u},\vec{v}) - <\gamma_1^{(\kappa)}\vec{u}, \gamma_0 \vec{v}> \tag{2.33}$$

b. *The second Green formula holds for all $\vec{u},\vec{v} \in V^1_P$.*

$$\int_\Omega \vec{u}\cdot(P\vec{v}) - \vec{v}\cdot(P\vec{u}) dx \;=\; <\gamma_1^{(\kappa)}\vec{u}, \gamma_0\vec{v}> - <\gamma_1^{(\kappa)}\vec{v}, \gamma_0\vec{u}> \tag{2.34}$$

c. *Let $\vec{u} \in (L^2(\mathbb{R}^2))^2$ be given with*

$$\vec{u}|_\Omega \in V^1_P \text{ and } \vec{u}|_{\Omega^c} \in V^1_P(\Omega^c).$$

Then

$$\begin{aligned} u(x) \;&=\; (GP\vec{u})(x) + <[\gamma_1^{(\kappa)}\vec{u}], G(\cdot,x)> \\ &\quad - \int_\Gamma (\mathcal{T}_\kappa(n_y) G(\cdot,x))^T [\gamma_0 \vec{u}]\, ds_y. \end{aligned} \tag{2.35}$$

Proof: Lemma 2.3 shows that for every $\vec{u} \in V^1_P$ there exists a sequence $(\vec{u}_n)_{n\in\mathbb{N}} \subset V^2$ for which

$$\lim_{n\to\infty} \|\vec{u} - \vec{u}_n\|_{V^1_P} \;=\; 0.$$

Now we have

$$
\begin{aligned}
(\alpha)\ P\vec{u}_n \overset{L^2}{\to} P\vec{u} \quad &\Rightarrow \quad \int_\Omega (P\vec{u}_n)\cdot\vec{v}\,dx \;\to\; \int_\Omega (P\vec{u})\cdot\vec{v}\,dx \\
(\beta)\ \vec{u}_n \overset{V^1}{\to} \vec{u} \quad &\Rightarrow \quad \Phi_\Omega^{(\kappa)}(\vec{u}_n,\vec{v}) \;\to\; \Phi_\Omega^{(\kappa)}(\vec{u},\vec{v}) \\
(\gamma)\ \vec{u}_n \overset{V_P^1}{\to} \vec{u} \quad &\Rightarrow \quad \gamma_1^{(\kappa)}\vec{u}_n \overset{H^{-\frac{1}{2}}}{\to} \gamma_1^{(\kappa)}\vec{u} \\
&\Rightarrow \quad <\gamma_1^{(\kappa)}\vec{u}_n,\gamma_0\vec{v}> \to <\gamma_1^{(\kappa)}\vec{u},\gamma_0\vec{v}>
\end{aligned}
$$

for every $\vec{v}\in V^1$.
This implies

$$
\begin{aligned}
<\gamma_1^{(\kappa)}\vec{u}_n,\gamma_0\vec{v}> \quad &\overset{Def.}{=} \quad \Phi_\Omega^{(\kappa)}(\vec{u}_n,\gamma_0^-\gamma_0\vec{v}) - \int_\Omega (P\vec{u}_n)\cdot(\gamma_0^-\gamma_0\vec{v})\,dx \\
&\overset{(2.6),(2.15)}{=} \quad \int_\Gamma (\mathcal{T}_\kappa(n_y)\vec{u}_n)\cdot(\gamma_0\vec{v})\,ds_y \\
&\overset{partial\ int.}{=} \quad \Phi_\Omega^{(\kappa)}(\vec{u}_n,\vec{v}) - \int_\Omega (P\vec{u}_n)\cdot(\vec{v})\,dx \\
&\overset{(\alpha),(\beta)}{\to} \quad \Phi_\Omega^{(\kappa)}(\vec{u},Vv) - \int_\Omega (P\vec{u})\cdot(\vec{v})\,dx
\end{aligned}
$$

Now (γ) proves a.
b. follows from a. and the symmetry of $\Phi_\Omega^{(\kappa)}$.
c. Let $\vec{v}\in V^2$, $x\in\Omega$. Formula (2.20) and the symmetry of $G(x,y)$ show

$$
\begin{aligned}
\vec{v}(x) \quad = \quad & \int_\Omega G(y,x)(P\vec{v})(y)\,dy \\
& + \int_\Gamma G(y,x)\left(\mathcal{T}_\kappa(n_y)\vec{v}(y)\right) - \left(\mathcal{T}_\kappa(n_y)G(y,x)\right)^T\vec{v}(y)ds_y.
\end{aligned}
$$

For $x\in\Omega^c$ we have $PG(y,x)|_\Omega\equiv 0$. This implies

$$
\begin{aligned}
0 \quad = \quad & \int_\Omega G(y,x)(P\vec{v})(y)\,dy \\
& + \int_\Gamma G(y,x)\left(\mathcal{T}_\kappa(n_y)\vec{v}(y)\right) - \left(\mathcal{T}_\kappa(n_y)G(y,x)\right)^T\vec{v}(y)ds_y.
\end{aligned}
$$

For $\vec{v}\in\left(H^2(\Omega^c)\right)^2$ with compact support we can apply the above two formulas for the domain $\widetilde{\Omega}:=\Omega^c\cap B_R(0)$ with $\mathrm{supp}(\vec{v})\subset B_{R/2}(0)$. We will denote by n_y the outer normal on Γ with respect to Ω. We get for $x\in\Omega^c$

$$
\begin{aligned}
\vec{v}(x) \quad = \quad & \int_\Omega G(y,x)(P\vec{v})(y)\,dy \\
& - \int_\Gamma G(y,x)\left(\mathcal{T}_\kappa(n_y)\vec{v}(y)\right) - \left(\mathcal{T}_\kappa(n_y)G(y,x)\right)^T\vec{v}(y)ds_y
\end{aligned}
$$

and for $x\in\Omega$

$$
\begin{aligned}
0 \quad = \quad & \int_\Omega G(y,x)(P\vec{v})(y)\,dy \\
& - \int_\Gamma G(y,x)\left(\mathcal{T}_\kappa(n_y)\vec{v}(y)\right) - \left(\mathcal{T}_\kappa(n_y)G(y,x)\right)^T\vec{v}(y)ds_y.
\end{aligned}
$$

For a function $\vec{v} \in \left(L^2(\mathbb{R}^2)\right)^2$ with compact support and $\vec{v}|_\Omega \in V^2$, $\vec{v}|_{\Omega^c} \in \left(H^2(\Omega^c)\right)^2$, we get by addition of the above formulas

$$\begin{aligned}
\vec{v}(x) &= \int_{\mathbb{R}^2} G(y,x)(P\vec{v})(y)\,dy \\
&\quad + \int_\Gamma G(y,x)\left[\mathcal{T}_\kappa(n_y)\vec{v}(y)\right] - \left(\mathcal{T}_\kappa(n_y)G(y,x)\right)^T [\vec{v}(y)]ds_y \\
&= \int_{\mathbb{R}^2} G(y,x)(P\vec{v})(y)\,dy \\
&\quad + < [\mathcal{T}_\kappa(n_y)\vec{v}]\,, G(\cdot,x) > - \textstyle\int_\Gamma \left(\mathcal{T}_\kappa(n_y)G(y,x)\right)^T [\vec{v}(y)]ds_y.
\end{aligned}$$

This implies formula (2.35), because every function $\vec{u}$ as in c. can be approximated by a sequence $(\vec{v}_n)_n$ of functions in V^2 by Lemma 2.4. □

Lemma 2.7. *The trace mapping*

$$(\gamma_0, \gamma_1^{(\kappa)}) \;:\; \vec{\phi} \;\to\; (\gamma_0\vec{\phi}, \gamma_1^{(\kappa)}\vec{\phi})$$

maps $\left(C_0^\infty(\mathbb{R}^2)\right)^2$ *onto a dense subset of*

$$\left(H^{\frac{1}{2}}(\Gamma)\right)^2 \times \left(H^{-\frac{1}{2}}(\Gamma)\right)^2.$$

Proof: We assume that for

$$(\vec{\chi}, \vec{\psi}) \;\in\; \left(H^{\frac{1}{2}}(\Gamma)\right)^2 \times \left(H^{-\frac{1}{2}}(\Gamma)\right)^2$$

the equation

$$< (\gamma_0\vec{\phi}, \gamma_1^{(\kappa)}\vec{\phi}), (-\vec{\psi}, \vec{\chi}) > = \; 0$$

holds for all $\vec{\phi}$. This implies

$$< \vec{\psi}, \gamma_0\vec{\phi} > \;\; = \;\; < \gamma_1^{(\kappa)}\vec{\phi}, \vec{\chi} >, \; \forall \vec{\phi} \in \left(C_0^\infty(\mathbb{R}^2)\right)^2. \tag{2.36}$$

Now let $T\vec{\chi}$ be the solution of

$$PT\vec{\chi} \;=\; 0,\; \gamma_0 T\vec{\chi} \;=\; \vec{\chi},$$

see Lemma 2.2.
For $\vec{f} \in V^0$ we denote by $S\vec{f}$ the solution of

$$PS\vec{f} \;=\; \vec{f},\; \gamma_0 S\vec{f} \;=\; 0,$$

see Corollary 2.1.

Because of $S\vec{f}, T\vec{\chi} \in V_P^1$ we can apply the second Green formula in Lemma 2.6 and we get

$$\int_\Omega S\vec{f} \cdot (PT\vec{\chi}) - (T\vec{\chi}) \cdot (PS\vec{f})\, dx = \ < \gamma_1^{(\kappa)} S\vec{f}, \gamma_0 T\vec{\chi} > - < \gamma_1^{(\kappa)} T\chi, \gamma_0 S\vec{f} >$$

This implies

$$-\int_\Omega \vec{f} \cdot T\vec{\chi} dx \ = < \gamma_1^{(\kappa)} S\vec{f}, \vec{\chi} >, \ \forall \vec{f} \in V^0.$$

Because $\left(C_0^\infty(\mathbb{R}^2)\right)^2$ is dense in V^2 and V^2 is dense in V_P^1, formula (2.36) holds for $\vec{\phi} = S\vec{f} \in V_P^1$. Now we have

$$0 \ = < \vec{\psi}, \gamma_0 S\vec{f} > = < \gamma_1^{(\kappa)} S\vec{f}, \vec{\chi} > .$$

Therefore we get

$$0 \ = \ \int_\Omega \vec{f} \cdot T\vec{\chi} dx, \ \forall \vec{f}.$$

But this means $T\vec{\chi} = 0$ and $\vec{\chi} = 0$ by Lemma 2.2. By (2.36) this implies

$$< \vec{\psi}, \gamma_0 \vec{\phi} > = \ 0, \ \forall \vec{\phi}.$$

We finally get $\vec{\psi} = 0$, because

$$\gamma_0 \ : \ V^1 \ \rightarrow \ \left(H^{\frac{1}{2}}(\Gamma)\right)^2$$

is surjective (it has a continuous right inverse by (2.15)) and the C^∞–functions are dense in V^1.
So we get $\vec{\psi} = 0$ and $\vec{\chi} = 0$, and this proves the Lemma. □

Lemma 2.8. *The trace operator*

$$\gamma_0 \ : \ u \ \rightarrow \ u|_\Gamma \ : \ H_{loc}^s(\mathbb{R}^2) \ \rightarrow \ H^{s-\frac{1}{2}}(\Gamma)$$

is continuous for $s \in (\frac{1}{2}, \frac{3}{2})$.

Proof: See [4, Lemma 3.6]. □

Lemma 2.9. *The Green operator G fulfills*

$$G \ : \ \left(H^s(\mathbb{R}^2)\right)^2 \ \rightarrow \ \left(H_{loc}^{s+2}(\mathbb{R}^2)\right)^2 .$$

Proof: We calculate the symbolmatrix σ of P with the Fourier transform.

$$P \ = \ \begin{pmatrix} \mu(\partial_{11}^2 + \partial_{22}^2) + (\lambda + \mu)\partial_{11}^2 & (\lambda + \mu)\partial_{12}^2 \\ (\lambda + \mu)\partial_{12}^2 & \mu(\partial_{11}^2 + \partial_{22}^2) + (\lambda + \mu)\partial_{22}^2 \end{pmatrix}$$

This implies for the symbol matrix σ

$$\begin{aligned}\sigma(\xi_1,\xi_2) &= -\begin{pmatrix} \mu(\xi_1^2+\xi_2^2)+(\lambda+\mu)\xi_1^2 & (\lambda+\mu)\xi_1\xi_2 \\ (\lambda+\mu)\xi_1\xi_2 & \mu(\xi_1^2+\xi_2^2)+(\lambda+\mu)\xi_2^2 \end{pmatrix} \\ &= -U\begin{pmatrix} \mu(\xi_1^2+\xi_2^2) & 0 \\ 0 & (\lambda+2\mu)(\xi_1^2+\xi_2^2) \end{pmatrix} U^T,\end{aligned}$$

with an orthogonal matrix

$$U = \frac{1}{(\xi_1^2+\xi_2^2)^{\frac{1}{2}}}\begin{pmatrix} -\xi_2 \mathrm{sgn}(\xi_1) & \xi_1 \mathrm{sgn}(\xi_2) \\ |\xi_1| & |\xi_2| \end{pmatrix}, (\xi_1,\xi_2) \neq 0.$$

This implies

$$\sigma^{-1}(\xi_1,\xi_2) = -\frac{1}{(\xi_1^2+\xi_2^2)} U \begin{pmatrix} \frac{1}{\mu} & 0 \\ 0 & \frac{1}{\lambda+2\mu} \end{pmatrix} U^T$$

and

$$\|\sigma^{-1}(\xi_1,\xi_2)\| \leq \frac{C}{\xi_1^2+\xi_2^2},$$

which proves the lemma. □

Now we define the single layer operator K_0 and the double layer operator K_1.

$$(K_0\vec{v})(x) = \int_\Gamma G(y,x)\vec{v}(y)ds_y, \quad x \in \mathbb{R}^2 \setminus \Gamma \tag{2.37}$$

$$(K_1^{(\kappa)}\vec{v})(x) = \int_\Gamma (\mathcal{T}_\kappa(n_y)G(y,x))^T \vec{v}(y)ds_y, \quad x \in \mathbb{R}^2 \setminus \Gamma \tag{2.38}$$

The following lemma repeats the results of Theorem 1.(i)(ii) of [4] for the Lamé operator.

Lemma 2.10.

a. The mapping $K_0 : \left(H^{-\frac{1}{2}+\sigma}(\Gamma)\right)^2 \to \left(H^{1+\sigma}_{loc}(\mathbb{R}^2)\right)^2$, $\sigma \in (-\frac{1}{2},\frac{1}{2})$, *is continuous.*

b. The mapping $K_1^{(\kappa)} : \left(H^{\frac{1}{2}}(\Gamma)\right)^2 \to (H^1_{loc}(\mathbb{R}^2))^2$, $\kappa \in [0,\mu]$, *is continuous.*

Proof:

a. Let $\vec{v} \in \left(H^{-\frac{1}{2}+\sigma}(\Gamma)\right)^2$. Then $\gamma_0'\vec{v}$ (where γ_0' is the adjoint of γ_0) is a distribution in $\mathbb{R}^2$ with compact support

$$< \gamma_0'\vec{v}, \vec{\phi} > \;:=\; < \vec{v}, \gamma_0\vec{\phi} >, \quad \forall \vec{\phi} \in \left(C_0^\infty(\mathbb{R}^2)\right)^2.$$

Now we have

$$K_0 \;=\; G \circ \gamma_0'$$

and Lemma 2.8 shows

$$\gamma_0' \;:\; \left(H^{\frac{1}{2}-s}(\Gamma)\right)^2 \;\rightarrow\; \left(H^{-s}_{loc}(\mathbb{R}^2)\right)^2, \; s \in (\frac{1}{2}, \frac{3}{2}).$$

Lemma 2.9 finally implies

$$G \circ \gamma_0' \;:\; \left(H^{\frac{1}{2}-s}(\Gamma)\right)^2 \;\rightarrow\; \left(H^{2-s}_{loc}(\mathbb{R}^2)\right)^2.$$

Define $\sigma = 1 - s \in (-\frac{1}{2}, \frac{1}{2})$. Then the above equation gives

$$K \;:\; \left(H^{-\frac{1}{2}+\sigma}(\Gamma)\right)^2 \;\rightarrow\; \left(H^{1+\sigma}_{loc}(\mathbb{R}^2)\right)^2.$$

b. Let $\vec{v} \in \left(H^{\frac{1}{2}}(\Gamma)\right)^2$ and $\vec{u} := T\vec{v} \in V_P^1$, where $T\vec{v}$ is the solution of (2.16), see Lemma 2.2. Formula (2.35), where we define $\vec{u}|_{\Omega^c} \equiv 0$, now gives us

$$\begin{aligned} T\vec{v} &= \int_\Omega G(y,x)T\vec{v}(y)ds_y + < \gamma_1^{(\kappa)}T\vec{v}, G(\cdot,x) > -(K_1^{(\kappa)}T\vec{v}) \\ &= K_0\gamma_1^{(\kappa)}T\vec{v} - K_1^{(\kappa)}\vec{v}. \end{aligned}$$

Therefore

$$\begin{aligned} K_1^{(\kappa)} &= K_0 \circ \gamma_1^{(\gamma)} \circ T - T \\ &= (K_0 \circ \gamma_1^{(\kappa)} - I) \circ T \end{aligned}$$

and we have

$$\begin{array}{llll} T: & \left(H^{\frac{1}{2}}(\Gamma)\right)^2 & \rightarrow V_P^1 & \text{(Lemma 2.2)} \\ \gamma_1^{(\kappa)}: & V_P^1 & \rightarrow \left(H^{-\frac{1}{2}}(\Gamma)\right)^2 & \text{(Lemma 2.5)} \\ K_0: & \left(H^{-\frac{1}{2}}(\Gamma)\right)^2 & \rightarrow V^1 & \text{(part a.)} \end{array}$$

□

In the next lemma we collect some smoothness properties for the double layer potential and for a special parameter $\overline{\kappa}$ we estimate the norm of $K_1^{(\overline{\kappa})}\vec{v}(x)$ and its derivatives.

Lemma 2.11. *For* $\vec{v} \in \left(H^{\frac{1}{2}}(\Gamma)\right)^2$ *and*

$$\overline{\kappa} \;:=\; \mu \frac{\lambda + \mu}{\lambda + 3\mu} \tag{2.39}$$

the following results hold:

a. $K_1^{(\overline{\kappa})}\vec{v} \;\in\; \left(C^\infty(\mathbb{R}^2 \setminus \Gamma)\right)^2$,

b.

$$\begin{aligned} \|(K_1^{(\overline{\kappa})}\vec{v})(x)\| &= \mathcal{O}_{\|x\|\to\infty}\left(\frac{1}{\|x\|}\right), \\ \|(\nabla K_1^{(\overline{\kappa})}\vec{v})(x)\| &= \mathcal{O}_{\|x\|\to\infty}\left(\frac{1}{\|x\|^2}\right), \end{aligned}$$

c. $K_1^{(\kappa)}\vec{v} \;\in\; V_P^1(\Omega^c)$, $\kappa \in [0, \mu]$.

Proof:

a. By a calculation we get

$$\mathcal{T}_{\overline{\kappa}}(n_y)G(y,x) \;=\; -\frac{1}{2\pi}\left((1-\overline{c})I_{2\times 2} + 2\overline{c}\frac{(y-x)(y-x)^T}{\|x-y\|^2}\right)\frac{y - x \cdot n_y}{\|x-y\|^2}, \tag{2.40}$$

with

$$\overline{c} \;:=\; \frac{\lambda+\mu}{\lambda+3\mu}. \tag{2.41}$$

This shows that all components of $\mathcal{T}_{\overline{\kappa}}(n_y)G$ are in $C^\infty(\Gamma \times (\mathbb{R}^2 \setminus \Gamma))$. The C^∞ property would also hold for every other value of κ, but if $\kappa \neq \overline{\kappa}$ a third term in formula (2.40) would appear, which has a stronger singularity for $x = y$, and this term does not allow the analysis in chapter 3 (see [14]).

b. Formula (2.40) gives us

$$\begin{aligned} \|\mathcal{T}_{\overline{\kappa}}(n_y)G(y,x)\| &= \|\frac{1}{2\pi}\left((1-\overline{c})I_{2\times 2} + 2\overline{c}\frac{(y-x)(y-x)^T}{\|x-y\|^2}\right)\|\,|(y-x)\cdot n_y|\,\frac{1}{\|x-y\|^2} \\ &\le C\frac{1}{\|x-y\|}. \end{aligned}$$

The calculation of the derivatives of the entries of $\mathcal{T}_{\overline{\kappa}}(n_y)G$ with respect to x_1 and x_2 shows

$$\|\nabla \mathcal{T}_{\overline{\kappa}}(n_y)G(y,x)\| \;\leq\; C\frac{1}{\|x-y\|^2}. \tag{2.42}$$

b. follows because the length of Γ is finite.

c. We will repeat here the arguments of Lemma 2.5 and Lemma 2.10 for the domain $\widetilde{\Omega} := (\mathbb{R}^2 \setminus \overline{\Omega}) \cap B_R(0)$, $R > R_0$. We have $\partial\widetilde{\Omega} = \Gamma \cup \partial B_R(0)$ and the properties of $\widetilde{\gamma}_1^{(\kappa)}$ in Lemma 2.5 and the properties of $\widetilde{K}_0$ in Lemma 2.10 for $\partial\widetilde{\Omega}$ instead of Γ are valid.

Let $\vec{v} \in \left(H^{\frac{1}{2}}(\Gamma)\right)^2$. By $\vec{u} := \widetilde{T}\vec{v} \in \left(H^1(\widetilde{\Omega})\right)^2$ we denote the unique solution (see Lemma 2.2) of

$$\begin{aligned} P\widetilde{T}\vec{v} &= 0, \quad in\, \widetilde{\Omega} \\ \gamma_0\widetilde{T}\vec{v} &= \vec{v}, \\ \widetilde{T}\vec{v}|_{\partial B_r(0)} &= 0. \end{aligned}$$

By Lemma 2.6.c. (where we define $\widetilde{T}\vec{v}|_{\widetilde{\Omega}^c} \equiv 0$) we get

$$\widetilde{T}\vec{v} \;=\; <\widetilde{\gamma}_1^{(\kappa)}\widetilde{T}\vec{v}, G(\cdot,x)> + K_1^{(\kappa)}\vec{v},$$

where we have used that $\vec{v}|_{\partial B_R(0)} = 0$ and that $K_1^{(\kappa)}$ depends on the outer normal of Γ with respect to Ω. This equation implies

$$K_1^{(\kappa)}\vec{v} \;=\; (I - \widetilde{K}_0 \circ \widetilde{\gamma}_1^{(\kappa)}) \circ \widetilde{T}.$$

As in Lemma 2.10 the last equation shows

$$K_1^{(\kappa)}\vec{v} \in \left(H^1(\widetilde{\Omega})\right)^2,$$

$R > R_0$ arbitrary. This proves c.

□

Lemma 2.12. *For $\vec{v} \in \left(H^{\frac{1}{2}}(\Gamma)\right)^2$, $\kappa \in [0,\mu]$, we get*

$$[\gamma_0 K_1^{(\kappa)}\vec{v}] \;=\; -\vec{v}, \qquad [\gamma_1^{(\kappa)} K_1^{(\kappa)}\vec{v}] \;=\; 0.$$

Proof: Let $\vec{v} \in \left(H^{\frac{1}{2}}(\Gamma)\right)^2$, $\vec{\phi} \in (C_0^\infty(\mathbb{R}^2))^2$, $\vec{u} := K_1^{(\kappa)}\vec{v}$. By Lemma 2.11 we can apply Lemma 2.6 and get

$$\begin{aligned}\int_\Omega \vec{u} \cdot P\vec{\phi}\, dx &= < \gamma_1^{(\kappa)}\vec{u}|_\Omega, \gamma_0\vec{\phi} > - < \gamma_1^{(\kappa)}\vec{\phi}|_\Omega, \gamma_0\vec{u}|_\Omega > \\ \int_{\Omega^c} \vec{u} \cdot P\vec{\phi}\, dx &= < -\gamma_1^{(\kappa)}\vec{u}|_{\Omega^c}, \gamma_0\vec{\phi} > + < \gamma_1^{(\kappa)}\vec{\phi}|_{\Omega^c}, \gamma_0\vec{u}|_{\Omega^c} > .\end{aligned}$$

The different signs in the second formula are caused by the choice of the outer normal for Ω in the definition of $\gamma_1^{(\kappa)}$. This implies

$$\int_{\mathbb{R}^2} \vec{u} \cdot P\vec{\phi}\, dx = < [\gamma_1^{(\kappa)}\vec{u}], \gamma_0\vec{\phi} > - < \gamma_1^{(\kappa)}\vec{\phi}, [\gamma_0\vec{u}] > . \tag{2.43}$$

On the other hand we have

$$\vec{u} = K_1^{(\kappa)}\vec{v} = G((\gamma_1^{(\kappa)})'\vec{v}),$$

where the distribution $(\gamma_1^{(\kappa)})'\vec{v}$ with compact support is defined by

$$< (\gamma_1^{(\kappa)})'\vec{v}, \vec{\phi} > = < \vec{v}, \gamma_1^{(\kappa)}\vec{\phi} >, \quad \vec{\phi} \in \left(C_0^\infty(\mathbb{R}^2)\right)^2 .$$

Now the left side of equation (2.43) can be rewritten

$$\begin{aligned}\int_\Omega \vec{u} \cdot P\vec{\phi}\, dx &= < G \circ (\gamma_1^{(\kappa)})'\vec{v}, P\vec{\phi} > \\ &= < (\gamma_1^{(\kappa)})'\vec{v}, \underbrace{G \circ P}_{=I}\vec{\phi} > \\ &= < \vec{v}, \gamma_1^{(\kappa)}\vec{\phi} > .\end{aligned} \tag{2.44}$$

Formula (2.43) and (2.44) give us

$$< [\gamma_1^{(\kappa)} K_1^{(\kappa)}\vec{v}], \gamma_0\vec{\phi} > = < [\gamma_0 K_1^{(\kappa)}\vec{v}] + \vec{v}, \gamma_1^{(\kappa)}\vec{\phi} >, \ \forall\vec{\phi} \in \left(C_0^\infty(\mathbb{R}^2)\right)^2 .$$

Lemma 2.8 proves the lemma. □

Now we can prove the uniqueness of the solution of the exterior Neumann problem, where instead of the normal derivative the generalized stress operator is used.

Lemma 2.13. *Let $\kappa \in [0, \mu)$ and $\vec{u} \in (H^1_{loc}(\Omega^c))^2$ with*

a. $P\vec{u} = 0$

b. $\gamma_1^{(\kappa)}\vec{u} = 0$

c. $\vec{u}|_{\mathbb{R}^2\setminus\overline{\Omega}} \in \left(C_0^\infty(\mathbb{R}^2 \setminus \overline{\Omega})\right)^2$ *and*

$$|\vec{u}(x)| \;=\; \mathcal{O}(\frac{1}{\|x\|}) \qquad |\nabla\vec{u}(x)| \;=\; \mathcal{O}(\frac{1}{\|x\|^2}).$$

Then we have $\vec{u} \;=\; 0$.

Proof: The first Green formula for $\vec{u}|_{\Omega^c\cap B_R(0)}$, $R > R_0$, and property b. give

$$\begin{aligned} 0 &= \int_{\Omega^c\cap B_R(0)} \vec{u}\cdot P\vec{u}\,dx \\ &= \Phi^{(\kappa)}_{\Omega^c\cap B_R(0)}(\vec{u},\vec{u}) + <\underbrace{\gamma_1^{(\kappa)}\vec{u}}_{=0},\vec{u}> - \int_{\partial B_R(0)} \mathcal{T}_\kappa\vec{u}\cdot\vec{u}\,ds_y. \end{aligned}$$

Because $\mathcal{T}_\kappa\vec{u}$ contains only first derivatives of $\vec{u}$, we get by the Cauchy–Schwarz inequality

$$|\mathcal{T}_\kappa\vec{u}(x)\cdot\vec{u}(x)| \;\le\; \frac{C}{\|x\|^3}.$$

Now we have

$$\begin{aligned} \left|\Phi^{(\kappa)}_{\Omega^c\cap B_R(0)}(\vec{u},\vec{u})\right| &\le C\,4\pi R^2\,\frac{1}{R^3} \\ &\overset{R\to\infty}{\to} 0. \end{aligned}$$

On the other hand we have that

$$\Phi^{(\kappa)}_{\Omega^c\cap B_R(0)}(\vec{u},\vec{u}) \;\ge\; 0$$

is a monotonically increasing function of R, and this finally implies

$$0 \;=\; \Phi^{(\kappa)}_{\Omega^c\cap B_R(0)}(\vec{u},\vec{u}), \qquad \forall R \ge R_0.$$

Now inequality (2.26) implies that $\vec{u}$ is constant and property c. proves the lemma. □

In the next section we will prove the injectivity of our boundary integral equation with the help of the last lemma. For the solution of the equation (2.2) we use the double layer potential (2.38) with $\kappa = \overline{\kappa}$ (2.39) and finally we need the boundary values of the double layer potential. For $\vec{u} \in (C(\Gamma))^2$ we get by direct calculation

$$\lim_{\Omega\ni x\to x_0\in\gamma} (K^{(\overline{\kappa})}\vec{u})(x) \;=\; -\frac{1}{2}\vec{u}(x_0) + \frac{1}{2}\left(\mathcal{K}_1^{(\overline{c})}\vec{u}\right)(x_0), \tag{2.45}$$

where $\mathcal{K}_1^{(\overline{c})}$ is defined by

$$\left(\mathcal{K}_1^{(\overline{c})}\vec{u}\right)(x_0) \;=\; \frac{1}{\pi}\int_\Gamma \left[\frac{(x-y)\cdot n_y}{\|x-y\|^2}\left((1-\overline{c})I_{2\times2} + 2\overline{c}\frac{(x-y)(x-y)^T}{\|x-y\|^2}\right)\vec{u}(y)\right] ds_y \tag{2.46}$$

See Lemma 2.11 for the definition of $\overline{c}$ (see (2.41)) and for the kernel of $\mathcal{K}_1^{(\overline{c})}$ (see (2.40)).

3 Results for the integral equation

In this section we prove the Fredholm property of the operator

$$I \;-\; \mathcal{K}_1^{(\overline{c})} \tag{3.1}$$

which was defined in (2.46). We show that the operator is invertible and we prove a regularity result, which we need for the numerical results in the next section and for the proof of the injectivity of the operator.
We will assume that the polygon Γ is parametrized by $\gamma : [0,T] \to \mathbb{R}^2$ in the following way.

$$0 \;=\; s_0 \;<\; s_1 < \ldots < s_n \;=\; T,$$

$\xi_i := \gamma(s_i)$, $i = 0(1)n$, are the corners of Γ and $\xi_0 = \xi_n$.

$$\gamma|_{[s_i,s_{i+1}]}(s) \;\;=\;\; \xi_i + (s - s_i)\zeta_i, \tag{3.2}$$

where

$$\zeta_i \;:=\; \begin{pmatrix} \cos(\alpha_i) \\ \sin(\alpha_i) \end{pmatrix}, \quad \text{and we define} \quad \eta_i \;:=\; \begin{pmatrix} \sin(\alpha_i) \\ -\cos(\alpha_i) \end{pmatrix} \tag{3.3}$$

the outer normal for Γ on $\gamma(s_i, s_{i+1})$.
In the following we will identify the functions on Γ and on $[0,T]$. So the study of (3.1) leads us to the study of the integral equation

$$\vec{u}(s) \;+\; \int_0^T k^{(\omega)}(s,\tau)\vec{u}(\tau)\,d\tau \;\;=\;\; \vec{f}(s), \qquad s \in [0,T], \tag{3.4}$$

where $k^{(\omega)}$ is given by (see (2.46))

$$\begin{aligned} k^{(\omega)}(s,\tau) \;\;&=\;\; -\frac{1}{\pi}\frac{(\gamma(s)-\gamma(\tau))\cdot n(\tau)}{\|\gamma(s)-\gamma(\tau)\|} \\ &\quad \Big((1-\omega)I_{2\times 2} + 2\omega\frac{(\gamma(s)-\gamma(\tau))(\gamma(s)-\gamma(\tau))^T}{\|\gamma(s)-\gamma(\tau)\|}\Big). \end{aligned} \tag{3.5}$$

$\vec{f} \in \left(L^2(0,T)\right)^2$ given, $\vec{u} \in \left(L^2(0,T)\right)^2$ is the solution. Finally we are only interested in $k^{(\overline{c})}$ (see (2.41)), but we will study $k^{(\omega)}$, $\omega \in [0,1]$.
In the following we will localize equation (3.4) around each corner. Therefore we have to introduce a couple of definitions.
We choose a real number $\delta > 0$ with

$$\delta \;\;<\;\; \min_{i=1}^{n-1}\{s_i - s_{i-1}\}/2. \tag{3.6}$$

We define J_{3i+j}, $0 \le i \le n-1$, $0 \le j \le 2$, by

$$J_{3i+j} \;:=\; \begin{cases} [s_i, s_i+\delta], & j=0 \\ [s_i+\delta, s_{i+1}-\delta], & j=1 \\ [s_{i+1}-\delta, s_{i+1}], & j=2 \end{cases}, \tag{3.7}$$

and $J_{3n} := [T, T+\delta]$.
Let

$$\vec{L}^2_\pi \;:=\; \prod_{i=1}^{3n} \left(L^2(J_i)\right)^2 . \tag{3.8}$$

We notice that the mapping

$$\Phi \,:\, \left(L^2(0,T)\right)^2 \;\to\; \vec{L}^2_\pi \quad \text{defined by} \quad (\Phi\vec{u})_i \;:=\; \vec{u}|_{J_i} \tag{3.9}$$

is an isomorphism, where we have extended $\vec{u}$ periodically on $\mathbb{R}$. It is clear that for functions $\vec{u} \in (L^2(0,T))^2$, $\vec{u}|_{[s_i,s_{i+1}]} \in (C_0^\infty([s_i,s_{i+1}]))^2$, the operators $\mathcal{K}^{(\omega)}$

$$(\mathcal{K}^{(\omega)}\vec{u})(s) \;:=\; \int_0^T k^{(\omega)}(s,\tau)\vec{u}(\tau)\,d\tau \tag{3.10}$$

and

$$\mathcal{B}^{(\omega)} \;:=\; I \,+\, \mathcal{K}^{(\omega)} \tag{3.11}$$

are well defined.
We then get

$$\mathcal{B}^{(\omega)} \;=\; \Phi^{-1} \circ (I_{\vec{L}^2_\pi} \,+\, \mathcal{K}^{(\omega)}_\pi) \circ \Phi \tag{3.12}$$

where $\mathcal{K}^{(\omega)}_\pi$ is defined by

$$\begin{aligned} \mathcal{K}^{(\omega)}_\pi \left((\vec{u}_j)_{j=1}^{3n}\right)_i (s) \;&=\; \sum_{j=1}^{3n} \left(\mathcal{K}^{(\omega)}_{i,j}\vec{u}_j\right)(s) \\ &=\; \sum_{j=1}^{3n} \int_{J_j} k^{(\omega)}(s,\tau)\vec{u}_j(\tau)\,d\tau,\; s \in J_i. \end{aligned} \tag{3.13}$$

Remark: For $i = 3i_0 + j_0$, $j_0 \in \{0,1,2\}$, we have

$$\mathcal{K}_{i,j}^{(\omega)} \quad = \quad 0, \qquad j \in 3i_0 + \{0,1,2\}. \tag{3.14}$$

Now we split up the matrix $\left(\mathcal{K}_{i,j}^{(\omega)}\right)_{i,j}$ in the following way

$$\mathcal{K}_{i,j}^{(\omega,1)} \quad := \quad \begin{cases} \mathcal{K}_{i,j}^{(\omega)}, & |i-j| \leq 1 \text{ or } i-j = 3n-1, \\ 0, & \text{else}, \end{cases} \tag{3.15}$$

$$\mathcal{K}_{i,j}^{(\omega,2)} \quad = \quad \mathcal{K}_{i,j}^{(\omega)} - \mathcal{K}_{i,j}^{(\omega,1)}, \qquad \forall i,j, \tag{3.16}$$

$$\mathcal{K}_{\pi}^{(\omega,1)} \quad := \quad \left(\mathcal{K}_{i,j}^{(\omega,1)}\right)_{i,j},$$

$$\mathcal{K}_{\pi}^{(\omega,1)} \quad := \quad \left(\mathcal{K}_{i,j}^{(\omega,1)}\right)_{i,j}.$$

Then we define

$$\mathcal{B}_{\pi}^{(\omega)} \quad := \quad I_{\vec{L}_{\pi}^2} + \mathcal{K}_{\pi}^{(\omega,1)}, \tag{3.17}$$

$$\mathcal{B}^{(\omega,1)} \quad := \quad \Phi^{-1} \circ \mathcal{B}_{\pi}^{(\omega)} \circ \Phi \tag{3.18}$$

$$\mathcal{K}^{(\omega,1)} \quad := \quad \Phi^{-1} \circ \mathcal{K}_{\pi}^{(\omega,1)} \circ \Phi \tag{3.19}$$

$$\mathcal{K}^{(\omega,2)} \quad := \quad \Phi^{-1} \circ \mathcal{K}_{\pi}^{(\omega,2)} \circ \Phi \tag{3.20}$$

We observe (see (3.14) and (3.15)) that the matrix $\mathcal{B}_{\pi}^{(\omega)}$ has the following structure

$$\mathcal{B}_{\pi}^{(\omega)} \quad = \quad \begin{pmatrix} L_1^{(\omega)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & L_n^{(\omega)} \end{pmatrix}, \text{ where} \tag{3.21}$$

$$L_i^{(\omega)} \quad = \quad \begin{pmatrix} I & 0 & 0 \\ 0 & I & \mathcal{K}_{3i-1,3i}^{(\omega)} \\ 0 & \mathcal{K}_{3i,3i-1}^{(\omega)} & I \end{pmatrix}. \tag{3.22}$$

First we calculate the kernels of the two operators $\mathcal{K}_{3i-1,3i}^{(\omega)}$ and $\mathcal{K}_{3i,3i-1}^{(\omega)}$, where

$$\mathcal{K}_{3i-1,3i}^{(\omega)} : \left(C_0^{\infty}(s_i, s_i+\delta)\right)^2 \quad \rightarrow \quad \left(L^2(s_i-\delta, s_i)\right)^2$$

$$\mathcal{K}_{3i,3i-1}^{(\omega)} : \left(C_0^{\infty}(s_i-\delta, s_i)\right)^2 \quad \rightarrow \quad \left(L^2(s_i, s_i+\delta)\right)^2.$$

We will assume $s_i = 0$.

(i) $\mathcal{K}^{(\omega)}_{3i-1,3i}$, $s \in [-\delta, 0]$, $\tau \in [0, \delta]$.

$$\begin{aligned}
\gamma(s) &= \xi_{i-1} + (s - s_{i-1})\zeta_{i-1} \\
&= \xi_{i-1} + (0 - s_{i-1})\zeta_{i-1} + (s - 0)\zeta_{i-1} \\
&= \xi_i + s\zeta_{i-1} \\
\gamma(\tau) &= \xi_i + \tau\zeta_i.
\end{aligned}$$

This implies the following equations:

$$\begin{aligned}
\gamma(s) - \gamma(\tau) &= s\zeta_{i-1} - \tau\zeta_i \\
&= s\begin{pmatrix} \cos(\alpha_{i-1}) \\ \sin(\alpha_{i-1}) \end{pmatrix} - \tau\begin{pmatrix} \cos(\alpha_i) \\ \sin(\alpha_i) \end{pmatrix} \\
\|\gamma(s) - \gamma(\tau)\|^2 &= s^2\cos^2(\alpha_{i-1}) - 2s\tau\cos(\alpha_{i-1})\cos(\alpha_i) + \tau^2\cos^2(\alpha_i) \\
&\quad + s^2\sin^2(\alpha_{i-1}) - 2s\tau\sin(\alpha_{i-1})\sin(\alpha_i) + \tau^2\sin^2(\alpha_i) \\
&= s^2 - 2s\tau(\cos(\alpha_{i-1})\cos(\alpha_i) + \sin(\alpha_{i-1})\sin(\alpha_i)) + \tau^2 \\
&= s^2 - 2s\tau\cos(\alpha_i - \alpha_{i-1}) + \tau^2 \\
(\gamma(s) - \gamma(\tau)) \cdot n(\tau) &= s\zeta_{i-1} \cdot \eta_i \\
&= s(\cos(\alpha_{i-1})\sin(\alpha_i) - \cos(\alpha_i)\sin(\alpha_{i-1})) \\
&= s\sin(\alpha_i - \alpha_{i-1})
\end{aligned}$$

$$(\gamma(s) - \gamma(\tau))(\gamma(s) - \gamma(\tau))^T = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix}$$

with

$$\begin{aligned}
\gamma_1 &= s^2\cos^2(\alpha_{i-1}) - 2s\tau\cos(\alpha_{i-1})\cos(\alpha_i) + \tau^2\cos^2(\alpha_i) \\
\gamma_2 &= s^2\sin(\alpha_{i-1})\cos(\alpha_{i-1}) - s\tau(\sin(\alpha_{i-1})\cos(\alpha_i) + \sin(\alpha_i)\cos(\alpha_{i-1})) \\
&\quad + \tau^2\cos(\alpha_i)\sin(\alpha_i) \\
\gamma_3 &= s^2\sin^2(\alpha_{i-1}) - 2s\tau\sin(\alpha_{i-1})\sin(\alpha_i) + \tau^2\sin^2(\alpha_i)
\end{aligned}$$

Now we further assume $\alpha_{i-1} = 0$. Then the last formula implies

$$\begin{aligned}
&(\gamma(s) - \gamma(\tau))(\gamma(s) - \gamma(\tau))^T \\
= &\begin{pmatrix} s^2 - 2s\tau\cos(\alpha_i) + \tau^2\cos^2(\alpha_i) & \tau\sin(\alpha_i)(\tau\cos(\alpha_i) - s) \\ \tau\sin(\alpha_i)(\tau\cos(\alpha_i) - s) & \tau^2\sin^2(\alpha_i) \end{pmatrix}
\end{aligned}$$

The kernel of $\mathcal{K}^{(\omega)}_{3i-1,3i}$ is given by

$$\begin{aligned} k^{(\omega)}(s,\tau) &= -\frac{1}{\pi}\frac{s\sin(\alpha_i)}{s^2-2s\tau\cos(\alpha_i)+\tau^2}\Big((1-\omega)I_{2\times 2}+\frac{2\omega}{s^2-2s\tau\cos(\alpha_i)+\tau^2} \\ &\quad \begin{pmatrix} s^2-2s\tau\cos(\alpha_i)+\tau^2\cos^2(\alpha_i) & \tau\sin(\alpha_i)(\tau\cos(\alpha_i)-s) \\ \tau\sin(\alpha_i)(\tau\cos(\alpha_i)-s) & \tau^2\sin^2(\alpha_i) \end{pmatrix}\Big) \\ &=: \widetilde{k}^{(\omega)}_{i,1}(\frac{s}{\tau})\frac{1}{\tau}, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} \widetilde{k}^{(\omega)}_{i,1}(z) &:= -\frac{1}{\pi}\frac{z\sin(\alpha_i)}{z^2-2z\cos(\alpha_i)+1}\Big((1-\omega)I_{2\times 2}+\frac{2\omega}{z^2-2z\cos(\alpha_i)+1} \\ &\quad \begin{pmatrix} z^2-2z\cos(\alpha_i)+\cos^2(\alpha_i) & \sin(\alpha_i)(\cos(\alpha_i)-z) \\ \sin(\alpha_i)(\cos(\alpha_i)-z) & \sin^2(\alpha_i) \end{pmatrix}\Big) \end{aligned} \tag{3.24}$$

(ii) $\mathcal{K}^{(\omega)}_{3i,3i-1}$, $s\in[0,\tau]$, $\tau\in[-\delta,0]$.

$$\begin{aligned} \gamma(s) &= \xi_i + s\zeta_i \\ \gamma(\tau) &= \xi_i + \tau\zeta_{i-1}. \end{aligned}$$

This implies the following equations:

$$\begin{aligned} \gamma(s)-\gamma(\tau) &= s\begin{pmatrix}\cos(\alpha_i)\\ \sin(\alpha_i)\end{pmatrix} - \tau\begin{pmatrix}\cos(\alpha_{i-1})\\ \sin(\alpha_{i-1})\end{pmatrix} \\ \|\gamma(s)-\gamma(\tau)\|^2 &= s^2-2s\tau(\cos(\alpha_{i-1})\cos(\alpha_i)+\sin(\alpha_{i-1})\sin(\alpha_i))+\tau^2 \\ &= s^2-2s\tau\cos(\alpha_i-\alpha_{i-1})+\tau^2 \\ (\gamma(s)-\gamma(\tau))\cdot n(\tau) &= s\zeta_i\cdot\eta_{i-1} \\ &= s(\cos(\alpha_i)\sin(\alpha_{i-1})-\cos(\alpha_{i-1})\sin(\alpha_i)) \\ &= -s(\cos(\alpha_{i-1})\sin(\alpha_i)-\sin(\alpha_{i-1})\cos(\alpha_i)) \\ &= -s\sin(\alpha_i-\alpha_{i-1}) \end{aligned}$$

$$(\gamma(s)-\gamma(\tau))(\gamma(s)-\gamma(\tau))^T = \begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_2 & \gamma_3\end{pmatrix}$$

with

$$
\begin{aligned}
\gamma_1 &= s^2\cos^2(\alpha_i) - 2s\tau\cos(\alpha_{i-1})\cos(\alpha_i) + \tau^2\cos^2(\alpha_{i-1}) \\
\gamma_2 &= s^2\sin(\alpha_i)\cos(\alpha_i) - s\tau(\sin(\alpha_{i-1})\cos(\alpha_i) + \sin(\alpha_i)\cos(\alpha_{i-1})) \\
&\quad + \tau^2\cos(\alpha_{i-1})\sin(\alpha_{i-1}) \\
\gamma_3 &= s^2\sin^2(\alpha_i) - 2s\tau\sin(\alpha_{i-1})\sin(\alpha_i) + \tau^2\sin^2(\alpha_{i-1})
\end{aligned}
$$

Now we again assume $\alpha_{i-1} = 0$. Then the last formula implies

$$
\begin{aligned}
&(\gamma(s) - \gamma(\tau))(\gamma(s) - \gamma(\tau))^T \\
= &\begin{pmatrix} s^2\cos^2(\alpha_i) - 2s\tau\cos(\alpha_i) + \tau^2 & s\sin(\alpha_i)(s\cos(\alpha_i) - \tau) \\ s\sin(\alpha_i)(s\cos(\alpha_i) - \tau) & s^2\sin^2(\alpha_i) \end{pmatrix}
\end{aligned}
$$

The kernel of $\mathcal{K}^{(\omega)}_{3i,3i-1}$ is given by

$$
\begin{aligned}
k^{(\omega)}(s,\tau) &= \frac{1}{\pi}\frac{s\sin(\alpha_i)}{s^2 - 2s\tau\cos(\alpha_i) + \tau^2}\Big((1-\omega)I_{2\times 2} + \frac{2\omega}{s^2 - 2s\tau\cos(\alpha_i) + \tau^2} \\
&\qquad \begin{pmatrix} s^2\cos^2(\alpha_i) - 2s\tau\cos(\alpha_i) + \tau^2 & s\sin(\alpha_i)(s\cos(\alpha_i) - \tau) \\ s\sin(\alpha_i)(s\cos(\alpha_i) - \tau) & s^2\sin^2(\alpha_i) \end{pmatrix}\Big) \\
&=: \widetilde{k}^{(\omega)}_{i,2}(\frac{s}{\tau})\frac{1}{\tau},
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
\widetilde{k}^{(\omega)}_{i,1}(z) &:= \frac{1}{\pi}\frac{z\sin(\alpha_i)}{z^2 - 2z\cos(\alpha_i) + 1}\Big((1-\omega)I_{2\times 2} + \frac{2\omega}{z^2 - 2z\cos(\alpha_i) + 1} \\
&\qquad \begin{pmatrix} \cos(\alpha_i)^2 z^2 - 2z\cos(\alpha_i) + 1 & z\sin(\alpha_i)(z\cos(\alpha_i) - 1) \\ z\sin(\alpha_i)(z\cos(\alpha_i) - 1) & z^2\sin^2(\alpha_i) \end{pmatrix}\Big)
\end{aligned}
\tag{3.26}
$$

If we now introduce the operator

$$
\widetilde{\mathcal{B}}^{(\omega)}_i : (C_0^\infty([-\delta,0]))^2 \times (C_0^\infty([0,\delta]))^2 \to \big(L^2(-\delta,0)\big)^2 \times \big(L^2(0,\delta)\big)^2
$$

by

$$\widetilde{\mathcal{B}}_i^{(\omega)} \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \end{pmatrix} := \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix}, \text{ where}$$

$$\vec{v}_1(s) := \vec{u}_1(s) + \int_0^\delta \widetilde{k}_{i,1}^{(\omega)}(\frac{s}{\tau})\vec{u}_2(\tau)\frac{d\tau}{\tau},\ s \in [-\delta, 0],$$

$$\vec{v}_2(s) := \vec{u}_2(s) + \int_{-\delta}^0 \widetilde{k}_{i,2}^{(\omega)}(\frac{s}{\tau})\vec{u}_1(\tau)\frac{d\tau}{\tau},\ s \in [0, \delta],$$

and $\widetilde{\Phi}_i : (L^2(0,T))^2 \to (L^2(-\delta,0))^2 \times (L^2(0,\delta))^2$ by

$$\left(\widetilde{\Phi}_i \vec{u}\right)(x) := \begin{pmatrix} \vec{u}(s_i + \cdot) \\ \vec{u}(s_i + \cdot) \end{pmatrix},$$

then we get

$$(\widetilde{\Phi}_i \circ \mathcal{B})\vec{u} = (\widetilde{\mathcal{B}}_i^{(\omega)} \circ \widetilde{\Phi}_i)\vec{u} \tag{3.27}$$

for all functions $\vec{u}$ for which both sides are well defined (we will later see that this is the case for all $\vec{u} \in (L^2(0,T))^2$).
Finally we transform the equation on the interval $(0,1)$ by

$$\overline{\Phi} : \left(L^2(0,1)\right)^4 \to \left(L^2(-\delta,0)\right)^2 \times \left(L^2(0,\delta)\right)^2$$

with

$$\overline{\Phi}\vec{u} := \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix}, \tag{3.28}$$

$$\vec{v}_1(s) = \begin{pmatrix} u_1(-\frac{s}{\delta}) \\ u_2(-\frac{s}{\delta}) \end{pmatrix}$$

$$\vec{v}_2(s) = \begin{pmatrix} u_3(\frac{s}{\delta}) \\ u_4(\frac{s}{\delta}) \end{pmatrix}$$

where $\vec{u} = (u_1, u_2, u_3, u_4)^T \in (L^2(0,T))^4$. We get

$$\begin{pmatrix} \vec{w}_1 \\ \vec{w}_2 \end{pmatrix} = \widetilde{\mathcal{B}}_i^{(\omega)}(\overline{\Phi}\vec{u}), \text{ where}$$

$$\vec{w}_1(s) = \begin{pmatrix} u_1(-\frac{s}{\delta}) \\ u_2(-\frac{s}{\delta}) \end{pmatrix} + \int_0^\delta \widetilde{k}_{i,1}^{(\omega)}(\frac{s}{\tau}) \begin{pmatrix} u_3(\frac{\tau}{\delta}) \\ u_4(\frac{\tau}{\delta}) \end{pmatrix} \frac{d\tau}{\tau},\ s \in [-\delta, 0],$$

$$\vec{w}_2(s) = \begin{pmatrix} u_3(\frac{s}{\delta}) \\ u_4(\frac{s}{\delta}) \end{pmatrix} + \int_{-\delta}^0 \widetilde{k}_{i,2}^{(\omega)}(\frac{s}{\tau}) \begin{pmatrix} u_1(-\frac{\tau}{\delta}) \\ u_2(-\frac{\tau}{\delta}) \end{pmatrix} \frac{d\tau}{\tau},\ s \in [0, \delta].$$

Now we apply $\overline{\Phi}_i^{-1}$ to $(\vec{w}_1, \vec{w}_2)^T$ and get

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \int_0^\delta \widetilde{k}_{i,1}^{(\omega)}(-\frac{\delta t}{\tau}) \begin{pmatrix} u_3(\frac{\tau}{\delta}) \\ u_4(\frac{\tau}{\delta}) \end{pmatrix} \frac{d\tau}{\tau},\ t \in [0,1],$$

$$\begin{pmatrix} u_3(t) \\ u_4(t) \end{pmatrix} + \int_{-\delta}^0 \widetilde{k}_{i,2}^{(\omega)}(\frac{\delta t}{\tau}) \begin{pmatrix} u_1(-\frac{\tau}{\delta}) \\ u_2(-\frac{\tau}{\delta}) \end{pmatrix} \frac{d\tau}{\tau},\ t \in [0,1].$$

Substituting $\nu := \frac{\tau}{\delta}$ in the first and $\nu := -\frac{\tau}{\delta}$ in the second integral gives

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \int_0^1 \widetilde{k}_{i,1}^{(\omega)}(-\frac{t}{\nu}) \begin{pmatrix} u_3(\nu) \\ u_4(\nu) \end{pmatrix} \frac{d\nu}{\nu},\ t \in [0,1],$$

$$\begin{pmatrix} u_3(t) \\ u_4(t) \end{pmatrix} - \int_0^1 \widetilde{k}_{i,2}^{(\omega)}(-\frac{t}{\nu}) \begin{pmatrix} u_1(-\nu) \\ u_2(-\nu) \end{pmatrix} \frac{d\nu}{\nu},\ t \in [0,1].$$

So we define

$$\overline{\mathcal{K}}_i^{(\omega)} : (C_0^\infty(0,1))^4 \rightarrow \left(L^2(0,1)\right)^4$$

by

$$(\overline{\mathcal{K}}_i^{(\omega)} \vec{u})(s) = \begin{pmatrix} \int_0^1 k_{i,1}^{(\omega)}(s/\tau)(u_3(\tau), u_4(\tau))^T \frac{d\tau}{\tau} \\ \int_0^1 k_{i,2}^{(\omega)}(s/\tau)(u_1(\tau), u_2(\tau))^T \frac{d\tau}{\tau} \end{pmatrix}, \tag{3.29}$$

where

$$\begin{aligned} k_{i,1}^{(\omega)}(z) &:= \widetilde{k}_{i,1}^{(\omega)}(-z) \\ &= \frac{1}{\pi} \frac{z \sin(\alpha_i)}{z^2 + 2z\cos(\alpha_i) + 1} \Big((1-\omega) I_{2\times 2} + \frac{2\omega}{z^2 + 2z\cos(\alpha_i) + 1} \\ &\quad \begin{pmatrix} z^2 + 2z\cos(\alpha_i) + \cos^2(\alpha_i) & \sin(\alpha_i)(\cos(\alpha_i) + z) \\ \sin(\alpha_i)(\cos(\alpha_i) + z) & \sin^2(\alpha_i) \end{pmatrix} \Big) \end{aligned} \tag{3.30}$$

$$
\begin{aligned}
k_{i,2}^{(\omega)}(z) &:= -\widetilde{k}_{i,2}^{(\omega)}(-z) \\
&= \frac{1}{\pi}\frac{z\sin(\alpha_i)}{z^2+2z\cos(\alpha_i)+1}\Bigg((1-\omega)I_{2\times 2}+\frac{2\omega}{z^2+2z\cos(\alpha_i)+1} \\
&\quad \begin{pmatrix} z^2\cos^2(\alpha_i)+2z\cos(\alpha_i)+1 & z\sin(\alpha_i)(z\cos(\alpha_i)+1) \\ z\sin(\alpha_i)(z\cos(\alpha_i)+1) & z^2\sin^2(\alpha_i)\end{pmatrix}\Bigg)
\end{aligned} \tag{3.31}
$$

For

$$
\overline{\mathcal{B}}_i^{(\omega)} := I + \overline{\mathcal{K}}_i^{(\omega)} \tag{3.32}
$$

we get

$$
(\overline{\Phi}_i \circ \mathcal{B}^{(\omega,1)})\vec{u} = (\overline{\mathcal{B}}_i^{(\omega)} \circ \overline{\Phi}_i)\vec{u}, \tag{3.33}
$$

with

$$
\overline{\Phi}_i := \overline{\Phi}^{-1} \circ \widetilde{\Phi}_i. \tag{3.34}
$$

In the next lemma we will prove some mapping properties of $\mathcal{K}^{(\omega,2)}$ (see (3.20)). Then we will study the properties of $\mathcal{B}^{(\omega,1)}$ with the help of $\overline{\mathcal{B}}_i^{(\omega)}$, $i = 1(1)n$.

Lemma 3.1. *For $\omega \in [0,1]$ we have*

a. $\mathcal{K}^{(\omega,2)} : \left(L^2(0,T)\right)^2 \to \left(L^2(0,T)\right)^2$ *is compact.*

b. $\vec{u} \in \left(L^2(0,T)\right)^2 \Rightarrow (\mathcal{K}^{(\omega,2)}\vec{u})|_{[s_i,s_{i+1}]} \in \left(C^\infty([s_i,s_{i+1}])\right)^2$, $\forall i$. *The mapping*

$$
\vec{u} \longrightarrow \mathcal{K}^{(\omega,2)}\vec{u}|_{[s_i,s_{i+1}]} \in \left(C^l([s_i,s_{i+1}])\right)^2
$$

is continuous for $l \in \mathbb{N}_0$.

c. $\vec{u} \in \left(L^2(0,T)\right)^2 \Rightarrow \mathcal{K}^{(\omega,2)}\vec{u} \in (C[0,T])^2$.

Proof: We first use the relation (3.20) and look at the components of the operator matrix $\mathcal{K}_\pi^{(\omega,2)}$. For $|i-j| \leq 1$ or $i-j = 3n-1$ we have $\mathcal{K}_{i,j}^{(\omega,2)} = 0$ and for $i,j \in \{1,\ldots,3n\}$, $|i-j| \geq 2$ we have $k(s,\tau) \in C^\infty(J_i \times J_j)^4$. This implies

a) $\mathcal{K}_{i,j}^{(\omega,2)}$ compact.

b) $(\mathcal{K}_{i,j}^{(\omega,2)}\vec{u})(s) \in (C^\infty(J_i))^2$, $\vec{u} \in (L^2(J_j))^2$.

c) $\mathcal{K}_{i,j}^{(\omega,2)} : \left(L^2(J_j)\right)^2 \longrightarrow \left(C^l(J_i)\right)^2$ is continuous, $l \in \mathbb{N}_0$.

Property a) shows that $\mathcal{K}_\pi^{(\omega,2)}$ is compact and (3.20) proves a. Property b) shows that $\mathcal{K}^{(\omega,2)}\vec{u} \in (C^\infty(J_i))^2$. This implies $\mathcal{K}^{(\omega,2)}\vec{u}|_{[s_i+\delta,s_{i+1}-\delta]}$ is a C^∞–function. $\delta > 0$ is fixed but arbitrary, so the first part of b. follows. Property c) shows the second part of b.
To prove c. we only have to show that

$$\left(\mathcal{K}^{(\omega,2)}\vec{u}\right)|_{J_{3i-1}\cup J_{3i}} \text{ is continuous, } \forall i \in \{1,\ldots,n\}.$$

Choose $i \in \{1,\ldots,n\}$. For $s \in J_{3i-1} = [s_i - \delta, s_i]$ we have

$$\begin{aligned}\left(\mathcal{K}^{(\omega,2)}\vec{u}\right)(s) &= \int_0^{s_{i-1}} k(s,\tau)\vec{u}(\tau)\,d\tau + \int_{s_i+\delta}^{T} k(s,\tau)\vec{u}(\tau)\,d\tau \\ &= \int_{(0,s_{i-1})\cup(s_{i+1},T)} k(s,\tau)\vec{u}(\tau)\,d\tau \\ &\quad + \int_{s_i+\delta}^{s_{i+1}} k(s,\tau)\vec{u}(\tau)\,d\tau.\end{aligned}$$

For $s \in J_{3i} = [s_i, s_i+\delta]$ we have

$$\begin{aligned}\left(\mathcal{K}^{(\omega,2)}\vec{u}\right)(s) &= \int_{(0,s_{i-1})\cup(s_{i+1},T)} k(s,\tau)\vec{u}(\tau)\,d\tau \\ &\quad + \int_{s_{i-1}}^{s_i-\delta} k(s,\tau)\vec{u}(\tau)\,d\tau.\end{aligned}$$

Now we note that $k \in C([s_i-\delta, s_i+\delta] \times ([0,s_{i-1}] \cup [s_{i+1},T]))^4$, and we also have

$$\begin{aligned} k(s,\tau) &\to 0 \quad \text{as } s \nearrow s_i, \text{ uniformly for } \tau \in [s_i+\delta, s_{i+1}], \\ k(s,\tau) &\to 0 \quad \text{as } s \searrow s_i, \text{ uniformly for } \tau \in [s_{i-1}, s_i-\delta].\end{aligned}$$

We define $\widetilde{k}$ on $[s_i-\delta, s_i+\delta] \times \Big([s_{i-1}, s_i-\delta] \cup [s_i+\delta, s_{i+1}]\Big)$ by

$$\widetilde{k}(s,\tau) := \begin{cases} k(s,\tau), & (s,\tau) \in ([s_i-\delta, s_i] \times [s_i+\delta, s_{i+1}]) \cup \\ & \qquad ([s_i, s_i+\delta] \times [s_{i-1}, s_i-\delta]) \\ 0, & else. \end{cases}$$

This implies $\widetilde{k} \in C([s_i-\delta, s_i+\delta] \times ([s_{i-1}, s_i-\delta] \cup [s_i+\delta, s_{i+1}]))^4$. Because of this

$$\int_{[s_{i-1},s_i-\delta]\cup[s_i+\delta,s_{i+1}]} \widetilde{k}(s,\tau)\vec{u}(\tau)\,d\tau$$

is continuous for $s \in [s_i-\delta, s_i+\delta]$. But for $s \in [s_i\delta, s_i+\delta]$ we have

$$\left(\mathcal{K}^{(\omega,2)}\vec{u}\right)(s) \;=\; \int_{[0,s_{i-1}]\cup[s_{i+1},T]} k(s,\tau)\vec{u}(\tau)\,d\tau + \int_{[s_{i-1},s_i-\delta]\cup[s_i+\delta,s_{i+1}]} \widetilde{k}(s,\tau)\vec{u}(\tau)\,d\tau$$

which proves the continuity. □

We now define four functions, which build up the functions $k_{i,1}^{(\omega)}$ and $k_{i,2}^{(\omega)}$.

$$\left.\begin{aligned} l_{i,1}(z) &:= \frac{z}{1+2\cos(\alpha_i)z+z^2} \\ l_{i,2}(z) &:= \frac{z}{(1+2\cos(\alpha_i)z+z^2)^2} \\ l_{i,3}(z) &:= \frac{z^2}{(1+2\cos(\alpha_i)z+z^2)^2} \\ l_{i,4}(z) &:= \frac{z^3}{(1+2\cos(\alpha_i)z+z^2)^2} \end{aligned}\right\}, \quad \alpha_i \in (-\pi,\pi) \tag{3.35}$$

The following properties are clear.

$l_{i,j} \in C^\infty([0,\infty)),\ l_{i,j}(0) = 0,\ l_{i,j}(x) > 0$, if $x > 0$ and

$$\begin{aligned} \int_0^\infty x^q l_{i,1}(x)\,dx &< \infty, \quad -2 < q < 0, \\ \int_0^\infty x^q l_{i,2}(x)\,dx &< \infty, \quad -2 < q < 2, \\ \int_0^\infty x^q l_{i,3}(x)\,dx &< \infty, \quad -3 < q < 1, \\ \int_0^\infty x^q l_{i,4}(x)\,dx &< \infty, \quad -4 < q < 0, \end{aligned}$$

This implies

$$\int_0^\infty x^q l_{i,j}(x)\,dx < \infty, \quad q \in (-2,0) \quad \forall\, i,j. \tag{3.36}$$

From now on we will omit the index i for $l_{i,j}$ and $k_{i,j}^{(\omega)}$. We get

$$\left.\begin{aligned} k_1^{(\omega)}(z) &= (1-\omega)k_D(z)I_{2\times 2} + \omega \begin{pmatrix} k_1^{(1)}(z) & k_2^{(1)}(z) \\ k_2^{(1)}(z) & k_3^{(1)}(z) \end{pmatrix} \\ k_2^{(\omega)}(z) &= (1-\omega)k_D(z)I_{2\times 2} + \omega \begin{pmatrix} k_1^{(2)}(z) & k_2^{(2)}(z) \\ k_2^{(2)}(z) & k_3^{(2)}(z) \end{pmatrix} \end{aligned}\right\} \tag{3.37}$$

with

$$
\begin{aligned}
k_D(z) &= \frac{1}{\pi}\frac{z\sin(\alpha_i)}{z^2+2\cos(\alpha_i)z+1} \\
&= \frac{\sin(\alpha_i)}{\pi} l_1(z)
\end{aligned} \tag{3.38}
$$

$$
\left.
\begin{aligned}
k_3^{(1)}(z) &= \frac{2}{\pi}\frac{z\sin(\alpha_i)^3}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= \frac{2}{\pi}\sin(\alpha_i)^3 l_2(z) \\
k_1^{(1)}(z) &= \frac{2}{\pi}z\sin(\alpha_i)\frac{z^2+2\cos(\alpha_i)z+\cos(\alpha_i)^2}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= -\frac{2}{\pi}\frac{\sin(\alpha_i)^3 z}{(z^2+2\cos(\alpha_i)z+1)^2}+\frac{2}{\pi}\frac{z\sin(\alpha_i)}{z^2+2\cos(\alpha_i)z+1} \\
&= 2k_D(z)-k_3^{(1)}(z) \\
k_2^{(1)}(z) &= \frac{2}{\pi}\frac{z\sin(\alpha_i)^2(z+\cos(\alpha_i))}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= \frac{2}{\pi}\sin(\alpha_i)^2 l_3(z)+\frac{2}{\pi}\cos(\alpha_i)\sin(\alpha_i)^2 l_2(z)
\end{aligned}
\right\} \tag{3.39}
$$

$$
\left.
\begin{aligned}
k_3^{(2)}(z) &= \frac{2}{\pi}\frac{z^3\sin(\alpha_i)^3}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= \frac{2}{\pi}\sin(\alpha_i)^3 l_4(z) \\
k_1^{(2)}(z) &= \frac{2}{\pi}z\sin(\alpha_i)\frac{\cos(\alpha_i)^2 z^2+2\cos(\alpha_i)z+1}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= -\frac{2}{\pi}\frac{\sin(\alpha_i)^3 z^3}{(z^2+2\cos(\alpha_i)z+1)^2}+\frac{2}{\pi}\frac{z\sin(\alpha_i)}{z^2+2\cos(\alpha_i)z+1} \\
&= 2k_D(z)-k_3^{(2)}(z) \\
k_2^{(2)}(z) &= \frac{2}{\pi}\frac{z^2\sin(\alpha_i)^2(z\cos(\alpha_i)+1)}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= \frac{2}{\pi}\sin(\alpha_i)^2 l_3(z)+\frac{2}{\pi}\cos(\alpha_i)\sin(\alpha_i)^2 l_4(z)
\end{aligned}
\right\} \tag{3.40}
$$

Remark: The mapping properties of Mellin convolutions with kernel $l_j(z)$ also hold for Mellin convolutions with kernel $k_1^{(\omega)}$ or $k_2^{(\omega)}$. The kernel $k_D(z)$ is the kernel of the double layer potential.

Now we recall some definitions from [5].
Let $\rho \geq 0,\ l \in \mathbb{N},\ p \in [1,\infty]$, be given. Then we define

$$
X_\rho^{p,l}(0,1) \ := \ \{u \in \mathcal{D}'(0,1) \,|\, x^{j-\rho}D^j u \in L^p(0,1),\ j = 0(1)l\,\} \tag{3.41}
$$

with the norm

$$\|u\|_{p,l,\rho,(0,1)} \ := \ \sum_{0\le j\le l} \|x^{j-\rho} D^j u\|_{L^p(0,1)}. \tag{3.42}$$

In [5] also appear the following two conditions for functions g on $[0,\infty)$.

$$\begin{array}{llll} (H1^p) & \displaystyle\int_0^\infty x^{\frac{1}{p}-1}\,|g(x)|\,dx & < & \infty \\ (H1^{p,l}_\rho) & \displaystyle\int_0^\infty x^{\frac{1}{p}-1-\rho}\,\left|x^j D^j g(x)\right|\,dx & < & \infty,\ j=0(1)l. \end{array}$$

Formula (3.36) shows that all l_j, $j \in \{1,\dots,4\}$ fulfill the conditions $(H1^p)$, $1 < p \le \infty$, $(H1^{p,1}_1)$, $1 \le p < \infty$, and $(H1^{2,1}_\rho)$, $\rho \in [0,3/2)$. Formulas (3.37)–(3.40) now show that $\overline{\mathcal{B}}_i^{(\omega)}$ (see (3.32)) maps $(L^p(0,1))^4$ continuously into $(L^p(0,1))^4$ for $1 < p \le \infty$ and $\left(X_1^{p,0}(0,1)\right)^4$ continuously into $\left(X_1^{p,0}(0,1)\right)^4$ for $1 \le p < \infty$ (see [5, p. 275 and the proof of theorem 1.10]). We have shown the following lemma.

Lemma 3.2. *For $\omega \in \mathbb{R}$, $i \in \{1,\dots,n\}$, we have*

$$\begin{array}{rcll} (L^p(0,1))^4 & \stackrel{\overline{\mathcal{B}}_i^{(\omega)}}{\longrightarrow} & (L^p(0,1))^4 & \textit{is continuous for}\ 1 < p \le \infty, \\ \left(X_1^{p,0}(0,1)\right)^4 & \stackrel{\overline{\mathcal{B}}_i^{(\omega)}}{\longrightarrow} & \left(X_1^{p,0}(0,1)\right)^4 & \textit{is continuous for}\ 1 \le p < \infty,\ \textit{and} \\ \left(X_\rho^{2,0}(0,1)\right)^4 & \stackrel{\overline{\mathcal{B}}_i^{(\omega)}}{\longrightarrow} & \left(X_\rho^{2,0}(0,1)\right)^4 & \textit{is continuous for}\ 0 \le \rho < 3/2. \end{array}$$

To calculate the Mellin symbol of the operator $\overline{\mathcal{B}}_i^{(\omega)}$ we first collect the Mellin transformations $\widehat{l_i}(s)$ of the $l_i(z)$ (see [13]).

$$\left.\begin{array}{rcl} \widehat{l}_1(s) & = & \dfrac{\pi}{\sin(\alpha_i)}\dfrac{\sin(\alpha_i s)}{\sin(\pi s)} \\ \widehat{l}_2(s) & = & -\dfrac{\pi}{2\sin(\alpha_i)^3}\dfrac{1}{\sin(\pi s)}\Big(s\cos(\alpha_i)\sin(\alpha_i)\cos(\alpha_i s)+ \\ & & \quad s\sin(\alpha_i)^2\sin(\alpha_i s)-\sin(\alpha_i s)\Big) \\ \widehat{l}_3(s) & = & -\dfrac{\pi}{2\sin(\alpha_i)^3}\dfrac{1}{\sin(\pi s)}\Big(-s\sin(\alpha_i)\cos(\alpha_i s)+\cos(\alpha_i)\sin(\alpha_i s)\Big) \\ \widehat{l}_4(s) & = & -\dfrac{\pi}{2\sin(\alpha_i)^3}\dfrac{1}{\sin(\pi s)}\Big(s\cos(\alpha_i)\sin(\alpha_i)\cos(\alpha_i s)- \\ & & \quad s\sin(\alpha_i)^2\sin(\alpha_i s)-\sin(\alpha_i s)\Big) \end{array}\right\} \tag{3.43}$$

In the next lemma we calculate the Mellin symbol matrix of the operator $\overline{\mathcal{B}}_i^{(\omega)}$.

Lemma 3.3. *The Mellin symbol matrix $\widehat{\mathcal{B}}_i^{(\omega)}(s)$ of $\overline{\mathcal{B}}_i^{(\omega)}$, $\omega \in \mathbb{R}$, is given by*

$$\widehat{\mathcal{B}}_i^{(\omega)}(s) = \begin{pmatrix} I_{2\times 2} & g_i(s)I_{2\times 2} + \omega h_i(s)\mathcal{S}_{i,1}(s) \\ g_i(s)I_{2\times 2} + \omega h_i(s)\mathcal{S}_{i,2}(s) & I_{2\times 2} \end{pmatrix}. \tag{3.44}$$

$$g_i(s) = \frac{\sin(\alpha_i s)}{\sin(\pi s)}, \quad h_i(s) = \sin(\alpha_i)\frac{s}{\sin(\pi s)} \tag{3.45}$$

$$\left.\begin{aligned} \mathcal{S}_{i,1} &= \begin{pmatrix} \cos(\alpha_i(s-1)) & -\sin(\alpha_i(s-1)) \\ -\sin(\alpha_i(s-1)) & -\cos(\alpha_i(s-1)) \end{pmatrix} \\ \mathcal{S}_{i,2} &= \begin{pmatrix} \cos(\alpha_i(s+1)) & \sin(\alpha_i(s+1)) \\ \sin(\alpha_i(s+1)) & -\cos(\alpha_i(s+1)) \end{pmatrix} \end{aligned}\right\} \tag{3.46}$$

Proof: We substitute the formulas (3.43) into the formula (3.37) and use (3.38)–(3.40).

$$\begin{aligned}
\widehat{k}_D(s) &\overset{(3.38)}{=} \frac{\sin(\alpha_i)}{\pi}\widehat{l}_1(s) \\
&\overset{(3.43)}{=} \frac{\sin(\alpha_i s)}{\sin(\pi s)} \\
\widehat{k}_3^{(1)}(s) &\overset{(3.39)}{=} \frac{2}{\pi}\sin(\alpha_i)^3\widehat{l}_2(s) \\
&\overset{(3.43)}{=} -\frac{1}{\sin(\pi s)}\Big(s\cos(\alpha_i)\sin(\alpha_i)\cos(\alpha_i s) + s\sin(\alpha_i)^2\sin(\alpha_i s) - \sin(\alpha_i s)\Big) \\
\widehat{k}_1^{(1)}(s) &\overset{(3.39)}{=} 2\widehat{k}_D(s) - \widehat{k}_3^{(1)}(s) \\
&= \frac{1}{\sin(\pi s)}\Big(s\cos(\alpha_i)\sin(\alpha_i)\cos(\alpha_i s) + s\sin(\alpha_i)^2\sin(\alpha_i s) + \sin(\alpha_i s)\Big) \\
\widehat{k}_2^{(1)}(s) &\overset{(3.39)}{=} \frac{2}{\pi}\sin(\alpha_i)^2\widehat{l}_3(s) + \frac{2}{\pi}\sin(\alpha_i)^2\cos(\alpha_i)\widehat{l}_2(s) \\
&= -\frac{1}{\sin(\alpha_i)\sin(\pi s)}\Big(-s\sin(\alpha_i)\cos(\alpha_i s) \\
&\qquad + \cos(\alpha_i)\sin(\alpha_i s) + s\cos(\alpha_i)^2\sin(\alpha_i)\cos(\alpha_i s) \\
&\qquad + s\sin(\alpha_i)^2\cos(\alpha_i)\sin(\alpha_i s) - \cos(\alpha_i)\sin(\alpha_i s)\Big) \\
&= -\frac{1}{\sin(\alpha_i)\sin(\pi s)}\Big(-s\sin(\alpha_i)\cos(\alpha_i s)(1-\cos(\alpha_i)^2) \\
&\qquad + s\sin(\alpha_i)^2\cos(\alpha_i)\sin(\alpha_i s)\Big) \\
&= -\sin(\alpha_i)\frac{s}{\sin(\pi s)}\Big(-\sin(\alpha_i)\cos(\alpha_i s) + \cos(\alpha_i)\sin(\alpha_i s)\Big) \\
&= -\sin(\alpha_i)\frac{s}{\sin(\pi s)}\sin(\alpha_i(s-1))
\end{aligned} \tag{3.47}$$

By formula (3.37) we have to calculate

$$\begin{aligned}
-\widehat{k}_D(s)+\widehat{k}_1^{(1)}(s) &= \sin(\alpha_i)\frac{s}{\sin(\pi s)}\Big(\cos(\alpha_i)\cos(\alpha_i s)+\sin(\alpha_i)\sin(\alpha_i s)\Big)\\
&= \sin(\alpha_i)\frac{s}{\sin(\pi s)}\cos(\alpha_i(s-1)) \qquad (3.48)\\
-\widehat{k}_D(s)+\widehat{k}_3^{(1)}(s) &= -\sin(\alpha_i)\frac{s}{\sin(\pi s)}\Big(\cos(\alpha_i)\cos(\alpha_i s)+\sin(\alpha_i)\sin(\alpha_i s)\Big)\\
&= -\sin(\alpha_i)\frac{s}{\sin(\pi s)}\cos(\alpha_i(s-1)) \qquad (3.49)
\end{aligned}$$

We repeat the whole procedure to calculate $\widehat{k}_2^{(\omega)}(s)$.

$$\begin{aligned}
\widehat{k}_3^{(2)}(s) &= \frac{2}{\pi}\sin(\alpha_i)^3\widehat{l}_4(s)\\
&= -\frac{1}{\sin(\pi s)}\Big(s\cos(\alpha_i)\sin(\alpha_i)\cos(\alpha_i s)-s^2\sin(\alpha_i)^2\sin(\alpha_i s)-\sin(\alpha_i s)\Big)\\
\widehat{k}_1^{(2)}(s) &= \frac{1}{\sin(\pi s)}\Big(s\cos(\alpha_i)\sin(\alpha_i)\cos(\alpha_i s)-s\sin(\alpha_i)^2\sin(\alpha_i s)+\sin(\alpha_i s)\Big)\\
\widehat{k}_2^{(2)}(s) &= \frac{2}{\pi}\sin(\alpha_i)^2\widehat{l}_3(s)+\frac{2}{\pi}\sin(\alpha_i)^2\cos(\alpha_i)\widehat{l}_4(s)\\
&= -\frac{1}{\sin(\alpha_i)\sin(\pi s)}\Big(-s\sin(\alpha_i)\cos(\alpha_i s)\\
&\qquad +\cos(\alpha_i)\sin(\alpha_i s)+s\cos(\alpha_i)^2\sin(\alpha_i)\cos(\alpha_i s)\\
&\qquad -s\sin(\alpha_i)^2\cos(\alpha_i)\sin(\alpha_i s)-\cos(\alpha_i)\sin(\alpha_i s)\Big)\\
&= -\frac{1}{\sin(\alpha_i)\sin(\pi s)}\Big(-s\sin(\alpha_i)\cos(\alpha_i s)(1-\cos(\alpha_i)^2)\\
&\qquad -s\sin(\alpha_i)^2\cos(\alpha_i)\sin(\alpha_i s)\Big)\\
&= \sin(\alpha_i)\frac{s}{\sin(\pi s)}\Big(\sin(\alpha_i)\cos(\alpha_i s)+\cos(\alpha_i)\sin(\alpha_i s)\Big)\\
&= \sin(\alpha_i)\frac{s}{\sin(\pi s)}\sin(\alpha_i(s+1)) \qquad (3.50)
\end{aligned}$$

We again have to calculate the following two terms

$$\begin{aligned}
-\widehat{k}_D(s)+\widehat{k}_1^{(2)}(s) &= \sin(\alpha_i)\frac{s}{\sin(\pi s)}\Big(\cos(\alpha_i)\cos(\alpha_i s)-\sin(\alpha_i)\sin(\alpha_i s)\Big)\\
&= \sin(\alpha_i)\frac{s}{\sin(\pi s)}\cos(\alpha_i(s+1)) \qquad (3.51)\\
-\widehat{k}_D(s)+\widehat{k}_3^{(2)}(s) &= -\sin(\alpha_i)\frac{s}{\sin(\pi s)}\Big(\cos(\alpha_i)\cos(\alpha_i s)-\sin(\alpha_i)\sin(\alpha_i s)\Big)\\
&= -\sin(\alpha_i)\frac{s}{\sin(\pi s)}\cos(\alpha_i(s+1)) \qquad (3.52)
\end{aligned}$$

Formula (3.32) and (3.37) together with (3.47)–(3.49), (3.50)–(3.52) and the formula for $\widehat{k}_D(s)$ show (3.44) with the notations (3.45) and (3.46). □

Remark (on reflection matrices): The matrices $\mathcal{S}_{i,1}$ and $\mathcal{S}_{i,2}$, which appear in Lemma 3.2 can be viewed as reflection matrices. A reflection matrix S_β in $\mathbb{R}^2$, which describes the reflection at the straight line orthogonal to $(\cos(\beta), \sin(\beta))^T$, has the following form

$$\begin{aligned} S_\beta &= I_{2\times 2} - 2\begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix} (\cos(\beta), \sin(\beta)) \\ &= \begin{pmatrix} 1-2\cos(\beta)^2 & -2\sin(\beta)\cos(\beta) \\ -2\sin(\beta)\cos(\beta) & 1-2\sin(\beta)^2 \end{pmatrix} \\ &= \begin{pmatrix} -\cos(2\beta) & -\sin(2\beta) \\ -\sin(2\beta) & \cos(2\beta) \end{pmatrix}, \end{aligned}$$

which shows $S_\beta = S_\beta^T = S_\beta^{-1}$. Define

$$\begin{aligned} \beta_1(s) &= \frac{\pi}{2} - \frac{(s-1)}{2}\alpha_i \text{ and} \\ \beta_2(s) &= \frac{\pi}{2} + \frac{(s+1)}{2}\alpha_i. \end{aligned}$$

Then one obtains

$$\mathcal{S}_{i,1}(s) = S_{\beta_1(s)} \quad \text{and} \quad \mathcal{S}_{i,2}(s) = S_{\beta_2(s)}.$$

This means that $\mathcal{S}_{i,1}(s)$ and $\mathcal{S}_{i,2}(s)$ are reflection matrices for real s . There is a further reflection matrix $\widetilde{S}_i$ independent of s by which the matrices $\mathcal{S}_{i,1}(s)$ and $\mathcal{S}_{i,2}(s)$ are conjugated:

$$\widetilde{S}_i := S_{\frac{\beta_1(s)+\beta_2(s)}{2}} = S_{\frac{\pi+\alpha_i}{2}} = \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix}. \tag{3.53}$$

We obtain

$$\widetilde{S}_i\, \mathcal{S}_{i,1}(s)\, \widetilde{S}_i = \widetilde{S}_{i,2}(s).$$

Now (3.44) can be written in the following way

$$\widehat{\mathcal{B}}_i^{(\omega)}(s) = \begin{pmatrix} I_{2\times 2} & g_i(s)I_{2\times 2} + \omega h_i(s)\mathcal{S}_{i,1}(s) \\ \widetilde{S}_i(g_i(s)I_{2\times 2} + \omega h_i(s)\mathcal{S}_{i,1}(s))\widetilde{S}_i & I_{2\times 2} \end{pmatrix} \tag{3.54}$$

As a next step we will prove the Fredholm property of $\overline{\mathcal{B}}_i^{(\omega)}$. We will closely follow [12] and we first recall Lemma 6.2 from [12].

Lemma 3.4. *We consider the equation*

$$(*) \qquad \frac{\sin(\gamma z)}{\gamma z} - \omega\frac{\sin(\gamma)}{\gamma} \;=\; 0, \qquad \gamma \in (0, 2\pi).$$

a. *Let $\omega = 1$. For $0 < \gamma \le \gamma_{crit}$ equation $(*)$ has no solution in $\Gamma_{0,1} \setminus \{0\}$, $\Gamma_{0,1} := \{z \in \mathbb{C} \,|\, 0 \le Re(z) < 1\}$, for $\gamma_{crit} < \gamma < 2\pi$ there is exactly one solution $z_0(1,\gamma) \in \Gamma_{0,1} \setminus \{0\}$. This solution is real and decreases monotonically from 1 to 1/2 if γ varies between γ_{crit} and 2π.*

b. *Let $-1 \le \omega < 1$. For $0 < \gamma \le \pi$ equation $(*)$ has no solution in $\Gamma_{0,1}$, for $\pi < \gamma < 2\pi$ there is exactly one solution $z_0(\omega,\gamma) \in \Gamma_{0,1}$, which decreases monotonically from 1 to 1/2 if γ runs from π to 2π.*

Remark: We define

$$\begin{aligned} \overline{z}(\omega) &:= \overline{z}(\omega, \alpha_1, \ldots, \alpha_N) \\ &:= \min_{i=1}^{n}\{z_0(\omega, \pi + |\alpha_i|), z_0(-\omega, \pi + |\alpha_i|)\} \\ &= \min_{i=1}^{n}\{z_0(-\omega, \pi + |\alpha_i|)\}, \qquad \omega \in [0,1], \end{aligned} \tag{3.55}$$

because $z_0(\omega,\alpha)$ is monotonically decreasing as a function of ω. The Lemma of Lewis shows

$$\frac{1}{2} < \quad \overline{z}(1) \;\le\; \overline{z}(\omega) \quad < \; 1, \tag{3.56}$$

and

$$\frac{\sin((\pi \pm \alpha_i)z)}{(\pi \pm \alpha_i)z} \pm \frac{\sin(\pi \pm \alpha_i)}{\pi \pm \alpha_i} \;\ne\; 0, \; i \in \{1, \ldots, n\}, \tag{3.57}$$

$z \in \{z \in \mathbb{C} \setminus \{0\} \,|\, 0 \le Re(z) < \overline{z}(\omega)\}$.

Lemma 3.5. *Let $\omega \in [0,1]$.*

a. *$\overline{\mathcal{B}}_i^{(\omega)} \,:\, (L^p(0,1))^4 \longrightarrow (L^p(0,1))^4$ is a Fredholm operator for all $p \in (1/\overline{z}(\omega), \infty]$, $i \in \{1, \ldots, n\}$.*

b. $\overline{\mathcal{B}}_i^{(\omega)} : \left(X_1^{p,0}(0,1)\right)^4 \longrightarrow \left(X_1^{p,0}(0,1)\right)^4$ *is a Fredholm operator for all* $p \in [1, 1/(1 - \overline{z}(\omega))\,)$,
$i \in \{1, \ldots, n\}$.

c. $\overline{\mathcal{B}}_i^{(\omega)} : \left(X_\rho^{2,0}(0,1)\right)^4 \longrightarrow \left(X_\rho^{2,0}(0,1)\right)^4$ *is a Fredholm operator for all* $\rho \in [0, 1/2 + \overline{z}(\omega))$,
$i \in \{1, \ldots, n\}$.

Proof: By the transformation $x \to e^{-x/p}$ we have the correspondence between the operator $\overline{\mathcal{B}}_i^{(\omega)}$ on $(0,1)$ and a Wiener–Hopf operator $W_i^{(\omega)}$ on $[0, \infty)$.
The Wiener–Hopf operator is a Fredholm operator if the determinant of its symbol $\widehat{W}_i^{(\omega)}(s)$ is different from zero on the real line [9, Theorem VIII,6.1]. If we consider $\overline{\mathcal{B}}_i^{(\omega)}$ as an operator on $(L^p(0,1))^4$ the corresponding Wiener–Hopf operator has the symbol $\widehat{\mathcal{B}}_i^{(\omega)}(s)$, $Re(s) = 1/p$ $(*_1)$. If we consider $\widehat{\mathcal{B}}_i^{(\omega)}(s)$ as an operator on $\left(X_\rho^{p,0}(0,1)\right)^4$ the corresponding Wiener–Hopf operator has the symbol $\widehat{\mathcal{B}}_i^{(\omega)}(s)$, $Re(s) = 1/p - \rho$ $(*_2)$, see [5].
This implies that we have to study the zeros of the function

$$\det(\widehat{\mathcal{B}}_i^{(\omega)}(s)), \quad \omega \in [0,1].$$

By formula (3.54) we have

$$\widehat{\mathcal{B}}_i^{(\omega)}(s) = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \widetilde{S}_i \end{pmatrix} \widetilde{\mathcal{B}}_i^{(\omega)}(s) \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \widetilde{S}_i \end{pmatrix}, \text{ where}$$

$$\widetilde{\mathcal{B}}_i^{(\omega)}(s) := \begin{pmatrix} I_{2\times 2} & \widehat{C}_i^{(\omega)}(s) \\ \widehat{C}_i^{(\omega)}(s) & I_{2\times 2} \end{pmatrix}, \text{ with}$$

$$\widehat{C}_i^{(\omega)}(s) = (g_i(s) I_{2\times 2} + \omega h_i(s) \mathcal{S}_{i,1}(s)) \widetilde{S}_i. \tag{3.58}$$

Because of $\det(\widehat{S}_i) = -1$ we get

$$\det(\widehat{\mathcal{B}}_i^{(\omega)}(s)) = \det(\widetilde{\mathcal{B}}_i^{(\omega)}(s)).$$

Now we follow the proofs of Lewis in [12].
First we have

$$\begin{aligned} \det(\widetilde{\mathcal{B}}_i^{(\omega)}(s)) &= \det \begin{pmatrix} I_{2\times 2} & \widehat{C}_i^{(\omega)}(s) \\ \widehat{C}_i^{(\omega)}(s) & I_{2\times 2} \end{pmatrix} \\ &= \det(I_{2\times 2} - \widehat{C}_i^{(\omega)}(s)) \det(I_{2\times 2} + \widehat{C}_i^{(\omega)}(s)). \end{aligned}$$

Furthermore we have

$$\begin{aligned}
I_{2\times 2} - \widehat{C}_i^{(\omega)}(s) &= \frac{1}{\sin(\pi s)}\Big[\begin{pmatrix} \sin(\pi s) & 0 \\ 0 & \sin(\pi s) \end{pmatrix} - \sin(\alpha_i s)\widetilde{S}_i \\
&\quad - \omega s \sin(\alpha_i)\begin{pmatrix} \cos(\alpha_i(s-1)) & -\sin(\alpha_i(s-1)) \\ -\sin(\alpha_i(s-1)) & -\cos(\alpha_i(s-1)) \end{pmatrix} \times \\
&\quad \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix}\Big] \\
&= \frac{1}{\sin(\pi s)}\Big[\begin{pmatrix} \sin(\pi s) & 0 \\ 0 & \sin(\pi s) \end{pmatrix} - \sin(\alpha_i s)\widetilde{S}_i \\
&\quad - \omega s \sin(\alpha_i)\begin{pmatrix} \cos(\alpha_i s) & \sin(\alpha_i s) \\ -\sin(\alpha_i s) & \cos(\alpha_i s) \end{pmatrix}\Big] \\
&= \frac{1}{\sin(\pi s)}\Big[- \sin(\alpha_i s)\widetilde{S}_i \\
&\quad \begin{pmatrix} \sin(\pi s) - \omega s \sin(\alpha_i)\cos(\alpha_i) & -\omega s \sin(\alpha_i)\sin(\alpha_i s) \\ \omega s \sin(\alpha_i)\sin(\alpha_i s) & \sin(\pi s) - \omega s \sin(\alpha_i)\cos(\alpha_i) \end{pmatrix}\Big] \\
&=: \frac{1}{\sin(\pi s)}(A_2 + A_1)
\end{aligned}$$

Here A_1 has the form

$$A_1 = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ -a_{12}^{(1)} & a_{11}^{(1)} \end{pmatrix}$$

(which is called antireflective by Lewis) and A_2 has the form

$$A_2 = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{12}^{(2)} & -a_{11}^{(2)} \end{pmatrix}$$

(called reflective by Lewis). This implies

$$
\begin{aligned}
\det(A_1 \pm A_2) &= \det(A_1) \ + \ \det(A_2) \\
&= ((a_{11}^{(1)})^2 + (a_{12}^{(1)})^2) - ((a_{11}^{(2)})^2 + (a_{12}^{(2)})^2)
\end{aligned}
$$

and we get

$$
\begin{aligned}
\det(I_{2\times 2} - \widehat{C}_i^{(\omega)}(s)) &= \frac{1}{\sin(\pi s)^2}\Big((\sin(\pi s) - \omega s \sin(\alpha_i)\cos(\alpha_i s))^2 \\
&\qquad + \omega^2 s^2 \sin(\alpha_i)^2 \sin(\alpha_i s)^2 - \sin(\alpha_i s)^2\Big) \\
&= \frac{1}{\sin(\pi s)^2}\Big(\sin(\pi s)^2 - 2\omega s \sin(\alpha_i)\cos(\alpha_i s)\sin(\pi s) \\
&\qquad + \omega^2 s^2 \sin(\alpha_i)^2 - \sin(\alpha_i s)^2\Big) \\
&= \frac{1}{\sin(\pi s)^2}\Big((\sin(\pi s)\cos(\alpha_i s) - \omega s \sin(\alpha_i))^2 \\
&\qquad - \sin(\pi s)^2 \cos(\alpha_i s)^2 + \sin(\pi s)^2 - \sin(\alpha_i s)^2\Big) \\
&= \frac{1}{\sin(\pi s)^2}\Big((\sin(\pi s)\cos(\alpha_i s) - \omega s \sin(\alpha_i))^2 + \\
&\qquad (\sin(\pi s)^2 - 1)\sin(\alpha_i s)^2\Big) \\
&= \frac{1}{\sin(\pi s)^2}\Big((\sin(\pi s)\cos(\alpha_i s) - \omega s \sin(\alpha_i))^2 - \\
&\qquad (\cos(\pi s)\sin(\alpha_i s))^2\Big) \\
&=: \frac{1}{\sin(\pi s)^2}(\alpha^2 - \beta^2) \\
&= \frac{1}{\sin(\pi s)^2}(\alpha - \beta)(\alpha + \beta),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= \sin(\pi s)\cos(\alpha_i s) - \omega s \sin(\alpha_i) \\
\beta &= \cos(\pi s)\sin(\alpha_i s).
\end{aligned}
$$

For $s = 0$, α and β have a simple zero. Now we have

$$
\begin{aligned}
& \alpha - \beta && = 0 \\
\Longleftrightarrow \quad & \sin(\pi s)\cos(\alpha_i s) - \cos(\pi s)\sin(\alpha_i s) - \omega s \sin(\alpha_i) && = 0 \\
\Longleftrightarrow \quad & \sin((\pi - \alpha_i)s) - \omega s \sin(\alpha_i) && = 0 \\
\Longleftrightarrow \quad & \sin((\pi - \alpha_i)s) - \omega s \sin(\pi - \alpha_i) && = 0 \\
\overset{s\neq 0}{\Longleftrightarrow} \quad & \frac{\sin((\pi - \alpha_i)s)}{(\pi - \alpha_i)s} - \omega \frac{\sin(\pi - \alpha_i)}{\pi - \alpha_i} && = 0
\end{aligned}
$$

By Lemma 3.4 of Lewis we get

$$\alpha - \beta \neq 0, \quad \text{for } s \in \mathbb{C},\ 0 \leq Re(s) < \overline{z}(\omega)\ s \neq 0.$$

On the other hand we have

$$\begin{aligned}
& \alpha + \beta &&= 0 \\
\Longleftrightarrow\ & \sin(\pi s)\cos(\alpha_i s) + \cos(\pi s)\sin(\alpha_i s) - \omega s \sin(\alpha_i) &&= 0 \\
\Longleftrightarrow\ & \sin((\pi + \alpha_i)s) - \omega s \sin(\alpha_i) &&= 0 \\
\Longleftrightarrow\ & \sin((\pi + \alpha_i)s) + \omega s \sin(\pi + \alpha_i) &&= 0 \\
\overset{s\neq 0}{\Longleftrightarrow}\ & \frac{\sin((\pi + \alpha_i)s)}{(\pi + \alpha_i)s} + \omega \frac{\sin(\pi + \alpha_i)}{\pi + \alpha_i} &&= 0.
\end{aligned}$$

Again by the lemma of Lewis we know

$$\alpha + \beta \neq 0, \quad \text{for } 0 \leq Re(s) < \overline{z}(\omega),\ s \neq 0.$$

This implies

$$\det(I_{2\times 2} - \widehat{C}_i^{(\omega)}(s)) \neq 0, \quad 0 \leq Re(s) < \overline{z}(\omega). \tag{3.59}$$

Analogous to $I_{2\times 2} - \widehat{C}_i^{(\omega)}(s)$ we now analyze $I_{2\times 2} + \widehat{C}_i^{(\omega)}(s)$.

$$\begin{aligned}
I_{2\times 2} + \widehat{C}_i^{(\omega)}(s) = \frac{1}{\sin(\pi s)}\Big[& \sin(\alpha_i s)\widetilde{S}_i + \\
& \begin{pmatrix} \sin(\pi s) + \omega s \sin(\alpha_i)\cos(\alpha_i s) & \omega s \sin(\alpha_i)\sin(\alpha_i s) \\ -\omega s \sin(\alpha_i)\sin(\alpha_i s) & \sin(\pi s) + \omega s \sin(\alpha_i)\cos(\alpha_i s) \end{pmatrix} \Big].
\end{aligned}$$

$$\begin{aligned}
\det(I_{2\times 2} + \widehat{C}_i^{(\omega)}(s)) &= \frac{1}{\sin(\pi s)^2}\Big((\sin(\pi s) + \omega s \sin(\alpha_i)\cos(\alpha_i s))^2 \\
&\quad + \omega^2 s^2 \sin(\alpha_i)^2 \sin(\alpha_i s)^2 - \sin(\alpha_i s)^2\Big) \\
&= \frac{1}{\sin(\pi s)^2}\Big(\sin(\pi s)^2 + 2\omega s \sin(\alpha_i)\cos(\alpha_i s)\sin(\pi s) \\
&\quad + \omega^2 s^2 \sin(\alpha_i)^2 - \sin(\alpha_i s)^2\Big) \\
&= \frac{1}{\sin(\pi s)^2}\Big((\sin(\pi s)\cos(\alpha_i s) + \omega s \sin(\alpha_i))^2 + \sin(\pi s)^2 \\
&\quad - \sin(\pi s)^2 \sin(\alpha_i s)^2 - \sin(\alpha_i s)^2\Big) \\
&= \frac{1}{\sin(\pi s)^2}\Big((\sin(\pi s)\cos(\alpha_i s) - \omega s \sin(\alpha_i))^2 \\
&\quad - (\cos(\pi s)\sin(\alpha_i s))^2\Big) \\
&=: \frac{1}{\sin(\pi s)^2}(\alpha^2 - \beta^2) \\
&= \frac{1}{\sin(\pi s)^2}(\alpha - \beta)(\alpha + \beta),
\end{aligned}$$

where

$$\begin{aligned}\alpha &= \sin(\pi s)\cos(\alpha_i s) + \omega s \sin(\alpha_i) \\ \beta &= \cos(\pi s)\sin(\alpha_i s).\end{aligned}$$

For $s = 0$, α and β have a simple zero. Now we have

$$\begin{aligned} & \alpha - \beta &= 0 \\ \Longleftrightarrow\ & \sin(\pi s)\cos(\alpha_i s) - \cos(\pi s)\sin(\alpha_i s) + \omega s \sin(\alpha_i) &= 0 \\ \Longleftrightarrow\ & \sin((\pi - \alpha_i)s) + \omega s \sin(\alpha_i) &= 0 \\ \Longleftrightarrow\ & \sin((\pi - \alpha_i)s) + \omega s \sin(\pi - \alpha_i) &= 0 \\ \overset{s\neq 0}{\Longleftrightarrow}\ & \frac{\sin((\pi - \alpha_i)s)}{(\pi - \alpha_i)s} + \omega \frac{\sin(\pi - \alpha_i)}{\pi - \alpha_i} &= 0 \end{aligned}$$

This implies $\alpha - \beta \neq 0$ for s with $0 \leq Re(s) < \overline{z}(\omega)$, $s \neq 0$.

$$\begin{aligned} & \alpha + \beta &= 0 \\ \Longleftrightarrow\ & \sin((\pi + \alpha_i)s) + \omega s \sin(\alpha_i) &= 0 \\ \Longleftrightarrow\ & \sin((\pi + \alpha_i)s) - \omega s \sin(\pi + \alpha_i) &= 0 \\ \overset{s\neq 0}{\Longleftrightarrow}\ & \frac{\sin((\pi + \alpha_i)s)}{(\pi + \alpha_i)s} - \omega \frac{\sin((\pi + \alpha_i)}{\pi + \alpha_i} &= 0. \end{aligned}$$

This proves $\alpha + \beta \neq 0$ for all s with $0 \leq Re(s) < \overline{z}(\omega)$, $s \neq 0$. The last results imply

$$\det(I_{2\times 2} + \widehat{C}_i^{(\omega)}(s)) \neq 0, \quad 0 \leq Re(s) < \overline{z}(\omega). \tag{3.60}$$

$\det(I_{2\times 2} \pm \widehat{C}_i^{(\omega)}(s))$ are even functions, therefore (3.59) and (3.60) show

$$\det(I_{2\times 2} \pm \widehat{C}_i^{(\omega)}(s)) \neq 0, \quad \text{if } |Re(s)| < \overline{z}(\omega).$$

This gives

$$\det(\widehat{\mathcal{B}}_i^{(\omega)}(s)) \neq 0, \quad \text{if } |Re(s)| < \overline{z}(\omega).$$

Lemma 3.2 now shows c. for $1/2 - \rho > -\overline{z}(\omega)$, and Lemma 3.2 and $(*_1)$ imply

$$\overline{\mathcal{B}}_i^{(\omega)} : (L^p(0,1))^4 \longrightarrow (L^p(0,1))^4$$

is a Fredholm operator for

$$0 \leq \frac{1}{p} < \overline{z}(\omega) \Longleftrightarrow \frac{1}{\overline{z}(\omega)} < p \leq \infty.$$

Lemma 3.2 and $(*_2)$ show

$$\overline{\mathcal{B}}_i^{(\omega)} \; : \; \left(X_1^{p,0}(0,1)\right)^4 \;\longrightarrow\; \left(X_1^{p,0}(0,1)\right)^4$$

is a Fredholm operator for

$$-\overline{z}(\omega) \;<\; \frac{1}{p} - 1 \;\leq\; 0 \iff 1 \;\leq\; p \;<\; \frac{1}{1-\overline{z}(\omega)}.$$

□

Lemma 3.6. *Let $\omega \in [0,1]$ and $i \in \{1,\dots,n\}$. We have that*

a. $\overline{\mathcal{B}}_i^{(\omega)} \; : \; (L^p(0,1))^4 \longrightarrow (L^p(0,1))^4$, $p \in (1/\overline{z}(\omega), \infty]$,

b. $\overline{\mathcal{B}}_i^{(\omega)} \; : \; \left(X_1^{p,0}(0,1)\right)^4 \longrightarrow \left(X_1^{p,0}(0,1)\right)^4$, $p \in [1, 1/(1-\overline{z}(\omega))\,)$, *and*

c. $\overline{\mathcal{B}}_i^{(\omega)} \; : \; \left(X_\rho^{2,0}(0,1)\right)^4 \longrightarrow \left(X_\rho^{2,0}(0,1)\right)^4$, $\rho \in [0, 1/2 + \overline{z}(\omega))$

are Fredholm operators with index 0.

Proof: $\overline{\mathcal{B}}_i^{(\omega)}$, $\omega \in [0,1]$, is a homotopy between $\overline{\mathcal{B}}_i^{(0)}$ and $\overline{\mathcal{B}}_i^{(1)}$ in case a. and b. and for p fixed and in case c. for fixed ρ. It remains in the set of Fredholm operators by Lemma 3.5 if p and ρ are restricted to the range which is given in Lemma 3.5 So it is sufficient to prove that the index of $\overline{\mathcal{B}}_i^{(0)}$ is 0.
We define

$$\begin{aligned}
\widetilde{B}_i(s) \;&:=\; \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix}^{-1} \widehat{\mathcal{B}}_1^{(0)}(s) \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix} \\
&=\; \frac{1}{2} \begin{pmatrix} I_{2\times 2} & I_{2\times 2} \\ -I_{2\times 2} & I_{2\times 2} \end{pmatrix} \begin{pmatrix} I_{2\times 2} & g_i(s) I_{2\times 2} \\ g_i(s) I_{2\times 2} & I_{2\times 2} \end{pmatrix} \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix} \\
&=\; \frac{1}{2} \begin{pmatrix} I_{2\times 2} & I_{2\times 2} \\ -I_{2\times 2} & I_{2\times 2} \end{pmatrix} \begin{pmatrix} (1+g_i(s)) I_{2\times 2} & (g_i(s)-1) I_{2\times 2} \\ (1+g_i(s)) I_{2\times 2} & (1-g_i(s)) I_{2\times 2} \end{pmatrix} \\
&=\; \begin{pmatrix} (1+g_i(s)) I_{2\times 2} & 0 \\ 0 & (1-g_i(s)) I_{2\times 2} \end{pmatrix}
\end{aligned}$$

So we see that $\overline{\mathcal{B}}_i^{(0)}$ can be diagonalized by a transformation not dependent on s. This means that we can apply the one dimensional theory. By Lemma 3.5 we already know that $1 \pm g_i(s) \neq 0$, $\forall s$, $|Re(s)| < \overline{z}(0)$.

But in this special case we can prove even more.

$$
\begin{aligned}
|g_i(s)| &= \left| \int_0^\infty k_D(t) t^{s-1}\, dt \right| \\
&\le \int_0^\infty |k_D(t)|\, t^{Re(s)-1}\, dt \\
&= \frac{\sin(|\alpha_i|\, Re(s))}{\sin(\pi Re(s))} \\
&< 1
\end{aligned}
$$

for all s with $Re(s) \in (-\overline{z}(0), \overline{z}(0))$, because $\sin(|\alpha_i|\,\overline{z}(0)) = \sin(\pi\overline{z}(0))$. If $x \in \mathbb{R}$, $|x| < \overline{z}(0)$, we get for the index of the functions $1 \pm g_i(x+iy)$, $y \in \mathbb{R}$,

$$
\operatorname{Index}_{y=-\infty}^{\infty}(1 \pm g_i(x+iy)) \;=\; 0.
$$

Now we obtain with the correspondence between $\overline{\mathcal{B}}_i^{(0)}$ and Wiener–Hopf operators on $\mathbb{R}^+$ (see the proof of Lemma 3.5) and by [9, Theorem I.8.1]:

a. $\overline{\mathcal{B}}_i^{(0)}$ is invertible on $(L^p(0,1))^4$, $p \in (1/\overline{z}(0), \infty]$ and

b. $\overline{\mathcal{B}}_i^{(0)}$ is invertible on $\left(X_1^{p,0}(0,1)\right)^4$, $p \in [1, 1/(1-\overline{z}(0))\,)$.

c. $\overline{\mathcal{B}}_i^{(0)}$ is invertible on $\left(X_\rho^{2,0}(0,1)\right)^4$, $\rho \in [0, 1/2 + \overline{z}(0))$.

This shows a., b. and c. for $\omega = 0$. Thus the remarks at the beginning of the proof and $\overline{z}(\omega) < \overline{z}(0)$ show the statement of the lemma. □

The transformation which we used in the proof of the last lemma is now applied again. With the help of $\widetilde{S}_i$ (3.53) and $\widehat{C}_i^{(\omega)}(s)$ (3.58) we construct a matrix $\widetilde{B}_i^{(\omega)}(s)$ which is similar to $\widehat{\mathcal{B}}_i^{(\omega)}(s)$ but has a simpler structure:

$$
\begin{aligned}
\widetilde{B}_i^{(\omega)}(s) &:= \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix}^{-1} \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \widetilde{S}_i \end{pmatrix} \widehat{\mathcal{B}}_1^{(\omega)}(s) \times \\
&\qquad \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \widetilde{S}_i \end{pmatrix} \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix} \\
&= \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix}^{-1} \begin{pmatrix} I_{2\times 2} & \widehat{C}_i^{(\omega)}(s) \\ \widehat{C}_i^{(\omega)}(s) & I_{2\times 2} \end{pmatrix} \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix} \\
&= \begin{pmatrix} I_{2\times 2} + \widehat{C}_i^{(\omega)}(s) & 0 \\ 0 & I_{2\times 2} - \widehat{C}_i^{(\omega)}(s) \end{pmatrix}
\end{aligned}
\tag{3.61}
$$

So $\widetilde{B}_i^{(\omega)}(s)$ and $\widehat{\mathcal{B}}_i^{(\omega)}(s)$ are similar with a transformation independent of s. In the next lemma we study the eigenvalues of $\widetilde{B}_i^{(\omega)}(s)$.

Lemma 3.7. *There is a $q > 0$ such that all eigenvalues of $Re(\widetilde{B}_i^{(\omega)}(s))$, $Re(s) = 0$, are greater than q for $\omega \in [0,1]$, $i \in \{1, \ldots, n\}$.*

Proof: We have

$$\widehat{C}_i^{(\omega)}(s) = g_i(s)\widetilde{S}_i + \omega h_i(s) \underbrace{\mathcal{S}_{i,1}(s)\widetilde{S}_i}_{=:\widetilde{\mathcal{S}}_{i,1}(s)}$$

For $\widetilde{\mathcal{S}}_{i,1}(s)$ we get

$$\widetilde{\mathcal{S}}_{i,1}(s) = \begin{pmatrix} \cos(\alpha_i s) & \sin(\alpha_i s) \\ -\sin(\alpha_i s) & \cos(\alpha_i s) \end{pmatrix}.$$

$$\begin{aligned} Re(\widehat{C}_i^{(\omega)}(s)) &= \frac{1}{2}\Big(g_i(s)\widetilde{S}_i + \overline{g_i(s)}\widetilde{S}_i + \omega h_i(s)\widetilde{\mathcal{S}}_{i,1}(s) \\ &\qquad \omega\overline{h_i(s)}\widetilde{\mathcal{S}}_{i,1}(s)^*\Big) \\ &= Re(g_i(s))\widetilde{S}_i + \omega \begin{pmatrix} Re(h_i(s)\cos(\alpha_i s)) & \sqrt{-1}\, Im(h_i(s)\sin(\alpha_i s)) \\ -\sqrt{-1}\, Im(h_i(s)\sin(\alpha_i s)) & Re(h_i(s)\cos(\alpha_i s)) \end{pmatrix} \end{aligned}$$

The first term is a reflective matrix and the second term is antireflective. This will be used in the calculation of the eigenvalues of $Re(\widehat{C}_i^{(\omega)}(s))$.

$$\begin{aligned} \det(Re(\widehat{C}_i^{(\omega)}(s)) - \lambda I_{2\times 2}) &= \det\Bigg(\begin{pmatrix} \omega Re(h_i(s)\cos(\alpha_i s)) - \lambda & \sqrt{-1}\omega\, Im(h_i(s)\sin(\alpha_i s)) \\ -\sqrt{-1}\omega\, Im(h_i(s)\sin(\alpha_i s)) & Re(h_i(s)\cos(\alpha_i s)) - \lambda \end{pmatrix} \\ &\qquad + Re(g_i(s))\widetilde{S}_i\Bigg) \\ &= (\omega Re(h_i(s)\cos(\alpha_i s)) - \lambda)^2 - \omega^2\, Im((h_i(s)\sin(\alpha_i s)))^2 \\ &\qquad - Re(g_i(s))^2 \\ &= \lambda^2 - 2\omega\, Re(h_i(s)\cos(\alpha_i s))\lambda + \\ &\qquad \omega^2(Re(h_i(s)\cos(\alpha_i s))^2 - Im(h_i(s)\sin(\alpha_i s))^2) \\ &\qquad - Re(g_i(s))^2 \end{aligned}$$

The two solutions for λ are given by

$$\begin{aligned} \lambda_{1/2}(s) &= \omega\, Re(h_i(s)\cos(\alpha_i s)) \pm \\ &\qquad \big(Re(g_i(s))^2 + \omega^2\, Im(h_i(s)\sin(\alpha_i s))^2\big)^{\frac{1}{2}} \end{aligned} \tag{3.62}$$

We recall that

$$\begin{aligned}
h_i(s)\sin(\alpha_i s) &= \sin(\alpha_i)\frac{s\sin(\alpha_i s)}{\sin(\pi s)},\\
h_i(s)\cos(\alpha_i s) &= \sin(\alpha_i)\frac{s\cos(\alpha_i s)}{\sin(\pi s)},\\
g_i(s) &= \frac{\sin(\alpha_i s)}{\sin(\pi s)}.
\end{aligned}$$

We substitute $s = \sqrt{-1}y$, $y \in \mathbb{R}$, and get

$$\begin{aligned}
h_i(s)\sin(\alpha_i s) &= \sqrt{-1}\,\sin(\alpha_i)\frac{y\sinh(\alpha_i y)}{\sinh(\pi y)},\\
h_i(s)\cos(\alpha_i s) &= \sin(\alpha_i)\frac{\sqrt{-1}\,y\cosh(\alpha_i y)}{\sqrt{-1}\,\sinh(\pi y)},\\
&= \sin(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)},\\
g_i(s) &= \frac{\sinh(\alpha_i y)}{\sinh(\pi y)}.
\end{aligned}$$

By (3.62) we get

$$\begin{aligned}
\lambda_{1/2}(y) &= \omega\sin(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)}\\
&\quad \pm\left(\frac{\sinh(\alpha_i y)^2}{\sinh(\pi y)^2} + \omega^2\sin(\alpha_i)^2\frac{\sinh(\alpha_i y)^2}{\sinh(\pi y)^2}\right)^{\frac{1}{2}}\\
&= \omega\sin(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)}\\
&\quad \pm\left|\frac{\sinh(\alpha_i y)}{\sinh(\pi y)}\right|\left(1+\omega^2\sin(\alpha_i)^2y^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Let $\alpha_i \geq 0$. We define

$$\begin{aligned}
f_1(\alpha_i, y) &:= \lambda_1(y)\\
&= \omega\sin(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)} + \frac{\sinh(\alpha_i y)}{\sinh(\pi y)}\left(1+\omega^2\sin(\alpha_i)^2y^2\right)^{\frac{1}{2}}\\
f_2(\alpha_i, y) &:= \lambda_2(y)\\
&= \omega\sin(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)} - \frac{\sinh(\alpha_i y)}{\sinh(\pi y)}\left(1+\omega^2\sin(\alpha_i)^2y^2\right)^{\frac{1}{2}}
\end{aligned}$$

For $\alpha_i < 0$ we have

$$\begin{aligned}
\lambda_1(y) &= -\omega\sin(-\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)} + \frac{\sinh(-\alpha_i y)}{\sinh(\pi y)}\left(1+\omega^2\sin(\alpha_i)^2 y^2\right)^{\frac{1}{2}} \\
&= -f_2(-\alpha_i, y) \\
\lambda_2(y) &= -\omega\sin(-\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)} - \frac{\sinh(-\alpha_i y)}{\sinh(\pi y)}\left(1+\omega^2\sin(\alpha_i)^2 y^2\right)^{\frac{1}{2}} \\
&= -f_1(-\alpha_i, y)
\end{aligned}$$

We are only interested in the absolute values of $\lambda_{1/2}(y)$, so we only have to consider $f_{1/2}(\alpha_i, y)$, $\alpha_i \in [0, \pi)$, $y \in \mathbb{R}$. But we further have

(i) $f_{1/2}(\alpha_i, y) = f_{1/2}(\alpha_i, -y)$, $\forall \alpha_i$ and $y \in \mathbb{R}$.

This means we only have to look at the case $\alpha_i \in [0, \pi)$, $y \geq 0$. But then we have

$$|f_1(\alpha_i, y)| \geq |f_2(\alpha_i, y)| \text{ and} f_1(\alpha_i, y) \geq 0.$$

So it remains to show

$$\exists q_i^* < 1 \text{ such that } f_1(\alpha_i, y) \leq q_i^*,\ \forall\, y \geq 0. \tag{3.63}$$

The statement of the lemma then follows with

$$q := 1 - \max_{i=1}^{n}\{q_i^*\}.$$

Define

$$f(\alpha_i, y) := \sin(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)} + \frac{\sinh(\alpha_i y)}{\sinh(\pi y)}\left(1+\sin(\alpha_i)^2 y^2\right)^{\frac{1}{2}}$$

(here ω is equal to 1). We have the following properties

(ii) $f_1(\alpha_i, y) \leq f(\alpha_i, y)$, $\forall \alpha_i \in [0, \pi)$, $y \geq 0$.

(iii) $\lim_{|y|\to\infty} f(\alpha_i, y) = 0$.

(iv) $f(0, y) \equiv 0$.

(v) $f(\pi, y) \equiv 1$.

From (ii) it follows that we have to prove (3.63) only for $f(\alpha_i, y)$. (iii)–(v) show that it is sufficient to prove that the mapping

$$\alpha_i \longrightarrow f(\alpha_i, y)$$

is monotonically increasing, $y \geq 0$ $(*_1)$. Then (3.63) is proved. But we have

$$\begin{aligned}
\frac{\partial f}{\partial \alpha_i}(\alpha_i, y) &= \cos(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)} + \sin(\alpha_i)\frac{y^2\sinh(\alpha_i y)}{\sinh(\pi y)} \\
&\quad + \frac{y\cosh(\alpha_i y)}{\sinh(\pi y)}\left(1+\sin(\alpha_i)^2 y^2\right)^{\frac{1}{2}} \\
&\quad + \sin(\alpha_i)\cos(\alpha_i)\frac{y^2\sinh(\alpha_i y)}{\sinh(\pi y)}\left(1+\sin(\alpha_i)^2 y^2\right)^{-\frac{1}{2}} \\
&= \frac{1}{\sinh(\pi y)}\Big(y\cosh(\alpha_i y)\underbrace{\left(\cos(\alpha_i) + (1+\sin(\alpha_i)^2y^2)^{\frac{1}{2}}\right)}_{>0} \\
&\quad + y^2\underbrace{\sin(\alpha_i)\sinh(\alpha_i y)}_{\geq 0}\underbrace{\left(1+\cos(\alpha_i)(1+\sin(\alpha_i)^2y^2)^{-\frac{1}{2}}\right)}_{\geq 0} \\
&\geq 0.
\end{aligned}$$

This proves $(*_1)$ and the lemma. □

Remark: The statement of Lemma 3.7 is wrong for $Re(s) = 1/2$. A numerical calculation shows that $Re(\widetilde{B}_i^{(1)}(1/2))$ has an eigenvalue greater than 1.05 if $\alpha_i = 0.6 * \pi$.

Theorem 3.8. *Let $\omega \in [0,1]$, $i \in \{1,\ldots,n\}$. Then*

a. $\overline{\mathcal{B}}_i^{(\omega)} : (L^p(0,1))^4 \longrightarrow (L^p(0,1))^4$, $p \in (1/\overline{z}(\omega), \infty]$, *and*

b. $\overline{\mathcal{B}}_i^{(\omega)} : \left(X_1^{p,0}(0,1)\right)^4 \longrightarrow \left(X_1^{p,0}(0,1)\right)^4$, $p \in [1, 1/(1-\overline{z}(\omega))\,)$,

c. $\overline{\mathcal{B}}_i^{(\omega)} : \left(X_\rho^{2,0}(0,1)\right)^4 \longrightarrow \left(X_\rho^{2,0}(0,1)\right)^4$, $\rho \in [1, 1/2+\overline{z}(\omega))$,

are invertible (see Lemma 3.4 and (3.55) for the definition of $\overline{z}(\omega)$).

Proof: Lemma 3.7 and (3.61) show that $\widehat{\mathcal{B}}_i^{(\omega)}(s)$ is strongly elliptic for $Re(s) = 0$, if the transformation (3.61) is applied. By the correspondence to the Wiener–Hopf operators we get that

(i) $\overline{\mathcal{B}}_i^{(\omega)} : (L^\infty(0,1))^4 \longrightarrow (L^\infty(0,1))^4$ is invertible and

(ii) $\overline{\mathcal{B}}_i^{(\omega)} : \left(X_1^{1,0}(0,1)\right)^4 \longrightarrow \left(X_1^{1,0}(0,1)\right)^4$ is invertible.

$\overline{\mathcal{B}}_i^{(\omega)} : (L^p(0,1))^4 \longrightarrow (L^p(0,1))^4$, $p \in (1/\overline{z}(0), \infty]$, is a Fredholm operator of index 0 by Lemma 3.6 and $L^\infty(0,1) \subset L^p(0,1)$ is dense. Then it follows by a standard argument for Fredholm operators (see [17]) that

$$N(\overline{\mathcal{B}}_i^{(\omega)}|_{(L^p(0,1))^4}) \subset N(\overline{\mathcal{B}}_i^{(\omega)}|_{(L^\infty(0,1))^4}) = \{0\},$$

where $N(L)$ denotes the kernel of the linear mapping L. This implies $\overline{\mathcal{B}}_i^{(\omega)}$ is invertible and this proves a.
Lemma 3.6 also shows that $\overline{\mathcal{B}}_i^{(\omega)} : \left(X_1^{p,0}(0,1)\right)^4 \longrightarrow \left(X_1^{p,0}(0,1)\right)^4$, $p \in [1, 1/(1-\overline{z}(0))\,)$, is a Fredholm operator with index 0. But we have $\left(X_1^{p,0}(0,1)\right)^4 \subset \left(X_1^{1,0}(0,1)\right)^4$ and so we get

$$N(\overline{\mathcal{B}}_i^{(\omega)}|_{\left(X_1^{p,0}(0,1)\right)^4}) \;\subset\; N(\overline{\mathcal{B}}_i^{(\omega)}|_{\left(X_1^{1,0}(0,1)\right)^4}) \;=\; \{0\}.$$

This proves b, and the inclusion

$$N(\overline{\mathcal{B}}_i^{(\omega)}|_{\left(X_\rho^{2,0}(0,1)\right)^4}) \;\subset\; N(\overline{\mathcal{B}}_i^{(\omega)}|_{\left(X_1^{2,0}(0,1)\right)^4}) \;=\; \{0\},\ \rho \in [1, 1/2 + \overline{z}(0)).$$

shows c. in a similiar way. □

Corollary 3.9. *The operators $\mathcal{B}_\pi^{(\omega,1)}$ and $\mathcal{B}^{(\omega,1)}$ (see (3.17), (3.18)) are invertible and $\mathcal{B}^{(\omega)}$ (see (3.11)) is a Fredholm operator with index 0.*

Proof: $\mathcal{B}_\pi^{(\omega,1)}$ consists of identical operators on J_{3i+1}, $i = 0(1)n-1$, and $\overline{\mathcal{B}}_i^{(\omega)}$, $i = 1(1)n$, see (3.21),(3.22). By Theorem 3.8 we get

$$\mathcal{B}_\pi^{(\omega,1)} : \vec{L}_\pi^2 \;\longrightarrow\; \vec{L}_\pi^2$$

is invertible. The relation (3.18) shows that $\mathcal{B}^{(\omega,1)}$ is invertible. Lemma 3.1.a. and $\mathcal{B}^{(\omega)} = \mathcal{B}^{(\omega,1)} + \mathcal{K}^{(\omega,2)}$ prove the statement for $\mathcal{B}^{(\omega)}$. □

Lemma 3.10. *Let $\vec{u} \in \left(L^2(0,1)\right)^4$, $\omega \in [0,1]$, $\rho \in [1, 1/2 + \overline{z}(\omega))$ and $l \in \mathbb{N}$. Then*

a. $\overline{\mathcal{B}}_i^{(\omega)}\vec{u} \in \left(X_\rho^{2,l}(0,1)\right)^4$ implies $\vec{u} \in \left(X_\rho^{2,l}(0,1)\right)^4$ and

b. $\overline{\mathcal{B}}_i^{(\omega)}\vec{u} \in \left(X_\rho^{2,l}(0,1)\right)^4 \dot{+} \mathbb{R}^4$ implies $\vec{u} \in \left(X_\rho^{2,l}(0,1)\right)^4 \dot{+} \mathbb{R}^4$.

Proof: We follow closely [5, Theorem 1.10] and define

$$T_i^{(\omega)}(z) \;:=\; \begin{pmatrix} 0 & k_{i,1}^{(\omega)}(z) \\ k_{i,2}^{(\omega)}(z) & 0 \end{pmatrix}$$

(see (3.37)). (3.38)–(3.40) and (3.35)–(3.36) show that all entries of $T_i^{(\omega)}$ are in $X_1^{2,m}(0,1)$, $\forall m \in \mathbb{N}$. A simple calculation shows

$$[zD\overline{\mathcal{B}}_i^{(\omega)}\vec{u}](z) \;=\; [\overline{\mathcal{B}}_i^{(\omega)}(zD\vec{u})](z) - T_i^{(\omega)}(z)\vec{u}(1),$$

for $\vec{u} \in \left(X_0^{2,1}(0,1)\right)^4$. By an induction we get

$$[(zD+2)^m \overline{\mathcal{B}}_i^{(\omega)} \vec{u}](z) \quad = \quad [\overline{\mathcal{B}}_i^{(\omega)} (zD+2)^m \vec{u})](z) + (T_{i,m}^{(\omega)}(z)\vec{u})(z),$$

$\vec{u} \in \left(X_0^{2,m}(0,1)\right)^4$, $m \in \mathbb{N}$. Here $T_{i,m}^{(\omega)}$ is a finite dimensional operator, which consists of linear combinations of $T_i^{(\omega)}(z)$ and its derivatives. In [6] it is shown that

$$\phi_m := (zD+2)^m \,:\, \left(X_\rho^{p,m}(0,1)\right)^4 \longrightarrow \left(X_\rho^{p,0}(0,1)\right)^4,\ 1 \leq p \leq \infty,$$

is an isomorphism for all $m \in \mathbb{N}$.
This implies

$$\overline{\mathcal{B}}_i^{(\omega)} \quad = \quad \phi_m^{-1} \circ \overline{\mathcal{B}} \circ \phi_m + \phi_m^{-1} \circ T_{i,m}^{(\omega)},$$

where $\overline{\mathcal{B}}_i^{(\omega)}$ is invertible on $\left(X_\rho^{2,m}(0,1)\right)^4$ by Theorem 3.8 and $\phi_m^{-1} \circ T_{i,m}^{(\omega)}$ is a finite dimensional operator and hence compact. Because of this we have that $\overline{\mathcal{B}}_i^{(\omega)}$ is a Fredholm operator and its index is 0. But we also have

$$N(\overline{\mathcal{B}}_i^{(\omega)}|_{\left(X_\rho^{2,m}(0,1)\right)^4}) \;\subset\; N(\overline{\mathcal{B}}_i^{(\omega)}|_{\left(X_\rho^{2,0}(0,1)\right)^4}) \;=\; \{0\}.$$

This shows that $\overline{\mathcal{B}}_i^{(\omega)}$ is an isomorphism on $\left(X_\rho^{2,m}(0,1)\right)^4$ and proves a.
We recall the formula

$$[zD\overline{\mathcal{B}}_i^{(\omega)} \vec{u}](z) \quad = \quad [\overline{\mathcal{B}}_i^{(\omega)} (zD\vec{u})](z) - T_i^{(\omega)}(z)\vec{u}(1)$$

from above. By [6] it follows that

$$zD \,:\, \left(X_\rho^{p,m}(0,1)\right)^4 \dot{+} \mathbb{R}^4 \quad \longrightarrow \quad \left(X_\rho^{p,m-1}(0,1)\right)^4$$

is surjective with kernel $\mathbb{R}^4$. This implies

$$\overline{\mathcal{B}}_i^{(\omega)}(\mathbb{R}^4) \;\subset\; \left(X_\rho^{p,m}(0,1)\right)^4 \dot{+} \mathbb{R}^4,$$

and $\overline{\mathcal{B}}_i^{(\omega)}$ is a Fredholm operator with index 0 on $\left(X_\rho^{p,m}(0,1)\right)^4 \dot{+} \mathbb{R}^4$. But the kernel of $\overline{\mathcal{B}}_i^{(\omega)}$ in $(L^2(0,1))^4$ is trivial by Theorem 3.8 and so b. follows. □

Lemma 3.11. *Let $\vec{u} \in (L^2(0,T))^2$ be a solution of the equation (3.4), with $\omega \in [0,1]$ and $\vec{f}|_{[s_i,s_{i+1}]} \in \left(C^l[s_i,s_{i+1}]\right)^2$, $i = 0(1)n-1$, $l \in \mathbb{N}_0$. Then we have*

a. $\vec{u}|_{(s_i,s_{i+1})} \in \left(C^l(s_i,s_{i+1})\right)^2$, $i = 0(1)n-1$.

b.

$$\vec v_+(t) \;:=\; \vec u(s_i + t\,\delta) \;\in\; \left(X^{2,l}_\rho(0,1)\right)^4 \dot{+}\,\mathbb{R}^4,\; i \in \{0,\dots,n-1\},$$
$$\vec v_-(t) \;:=\; \vec u(s_i - t\,\delta) \;\in\; \left(X^{2,l}_\rho(0,1)\right)^4 \dot{+}\,\mathbb{R}^4,\; i \in \{1,\dots,n\},$$

$\rho \in [1, 1/2 + \overline{z}(\omega))$. *(see (3.6) for the definition of* δ *and Lemma 3.4 and (3.55) for the definition of* $\overline{z}(\omega)$*).*

Proof:

a. For $\vec u$ we have (see (3.9), (3.17)–(3.20))

$$\begin{aligned} \mathcal{B}^{(\omega,1)}\vec u &= \vec f - \mathcal{K}^{(\omega,2)}\vec u \\ \Longleftrightarrow \quad \mathcal{B}^{(\omega)}_\pi(\Phi\vec u) &= \Phi(\vec f) - \mathcal{K}^{(\omega,2)}_\pi(\Phi\vec u) \end{aligned}$$

By the definition of $\overline{\mathcal{B}}^{(\omega)}_\pi$ (see (3.17) and (3.15)) we have for $j \equiv 1 \bmod 3$:

$$\vec u|_{J_j} \;=\; \vec f|_{J_j} - \mathcal{K}^{(\omega,2)}_\pi(\Phi\vec u)|_{J_j}.$$

The assumption on $\vec f$ and Lemma 3.1.b. show

$$\vec u|_{[s_j+\delta, s_{j+1}-\delta]} \;\in\; \left(C^l(J_j)\right)^2,\ \forall j.$$

$\delta > 0$ is fixed but arbitrary, so we get

$$\vec u|_{[s_j+\delta, s_{j+1}-\delta]} \text{ continuous } \forall\delta > 0.$$

This implies property a.

b. By (3.64) and (3.20) we see that the function $\vec v := (\vec v_-, \vec v_+)^T \in \left(L^2(0,1)\right)^4$ fulfills the equation

$$\overline{\mathcal{B}}^{(\omega)}_i \vec v \;=\; \begin{pmatrix} \vec g_- \\ \vec g_+ \end{pmatrix},$$

where

$$\begin{aligned} \vec g_-(x) &= \vec f(s_i - x\,\delta) - (\mathcal{K}^{(\omega,2)}\vec u)(s_i - x\,\delta), \\ \vec g_+(x) &= \vec f(s_i + x\,\delta) - (\mathcal{K}^{(\omega,2)}\vec u)(s_i + x\,\delta), \end{aligned}$$

The assumptions on $\vec{f}$ and Lemma 3.1.b. imply

$$\vec{g} \in \left(C^l[0,1]\right)^4.$$

For $l \geq 1$ and $\rho \in [1, 1/2 + \overline{z}(\omega))$ we have

$$\vec{g}(x) \;=\; (\vec{g}(x) - \vec{g}(0)) + \vec{g}(0) \;\in \left(X_\rho^{2,l}(0,1)\right)^4 \dot{+} \mathbb{R}^4.$$

By Lemma 3.10 we get

$$\vec{v} \;\in\; \left(X_\rho^{2,l}(0,1)\right)^4 \dot{+} \mathbb{R}^4,$$

and this proves b. □

Corollary 3.12. *Let $\vec{u} \in \left(L^2(0,T)\right)^2$ be the solution of (3.4), $\omega \in [0,1]$, $\vec{f}|_{[s_i,s_{i+1}]} \in \left(C^1[0,1]\right)^2$, $\forall i$. Then we have*

$$\vec{u}|_{[s_i,s_{i+1}]} \;\in\; \left(H^1[s_i, s_{i+1}]\right)^2.$$

Proof: Because of $X_1^{2,1}(0,1)\dot{+}\mathbb{R} \subset H^1[0,1]$ and by Lemma 3.11 it follows $\vec{u}|_{J_j} \in \left(H^1(J_j)\right)^2$, $\forall j \in \{1, \ldots, 3n\}$. The continuity of $\vec{u}$ on $J_{3i} \cup J_{3i+1} \cup J_{3i+2}$, $i \in \{1, \ldots, n\}$, follows from 3.11.a. □

Lemma 3.13. *Let $\vec{u} \in (C[0,T])^4$. Then we have*

$$\lim_{x \searrow 0} \left(\overline{\mathcal{B}}_i^{(\omega)} \vec{u}\right)(x) \;=\; E_i^{(\omega)} \vec{u}(0),$$

where

$$E_i^{(\omega)} \;=\; I_{4\times 4} + \begin{pmatrix} 0 & C_{E_i}^{(\omega)} \\ C_{E_i}^{(\omega)} & 0 \end{pmatrix}, \tag{3.64}$$

and

$$C_{E_i}^{(\omega)} \;=\; \frac{\alpha_i}{\pi} I_{2\times 2} + \frac{\omega}{\pi}\sin(\alpha_i) \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix}.$$

Proof: It is proved in [2] that

$$\lim_{x \searrow 0} \left(\overline{\mathcal{B}}_i^{(\omega)} \vec{u} \right)(x) \ = \ \widehat{\mathcal{B}}_i^{(\omega)}(0) \vec{u}(0),$$

for continuous $\vec{u}$ (only in the scalar case, but this is sufficient). We get:

$$\begin{aligned}
\lim_{s \to 0} g_i(s) &= \frac{\alpha_i}{\pi} \\
\lim_{s \to 0} \omega h_i(s) &= \frac{\omega}{\pi} \sin(\alpha_i) \\
\lim_{s \to 0} \widetilde{S}_{i,1}(s) &= \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix} \\
\lim_{s \to 0} \widetilde{S}_{i,2}(s) &= \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix}
\end{aligned}$$

(see (3.45) and (3.46) for the definitions). But then (3.44) gives the result. □

Lemma 3.14. *Let $\omega \in [0,1]$, $\alpha_i \in (-\pi, \pi)$. The matrix $E_i^{(\omega)}$ has the factorization*

$$E_i^{(\omega)} \ = \ \frac{1}{\sqrt{2}} \begin{pmatrix} U_i & -U_i \\ U_i & U_i \end{pmatrix} \begin{pmatrix} D_{i,1}^{(\omega)} & 0 \\ 0 & D_{i,2}^{(\omega)} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} U_i^T & U_i^T \\ -U_i^T & U_i^T \end{pmatrix}, \tag{3.65}$$

where U_i is a unitary matrix

$$U_i \ = \ \begin{pmatrix} \cos(\alpha_i/2) & -\sin(\alpha_i/2) \\ \sin(\alpha_i/2) & \cos(\alpha_i/2) \end{pmatrix} \tag{3.66}$$

and $D_{i,1}^{(\omega)}$ and $D_{i,2}^{(\omega)}$ are diagonal matrices

$$\left.\begin{aligned}
D_{i,1}^{(\omega)} &= \begin{pmatrix} 1 + \dfrac{\alpha_i + \omega \sin(\alpha_i)}{\pi} & 0 \\ 0 & 1 + \dfrac{\alpha_i - \omega \sin(\alpha_i)}{\pi} \end{pmatrix} \\
D_{i,2}^{(\omega)} &= \begin{pmatrix} 1 - \dfrac{\alpha_i + \omega \sin(\alpha_i)}{\pi} & 0 \\ 0 & 1 - \dfrac{\alpha_i - \omega \sin(\alpha_i)}{\pi} \end{pmatrix}
\end{aligned}\right\}, \tag{3.67}$$

which are non singular.

Proof: We first get

$$\frac{1}{2}\begin{pmatrix} I_{2\times 2} & I_{2\times 2} \\ -I_{2\times 2} & I_{2\times 2} \end{pmatrix}\begin{pmatrix} I_{2\times 2} & C^{(\omega)}_{E_i} \\ C^{(\omega)}_{E_i} & I_{2\times 2} \end{pmatrix}\begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix} = \begin{pmatrix} I + C^{(\omega)}_{E_i} & 0 \\ 0 & I - C^{(\omega)}_{E_i} \end{pmatrix},$$

$$\begin{aligned} I + C^{(\omega)}_{E_i} &= \left(1 + \frac{\alpha_i}{\pi}\right) I_{2\times 2} + \frac{\omega}{\pi}\sin(\alpha_i)\, S_1, \\ I - C^{(\omega)}_{E_i} &= \left(1 - \frac{\alpha_i}{\pi}\right) I_{2\times 2} - \frac{\omega}{\pi}\sin(\alpha_i)\, S_1, \end{aligned}$$

where S_1 is defined by

$$S_1 = \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix}.$$

S_1 is a reflection at the straight line parallel to

$$\vec{u}_1 := \begin{pmatrix} -\sin((\alpha_i+\pi)/2) \\ \cos((\alpha_i+\pi)/2) \end{pmatrix} = -\begin{pmatrix} \cos(\alpha_i/2) \\ \sin(\alpha_i/2 \end{pmatrix}$$

and orthogonal to

$$\vec{u}_2 := \begin{pmatrix} \cos((\alpha_i+\pi)/2) \\ \sin((\alpha_i+\pi)/2) \end{pmatrix} = \begin{pmatrix} -\sin(\alpha_i/2) \\ \cos(\alpha_i/2) \end{pmatrix}$$

(see the remark following Lemma 3.3). We get

$$S_1\vec{u}_1 = \vec{u}_1, \quad S_1\vec{u}_2 = -\vec{u}_2.$$

We define

$$U_i := (-\vec{u}_1, \vec{u}_2)$$

and get

$$U_i^T\, S_1\, U_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This implies

$$\begin{aligned} U_i^T(I + C^{(\omega)}_{E_i})U_i &= \left(1 + \frac{\alpha_i}{\pi}\right) I_{2\times 2} + \frac{\omega}{\pi}\sin(\alpha_i)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= D^{(\omega)}_{i,1} \\ U_i^T(I - C^{(\omega)}_{E_i})U_i &= \left(1 - \frac{\alpha_i}{\pi}\right) I_{2\times 2} - \frac{\omega}{\pi}\sin(\alpha_i)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= D^{(\omega)}_{i,2} \end{aligned}$$

$D_{i,1}^{(\omega)}$ and $D_{i,2}^{(\omega)}$ are non singular because $E_i^{(\omega)}$ is non singular by Lemma 3.7.
The above calculations show

$$\begin{aligned} T_i^T E_i^{(\omega)} T_i &= \begin{pmatrix} D_{i,1}^{(\omega)} & 0 \\ 0 & D_{i,2}^{(\omega)} \end{pmatrix}, \\ T_i &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix} \begin{pmatrix} U_i & 0 \\ 0 & U_i \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} U_i & -U_i \\ U_i & U_i \end{pmatrix}, \end{aligned}$$

T_i is orthogonal. This gives

$$E_i^{(\omega)} = T_i \begin{pmatrix} D_{i,1}^{(\omega)} & 0 \\ 0 & D_{i,2}^{(\omega)} \end{pmatrix} T_i^T,$$

which is (3.65). □

Theorem 3.15. *Let $\omega \in [0,1]$ and $\vec{u} \in \left(L^2(0,T)\right)^2$ a solution of the equation (3.4), $\vec{f} \in (C[0,T])^2$, $\vec{f}|_{[s_i,s_{i+1}]} \in (C^1[s_i,s_{i+1}])^2$. Then we have*

$$\vec{u} \in (C[0,T])^2.$$

Proof: Because of Lemma 3.11 we only have to show the continuity of $\vec{u}$ in s_i, $i \in \{1,\dots,n\}$. We choose $i \in \{1,\dots,n\}$ and define

$$\begin{aligned} \vec{v}_-(x) &:= \vec{u}(s_i - x\,\delta),\ x \in [0,1], \\ \vec{v}_+(x) &:= \vec{u}(s_i + x\,\delta),\ x \in [0,1], \text{ and} \\ \vec{v} &:= \begin{pmatrix} \vec{v}_- \\ \vec{v}_+ \end{pmatrix}. \end{aligned}$$

We have by (3.17) and (3.19)

$$\begin{aligned} \mathcal{B}^{(\omega)}\vec{u} &= (\mathcal{B}^{(\omega,1)} + \mathcal{K}^{(\omega,2)})\vec{u} \\ &= \vec{f} \end{aligned}$$

and this implies

$$\overline{\Phi}_i \mathcal{B}^{(\omega,1)} \vec{u} \overset{(3.20)}{=} \overline{\mathcal{B}}_i^{(\omega)} (\overline{\Phi} \vec{u}) = \overline{\Phi}_i (\vec{f} - \mathcal{K}^{(\omega,2)} \vec{u}).$$

We notice $\vec{v} = \overline{\Phi}_i \vec{u}$ and define

$$\vec{w} := \overline{\Phi}_i (\vec{f} - \mathcal{K}^{(\omega,2)} \vec{u}) =: \begin{pmatrix} \vec{w}_1 \\ \vec{w}_2 \end{pmatrix}.$$

By our assumption on $\vec{f}$ and by Lemma 3.1.c. we have $\vec{w} \in (C^1[0,1])^4$ and $\vec{w}_1(0) = \vec{w}_2(0)$. By Corollary 3.12 we get $\vec{v} \in (H^1[0,1])^4 \subset (C[0,1])^4$. Lemma 3.13 gives

$$E_i^{(\omega)} \vec{v}(0) = \begin{pmatrix} \vec{w}_1(0) \\ \vec{w}_1(0) \end{pmatrix}.$$

Lemma 3.14 shows

$$\begin{aligned} \vec{v}(0) &= (E_i^{(\omega)})^{-1} \begin{pmatrix} \vec{w}_1(0) \\ \vec{w}_1(0) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} U_i & -U_i \\ U_i & U_i \end{pmatrix} \begin{pmatrix} (D_{i,1}^{(\omega)})^{-1} & 0 \\ 0 & (D_{i,2}^{(\omega)})^{-1} \end{pmatrix} \begin{pmatrix} U_i^T & U_i^T \\ -U_i^T & U_i^T \end{pmatrix} \begin{pmatrix} \vec{w}_1(0) \\ \vec{w}_1(0) \end{pmatrix} \\ &= \begin{pmatrix} U_i (D_{i,1}^{(\omega)})^{-1} U_i^T \vec{w}_1(0) \\ U_i (D_{i,1}^{(\omega)})^{-1} U_i^T \vec{w}_1(0) \end{pmatrix}. \end{aligned}$$

This shows $\vec{v}_-(0) = \vec{v}_+(0) = U_i (D_{i,1}^{(\omega)})^{-1} U_i^T \vec{w}_1(0)$, and we have proved the continuity in s_i. □

Corollary 3.16. *Let $\omega \in [0,1]$, $\vec{u} \in \left(L^2(0,T)\right)^2$ a solution of the equation (3.4) and $\vec{f} \in (C[0,T])^2$, $\vec{f}|[s_i, s_{i+1}] \in (C^1[s_i, s_{i+1}])^2$. Then we have*

$$\vec{u} \in (H^1[0,T])^2.$$

Proof: The proof follows from Corollary 3.12 and Theorem 3.15 because $\vec{u}$ is a piecewise H^1–function, which is continuous on the whole interval. □

Theorem 3.17.

$$\mathcal{B}^{(\overline{c})} \,:\, \left(L^2(0,T)\right)^2 \longrightarrow \left(L^2(0,T)\right)^2$$

is an isomorphism (see (2.46) for the value of $\overline{c}$ *).*

Proof: By Corollary 3.9 $\mathcal{B}^{(\overline{c})}$ is a Fredholm operator with index 0. If $\vec{u} \in \left(L^2(0,T)\right)^2$ is a solution of

$$\mathcal{B}^{(\overline{c})}\vec{u} \;=\; 0,$$

then we know by Corollary 3.16 that $\vec{u} \in (H^1[0,T])^2$. We define the double layer potential $\vec{U}(x) := (K_1^{(\omega)}u)(x)$ (see (2.38) and (2.39)). By Lemma 2.10 we get

$$P\vec{U}|_\Omega \;=\; 0, \qquad \vec{U} \,\in\, V_P^1,$$

and the relation (2.45) implies $\gamma_0\vec{U}|_\Omega = 0$. By Lemma 2.2 we know $\vec{U}|_\Omega = 0$ and this implies $\gamma_1^{(\omega)}\vec{U}|_\Omega = 0$. By Lemma 2.11 $\vec{U}$ is also a weak solution of

$$P\vec{U} \;=\; 0 \,\text{in}\, \Omega^c.$$

Lemma 2.12 gives $\gamma_1^{(\omega)}\vec{U}|_{\Omega^c} = 0$ and Lemma 2.13 implies $\vec{U}|_{\Omega^c} = 0$. This shows $\gamma_0\vec{U}|_{\Omega^c} = 0$. But by Lemma 2.12 we obtain

$$0 \;=\; [\gamma_0\vec{U}] \;=\; [\gamma_0 K_1^{(\omega)}\vec{u}] \;=\; -\vec{u}.$$

So $\mathcal{B}^{(\overline{c})}$ is injective and this proves the theorem. □

Remark: Theorem 3.8 shows in an analogous way that

$$\mathcal{B}^{(\overline{c})} \,:\, (L^p(0,T))^2 \longrightarrow (L^p(0,T))^2$$

is a Fredholm operator with index 0 for $p \in [1/\overline{z}(\overline{c}), \infty]$, see Lemma 3.4 and (3.55) for the definition of $\overline{z}(\overline{c})$. The inclusion $L^p(0,T) \subset L^2(0,T)$ proves that $\mathcal{B}^{(\overline{c})} \,:\, (L^p(0,T))^2 \longrightarrow (L^p(0,T))^2$ is an isomorphism for $p \in [2, \infty]$.

Theorem 3.18. *Let* $\vec{f} \in (C[0,T])^2$, $\vec{f}|_{[s_i,s_{i+1}]} \in \left(C^l[s_i,\, s_{i+1}]\right)^2$, $i = 0(1)n-1$, $l \in \mathbb{N}$, *and* $\vec{u} \in \left(L^2(0,T)\right)^2$ *be the solution of*

$$\mathcal{B}^{(\overline{c})}\vec{u} \;=\; \vec{f}.$$

Then $\vec{u}$ *has the following properties :*

a. $\vec{u} \in (C[0,T])^2$,

b. $\vec{u}|_{(s_i,s_{i+1})} \in \left(C^l(s_i,s_{i+1})\right)^2$, $i \in \{0,\ldots,n-1\}$,

c.

$$\begin{aligned}\vec{u}(s_i + t\,\delta) &\in \left(X^{2,l}_\rho(0,1)\right)^2 \dot{+}\mathbb{R}^2, \quad i \in \{0,\ldots,n-1\},\\ \vec{u}(s_i - t\,\delta) &\in \left(X^{2,l}_\rho(0,1)\right)^2 \dot{+}\mathbb{R}^2, \quad i \in \{1,\ldots,n\},\end{aligned}$$

$\rho \in [1/2 + \overline{z}((\overline{c})\,)$. *See Lemma 3.4 and (3.55) for the definition of* $\overline{z}(\overline{c})$.

Proof: Corollary 3.6 shows a. and Lemma 3.11 implies b. and c. □

4 On the numerical approximation of the solution of the Lamé equation

In this section we use a collocation method to approximate the solution of equation (3.4). We use piecewise polynomials on $[0,T]$, which are continuous and periodic on $[0,T]$. The meshes must be graded near the corners to get a good convergence rate and a cut off technique (i^*-trick, see [7] and [3]) has to be used to guarantee the stability of the method. The prove of stability here is not standard, because the operator is not strongly elliptic in L^2 (see the remark after Lemma 3.6).
First we introduce projections P_h on $(L^2(0,T))^2$ and projections Q_h on the reference space $(L^2(0,1))^4$ to construct the finite section approximations.
Let $h \in (0,\delta)$. The projector $P_h : (L^2(0,T))^2 \to (L^2(0,T))^2$ is defined by

$$(P_h\vec{u})(x) \;=\; \begin{cases} \vec{u}(x), & |x - s_i| > h,\ \forall i \in \{0,\ldots,n\} \\ 0, & \text{else} \end{cases} \tag{4.1}$$

This implies

$$\lim_{h\to 0} P_h \;=\; I_{(L^2(0,T))^2}, \text{ strongly.}$$

The finite section approximation for $\mathcal{B}^{(\omega)}$ (3.11) is defined by

$$\mathcal{B}^{(\omega)}_h \;:=\; P_h\,\mathcal{B}^{(\omega)}\,P_h. \tag{4.2}$$

Our first aim is to prove that $\mathcal{B}^{(\omega)}_h$ has an bounded inverse for $h < h_0$, where $h_0 > 0$ is some constant. To prove the stability we have to study the corresponding finite section approximation for $\overline{\mathcal{B}}^{(\omega)}_i$ (3.32). The projector $Q_h : (L^2(0,1))^4 \to (L^2(0,1))^4$, $h \in (0,1)$, is given by

$$(Q_h\vec{u})(x) \;=\; \begin{cases} \vec{u}(x), & x \geq h \\ 0, & x \leq h \end{cases}, \tag{4.3}$$

and the finite section approximation for $\overline{\mathcal{B}}_i^{(\omega)}$ by

$$\overline{\mathcal{B}}_{i,h}^{(\omega)} \;=\; Q_h\,\overline{\mathcal{B}}_i^{(\omega)}\,Q_h. \tag{4.4}$$

Lemma 4.1. *Let $\omega \in [0,1]$. There exists $h_0 > 0$ such that $\overline{\mathcal{B}}_{i,h}^{(\omega)}$ has an inverse in $Q_h\left(L^2(0,1)\right)^4$ for all $h < h_0$ and $i \in \{1,\dots,n\}$. There is a constant C for which*

$$\|(\overline{\mathcal{B}}_{i,h}^{(\omega)})^{-1}\|_{Q_h(L^2(0,1))^4} \;\leq\; C,\; h < h_0,\; i \in \{1,\dots,n\}.$$

Proof: The proof is based on the stability results for the finite section approximation for Wiener–Hopf operators in [9]. The finite section approximation of a Wiener–Hopf operator W is stable if the symbol matrix $\widehat{W}(s)$ has determinant different from zero, $s \in \mathbb{R}$, and if the left and right partial indices of the symbol matrix $\widehat{W}(s)$ are all zero ([9, VIII.6.2]). The left partial indices of $\widehat{W}(s)$ are the right partial indices of $\widehat{W}(-s)$ ([9, p. 222]). The vanishing of the right partial indices of $\widehat{W}(s)$ is equivalent to the invertibility of the operator W ([9, Theorem VIII.6.1]) .
Now we denote by W the Wiener–Hopf operator which corresponds to $\overline{\mathcal{B}}_i^{(\omega)}$. Then $\widehat{W}(s) = \widehat{\mathcal{B}}_i^{(\omega)}(1/2 + i\,s)$, $s \in \mathbb{R}$. Because $\overline{\mathcal{B}}_i^{(\omega)}$ is invertible all right partial indices of $W(s)$ vanish. We denote by $C_i^{(\omega)}$ the operator on $\left(L^2(0,1)\right)^4$ which has the symbol matrix $\widehat{\mathcal{B}}_i^{(\omega)}(1-s)$. Then the Wiener–Hopf operator W_1 which corresponds to $C_i^{(\omega)}$ has the symbol matrix $W(-s)$. If we can show that $C_i^{(\omega)}$ is invertible then the right indices of $W(-s)$ are zero and then the finite section approximation for W and so for $\overline{\mathcal{B}}_i^{(\omega)}$ is stable.
It remains to show that the opertor $C_i^{(\omega)} : \left(L^2(0,1)\right)^4 \to \left(L^2(0,1)\right)^4$ is invertible. If $Re(s) = 1$ then $\widehat{C}_i^{(\omega)}(s) = \widehat{\mathcal{B}}_i^{(\omega)}(1-s)$ is strongly elliptic (Lemma 3.6), i.e. all eigenvalues of $Re(\widehat{C}_i^{(\omega)}(s))$, $Re(s) = 1$, are greater than some positive constant. This implies

$$C_i^{(\omega)} \;:\; \left(L^1(0,1)\right)^4 \;\to\; \left(L^1(0,1)\right)^4$$

is invertible. Now we remark that $\det(\widehat{C}_i^{(\omega)}(s)) \neq 0$, for all s with $Re(s) \in [1/2, 1]$ (see Lemma 3.5). Therefore the operator $C_i^{(\omega)}$ is a Fredholm operator on $\left(L^p(0,1)\right)^4$ for all $p \in [1,2]$ and $\omega \in [0,1]$. For $\omega = 0$ we get the operator, which corresponds to the double layer potential (see Lemma 3.6). This operator is invertible in $\left(L^p(0,1)\right)^4$ and so $C_i^{(\omega)}$ is a Fredholm operator with index 0 for all $p \in [1,2]$, $\omega \in [0,1]$. But

$$N\left(C_i^{(\omega)}|_{(L^p(0,1))^4}\right) \;\subset\; N\left(C_i^{(\omega)}|_{(L^1(0,1))^4}\right) \;=\; \{0\},$$

which proves the invertibility of $C_i^{(\omega)}$, especially for $p = 2$. □

Theorem 4.2. *There exists an $h_0 > 0$ and a constant $C > 0$ such that for $h \in (0, h_0)$ the operator $\mathcal{B}_h^{(\overline{c})}$ is invertible in $P_h((L^2(0,T))^2)$ and*

$$\|(\mathcal{B}_h^{(\overline{c})})^{-1}\|_{P_h((L^2(0,T))^2)} \leq C.$$

Proof: By (3.11), (3.17)–(3.19) we get

$$\mathcal{B}_h^{(\overline{c})} = P_h\mathcal{B}^{(\overline{c},1)}P_h + P_h\mathcal{K}^{(\overline{c},2)}P_h$$

with compact $\mathcal{K}^{(\overline{c},2)}$, by Lemma 3.1 $\mathcal{B}^{(\overline{c})}$ is invertible by Theorem 3.17 and [9, II.3.1] shows that we only have to prove the invertibility of $P_h\mathcal{B}^{(\overline{c},1)}P_h$. We have

$$\begin{aligned} P_h\mathcal{B}^{(\overline{c},1)}P_h &= \Phi^{-1} \circ \mathcal{B}_{\pi,h}^{(\overline{c},1)} \circ \Phi, \text{ where} \\ \mathcal{B}_{\pi,h}^{(\overline{c},1)} : \vec{L}_\pi^2 &\to \vec{L}_\pi^2 \end{aligned}$$

has the following structure

$$\mathcal{B}_{\pi,h}^{(\overline{c},1)} = \begin{pmatrix} L_{1,h}^{(\overline{c})} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & L_{n,h}^{(\overline{c})} \end{pmatrix},$$

$$L_{i,h}^{(\overline{c})} = \begin{pmatrix} I & 0 & 0 \\ 0 & P_{i,h}^- & P_{i,h}^- K_{3i-1,3i}^{(\overline{c})} P_{i,h}^+ \\ 0 & P_{i,h}^+ K_{3i,3i-1}^{(\overline{c})} P_{i,h}^- & P_{i,h}^+ \end{pmatrix},$$

$$\begin{aligned} P_{i,h}^- \ (L^2(s_i-\delta,s_i))^2 &\to (L^2(s_i-\delta,s_i))^2 \\ P_{i,h}^+ \ (L^2(s_i,s_i+\delta))^2 &\to (L^2(s_i,s_i+\delta))^2 \end{aligned}$$

$$(P_{i,h}^-\vec{u})(x) = \begin{cases} \vec{u}(x), & x \in [s_i-\delta, s_i-h) \\ 0, & x \in [s_i-h, s_i] \end{cases}$$

$$(P_{i,h}^+\vec{u})(x) = \begin{cases} \vec{u}(x), & x \in [s_i+h, s_i+\delta) \\ 0, & x \in [s_i, s_i+h] \end{cases}.$$

It follows that we only have to show that

$$\begin{pmatrix} P_{i,h}^- & P_{i,h}^- K_{3i-1,3i}^{(\overline{c})} P_{i,h}^+ \\ P_{i,h}^+ K_{3i,3i-1}^{(\overline{c})} P_{i,h}^- & P_{i,h}^+ \end{pmatrix}$$

is invertible for h sufficiently small. But by the transformation Φ_i (3.28) this is equivalent to the invertibility of

$$\overline{\mathcal{B}}^{(\overline{c})}_{i,\frac{h}{\delta}}.$$

The invertibility of $\overline{\mathcal{B}}^{(\overline{c})}_{i,\frac{h}{\delta}}$ follows from Lemma 4.1 for all sufficiently small h. □

Lemma 4.3. *Let $\omega \in \mathbb{R}$, $i \in \{1, \ldots, n\}$. The operator*

$$xD\mathcal{K}_i^{(\omega)} : \left(L^2(0,1)\right)^4 \longrightarrow \left(L^2(0,1)\right)^4$$

is continuous.

Proof: The definition of $\mathcal{K}_i^{(\omega)}$ in (3.29) and the formulas (3.37)–(3.40) show that we only have to prove that

$$xDL_j : L^2(0,1) \longrightarrow L^2(0,1),\ j = 1(1)4,$$

is continuous, where

$$(L_j u)(x) := \int_0^1 l_j(\frac{x}{\tau})u(\tau)\,\frac{d\tau}{\tau},$$

(see (3.35) for the definitions of the l_j). For $u \in C_0^\infty(0,1)$ we have

$$\begin{aligned}(\widetilde{L}_j u)(x) &:= [xD(L_j u)](x) \\ &= \int_0^1 l_j'(\frac{x}{\tau})(\frac{x}{\tau})u(\tau)\frac{d\tau}{\tau}.\end{aligned}$$

This shows that $\widetilde{L}_j$ is a Mellin convolution with kernel $l_j'(s)s$. But $l_j'(s)s$ fulfills $(H1^p)$, $p \in (-1,1)$. This shows [5]

$$\widetilde{L}_j : L^2(0,1) \rightarrow L^2(0,1)$$

is continuous and the lemma is proved. □

Let $\Theta_{m,q} := (x_j^{(m,q)})_{j=0}^m$, $m \in \mathbb{N}$, $q > 0$, be the partition of the interval $[0,1]$ given by

$$x_j^{(m,q)} := \left(\frac{j}{m}\right)^q \tag{4.5}$$

and define

$$h_j^{(m,q)} := x_j^{(m,q)} - x_{j-1}^{(m,q)}, \quad j = 1(1)m. \tag{4.6}$$

Remark: We assume this special partition only for simplicity. All of the following statements are true for partitions $(x_j^{(m)})_{j=1}^m$, which fulfill

$$c_1 \left(\frac{j}{m}\right)^q \le x_j^{(m)} \le c_2 \left(\frac{j}{m}\right)^q,$$

$c_1, c_2, q > 0$, and c_1, c_2, q independent of j and m.

For a sequence $\Theta_{m,q}$, $q > 0$, we define by

$$\Pi_m^d := \Pi_m^d(\Theta_{m,q}) \tag{4.7}$$

the space of all continuous functions, which are piecewise polynomials with respect to the partition $\Theta_{m,q}$ and of degree smaller or equal to d. If we choose $(\xi_k)_{k=0}^d \subset [0,1]$, $0 = \xi_0 < \ldots < \xi_d = 1$, then the projector $P_m^d : C[0,1] \to \Pi_m^d$, $P_m^d = P_m^d(\Theta_{m,q}, (\xi_k)_{k=0}^d)$ is defined by

$$(P_m^d u)(x_j^{(m,q)} + \xi_k h_{j+1}^{(m,q)}) = u(x_j^{(m,q)} + \xi_k h_{j+1}^{(m,q)}), \tag{4.8}$$

$j = 0(1)m-1$, $k = 0(1)d$. For $j \ge 1$ we define

$$\Pi_{m,j}^d := \{u \in \Pi_m^d \mid u|_{[0,x_j^{(m,q)}]} = 0\} \tag{4.9}$$

and $P_{m,j}^d : C[0,1] \to C[0,1]$ by

$$(P_{mj}^d u)(x_l^{(m,q)} + \xi_k h_{l+1}^{(m,q)}) = \begin{cases} u(x_l^{(m,q)} + \xi_k h_{l+1}^{(m,q)}), & l = j+1(1)m-1, k = 0(1)d, \\ u(x_j^{(m,q)} + \xi_k h_{j+1}^{(m,q)}), & k = 1(1)d, \\ 0, & \text{else} \end{cases} \tag{4.10}$$

As a next step we introduce partitions of $[0,T]$ (see the beginning of section 3). For $q > 1$ we define a sequence of partitions $\Delta_{m,q} := (s_j^{(m,q)})_{j=0}^{3mn+1}$, $m \in \mathbb{N}$, of $[0,T]$ with

$$0 = s_0^{(m,q)} < \ldots < s_{3mn+1}^{(m,q)} = T, \tag{4.11}$$

by the demand that the $3mn+1$ real numbers

$$s_i + \delta \left(\frac{j}{m}\right)^q, \quad j = 0(1)m, \quad i = 0(1)n-1,$$
$$s_i + \delta + \frac{j}{m}(s_{i+1} - s_i - 2\delta), \quad j = 0(1)m, \quad i = 0(1)n-1,$$
$$s_i - \delta \left(\frac{j}{m}\right)^q, \quad j = 0(1)m, \quad i = 1(1)n,$$

are elements of $\{s_j^{(m,q)}\,|\,j=0(1)3mn+1\}$. The stepwidth $\delta_j^{(m,q)}$ is defined by

$$\delta_j^{(m,q)} \;:=\; s_j^{(m,q)} - s_{j-1}^{(m,q)}, \qquad j = 1(1)3mn+1. \tag{4.12}$$

Remark: Here we also consider this special mesh only for simplicity. For a sequence of partitions $(s_j^{(m)})_{j=0}^{M(m)}$, $M(m)\in\mathbb{N}$, $M(m)\sim m$, it is sufficient that the greatest stepwidth goes to zero like $1/m$ and that near the points s_i the mesh behaves like $s_i \pm (j/m)^q$. All results in this section are valid if this is fullfilled.

We define by

$$\widetilde{\Pi}_m^d \;=\; \widetilde{\Pi}_m^d(\Delta_{m,q}), \qquad d\in\mathbb{N}, \tag{4.13}$$

the space of all continuous functions on $C[0,T]$, which are piecewise polynomials with respect to $\Delta_{m,q}$ and of degree smaller or equal to d. Given $0=\xi_0<\xi_1<\ldots<\xi_d=1$ we define the projector $\widetilde{P}_m^d\,:\,C[0,T]\to\widetilde{\Pi}_m^d$ by

$$(\widetilde{P}_m^d u)(s_j^{(m,q)}+\xi_k\delta_{j+1}^{(m,q)}) \;=\; u(s_j^{(m,q)}+\xi_k\delta_{j+1}^{(m,q)}),\; j=0(1)3mn,\; k=0(1)d. \tag{4.14}$$

For $j\in\mathbb{N}$, $j\le m$, we define

$$\Xi_j^{(m,q)} \;:=\; [0,\left(\frac{j}{m}\right)^q]\cup\bigcup_{l=1}^{n-1}[s_l-\left(\frac{j}{m}\right)^q, s_l+\left(\frac{j}{m}\right)^q]\cup[T-\left(\frac{j}{m}\right)^q, T]$$

and a further projector R_j^m in $\left(L^2(0,T)\right)^2$ by

$$(R_j^m\vec{u})(x) \;=\; \begin{cases} \vec{u}(x), & x\in[0,T]\setminus\Xi_j^{(m,q)}, \\ 0, & x\in\Xi_j^{(m,q)}. \end{cases} . \tag{4.15}$$

Finally we define the modifications of the space $\widetilde{\Pi}_m^d$ and its projector

$$\widetilde{\Pi}_{m,j}^d \;:=\; \{u\in\widetilde{\Pi}_m^d\,|\,u|_{\Xi_j^{(m,q)}}\;\equiv\;0\,\} \tag{4.16}$$

and $\widetilde{P}_{m,j}^d\,:\,C[0,T]\to\widetilde{\Pi}_{m,j}^d$ by

$$(\widetilde{P}_{m,j}^d u)(s_l^{(m,q)}+\xi_k\delta_{l+1}^{(m,q)}) \;=\; \begin{cases} u(s_l^{(m,q)}+\xi_k\delta_{l+1}^{(m,q)}), & s_l^{(m,q)}+\xi_k\delta_{l+1}^{(m,q)}\in[0,T]\setminus\Xi_j^{(m,q)} \\ 0, & \text{else.} \end{cases} \tag{4.17}$$

$\forall l, k$. All of the above spaces and projectors can be defined for functions with values in $\mathbb{R}^l$, $l \in \mathbb{N}$.

Remark: If we look at the proof of Theorem 4.2 then it is clear that for a fixed $j \geq 1$ we have

$$\|R_j^m(I + \mathcal{K}^{(\overline{c})})\vec{u}\|_{\left(L^2([0,T]\setminus\Xi_j^{(m,q)})\right)^2} \;\; \geq \;\; c\,\|\vec{u}\|_{\left(L^2([0,T]\setminus\Xi_j^{(m,q)})\right)^2}, \tag{4.18}$$

$\forall \vec{u} \in \left(L^2([0,T] \setminus \Xi_j^{(m,q)})\right)^2$, m sufficiently large.

Lemma 4.4. *Let $q > 0$ and $0 = \xi_0 < \xi_1 < \ldots < \xi_d = 1$ be given. For every $\varepsilon > 0$ there is an $\widetilde{i} \geq 1$ and $\widetilde{m} \geq 1$ such that*

$$\|(I - P_{m,j}^d)u\|_{L^2(x_j^{(m,q)},1)} \;\; \leq \;\; \varepsilon\|u\|_{X_0^{2,1}(0,1)}, \; u \in X_0^{2,1}(0,1),$$

$j \geq \widetilde{i}$, $m \geq \widetilde{m}$.

Proof: For $u \in X_0^{2,1}(0,1)$ we have $u \in H^1[\alpha, 1]$, $\forall \alpha > 0$. $P_{m,j}^d u$ is well defined for $j \geq 1$. If $m \in \mathbb{N}$, $j \geq 1$, we get

$$\begin{aligned} \|(I - P_{m,j}^d)u\|^2_{L^2(x_l^{(m,q)}, x_{l+1}^{(m,q)})} \;\; &\leq \;\; c(h_{l+1}^{(m,q)})^2 \int_{x_l^{(m,q)}}^{x_{l+1}^{(m,q)}} u'(x)^2 \, dx \\ &\leq \;\; c\left(\frac{h_{l+1}^{(m,q)}}{x_l^{(m,q)}}\right)^2 \int_{x_l^{(m,q)}}^{x_{l+1}^{(m,q)}} (xu'(x))^2 \, dx, \end{aligned}$$

where c depends only on $(\xi_k)_k$ (see [5, Section 2]). For $\varepsilon > 0$ there exists an $i^*(\varepsilon) \geq 1$ and $m^*(\varepsilon)$ such that

$$\left(\frac{h_{l+1}^{(m,q)}}{x_l^{(m,q)}}\right)^2 \;\; \leq \;\; \frac{\varepsilon^2}{c}, \qquad l \geq i^* \text{ and } m \geq m^*,$$

and therefore

$$\int_{x_l^{(m,q)}}^{x_{l+1}^{(m,q)}} \left((I - P_{m,j}^d)u\right)^2 \, dx \;\; \leq \;\; \varepsilon^2 \int_{x_l^{(m,q)}}^{x_{l+1}^{(m,q)}} (xu'(x))^2 \, dx.$$

Summation over l gives

$$\int_{x_j^{(m,q)}}^1 \left((I-P_{m,j}^d)u\right)^2\,dx \;\leq\; \varepsilon^2 \int_{x_j^{(m,q)}}^1 (xu'(x))^2\,dx,\; j\geq i^*,$$

and this implies

$$\begin{aligned}\|(I-P_{m,j}^d)u\|_{L^2(x_j^{(m,q)},1)} &\leq \varepsilon\|xu'\|_{L^2(x_j^{(m,q)},1)}\\ &\leq \varepsilon\|u\|_{X_0^{2,1}(0,1)},\end{aligned}$$

$\forall j\geq i^*$, $m\geq m^*$. This shows the lemma with $\widetilde{i}=i^*$ and $\widetilde{m}=m^*$. □

Lemma 4.5. *Let $\omega\in\mathbb{R}$, $q>0$, and $0=\xi_0<\xi_1<\ldots<\xi_d=1$. For every $\varepsilon>0$ there are $\widetilde{i}(\varepsilon)\geq 1$ and $\widetilde{m}(\varepsilon)\geq 1$ such that*

$$\|(I-\widetilde{P}_{m,j}^d)\mathcal{K}^{(\omega)}\vec{u}\|_{\left(L^2([0,T]\setminus\Xi_j^{(m)})\right)^2} \;\leq\; \varepsilon\|\vec{u}\|_{(L^2(0,T))^2},$$

$j\geq\widetilde{i}$, $m\geq\widetilde{m}$.

Proof: First we write

$$(I-\widetilde{P}_{m,j}^d)\mathcal{K}^{(\omega)}\vec{u} \;=\; (I-\widetilde{P}_{m,j}^d)\mathcal{K}^{(\omega,1)}\vec{u}+(I-\widetilde{P}_{m,j}^d)\mathcal{K}^{(\omega,2)}\vec{u}$$

(see (3.11), (3.19), (3.20)). By Lemma 3.1.b. we know that

$$\mathcal{K}^{(\omega,2)}\;:\;\left(L^2(0,T)\right)^2 \;\longrightarrow\; \prod_{i=0}^{n-1}\left(C^1[s_i,s_{i+1}]\right)^2$$

is continuous. By the definition of $\Delta_{m,q}$ we know that there is a $m_0^*(\varepsilon)\geq 1$, such that

$$\|(I-\widetilde{P}_{m,j}^d)\mathcal{K}^{(\omega,2)}\vec{u}\|_{\left(L^2(\Xi_j^{(m,q)})\right)^2} \;\leq\; \frac{\varepsilon}{2}\|\vec{u}\|_{(L^2(0,T))^2},\quad m\geq m_0^*(\varepsilon). \tag{4.19}$$

But $(\mathcal{K}^{(\omega,1)}\vec{u})(x)$ is different from zero only for $x\in[s_i-\delta,s_i+\delta]$. By the transformation (3.28) we can apply Lemma 4.4 and get for $\eta>0$ an $i_1^*(\eta)\geq 1$ and $m_1^*(\eta)\geq 1$ such that

$$\begin{aligned}\|(I-\widetilde{P}_{m,j}^d)\mathcal{K}^{(\omega,1)}\vec{u}\|_{(L^2(s_i-\delta,s_i+\delta))^2} &\leq \eta\|\overline{\mathcal{K}}_i^{(\omega)}\overline{\Phi}_i\vec{u}\|_{\left(X_0^{2,1}(0,1)\right)^4}\\ &\leq \eta c\|\overline{\Phi}_i\vec{u}\|_{(L^2(0,1))^4}\\ &\leq \eta\widetilde{c}\|\vec{u}\|_{(L^2(s_i-\delta,s_i+\delta))^2},\;\; i\geq i_1^*,\, m\geq m_1^*.\end{aligned}$$

Here the constant c is given by Lemma 4.3 and the constant $\widetilde{c}$ depends on c and δ, because the transformation $\overline{\Phi}_i$ (see (3.12)) depends on δ. So we can choose $i^*(\varepsilon) \geq 1$ and $m_2^*(\varepsilon) \geq 1$ with

$$\|(I - \widetilde{P}^d_{m,j})\mathcal{K}^{(\omega,1)}\vec{u}\|_{(L^2(s_i-\delta,s_i+\delta))^2} \leq \frac{\varepsilon}{2}\|\vec{u}\|_{(L^2(s_i-\delta,s_i+\delta))^2}, \quad i \geq i^*(\varepsilon),\ m \geq m_2^*(\varepsilon).$$

(4.19) and (4.20) together prove the theorem for $m^*(\varepsilon) := \{m_1^*(\varepsilon), m_2^*(\varepsilon)\}$. □

Now we can prove the stability of our modified collocation method $\widetilde{P}^d_{m,j}(I + \mathcal{K}^{(\omega)})\widetilde{P}^d_{m,j}$ if j is sufficiently large.

Theorem 4.6. *Let $q > 0$ and $0 = \xi_0 < \ldots < \xi_d = 1$. There exist i^*, $m^* \in \mathbb{N}$, such that for all $i \geq i^*$*

$$\|\widetilde{P}^d_{m,j}(I + \mathcal{K}^{(\overline{c})})\vec{u}\|_{(L^2(0,T))^2} \geq c\|\vec{u}\|_{(L^2(0,T))^2},$$

$\vec{u} \in \widetilde{\Pi}^d_{m,i}$, $m \geq m^*$, where the constant $c > 0$ does not depend on $\vec{u}$ or m.

Proof: By Theorem 4.2 there is a $h_0 > 0$, such that

$$\|P_h(I + \mathcal{K}^{(\overline{c})})P_h\vec{u}\|_{(L^2(h,T))^2} \geq c\|\vec{u}\|_{(L^2(h,T))^2},\ h \in (0, h_0), \tag{4.20}$$

where c is independent of $\vec{u}$ and h. By Lemma 4.5 exist i^* and m^* with

$$\|(I - \widetilde{P}^d_{m,j})\mathcal{K}^{(\overline{c})}\vec{u}\|_{\left(L^2([0,T]\setminus\Xi_j^{(m)})\right)^2} \leq \frac{c}{2}\|\vec{u}\|_{(L^2(0,T))^2},$$

$m \geq m^*$, $j \geq i^*$. Now we fix j. For $m \geq m_1^*$ the projector R_j^m (see (4.15)) also fulfills the inequality (4.20) (see the remark before Lemma 4.4). We further have

$$R_j^m \circ \widetilde{P}^d_{m,j} = \widetilde{P}^d_{m,j}. \tag{4.21}$$

For $m \geq \max\{m^*, m_1^*\}$ we get for $\vec{u} \in \left(\widetilde{\Pi}^d_{m,j}\right)^2$

$$\begin{aligned}
\|\widetilde{P}^d_{m,j}(I + \mathcal{K}^{(\overline{c})})\vec{u}\|_{(L^2(0,T))^2} &\overset{(4.21)}{\geq} \|R_j^m(I + \mathcal{K}^{(\overline{c})})R_j^m\vec{u}\|_{(L^2(0,T))^2} \\
&\qquad - \|(\widetilde{P}^d_{m,j} - R_j^m)(I + \mathcal{K}^{(\overline{c})})R_j^m\vec{u}\|_{(L^2(0,T))^2} \\
&\overset{(4.20)}{\geq} c\|R_j^m\vec{u}\|_{(L^2(0,T))^2} - \|(\widetilde{P}^d_{m,j} - R_j^m)\mathcal{K}^{(\overline{c})}R_j^m\vec{u}\|_{(L^2(0,T))^2} \\
&= c\|\vec{u}\|_{(L^2(0,T))^2} - \|(\widetilde{P}^d_{m,j} - I)\mathcal{K}^{(\overline{c})}R_j^m\vec{u}\|_{\left(L^2([0,T]\setminus\Xi_j^{(m)})\right)^2} \\
&\overset{\text{Lemma 4.5}}{\geq} \frac{c}{2}\|\vec{u}\|_{(L^2(0,T))^2}.
\end{aligned}$$

$\square$

Lemma 4.7. *Let* $\vec{u} \in (L^2(0,T))^2$,

$$\begin{aligned} \vec{u}|_{[s_i,s_{i+1}]} &\in \left(C^l[s_i,s_{i+1}]\right)^2, \; i = 0(1)n-1, \\ \vec{v}_i^+(x) := \vec{u}(s_i + x\,\delta) &\in X^{2,l}_{\widetilde{\rho}_i}(0,1)\dot{+}\mathbb{R}^2, \; i = 0(1)n-1, \\ \vec{v}_i^-(x) := \vec{u}(s_i - x\,\delta) &\in X^{2,l}_{\widetilde{\rho}_i}(0,1)\dot{+}\mathbb{R}^2, \; i = 1(1)n, \end{aligned}$$

(see the beginning of section 3. for the meaning of s_i *and* δ*)* $\widetilde{\rho}_i \in (1, 3/2)$, $i = 0(1)n$, $0 = \xi_0 < \ldots < \xi_d = 1$, $d \in \mathbb{N}$, $r := \min\{l, d+1\}$, *and* $q \geq 2r$. *For* $j^* \in \mathbb{N}$ *we have*

$$\|(I - \widetilde{P}^d_{m,j^*})\vec{u}\|_{(L^2(0,T))^2} \leq \frac{c(\vec{u}, j^*)}{m^r}.$$

Proof: By the triangle inequality we get

$$\begin{aligned} \|(I - \widetilde{P}^d_{m,j^*})\vec{u}\|_{(L^2(0,T))^2} \leq & \sum_{j=0}^{n-1} \|(I - \widetilde{P}^d_{m,j^*})\vec{u}\|_{(L^2(s_j+\delta, s_{j+1}-\delta))^2} \\ & + \sum_{j=0}^{n-1} \|(I - \widetilde{P}^d_{m,j^*})\vec{u}\|_{(L^2(s_j, s_j+\delta))^2} \\ & + \sum_{j=1}^{n} \|(I - \widetilde{P}^d_{m,j^*})\vec{u}\|_{(L^2(s_j-\delta, s_j))^2}. \end{aligned}$$

For the first summand we get

$$\left|(I - \widetilde{P}^d_{m,j^*})\vec{u}(x)\right| \leq \frac{c_1(\vec{u})}{m^r}, \quad x \in [s_i + \delta, s_{i+1} - \delta], \tag{4.22}$$

because here $\vec{u}$ is a C^l–function, $l \geq r$.
The terms in the second and third summand can all be estimated by the approximation error for $\vec{v}_i^{\pm}(x)$ on $(0,1)$. We look at one term in the second summand and because of $2r \geq r/\widetilde{\rho}_i$ we get by [5, Lemma 2.20]

$$\|(I - P^d_{m,j^*})\vec{v}_i^+\|_{\left(L^2(x^{(m,q)}_{j^*},1)\right)^2} \leq \frac{c_2}{m^r}\|\vec{v}_i^+\|_{X^{2,l}_{\widetilde{\rho}_i}(0,1)}, \tag{4.23}$$

where the constant c_2 depends only on $(\xi_k)_{k=0}^d$. Now we notice

$$\vec{v}_i^+ = \vec{w}_i^+ + v_0^+, \; \vec{w}_i^+ \in X^{2,l}_{\widetilde{\rho}_i}(0,1), \; v_0^+ \in \mathbb{R}^2.$$

We get

$$
\begin{aligned}
\|(I-P^d_{m,j^*})\vec{v}_i^+\|_{\left(L^2(0,x_{j^*}^{(m,q)})\right)^2} &= \|\vec{v}_i^+\|_{\left(L^2(0,x_{j^*}^{(m,q)})\right)^2} \\
&\leq \left(\int_0^{x_{j^*}^{(m,q)}} \|\vec{w}_i^+(x)\|^2\,dx\right)^{\frac{1}{2}} + \left(\int_0^{x_{j^*}^{(m,q)}} \|\vec{v}_0^+\|^2\,dx\right)^{\frac{1}{2}} \\
&\leq \left((x_{j^*}^{(m,q)})^{2\widetilde{\rho}_i} \int_0^{x_{j^*}^{(m,q)}} \left(x^{-\widetilde{\rho}_i}\|\vec{w}_i^+(x)\|\right)^2\,dx\right)^{\frac{1}{2}} + \\
&\quad\ (x_{j^*}^{(m,q)})^{\frac{1}{2}}\|\vec{v}_0^+\| \\
&\leq c_3(\vec{v}_i^+)(x_{j^*}^{(m,q)})^{\frac{1}{2}} \\
&\overset{q\geq 2r}{\leq} c_4(\vec{v}_i^+)\frac{1}{m^r}.
\end{aligned} \tag{4.24}
$$

Now (4.22)–(4.24) prove the lemma. □

Remark: In the proof of Lemma 4.7 we see that the high grading exponent $2r$ is only necessary for the proof of (4.24). If it is possible to prove stability for a modified projector P^d_{m,j^*} where the functions are constant (but not necessarily zero) in a vicinity of zero, then a grading exponent $r/\widetilde{\rho}_i$ would be sufficient for the approximation result in Lemma 4.7.

For $\vec{f} \in (C[0,T])^2$ and $q > 0$ we denote by $\vec{u}_m$ the solution of the collocation equation

$$\widetilde{P}^d_{m,j^*}(I+\mathcal{K}^{(\overline{c})})\vec{u}_m = \widetilde{P}^d_{m,j^*}\vec{f}. \tag{4.25}$$

The next Theorem shows that for m large enough $\vec{u}_m$ is well defined and we get an estimate for the error.

Theorem 4.8. *Let $\vec{f} \in (C[0,T])^2$, $\vec{f}|_{[s_i,s_{i+1}]} \in \left(C^l[s_i,s_{i+1}]\right)^2$, $i = 0(1)n-1$, $l \in \mathbb{N}$. We denote by $\vec{u}$ the solution of*

$$(I+\mathcal{K}^{(\overline{c})})\vec{u} = \vec{f}$$

(see Theorem 3.17). Let $0 = \xi_0 < \xi_1 < \ldots < \xi_d = 1$, $d \in \mathbb{N}$, $r := \min\{l, d+1\}$, and $q \geq 2r$. There exists an $i^* \in \mathbb{N}$, such that for $j^* \geq i^*$ and all sufficiently large m the equation (4.25) has a solution $\vec{u}_m$ and we get

$$\|\vec{u}-\vec{u}_m\|_{(L^2(0,T))^2} \leq \frac{c}{m^r}.$$

Proof: By Theorem 4.6 there exists an $i^* \in \mathbb{N}$, such that for $j^* \geq i^*$

$$\|\widetilde{P}^d_{m,j^*}(I+\mathcal{K}^{(\overline{c})})\vec{v}\|_{(L^2(0,T))^2} \geq c\|\vec{v}\|_{(L^2(0,T))^2},\ \forall \vec{v} \in \widetilde{\Pi}^d_{m,j^*}, \tag{4.26}$$

$m \geq m^*$, $c > 0$. Because $\widetilde{\Pi}^d_{m,j^*}$ is finite dimensional this shows the solvability of (4.25) and we get by the triangle inequality

$$\|\vec{u}-\vec{u}_m\|_{(L^2(0,T))^2} \leq \|\vec{u}-\widetilde{P}^d_{m,j^*}\vec{u}\|_{(L^2(0,T))^2} + \|\widetilde{P}^d_{m,j^*}\vec{u}-\vec{u}_m\|_{(L^2(0,T))^2}. \tag{4.27}$$

For the second summand we get by (4.26)

$$\begin{aligned}
\|\widetilde{P}^d_{m,j^*}\vec{u} - \vec{u}_m\|_{(L^2(0,T))^2} &\le \frac{1}{c}\|\widetilde{P}^d_{m,j^*}(I+\mathcal{K}^{(\overline{c})})\widetilde{P}^d_{m,j^*}\vec{u} - \widetilde{P}^d_{m,j^*}(I+\mathcal{K}^{(\overline{c})})\vec{u}_m\|_{(L^2(0,T))^2} \\
&= \frac{1}{c}\|\widetilde{P}^d_{m,j^*}(I+\mathcal{K}^{(\overline{c})})\widetilde{P}^d_{m,j^*}\vec{u} - \widetilde{P}^d_{m,j^*}\vec{f}\|_{(L^2(0,T))^2} \\
&= \frac{1}{c}\|\widetilde{P}^d_{m,j^*}(I+\mathcal{K}^{(\overline{c})})\widetilde{P}^d_{m,j^*}\vec{u} - \widetilde{P}^d_{m,j^*}(I+\mathcal{K}^{(\overline{c})})\vec{u}\|_{(L^2(0,T))^2} \\
&= \frac{1}{c}\|\widetilde{P}^d_{m,j^*}\mathcal{K}^{(\overline{c})}(\widetilde{P}^d_{m,j^*}\vec{u} - \vec{u})\|_{(L^2(0,T))^2} \\
&\le c_1\|\widetilde{P}^d_{m,j^*}\vec{u} - \vec{u}\|_{(L^2(0,T))^2} \qquad (4.28)
\end{aligned}$$

Here the continuity of $\widetilde{P}^d_{m,j^*}\mathcal{K}^{(\overline{c})}$ (see Lemma 4.5) has been used. (4.27) and (4.28) now give

$$\|\vec{u} - \vec{u}_m\|_{(L^2(0,T))^2} \le (1+c_1)\|\vec{u} - \widetilde{P}^d_{m,j^*}\vec{u}\|_{(L^2(0,T))^2},$$

but $\vec{u}$ fulfills the assumptions of Lemma 4.7 by Lemma 3.11 ($\widetilde{\rho}_i = \rho$, $i = 1(1)n$, $\rho \in [1, 1/2 + \overline{z}(\overline{c}))$) and this proves Theorem 4.8. □

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