TYPICAL PROFILES OF THE KAC-HOPFIELD MODEL

A. Bovier†, V. Gayrard‡ and P. Picco||

1. Introduction

Mean Field models, random or not, are very important to explain in a simple way the general phenomenon of phase transitions. However, for random systems, in general, their analysis is, as many of the contributions in this volume confirm, not simple at all, a fact which may justify the amount of effort spent on them. In spite of all that, mean fields models are in many respects only poor caricatures of realistic systems[1]and are unable to feature some of the most important aspects of those; in particular, in a phase transition regime, they are unable to properly account for the phenomenon of *phase separation*, i.e. the observed feature that states of the system where two or more phases coexist in seperate regions of space. This deficiency manifests itself also in the fact that the canonical free energy is generaly *not* a convex function of the order parameters, which in term means that the usual formalism of thermodynamics cannot imediately used (e.g. the isotherms are not monotone and thus cannot directly be used to determine the equations of state, and insisting on doing so would produce totally unphysical effect, like regions of parameters where the pressure is a decreasing function of the density) is solved in by the Maxwell-construction, by which the free energy is simply replaced ad hoc by its convex hull.

A step beyond mean field theory that allows one to incorporate the phase separation phenomenon and more generally geometric effects in phase transitions are Ginzburg-Landau or "phase-field models" (for a recent exposition see e.g. [BS]). While they are of immense practical importance, they are derived in an ad hoc way as models on a mesoscopic scale, with general thermodynamic and symmetry consideration as

† WIAS, Mohrenstr. 39 D-10117 Berlin, Germany.
e-mail: bovier@wias-berlin.de

‡ CPT CNRS Case 907, Luminy, 13288 Marseille, France.
e-mail: gayrard@cpt.univ-mrs.fr

|| CPT CNRS Case 907, Luminy, 13288 Marseille, France.
e-mail: picco@cpt.univ-mrs.fr

[1] This is not only true if physical systems are modelled, but also for models of neural or other networks. These typically have some underlying "geometry" or "architecture" that mean field models do not take into account properly.

main guiding principles and are not derived from microscopic Gibbsian theories. For disordered systems, such theories are still in an embryonic state. It is thus greatly desirable to have microscopic models at hand which allow the exact and rigorous computation of the Ginburg-Landau free energy functionals, just as the Curie-Weiss model allows the derivation of the van der Waals free energy. These models have been introduced by M. Kac in the mid sixties, originally with the main intention to give a rigorous derivation for Maxwell-construction [KUH, LP]. Kac models are characterized by interactions of strength of order γ but of range of order γ^{-1}, γ being a small parameter. But by solving the non-convexity pathology, the possibility of phase separation was as well re-established, and in fact the most appealing feature of Kac-models from a modern perspective is their close relation to Ginzburg-Landau type theories. This aspect was investigated in great depth over the last years, both from a static and dynamic point of view by Cassandro, de Masi, Orlandi, Presutti, Triolo and others [ABCP, BCP, BPRS, CP, CMP, COP, DGP, DMOPT1-6, LOP, OP, P1, P2]. In [COP], in particular, the structure of the typical mesoscopic configurations of the system was analyzed in great detail and a large deviation principle was proven where a Ginzburg-Landau free energy functional appears as rate function. The multidimensional ferromagnetic case is as to now not so well understood, but is a very active line of research [ABCP, BCP, BZ, CP]. There are a lot of people working on this subject and new results will certainly come soon.

Kac models are thus the natural candidates to study if results on disordered mean field systems are to be extended to more realistic situations. As far as we know the first Kac version of disordered system was considered by Pastur and Figotin [FP] for what is known as the Hopfield model [Ho]. However they considered a finite number of patterns and only obtained the convergence of the free energy to that of the mean field model as γ tends to zero. We studied the extension to a number of pattern that diverge in [BGP3] and also proved a Lebowitz-Penrose theorem, i.e. we showed that the free energy function (as a function of the overlap parameter) converges to the convex hull of that of the Hopfield model as γ tends to one. A first step in the study of typical configurations was done in [BGP4]. There is however a lot of work to do on this model and there are more open questions than real problems solved[2]. In the present paper we focus on reviewing the results and methods in [BGP4].

[2] While this review was being written, a number of papers on other disordered Kac models has appeared. We mention a site-diluted model [B].

Let us start by defining our model. Let $(\Omega, \mathcal{F}, I\!P)$ be a probability space. Let $\xi \equiv \{\xi_i^\mu\}_{i\in \mathbb{Z}, \mu\in I\!N}$ be a two-parameter family of independent, identically distributed random variables defined on this space such that $I\!P(\xi_i^\mu = 1) = I\!P(\xi_i^\mu = -1) = \frac{1}{2}$.

We denote by σ a function $\sigma : \mathbb{Z} \to \{-1, 1\}$ and call σ_i, $i \in \mathbb{Z}$ the spin at site i. We denote by $\mathcal{S}$ the space of all such functions, equipped with the product topology of the discrete topology in $\{-1, 1\}$.

Let $J_\gamma(i-j) \equiv \gamma J(\gamma|i-j|)$, and

$$J(x) = \begin{cases} 1, & \text{if } |x| \leq 1/2 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

Note that other choices for the function $J(x)$ are possible. They must satisfy the conditions $J(x) \geq 0$, $\int dx J(x) = 1$, and must decay rapidly to zero on a scale of order unity. For example, the original choice of Kac [KUH] was $J(x) = e^{-|x|}$ and he used in a crucial way the fact that it is the covariance of the Ornstein-Ulhenbeck Process to write the Boltzmann factor as the Laplace transform of this process. That is he used what is called the Hubbard-Stratonovich transformation.

The interaction between two spins at sites i and j will be chosen for given $\omega \in \Omega$, as

$$-\frac{1}{2} \sum_{\mu=1}^{M(\gamma)} \xi_i^\mu[\omega]\xi_j^\mu[\omega] J_\gamma(i-j)\sigma_i\sigma_j \tag{1.2}$$

and the *formal* Hamiltonian will be

$$H_\gamma[\omega](\sigma) = -\frac{1}{2} \sum_{(i,j)\in \mathbb{Z}\times\mathbb{Z}} \sum_{\mu=1}^{M(\gamma)} \xi_i^\mu[\omega]\xi_j^\mu[\omega] J_\gamma(i-j)\sigma_i\sigma_j \tag{1.3}$$

Note that the parameter γ introduces a natural length scale γ^{-1} into our model which is the distance over which spins interact directly. We will be interested later in the behavior of the system on that and larger scales and will refer to it as the *macroscopic* scale, whereas the sites i of the underlying lattice $\mathbb{Z}$ are referred to as the *microscopic* scale. In the course of our analysis we will have to introduce two more intermediate, *mesoscopic* scales, as it shall be explained later. We find it convenient to measure distances and to define finite volumes in the macroscopic rather than the microscopic scale, as this allows to deal with volumes that actually do not change with γ.

Let $\Lambda = [\lambda_-, \lambda_+] \subset I\!R$ be a *macroscopic* interval on the real line. For points $i \in Z\!\!Z$ referring to sites on the microscopic scale we will write

$$i \in \Lambda \quad \textit{iff} \quad \lambda_- \leq \gamma i \leq \lambda_+ \tag{1.4}$$

Note that we will stick very strictly to the convention that the letters i, j, k *always* refer to microscopic sites. The Hamiltonian corresponding to a volume Λ (with free boundary conditions) can then be written as

$$H_{\gamma,\Lambda}[\omega](\sigma) = -\frac{1}{2} \sum_{(i,j)\in\Lambda\times\Lambda} \sum_{\mu=1}^{M(\gamma)} \xi_i^\mu[\omega]\xi_j^\mu[\omega] J_\gamma(i-j)\sigma_i\sigma_j \tag{1.5}$$

We shall also write $\mathcal{S}_\Lambda \equiv \times_{i\in\Lambda}\{-1,1\}$ and denote its elements by σ_Λ. The *interaction* between the spins in Λ and those outside Λ will be written as

$$W_{\gamma,\Lambda}[\omega](\sigma_\Lambda, \sigma_{\Lambda^c}) = -\sum_{i\in\Lambda}\sum_{j\in\Lambda^c}\sum_{\mu=1}^{M(\gamma)} \xi_i^\mu[\omega]\xi_j^\mu[\omega] J_\gamma(i-j)\sigma_i\sigma_j \tag{1.6}$$

The finite volume Gibbs measure for such a volume Λ with fixed external configuration σ_{Λ^c} is then defined by assigning to each $\sigma_\Lambda \in \mathcal{S}_\Lambda$ the mass

$$\mathcal{G}_{\beta,\gamma,\Lambda}^{\sigma_{\Lambda^c}}[\omega](\sigma_\Lambda) \equiv \frac{1}{Z_{\beta,\gamma,\Lambda}^{\sigma_{\Lambda^c}}[\omega]} e^{-\beta[H_{\gamma,\Lambda}[\omega](\sigma_\Lambda)+W_{\gamma,\Lambda}[\omega](\sigma_\Lambda,\sigma_{\Lambda^c})]} \tag{1.7}$$

where $Z_{\beta,\gamma,\Lambda}^{\sigma_{\Lambda^c}}[\omega]$ is a normalizing factor usually called *partition function.* We will also denote by

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](\sigma_\Lambda) \equiv \frac{1}{Z_{\beta,\gamma,\Lambda}[\omega]} e^{-\beta H_{\gamma,\Lambda}[\omega](\sigma_\Lambda)} \tag{1.8}$$

the Gibbs measure with free boundary conditions. It is crucial to keep in mind that we are always interested in taking the infinite volume limit $\Lambda \uparrow I\!R$ first for fixed γ and to study the asymptotic of the result as $\gamma \downarrow 0$ (this is sometimes referred to as the 'Lebowitz-Penrose limit' even if 'Kac limit' is more appropriate from an historical point of view). In [BGP2] we have studied the distribution of the global 'overlaps' $m_\Lambda^\mu(\sigma) \equiv \frac{\gamma}{|\Lambda|}\sum_{i\in\Lambda}\xi_i^\mu\sigma_i$ under the Gibbs measure (1.7). Here we want to analyze the distribution of *local* overlaps. To do this we will actually have to introduce *two* intermediate *mesoscopic* length scales, $1 \ll \ell(\gamma) \ll L(\gamma) \ll \gamma^{-1}$. Note that both $\ell(\gamma)$ and $L(\gamma)$ will tend to infinity as $\gamma \downarrow 0$ while $\ell(\gamma)/L(\gamma)$ as well as $\gamma L(\gamma)$ tend to zero. We will assume to avoid an exaggerate use of integer part that ℓ, L and γ^{-1} are integer multiples of each other.

To simplify notations, the dependence on γ of ℓ and L will not be made explicit in the sequel. We now divide the real line into boxes of length $\gamma\ell$ and γL, respectively, with the first box, called 0 being centered at the origin. The boxes of length $\gamma\ell$ will be called x, y, or z, and labeled by the integers. That is, the box x is the interval of length $\gamma\ell$ centered at the point $\gamma\ell x$. No confusion should arise from the fact that we use the symbol x as denoting both the box *and* its label, since again x, y, z are used *exclusively* for this type of boxes. In the same way, the letters r, s, t are reserved for the boxes of length γL, centered at the points $\gamma L \mathbb{Z}$, and finally we reserve u, v, w for boxes of length one centered at the integers. With these conventions, it makes sense to write e.g. $i \in x$ shorthand for $\ell x - \ell/2 \leq i \leq \ell x + \ell/2$, etc.

In this spirit we define the $M(\gamma)$ dimensional vector $m_\ell(x,\sigma)$ and $m_L(r,\sigma)$ whose μ-th components are

$$m_\ell^\mu(x,\sigma) \equiv \frac{1}{\ell} \sum_{i\in x} \xi_i^\mu \sigma_i \tag{1.9}$$

and

$$m_L^\mu(r,\sigma) \equiv \frac{1}{L} \sum_{i\in r} \xi_i^\mu \sigma_i \tag{1.10}$$

respectively. They are called the *local* overlaps. Note that we have, for instance, that

$$m_L^\mu(r,\sigma) = \frac{\ell}{L} \sum_{x\in r} m_\ell^\mu(x,\sigma) \tag{1.11}$$

We will also have to be able to indicate the box on some larger scale containing a specified box on the smaller scale. Here we write simply, e.g., $r(x)$ for the unique box of length L that contains the box x of length ℓ. Expressions like $x(i)$, $u(y)$ or $s(k)$ have corresponding meanings.

The rôle of the different scales will be the following. We are interested in the typical (with respect to the Gibbs measure) profiles of the overlaps on the scale L, *i.e.* the function $r \to m_L(r,\sigma)$ for configurations of σ that are typical for the Gibbs measure. We will control these functions within volumes on the macroscopic scale γ^{-1}. The smaller mesoscopic scale ℓ enters here to express our system, on this scale, up to some errors as an Hopfield model on each block of length ℓ with interactions between these blocks. We will see that it is quite crucial to use a much smaller scale for that approximation than the scale on which we want to control the local overlaps. This was noted already in [COP].

We want to study the probability distribution induced by the Gibbs measure on the functions $r \to m_L(r)$ through the map defined by (1.10). The corresponding measure space is for fixed γ simply the discrete space $\mathcal{T}_\gamma \equiv \{-1,-1+2/L,\dots,1-2/L,1\}^{M(\gamma)\times \mathbb{Z}}$, which should be equipped with the product topology. Since this topology is quite non-uniform with respect to γ (note that both L and M tend to infinity as $\gamma \downarrow 0$), this is, however, not well adapted to take the limit $\gamma \downarrow 0$. Thus we replace the discrete topology on $\{-1,-1+2/L,\dots,1-2/L,1\}^{M(\gamma)}$ by the Euclidean ℓ_2-topology (which remains meaningful in the limit $\gamma \downarrow 0$) and the product topology corresponding to $\mathbb{Z}$ is replaced by the weak local L_2 topology w.r.t. the measure $\gamma L \sum_{r\in \cdot}$; that is to say, a family of profiles $m_L^n(r)$ converges to the profile $m_L(r)$, *iff* for all finite $R \in \mathbb{R}$, $\gamma L \sum_{r\in[-R,R]} \|m_L^n(r) - m_L(r)\|_2 \downarrow 0$ as $n \uparrow \infty$. While for all finite γ this topology is completely equivalent to the product topology of the discrete topology, the point here is that it is meaningful to ask for *uniform* convergence with respect to the parameter γ.

2. Block-spin approximations

This chapter is the first step to make clear the link between the Kac-Hopfield model and the Hopfield model. Models, as the usual Ising model, are not well adapted to what is called in the physics litterature the "block spin transformation" in the sense that the resulting effective interactions has a complicated expression. In Kac models this is usually not too difficult as far as the system is in a volume which is not too large.

Expressing the model in term of block spins, it is natural to introduce the following quantities:

$$E^{\ell}_{\gamma,\Lambda}(m) \equiv -\frac{1}{2}\gamma\ell \sum_{(x,y)\in\Lambda\times\Lambda} J_{\gamma\ell}(x-y)(m(x),m(y)) \tag{2.1}$$

and

$$E^{\ell,L}_{\gamma,\Lambda}(m,\tilde{m}) \equiv -\gamma\ell L \sum_{x\in\Lambda}\sum_{r\in\Lambda^c} J_{\gamma}(\ell x - Lr)(m(x),\tilde{m}(r)) \tag{2.2}$$

Note the $J_{\gamma\ell}$ in (2.1) to put everything in the mesoscopic scale ℓ. These quantities are related to the original Hamiltonian via the following two formulas where we introduce the relative errors ΔH and ΔW

$$H_{\gamma,\Lambda}(\sigma_\Lambda) = \gamma^{-1}E^{\ell}_{\gamma,\Lambda}(m_\ell(\sigma)) + \Delta H^{\ell}_{\gamma,\Lambda}(\sigma_\Lambda) \tag{2.3}$$

and

$$W_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\Lambda^c}) = \gamma^{-1}E^{\ell,L}_{\gamma,\Lambda}(m_\ell(\sigma),m_L(\sigma)) + \Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\Lambda^c}) \tag{2.4}$$

We have exhibited a γ^{-1} factor in front of $E^{\ell}_{\gamma,\Lambda}(m_\ell(\sigma))$ to make clear the scaling involved in the problem.

We consider only *macroscopic* volumes Λ of the form $\Lambda = [\lambda^-,\lambda^+]$ with $\lambda^\pm \in \mathbb{Z}$ with $|\Lambda| > 1$. For such volumes we set $\partial\Lambda \equiv \partial^-\Lambda \cup \partial^+\Lambda$, $\partial^-\Lambda \equiv [\lambda^- - \frac{1}{2}, \lambda^-)$, and $\partial^+\Lambda \equiv (\lambda^+, \lambda^+ + \frac{1}{2}]$. Thus, since the interaction range is γ^{-1} we have $W_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\Lambda^c}) = W_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial\Lambda})$ and $\Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\Lambda^c}) = \Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial\Lambda})$.

The following lemma is the basic result to control the block spin approximation.

Lemma 2.1: *For all* $\delta > 0$

i)

$$I\!P\left[\sup_{\sigma\in\mathcal{S}_\Lambda}\frac{\gamma}{|\Lambda|}|\Delta H_\Lambda(\sigma)|\geq\gamma\ell(\gamma)8\sqrt{2}(\log 2+\delta)+2\sqrt{2}\gamma M(\gamma)\right]$$
$$\leq 16e^{-\delta\frac{|\Lambda|}{\gamma}} \tag{2.5}$$

ii)

$$I\!P\left[\sup_{\sigma\in\mathcal{S}_{\Lambda\cup\partial\Lambda}}\gamma|\Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial\Lambda})|>(4\gamma L(\gamma)((\log 2+\delta)+\gamma M(\gamma))\right]$$
$$\leq 8e^{-\frac{\delta}{\gamma}} \tag{2.6}$$

The proof of (2.5) can be found in [BGP2], the one of (2.6) in [BGP4]. Let us mention the important fact that since the parameter $M(\gamma)$, $\ell(\gamma)$ and $L(\gamma)$ are chosen in such a way that $\alpha(\gamma)\equiv\gamma M(\gamma)\downarrow 0$, $\gamma\ell(\gamma)\downarrow 0$ and $\gamma L(\gamma)\downarrow 0$, it follows from (2.5) and (2.6) that with $I\!P$-probability very close to one the errors of the block spin approximations is of order a small parameter times the volume (expressed in the *macroscopic* unit). This will allows us to control only the Gibbs-probability of cylindrical events that have a basis with a diameter uniformly bounded. The main problem is to obtain estimates for the infinite volume Gibbs measure. In Kac models there are two ways of doing that. One is to consider the infinite volume limit and after to take after the $\gamma\downarrow 0$ namely the true 'Kac-limit'. The other possible way is to take the infinite volume in a γ dependent way, usually in a relatively slow way, but at least the macroscopic volume are going to infinity. That is the interaction length is negligible with respect to the volume, see [HL]. Depending on the events we consider this could be equivalent or not. For the Lebowitz and Penrose theorem where events related to the global overlaps is considered this is equivalent, a fact already noticed some years ago by [COPi] in the context of unbounded spins systems. For local events the situation is not so clear.

3. Local effective measures

In this chapter we present a very clever way, introduced by [COP], to deal with the infinite volume problem mentioned above. It was inspired by a fundamental work of Ruelle on superstability estimates. The point is that if we are interested only in local observables, say a cylindrical event with a base in a macroscopic box V, we will show that at a distance R, with a Gibbs probability growing exponentially to one when $R \uparrow \infty$, there are two (macroscopic) blocks of length 1, one on the left and one on the right of V, where the profiles are near an equilibrium value of the Hopfield model. This will allow us to decouple the system in an inside finite volume system and an outside infinite volume one. To make this precise we imitate [COP] and define the following random variables that will be crucial to describe the typical configurations on the set $\mathcal{T}_\gamma$: Given ζ, L, $u \in \mathbb{Z}$ and $\sigma \in S$ let

$$\eta(u,\sigma) \equiv \eta_{\zeta,L}(u,\sigma) = \begin{cases} se^{\mu} & \text{if } \forall_{r\in u}\, \|m^{(\mu,s)} - m_L(r,\sigma)\|_2 \le \zeta \\ 0 & \text{if } \forall_{\mu,s}\, \exists_{r\in u} : \|m^{(\mu,s)} - m_L(r,\sigma)\|_2 > \zeta \end{cases} \tag{3.1}$$

This definition is unequivocal if ζ is chosen small enough i.e. $\zeta < \sqrt{2}a(\beta)$. We do not write the explicit dependence of ζ, L when there is no risk of confusion. For a given configuration σ, $\eta(u,\sigma)$ determines whether in the unit interval centered at u all the local overlap on the scale L are within a ζ-neighborhood of the equilibrium. Note the fundamental fact that we ask that *all* blocks of length L within the block of (*microscopic*) length γ^{-1} are near equilibrium. This is crucial to have a good control of the system on this scale.

For a given volume $V \equiv [v_-, v_+] \subset \Lambda$, with $|V| > 1$, we set

$$\tau^+ = \begin{cases} \inf\{u \ge v_+ : \eta(u,\sigma) \neq 0\} \\ \infty \ \text{ if no such } u \ \text{ exists} \end{cases} \tag{3.2}$$

and

$$\tau^- = \begin{cases} \sup\{u \le v_- : \eta(u,\sigma) \neq 0\} \\ -\infty \ \text{ if no such } u \ \text{ exists} \end{cases} \tag{3.3}$$

for a given configuration σ, $\tau^\pm$ indicates the position of the first unit interval to the right, resp. the left, of V where the configurations σ is close to equilibrium. There are analogous of stopping times, in Markov chains theory, if we imagine the space $\mathbb{Z}$ of our process as the time variable of a Markov chain. We define a partition of our configuration space S according to the possible values of $\tau^\pm$ and the possible values of $\eta(\tau^{\pm})$. That is the atoms of the partition are

$$\mathcal{A}(\mu^{\pm}, s^{\pm}, w_{\pm}) \equiv \left\{\sigma \in \mathcal{S} : \tau^{\pm} = w_{\pm}, \eta(\tau^{\pm}, \sigma) = s^{\pm} e^{\mu^{\pm}}\right\} \tag{3.4}$$

For a given integer R, the indices $\mu^{\pm}$,$s^{\pm}$ $w_{\pm}$ will run over the sets: $\mu^{\pm} \in \{1, \dots, M(\gamma)\}$, $s^{\pm} \in \{-1, 1\}$ and $w_{+} \in [v_{+}, v_{+} + R]$, $w_{-} \in [v_{-} - R, v_{-}]$. In the sequel, if not otherwise specified, all sums and unions over these indices run over the above sets. With an little abuse of notation we denote by

$$\mathcal{S}_R = \bigcup_{\substack{\mu^{\pm}, s^{\pm}, w_{\pm} \\ 0 \le \pm(w_{\pm} - v_{\pm}) \le R}} \mathcal{A}(\mu^{\pm}, s^{\pm}, w_{\pm}) \tag{3.5}$$

Notice that

$$\mathcal{S}_R^c = A^{+}(R) \cup A^{-}(R) \tag{3.6}$$

where

$$\begin{aligned} A^{+}(R) &\equiv \left\{\sigma \in \mathcal{S} : \tau^{+} > v_{+} + R\right\} \\ &= \left\{\sigma \in \mathcal{S} : \forall_{v_{+} \le w \le v_{+} + R} \; \eta(w, \sigma) = 0\right\} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} A^{-}(R) &\equiv \left\{\sigma \in \mathcal{S} : \tau^{-} < v_{-} - R\right\} \\ &= \left\{\sigma \in \mathcal{S} : \forall_{v_{-} - R \le w \le v_{-}} \; \eta(w, \sigma) = 0\right\} \end{aligned} \tag{3.8}$$

For given indices $\mu^{\pm}, s^{\pm}, w_{\pm}$, it will be useful for the future to introduce the following sets that contain $\mathcal{A}(\mu^{\pm}, s^{\pm}, w_{\pm})$: let

$$\widehat{\mathcal{A}}(\mu^{\pm}, s^{\pm}, w_{\pm}) \equiv \left\{\sigma \in \mathcal{S} : \eta(w_{\pm}, \sigma) = s^{\pm} e^{\mu^{\pm}}\right\} \tag{3.9}$$

where the difference with $\mathcal{A}$ is just that on $\widehat{\mathcal{A}}$ we do not specify that the equilibrium is reached for the *first time* moving on the left and on the right of V, that is we specify only that at the points $w_{\pm}$ we are at the equilibrium and this could have happen before! We introduce for future use the set

$$\begin{aligned} &\mathcal{A}^{o}(\mu^{\pm}, s^{\pm}, w_{\pm}) \\ &\equiv \{\sigma \in \mathcal{S} : \eta(u_{\pm}, \sigma) = 0, \forall u_{\pm}, v_{+} \le u_{+} < w_{+}, w_{-} < u_{-} \le v_{-}\} \end{aligned} \tag{3.10}$$

which is nothing but the set $\mathcal{A}(\mu^{\pm}, s^{\pm}, w_{\pm})$ restricted to the volume $]w_-, w_+[$.

We define also $\Delta \equiv [w_- + \frac{1}{2}, w_+ - \frac{1}{2}]$ and, associated to these volumes, we define the Gibbs measure on Δ with *mesoscopic boundary conditions* $m^{(\mu^{\pm},s^{\pm})}$ as the measure that assigns, to each $\sigma_\Delta \in \mathcal{S}_\Delta$, the mass,

$$\mathcal{G}^{\mu^{\pm},s^{\pm}}_{\beta,\gamma,\Delta}[\omega](\sigma_\Delta) = \frac{1}{Z^{\mu^{\pm},s^{\pm}}_{\beta,\gamma,\Delta}[\omega]} e^{-\beta\left\{H_{\gamma,\Delta}[\omega](\sigma_\Delta)+W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^{\pm},s^{\pm})})\right\}} \tag{3.11}$$

where $Z^{\mu^{\pm},s^{\pm}}_{\beta,\gamma,\Delta}[\omega]$ is the corresponding normalization factor and

$$\begin{aligned} W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^{\pm},s^{\pm})}) \equiv & -\sum_{i\in\Delta} s^- a(\beta)\xi_i^{\mu^-}\sigma_i \sum_{j\in\partial^-\Delta} J_\gamma(i-j) \\ & -\sum_{i\in\Delta} s^+ a(\beta)\xi_i^{\mu^+}\sigma_i \sum_{j\in\partial^+\Delta} J_\gamma(i-j) \end{aligned} \tag{3.12}$$

The next proposition will make precise the above mentioned decoupling between the inside and the outside.

Proposition 3.1. *Let F be a cylinder event with base contained in $[v_-, v_+]$. Then*

i) *there exists a positive constant c such that, for all integer R, for all $\epsilon > 0$ there exists $\Omega_{R,\epsilon}$ with $I\!P(\Omega_{R,\epsilon}) \geq 1 - 8R^2M^2e^{-c\epsilon^2\gamma^{-1}}$ such that for all $\mu^{\pm}, s^{\pm}, w_{\pm}, v_+ \leq w_+ \leq v_+ + R, v_- - R \leq w \leq v_-$ and $\omega \in \Omega_{R,\epsilon}$ For all $\Lambda \supset [v_- - R, v_+ + R]$*

$$\begin{aligned} &\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\Big(F \cap \mathcal{A}(\mu^{\pm}, s^{\pm}, w_{\pm})\Big) \\ &\leq \mathcal{G}^{\mu^{\pm},s^{\pm}}_{\beta,\gamma,\Delta}[\omega]\left(F \cap \mathcal{A}^o(\mu^{\pm}, s^{\pm}, w_{\pm})\right) \\ &\times \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\Big(\widehat{\mathcal{A}}(\mu^{\pm}, s^{\pm}, w_{\pm})\Big) \\ &\quad e^{(8\beta\gamma^{-1}(\hat{\zeta}+2\gamma L))} \end{aligned} \tag{3.13}$$

Moreover

$$
\begin{aligned}
&\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\Big(F\cap\mathcal{A}(\mu^{\pm},s^{\pm},u_{\pm})\Big)\\
&\geq \mathcal{G}^{\mu^{\pm},s^{\pm}}_{\beta,\gamma,[u_-,u_+]}[\omega]\left(F\cap\mathcal{A}^o(\mu^{\pm},s^{\pm},w_{\pm})\right)\\
&\times\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^{\pm},s^{\pm},u_{\pm})\right)\\
&\quad e^{(-8\beta\gamma^{-1}(\hat{\zeta}+2\gamma L))}
\end{aligned}
\tag{3.14}
$$

here $\Delta\equiv[w_-+\frac{1}{2},w_+-\frac{1}{2}]$ *and* $\hat{\zeta}\equiv\hat{\zeta}(\epsilon)=\zeta(1+\sqrt{\gamma M})(1+\epsilon)$.

ii) *There exist a positive constant* c' *and an* $\epsilon_0>0$ *such that for all integer* R*, for all* $\epsilon<\epsilon_0$ *such that* $\epsilon>\max(\gamma\ell,\gamma\log\frac{1}{\gamma},\sqrt{\frac{\alpha}{\gamma\ell}},\alpha^{1/3})$ *, there exists* $\Omega_{R,\epsilon}$ *with* $I\!P(\Omega_{R,\epsilon})\geq 1-\gamma^{-1}Re^{-c'\epsilon^2\ell}$ *and there exist a finite positive constant* c_1 *and a positive constant* $c(\beta,\epsilon)$ *such that if* L *and* ζ *are such that* $\zeta^3\gamma Lc(\beta,\epsilon)>2c_1\epsilon$ *then for all* $\omega\in\Omega_{R,\epsilon}$ *and* $\Lambda\supset[v_--R,v_++R]$

$$
\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F\cap S_R^c)\leq\exp\left(-\beta LRc(\beta,\epsilon)\zeta^3\right)
\tag{3.15}
$$

Corollary 3.2. *Let* F *be a cylinder event with base contained in* $[v_-,v_+]$*. Then there exist a positive constant* c' *and an* $\epsilon_0>0$ *such that for all integer* R*, for all* $\epsilon<\epsilon_0$ *such that* $\epsilon>\max(\gamma\ell,\gamma\log\frac{1}{\gamma},\sqrt{\frac{\alpha}{\gamma\ell}},\alpha^{1/3})$*, there exists* $\Omega_{R,\epsilon}$ *with* $I\!P(\Omega_{R,\epsilon})\geq 1-8R^2M^2e^{-c'\epsilon\gamma^{-1}}$ *and there exist a finite positive constant* c_1 *and* $c(\beta,\epsilon)$ *such that if* $\zeta^3\gamma Lc(\beta,\epsilon)>c_1\epsilon$ *then for all* $\omega\in\Omega_{R,\epsilon}$ *and*

$\Lambda\supset[v_--R,v_++R]$

$$
\begin{aligned}
&\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F)\\
&\leq\sum_{\substack{\mu^{\pm},s^{\pm}\\ -R<w_-\leq v_-\\ v_+\leq w_+<R}}\mathcal{G}^{\mu^{\pm},s^{\pm}}_{\beta,\gamma,\Delta}[\omega]\left(F\cap\mathcal{A}^o(\mu^{\pm},s^{\pm},w_{\pm})\right)\\
&\times\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^{\pm},s^{\pm},w_{\pm})\right)\\
&\times e^{(8\beta\gamma^{-1}(\zeta+2\gamma L))}\\
&+\exp\left(-\beta LRc(\beta,\epsilon)\zeta^3\right)
\end{aligned}
\tag{3.16}
$$

and

$$\begin{aligned}
&\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F)\\
&\geq \sum_{\substack{\mu^\pm,s^\pm\\ -R<w_-\leq v_-\\ v_+\leq w_+<R}} \mathcal{G}^{\mu^\pm,s^\pm}_{\beta,\gamma,\Delta}[\omega]\left(F\cap\mathcal{A}^o(\mu^\pm,s^\pm,w_\pm)\right)\\
&\times \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm,s^\pm,u_\pm)\right)\\
&\times e^{(-8\beta\gamma^{-1}(\zeta+2\gamma L))}
\end{aligned} \tag{3.17}$$

and there exists $u^\pm,(\mu^\pm,s^\pm)$ *such that*

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm,s^\pm,u_\pm)\right)\geq\frac{1}{8R^2M^2} \tag{3.18}$$

Remark: As it become clear from its proof, the estimate (3.15), depends only on the fact that there is a run of length R of $\eta=0$ and is rather independent of the volume where it happens. However, the important fact to notice is that the $I\!\!P$-probability that such an event occurs is of the form $1-\exp(-c\epsilon^2\gamma^{-1})$. We have put the *entropy* factors R^2M^2, that are clearly irrelevant here and can be neglected by changing c in $c(1-\epsilon)$ for some $\epsilon>0$ as small as we want. An important part of this work is to control precisely all these $I\!\!P$-probability and the constraints on the volume that will appear later are coming precisely from the control of similar $I\!\!P$-probabilities that is uniform with respect to the various volumes that appear in the problem . In particular an estimate like (3.15) is valid uniformly with respect to volumes Λ that are of order $\exp-(c'\epsilon^2\gamma^{-1})$ for some positive constant c'.
Remark: The two estimates in (3.16) and (3.17) have the important property that the upper bound and the lower bound have the same order of magnitude. However the point is that the presence of the term $\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm,s^\pm,u_\pm)\right)$ shows that to get a full large deviation principle as in [COP] we need to control these terms in the infinite volume limit. The lower bound (3.18) is rather weak but it occurs with $I\!\!P$-probability which is also of order $1-\exp(-c\epsilon^2\gamma^{-1})$ and therefore is true for all volumes Λ that are of order $\exp(c'\epsilon^2\gamma^{-1})$ for some positive constant c'.
Proof. The first assertion of Corollary 3.2 is immediate from (3.13) and (3.15). To prove (3.18), we need to show that

$$\sup_{\mu^\pm,s^\pm}\ \sup_{\pm(u_\pm-v_\pm)\leq R}\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm,s^\pm,u_\pm)\right)\geq\frac{1}{8R^2M^2} \tag{3.19}$$

But from (3.15) we see that

$$
\begin{aligned}
\frac{1}{2} &\le 1-\exp\left(-\beta L R c_2 \zeta \epsilon(\zeta)\right) \le 1-\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](S_R^c) \\
&\le \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\tau_+ \le v_+ + R, \tau_- \ge v_- - R\right) \\
&\le \sum_{\pm(u_\pm - v_\pm)\le R} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\tau_- = u_-, \tau_+ = u_+\right) \\
&\le \sum_{\pm(u_\pm - v_\pm)\le R} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\eta(u_-,\sigma)\neq 0, \eta(u_+,\sigma)\neq 0\right) \\
&\le 4R^2M^2 \sup_{\pm(u_\pm - v_\pm)\le R} \sup_{\mu^\pm, s^\pm} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\hat{\mathcal{A}}(\mu^\pm, s^\pm, u_\pm)\right)
\end{aligned}
\tag{3.20}
$$

which gives (3.19).

Proof of Proposition 3.1 part i): Let us set $\Delta^c \equiv \Lambda\setminus\Delta$. To prove (3.13), we start by integrating on the spins configurations in Δ^c:

$$
\begin{aligned}
&\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F\cap\mathcal{A}(\mu^\pm, s^\pm, w_\pm)) \\
&= \frac{1}{Z_{\beta,\gamma,\Lambda}[\omega]} I\!\!E_{\sigma_\Delta}\Bigg[e^{-\beta\left[H_{\gamma,\Delta}[\omega](\sigma_\Delta)+W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm,s^\pm)})\right]} \\
&\times I\!\!E_{\sigma_{\Delta^c}} e^{-\beta\left[H_{\gamma,\Delta^c}[\omega](\sigma_{\Delta^c})+\left[W_{\gamma,\Delta}[\omega](\sigma_\Delta,\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm,s^\pm)})\right]\right]} \\
&\times 1\!\!\mathrm{I}_{\{\sigma\in F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm)\}}\Bigg]
\end{aligned}
\tag{3.21}
$$

To continue we multiply and divide into the $I\!\!E_{\sigma_\Delta}$ expectation by a partition function on a volume Δ with mesoscopic boundary conditions compatible with $\mathcal{A}(\mu^\pm, s^\pm, w_\pm)$ to get

$$
\begin{aligned}
&\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F\cap\mathcal{A}(\mu^\pm, s^\pm, w_\pm)) \\
&= I\!\!E_{\sigma_\Delta}\Bigg[\frac{1}{Z^{\mu^\pm,s^\pm}_{\beta,\gamma,\Delta}[\omega]} e^{-\beta\left[H_{\gamma,\Delta}[\omega](\sigma_\Delta)+W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm,s^\pm)})\right]} \\
&\times I\!\!E_{\sigma_{\Delta^c}} I\!\!E_{\tilde\sigma_\Delta} \frac{1}{Z_{\beta,\gamma,\Lambda}[\omega]} e^{-\beta\left[H_{\gamma,\Delta^c}[\omega](\sigma_{\Delta^c})+H_{\gamma,\Delta}[\omega](\tilde\sigma_\Delta)+W_{\gamma,\Delta}[\omega](\tilde\sigma_\Delta,\sigma_{\Delta^c})\right]} \\
&\quad 1\!\!\mathrm{I}_{\{\sigma\in F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm)\}} e^{-\beta\left[W_{\gamma,\Delta}[\omega](\sigma_\Delta,\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm,s^\pm)})\right]} \\
&\quad\times e^{+\beta\left[W_{\gamma,\Delta}[\omega](\tilde\sigma_\Delta, m^{(\mu^\pm,s^\pm)})-W_{\gamma,\Delta}[\omega](\tilde\sigma_\Delta,\sigma_{\Delta^c})\right]}\Bigg]
\end{aligned}
\tag{3.22}
$$

Using the fact that

$$\mathbb{1}_{\{\sigma\in\mathcal{A}(\mu^\pm,s^\pm,w_\pm)\}} = \mathbb{1}_{\{\sigma_\Delta\in\mathcal{A}^o(\mu^\pm,s^\pm,w_\pm)\}}\mathbb{1}_{\{\sigma^c_\Delta\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)\}} \tag{3.23}$$

see (3.88).

We can reconstruct the Gibbs measures in Δ, to get:

$$\begin{aligned}
&\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm))\\
&= I\!\!E_{\sigma_\Delta}\Bigg[\mathcal{G}^{\mu^\pm,s^\pm}_{\beta,\gamma,\Delta}[\omega](\sigma_\Delta)\mathbb{1}_{\{\sigma_\Delta F\cap\mathcal{A}^o mu^\pm,s^\pm,w_\pm)\}}\\
&\quad I\!\!E_{\bar\sigma_\Lambda}\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](\bar\sigma_\Lambda)\mathbb{1}_{\{\bar\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)\}}\\
&\quad\times e^{-\beta\left[W_{\gamma,\Delta}[\omega](\sigma_\Delta,\bar\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right]}\\
&\quad\times e^{+\left[W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m^{(\mu^\pm,s^\pm)})-W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,\bar\sigma_{\Delta^c})\right]}\Bigg]
\end{aligned} \tag{3.24}$$

Where $\bar\sigma_\Lambda$ is the configuration that coincide with σ on Δ^c and $\tilde\sigma$ on Δ.The important fact is that we have exhibited terms like:

$$\left[W_{\gamma,\Delta}[\omega](\sigma_\Delta,\bar\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right] \tag{3.25}$$

where the same configuration σ_Δ appears in the two terms of the previous difference and where $\bar\sigma_{\Delta^c}$ are such that $\bar\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)$ which implies in particular that $\eta(w_\pm)=s^\pm e^{\mu^\pm}$ and therefore we can expect that these terms are small. In fact if $\bar\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)$

$$\begin{aligned}
&\Big|\left[W_{\gamma,\Delta}[\omega](\sigma_\Delta,\bar\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right]\\
&+\left[W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m^{(\mu^\pm,s^\pm)})-W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,\bar\sigma_{\Delta^c})\right]\Big|\\
&\le 2\sup_{\bar\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)}\left|W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,\bar\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right|\\
&\le 2\sup_{\bar\sigma\in\mathcal{S}}\left|\Delta W^{1,L}_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,\bar\sigma_{\partial\Delta})\right|+2\times\\
&\sup_{\bar\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)}\left|\gamma^{-1}E^{1,L}_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m_L(\bar\sigma_{\partial\Delta}))-W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right|
\end{aligned} \tag{3.26}$$

and we have a very good control of the second term by Lemma 2.1 and of the first one by the following one:

Lemma 3.3: *For any given v_- and v_+, there exist positive constant c and K such that, for all $\epsilon > 0$, for all integer R, there exists $\Omega_{R,\epsilon}$ with $I\!P[\Omega_{R,\epsilon}] \geq 1 - 2KR^2 e^{-c\epsilon^2\gamma^{-1}}$ such that uniformly in $\mu^\pm, s^\pm$, $w_\pm$ such that $0 \leq \pm(w_\pm - v_\pm) \leq R$ and $\sigma : \eta(w_\pm, \sigma) = s^\pm e^{\mu^\pm}$ we have*

$$\begin{aligned}&\left|\gamma^{-1} E^{1,L}_{\gamma,\Delta}[\omega](\sigma_\Delta, m_L(\sigma_{\partial\Delta})) - W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm,s^\pm)})\right| \\ &\leq \zeta\gamma^{-1}(1 + \sqrt{\gamma M(\gamma)})\sqrt{1+\epsilon}\end{aligned} \tag{3.27}$$

where $\Delta = [w_- + \frac{1}{2}, w_+ - \frac{1}{2}]$.
Remark: The point is that the set $\Omega_{R,\epsilon}$ is independent of $\mu^\pm, s^\pm, w_\pm$. It would be clear later that R can be chosen not too large and will be bounded by some power of γ^{-1}. Moreover, such a result is true also uniformly with respect to the points $v_\pm$ as far as there are in a volume say centered at the origin of length bounded by $\exp\{+c\gamma^{-1}(1-\epsilon)|\}$ for some $\epsilon > 0$.

From this lemma and (3.25) we get immediately (3.13) and (3.14).
Proof. Let us set

$$\begin{aligned}W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm,s^\pm)}) &= W^+_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^+,s^+)}) \\ &\quad + W^-_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^-,s^-)})\end{aligned} \tag{3.28}$$

where

$$W^-_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^-,s^-)}) \equiv -L \sum_{i\in\Delta} s^- a(\beta)\xi_i^{\mu^-} \sigma_i \sum_{r\in\partial^-\Delta} J_\gamma(i - Lr) \tag{3.29}$$

and

$$W^+_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^+,s^+)}) \equiv -L \sum_{i\in\Delta} s^+ a(\beta)\xi_i^{\mu^+} \sigma_i \sum_{r\in\partial^+\Delta} J_\gamma(i - Lr) \tag{3.30}$$

We will consider only the terms corresponding to the interaction with the right part of Δ, the other one being similar. We have, using the Schwarz inequality and the definition of η (3.1): On the set

$\{\sigma \in \widehat{\mathcal{A}}(\mu^{\pm}, s^{\pm}, w_{\pm})\}$, we have

$$
\begin{aligned}
&\left|\gamma^{-1} E^{1,L}_{\gamma,\Delta}[\omega](\sigma_\Delta, m_L(\sigma_{\partial^+\Delta})) - W^+_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^+,s^+)})\right| \\
&\leq L \left| \sum_{i\in\Delta} \sum_{r\in\partial^+\Delta} J_\gamma(i-Lr)\sigma_i \left(\xi_i, \left[m_L(r,\sigma_{\partial^+\Delta}) - m^{(\mu^+,s^+)}\right]\right) \right| \\
&\leq L \sum_{r\in\partial^+\Delta} \left\| \sum_{i\in\Delta} J_\gamma(i-Lr)\xi_i\sigma_i \right\|_2 \left\| m_L(r,\sigma_{\partial^+\Delta}) - m^{(\mu^+,s^+)} \right\|_2 \\
&\leq \zeta L \sum_{r\in\partial^+\Delta} \left\| \sum_{i\in\Delta} J_\gamma(i-Lr)\xi_i\sigma_i \right\|_2 \equiv T^+(\sigma)
\end{aligned}
\tag{3.31}
$$

Define the $\gamma^{-1} \times \gamma^{-1}$ matrix with entries

$$
B_{i,j} \equiv \gamma \sum_{\mu=1}^{M(\gamma)} \xi_i^\mu \xi_j^\mu \tag{3.32}
$$

for $i, j \in \Delta$. Using again the Schwarz inequality, we have

$$
\begin{aligned}
T^+(\sigma) = \zeta L &\sum_{r\in\partial^+\Delta} \\
&\left(\sum_{i\in[w_+-1, w_+-\frac{1}{2}]} \sum_{j\in[w_+-1, w_+-\frac{1}{2}]} (\xi_i,\xi_j)\sigma_i\sigma_j J_\gamma(i-Lr) J_\gamma(j-Lr) \right)^{\frac{1}{2}} \\
&\leq \zeta L \sum_{r\in\partial^+\Delta} \left(\gamma^{-1}\|B\| \sum_{i\in[w_+-1, w_+-\frac{1}{2}]} (\sigma_i J_\gamma(i-Lr))^2 \right)^{\frac{1}{2}} \\
&\leq \zeta L \sum_{r\in\partial^+\Delta} \|B\|^{\frac{1}{2}} \\
&\leq \zeta (2\gamma)^{-1} \|B\|^{\frac{1}{2}}
\end{aligned}
\tag{3.33}
$$

where we have used in the last inequality that $\#\{r \in \partial^+\Delta\} = (2\gamma L)^{-1}$. Thus, using the Theorem 2.1 in [BG3] we get immediatly, for all $\epsilon > 0$,

$$
I\!P \left[\sup_{\sigma\in\mathcal{S}} T^+(\sigma) \geq \zeta (2\gamma)^{-1} (1 + \sqrt{\gamma M}) \sqrt{1+\epsilon} \right] \leq 2K \exp\left(-\frac{\epsilon^2}{2K\gamma} \right) \tag{3.34}
$$

for some absolute constant K from which (3.27) follows.

Proof of Proposition 3.1 part ii):

Using (3.6) the l.h.s. of (3.15) is bounded from above by $\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](A^+(R)) + \mathcal{G}_{\beta,\gamma,\Lambda}[\omega](A^-(R))$. We estimate the first term, the second one being similar. Since the spin configuration are away from the equilibria on a length R, we can decouple the interaction between this part and the rest of the volume Λ, by making a rough estimate of those interaction terms which are of order $c\gamma^{-1}$ as we will prove later. The fact that we have a run of $\eta = 0$ will give terms proportional to R that will be dominant if R is chosen large enough. Let us first state as a Lemma the fact that the interaction between a given volume and its complementary is bounded by 2 (one for the interaction with the left part and one for the interaction with the right part).

Lemma 3.4: *For any given v_- and v_+, there exists a positive constant c such that, for all $\epsilon > 0$, for all integer R, there exists $\Omega_{R,\epsilon}$ with $I\!P[\Omega_{R,\epsilon}] \geq 1 - 2Ke^{-c\epsilon\gamma^{-1}}$ such that for all $\mu^\pm, s^\pm, w_\pm, v_+ \leq w_+ \leq v_+ + R, v_- - R \leq w_- \leq v_-$ and $\omega \in \Omega_R$*

$$\sup_\sigma |W_{\gamma,\Delta}[\omega](\sigma_\Delta, \sigma_{\partial\Delta})| \leq \gamma^{-1} 2(1 + \sqrt{M/\ell})^2 (1+\epsilon) \tag{3.35}$$

where $\Delta = [w_- + \frac{1}{2}, w_+ - \frac{1}{2}]$.

Remark: Note that here also such an estimate is valid for volume that are of order $\exp +c(1-\epsilon)\gamma^{-1}$ for $\epsilon > 0$. Also, we assume that $M/\ell = \alpha(\gamma\ell)^{-1}$ goes to zero. The proof of this lemma is simple, using similar arguments as in the proof of the Lemma 3.18. It can be found in [BGP4].

With this in mind, calling $\Delta_R \equiv [v_+, v_+ + R]$, we have, for all fixed R,

$$\begin{aligned} &\mathcal{G}_{\beta,\gamma,\Lambda}\left(A^+(R)\right) \\ &= \frac{1}{Z_{\beta,\gamma,\Lambda}} I\!E_{\sigma_\Lambda}\Big[e^{-\beta H_{\gamma,\Lambda\setminus\Delta_R}(\sigma_{\Lambda\setminus\Delta_R})} \\ &\times e^{-\beta\left[H_{\gamma,\Delta_R}(\sigma_{\Delta_R}) + W_{\gamma,\Delta_R}(\sigma_{\Delta_R},\sigma_{\Lambda\setminus\Delta_R})\right]} 1\!\!I_{\{\sigma\in A^+(R)\}}\Big] \\ &\leq e^{4\beta\gamma^{-1}} \frac{1}{Z_{\beta,\gamma,\Delta_R}} I\!E_{\sigma_{\Delta_R}}\left[e^{-\beta H_{\gamma,\Delta_R}(\sigma_{\Delta_R})} 1\!\!I_{\{\sigma\in A^+(R)\}}(\sigma)\right] \\ &= e^{4\beta\gamma^{-1}} \mathcal{G}_{\beta,\gamma,\Delta_R}\left(A^+(R)\right) \end{aligned} \tag{3.36}$$

with $I\!P$-probability greater than $1 - Ke^{-c\gamma^{-1}}$ for some positive constants c and K, where we have used the previous lemma to bound the interaction between Δ_R and $\Lambda \setminus \Delta_R$.

To estimate the last term in (3.36), we express it in terms of block spin variables on the scale ℓ. Using (2.5) we get

$$\begin{aligned}&\mathcal{G}_{\beta,\gamma,\Delta_R}\left(A^+(R)\right)\\&\leq e^{2\beta\gamma^{-1}|\Delta_R|(4\gamma\ell+\gamma M)}\frac{\mathbb{E}_{\sigma_{\Delta_R}}e^{-\beta\gamma^{-1}E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma))}\mathbb{1}_{\{\sigma\in A^+(R)\}}}{\mathbb{E}_{\sigma_{\Delta_R}}e^{-\beta\gamma^{-1}E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma))}}\end{aligned}\tag{3.37}$$

with $\mathbb{P}$-probability greater than $1-e^{-c\gamma^{-1}|\Delta_R|}$

We derive first a lower bound on the denominator in (3.37) which will be given effectively by restricting the configurations to be in the neighborhood of a constant profile near one of the equilibrium positions $sa(\beta)e^{\mu}$. We can choose without lost of generality to be $s=1,\mu=1$. To make this precise, we define for any given $\rho>0$ the balls

$$\mathcal{B}_{\rho}^{(\mu,s)}\equiv\left\{m\in\mathbb{R}^M\,;\,\|m-m^{(\mu,s)}\|_2\leq\rho\right\}\tag{3.38}$$

Moreover,we will denote

$$\mathcal{B}_{\rho}\equiv\bigcup_{(\mu,s)\in\{1,\dots,M\}\times\{-1,1\}}\mathcal{B}_{\rho}^{(\mu,s)}\tag{3.39}$$

Obviously,

$$\begin{aligned}&\mathbb{E}_{\sigma_{\Delta_R}}e^{-\beta\gamma^{-1}E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma))}\\&\geq\mathbb{E}_{\sigma_{\Delta_R}}e^{-\beta\gamma^{-1}E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma))}\prod_{x\in\Delta_R}\mathbb{1}_{\{m_\ell(x,\sigma)\in\mathcal{B}_{\rho}^{(1,1)}\}}\end{aligned}\tag{3.40}$$

It can easily be shown that, on the set $\{m_\ell(x,\sigma)\in\mathcal{B}_\rho,\forall x\in\Delta_R\}$,

$$-\gamma^{-1}E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma))\geq\frac{\ell}{2}\sum_{x\in\Delta_R}(\|m_\ell(x,\sigma)\|_2^2-4\rho^2)\tag{3.41}$$

from which (3.40) yields

$$\begin{aligned}&\mathbb{E}_{\sigma_{\Delta_R}}e^{-\beta\gamma^{-1}E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma))}\\&\geq e^{-2\beta\gamma^{-1}|\Delta_R|\rho^2}\prod_{x\in\Delta_R}\mathbb{E}_{\sigma_x}e^{\frac{\beta\ell}{2}\|m_\ell(x,\sigma)\|_2^2}\mathbb{1}\{m_\ell(x,\sigma)\in\mathcal{B}_{\rho}^{(1,1)}\}\\&=e^{-2\beta\gamma^{-1}|\Delta_R|\rho^2}\prod_{x\in\Delta_R}Z_{x,\beta,\rho}\left(a(\beta)e^1\right)\end{aligned}\tag{3.42}$$

Next we derive an upper bound for the numerator of the ratio in (3.37). Using the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ we get

$$-\gamma^{-1} E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma)) \leq \frac{\ell}{2} \sum_{x \in \Delta_R} \|m_\ell(x,\sigma)\|_2^2 \tag{3.43}$$

and whence

$$\begin{aligned} & I\!\!E_{\sigma_{\Delta_R}} e^{-\beta\gamma^{-1} E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma))} 1\!\!I_{\{\sigma \in A^+(R))\}} \\ & \leq I\!\!E_{\sigma_{\Delta_R}} e^{\frac{\beta\ell}{2} \sum_{x \in \Delta_R} \|m_\ell(x,\sigma)\|_2^2} 1\!\!I_{\{\sigma \in A^+(R)\}} \end{aligned} \tag{3.44}$$

Let us now recall that, by definition,

$$A^+(R) = \left\{ \sigma \in \mathcal{S} \,\middle|\, \forall_{u \in \Delta_R} \exists_{r \in u} : \inf_{\mu,s} \|m^{(\mu,s)} - m_L(r,\sigma)\|_2 > \zeta \right\} \tag{3.45}$$

Using that $m_L(r,\sigma) = \frac{\ell}{L} \sum_{x \in r} m_\ell(x,\sigma)$ we have, by convexity

$$\|m^{(\mu,s)} - m_L(r,\sigma)\|_2 \leq \frac{\ell}{L} \sum_{x \in r} \|m^{(\mu,s)} - m_\ell(x,\sigma)\|_2 \tag{3.46}$$

so that

$$A^+(R) \subset \left\{ \sigma \in \mathcal{S} \,\middle|\, \forall_{u \in \Delta_R} \exists_{r \in u} : \inf_{\mu,s} \frac{\ell}{L} \sum_{x \in r} \|m^{(\mu,s)} - m_\ell(x,\sigma)\|_2 > \zeta \right\} \tag{3.47}$$

We will use the following fact

Lemma 3.6: *Let $\{X_k, k = 1, 2, \dots, K\}$ be a sequence of real numbers satisfying $0 \leq X_k \leq c$ for some $c < \infty$. Let $0 \leq \zeta < c$ and assume that*

$$\frac{1}{K} \sum_{k=1}^{K} X_k > \zeta \tag{3.48}$$

For $0 \leq \delta \leq \zeta$, define the set $V_{\delta,\zeta} \equiv \{k | X_k \leq \delta\zeta\}$. Then

$$|\{1 \leq k \leq K : X_k > \delta\zeta\}| \geq K \frac{\zeta(1-\delta)}{c - \delta\zeta} \tag{3.49}$$

Proof : Set $V^c_{\delta,\zeta} \equiv \{1,\dots,K\} \setminus V_{\delta,\zeta}$. Then

$$\begin{aligned}\frac{1}{K}\sum_{k=1}^{K} X_k &\leq \frac{1}{K}\sum_{k\in V_{\delta,\zeta}} X_k + \frac{1}{K}\sum_{k\in V^c_{\delta,\zeta}} X_k \\ &\leq \frac{1}{K} c|V^c_{\delta,\zeta}| + \frac{1}{K}\delta\zeta|V_{\delta,\zeta}| \\ &= \frac{1}{K}(c-\delta\zeta)|V^c_{\delta,\zeta}| + \delta\zeta\end{aligned} \tag{3.50}$$

which, together with (3.49) implies the bound (3.50).

Let us denote by $\mathcal{V}_{\delta,\zeta}(r)$ the set of all subsets $S \subset \{x \in r\}$ with cardinality $\frac{L}{\ell}\frac{\zeta(1-\delta)}{2-\delta\zeta}$, respectively volume

$$|S| \geq \gamma L \frac{\zeta(1-\delta)}{2-\delta\zeta} \tag{3.51}$$

Then, since $\|m^{(\mu,s)} - m_\ell(x,\sigma)\|_2 < 2$, Lemma 4.7 implies

$$A^+(R) \subset \left\{\sigma \in \mathcal{S} \,\middle|\, \forall_{u\in\Delta_R}\exists_{r\in u}\exists_{S\in\mathcal{V}_{\delta,\zeta}(r)} : \forall_{x\in S}\,,\, m_\ell(x,\sigma) \in \mathcal{B}^c_{\delta\zeta}\right\} \tag{3.52}$$

Therefore

$$\begin{aligned}&I\!\!E_{\sigma_{\Delta_R}} e^{-\beta\gamma^{-1}E^\ell_{\gamma,\Delta_R}(m_\ell(\sigma))} 1\!\!I_{\{\sigma\in A^+(R)\}} \\ &\leq \prod_{u\in\Delta_R} I\!\!E_{\sigma_u} e^{\frac{\beta\ell}{2}\sum_{x\in u}\|m_\ell(x,\sigma)\|_2^2} \\ &\quad\times 1\!\!I_{\{\exists_{r\in u}\exists_{S\in\mathcal{V}_{\delta,\zeta}(r)} : \forall x\in S\,,\, m_\ell(x,\sigma)\in\mathcal{B}^c_{\delta\zeta}\}} \\ &\leq \prod_{u\in\Delta_R}\sum_{r\in u}\sum_{S\in\mathcal{V}_{\delta,\zeta}(r)} I\!\!E_{\sigma_u} e^{\frac{\beta\ell}{2}\sum_{x\in u}\|m_\ell(x,\sigma)\|_2^2}\prod_{x\in S} 1\!\!I_{\{m_\ell(x,\sigma)\in\mathcal{B}^c_{\delta\zeta}\}}\end{aligned} \tag{3.53}$$

Inserting this and (3.42) into (3.37) we have

$$\begin{aligned}&\mathcal{G}_{\beta,\gamma,\Delta_R}[\omega]\left(A^+(R)\right) \\ &\leq e^{\beta\gamma^{-1}|\Delta_R|(16\gamma\ell+4\gamma M+4\rho^2)} \\ &\times \prod_{u\in\Delta_R}\sum_{r\in u}\sum_{S\in\mathcal{V}_{\delta,\zeta}(r)}\prod_{x\in u\setminus S}\frac{Z_{x,\beta}}{Z_{x,\beta,\rho}(a(\beta)e^1)}\prod_{x\in S}\frac{Z^c_{x,\beta,\delta\zeta}}{Z_{x,\beta,\rho}(a(\beta)e^1)} \\ &\equiv e^{\beta\gamma^{-1}|\Delta_R|(16\gamma\ell+4\gamma M+4\rho^2)}\prod_{u\in\Delta_R}\sum_{r\in u}\sum_{S\in\mathcal{V}_{\delta,\zeta}(r)} T^{(1)}_S T^{(2)}_S\end{aligned} \tag{3.54}$$

where we have defined

$$Z^c_{x,\beta,\delta\zeta} \equiv \mathbb{E}_{\sigma_x} e^{\frac{\beta\ell}{2}\|m_\ell(x,\sigma)\|_2^2} \mathbb{1}_{\{m_\ell(x,\sigma)\in\mathcal{B}^c_{\delta\zeta}\}} \tag{3.55}$$

To bound from above $Z_{x,\beta}$ we use an argument of Koch [K], see also [BG1]:

We have

$$\begin{aligned} Z_{x,\beta} &= \mathbb{E}_{\sigma_x} e^{\frac{\beta\ell}{2}\|m_\ell(x,\sigma)\|_2^2} \\ &= \left(\frac{\beta\ell}{2\pi}\right)^{M/2} \int_{\mathbb{R}^M} e^{-\frac{1}{2}\beta\ell\|z\|_2^2+\sum_{i=1}^{\ell}\log\cosh(\beta(z,\xi_i))} dz \end{aligned} \tag{3.56}$$

Given $\hat\delta > 0$ to be chosen later, we define

$$\phi_{\ell,\beta,\hat\delta}(\xi,z) \equiv \frac{1}{\beta\ell}\sum_{i=1}^{\ell} \frac{1-\hat\delta}{2}\beta(z,\xi_i)^2 - \log\cosh(\beta(z,\xi_i)) \tag{3.57}$$

therefore denoting $A(x) \equiv \frac{\sum_{i\in x}\xi_i\xi_i}{\ell}$ the $M\times M$ matrix we get

$$\begin{aligned} Z_{x,\beta} &= \left(\frac{\beta\ell}{2\pi}\right)^{M/2} \int_{\mathbb{R}^M} e^{-\beta\ell(z,\mathbb{1}-(1-\delta)A(x)z)} e^{-\beta\ell\phi_{\ell,\beta,\delta}(z)} dz \\ &\le e^{-\beta\ell\inf_{z\in\mathbb{R}^M}\phi_{\ell,\beta,\delta}(z)} (\det(\mathbb{1}-(1-\delta)A(x)))^{-1/2} \end{aligned} \tag{3.58}$$

it follows from the Theorem 4.1 in [BG6] that for $\epsilon > 0$ small enough,

$$\begin{aligned} (\det(\mathbb{1}-(1-\hat\delta)A(x)))^{-1/2} &\le (1-(1-\hat\delta)\|A(x)\|)^{-M/2} \\ &\le e^{\beta\ell(\frac{M}{2\ell}\log(1-(1-\hat\delta)[(1+\sqrt{\frac{M}{\ell}})^2+\epsilon]))} \end{aligned} \tag{3.59}$$

with $\mathbb{P}$-probability greater than $1-Ke^{-\frac{\ell\epsilon^2}{K}}$ for some absolute constant K. Choosing

$$\hat\delta \equiv \hat\delta(\epsilon,\ell) = (2\sqrt{\tfrac{M}{\ell}} + \tfrac{M}{\ell} + 2\epsilon)((1+\sqrt{\tfrac{M}{\ell}})^2+\epsilon)^{-1} \tag{3.60}$$

we get

$$\log(1-(1-\hat\delta)[(1+\sqrt{\frac{M}{\ell}})^2+\epsilon])) = -\log\epsilon \tag{3.61}$$

We assume that ℓ is chosen such that (recalling $\alpha \equiv \gamma M$), then

$$\frac{M}{\ell} = \frac{\alpha}{\gamma \ell} \downarrow 0 \tag{3.62}$$

when $\gamma \downarrow 0$. We impose that

$$\sqrt{\frac{\alpha}{\gamma \ell}} < c\epsilon \tag{3.63}$$

this implies that for γ and ϵ small enough, $\hat{\delta} \approx 2\epsilon$. Moreover, if ϵ and γ are small enough, it is easy to check that

$$\inf_{z \in I\!\!R^M} \phi_{\ell,\beta,\delta}(z) \geq \phi_\beta(a(\beta)) - c\epsilon \tag{3.64}$$

for some positive constant c. Therefore, if ϵ is small enough

$$Z_{x,\beta} \leq \exp\left(-\beta\ell\left[\phi_\beta(a(\beta)) - c\epsilon - c\frac{\alpha}{\gamma\ell}(1-\log\epsilon)\right]\right) \tag{3.65}$$

with $I\!\!P$-probability greater than $1 - Ke^{-\frac{\ell\epsilon^2}{K}}$.

We need a lower bound on $Z_{x,\beta,\rho}(a(\beta)e^1)$ we use the method of [BGP4] with a little modification. Defining as it is standard for finding lower bound for large deviations the so-called associated measure (corresponding to associated random variables, see [CT]): Let $\widetilde{I\!\!P}_{\sigma,x}$ be the measure defined on $\{-1,+1\}^\ell$ through their expectation $\widetilde{I\!\!E}_{\sigma,x}$, given by

$$\widetilde{I\!\!E}_{\sigma,x}(.) \equiv \frac{I\!\!E_\sigma\left(e^{\beta\ell(a(\beta)e^1, m_\ell(x,\sigma))}.\right)}{I\!\!E_\sigma e^{\beta\ell(a(\beta)e^1, m_\ell(x,\sigma))}} \tag{3.66}$$

Note the important fact that

$$I\!\!E_\sigma e^{\beta\ell(a(\beta)e^1, m_\ell(x,\sigma))} = (\cosh\beta a(\beta))^\ell \tag{3.67}$$

It is easy to check that

$$\begin{aligned} &Z_{x,\beta,\rho}(a(\beta)e^1) \\ &= e^{-\beta\ell\Phi_{\ell,\beta}(a(\beta)e^1)}\widetilde{I\!\!E}_{\sigma,x} e^{\frac{\beta\ell}{2}\|m_\ell(x,\sigma)-a(\beta)e^1\|_2^2} 1\!\!1_{\{\|m_\ell(x,\sigma)-a(\beta)e^1\|_2 \leq \rho\}} \end{aligned} \tag{3.68}$$

Since

$$e^{\frac{\beta\ell}{2}\|m_\ell(x,\sigma)-a(\beta)e^1\|_2^2}\mathbb{I}_{\{\|m_\ell(x,\sigma)-a(\beta)e^1\|_2\le\rho\}} \ge \mathbb{I}_{\{\|m_\ell(x,\sigma)-a(\beta)e^1\|_2\le\rho\}} \tag{3.69}$$

we get the lower bound

$$Z_{x,\beta,\rho}(a(\beta)e^1) \ge e^{-\beta\ell\Phi_{\ell,\beta}(a(\beta)e^1)}\widetilde{I\!P}_{\sigma,x}\left[\|m_\ell(x,\sigma)-a(\beta)e^1\|_2 \le \rho\right] \tag{3.70}$$

To estimate from below the previous $\widetilde{I\!P}_{x,\sigma}$ probability, we start by bounding from above

$$\begin{aligned}&\widetilde{I\!P}_{\sigma,x}\left[\|m_\ell(x,\sigma)-a(\beta)e^1\|_2 \ge \rho\right]\\ &=\frac{I\!E_{\sigma,x}\left[e^{\sum_{i\in x}\beta a(\beta)\xi_i^1\sigma_i}\mathbb{I}_{\{\|m_\ell(x,\sigma)-a(\beta)e^1\|_2^2\ge\rho^2\}}\right]}{(\cosh(\beta a(\beta)))^\ell}\end{aligned} \tag{3.71}$$

It is easy to check that

$$\begin{aligned}\mathbb{I}_{\|m_\ell(x,\sigma)-a(\beta)e^1\|_2^2\ge\rho^2} &\le e^{\beta\ell(-\frac{\rho^2}{2}+\frac{1}{2}\|m_\ell(x,\sigma)-a(\beta)e^1\|_2^2)}\\ &= e^{-\beta\ell\frac{\rho^2}{2}}e^{\beta\ell\frac{\|m_\ell(x,\sigma)\|_2^2}{2}}e^{-\sum_{i\in x}\beta a(\beta)\xi_i^1\sigma_i}e^{\frac{\beta\ell}{2}(a(\beta))^2}\end{aligned} \tag{3.72}$$

therefore, inserting this in (3.71) and regrouping the term $e^{\frac{\beta\ell}{2}(a(\beta))^2}$ with the term $(\cosh(\beta a(\beta)))^\ell$ in the denominator of (3.71) we get

$$\widetilde{I\!P}_{\sigma,x}\left[\|m_\ell(x,\sigma)-a(\beta)e^1\|_2 \ge \rho\right] \le e^{-\beta\ell\frac{\rho^2}{2}}e^{\beta\ell\Phi_\beta(a(\beta))}Z_{x,\beta} \tag{3.73}$$

Using the upper bound (3.65) we get

$$\tilde{I\!P}_{\sigma,x}\left[\|m_\ell(x,\sigma)-a(\beta)e^1\|_2 \ge \rho\right] \le e^{-\beta\ell\left[\frac{\rho^2}{2}-c\epsilon-c\frac{\alpha}{\gamma\ell}(1-\log\epsilon)\right]} \tag{3.74}$$

with $I\!P$-probability greater than $1-Ke^{-\frac{\ell\epsilon^2}{K}}$. From which we get,

$$\inf_{x\in\Delta_R} Z_{x,\beta,\rho}(a(\beta)e^1) \ge e^{-\beta\ell\Phi_\beta(a(\beta))}\left(1-e^{-\beta\ell\left[\frac{\rho^2}{2}-c\epsilon-c\frac{\alpha}{\gamma\ell}(1-\log(\epsilon))\right]}\right) \tag{3.75}$$

with $I\!P$-probability greater than $1 - KR(\gamma\ell)^{-1}e^{-\frac{\ell\epsilon^2}{K}}$. Therefore, assuming that ρ is such that

$$\frac{\rho^2}{4} > c\epsilon + c\frac{\alpha}{\gamma\ell}(1 - \log\epsilon) \tag{3.76}$$

we get,

$$\sup_{u\in\Delta_R}\sup_{r\in u}\sup_{S\in\mathcal{V}_{\delta,\zeta}(r)} T_S^{(1)} \leq \prod_{x\in u\setminus S}\frac{e^{\left(+\beta\ell c\frac{\alpha}{\gamma\ell}(1-\log\epsilon)\right)}}{1-e^{-\beta\ell\frac{\rho^2}{4}}} \tag{3.77}$$

with $I\!P$-probability $\geq 1 - KR(\gamma\ell)^{-1}e^{-\frac{\ell\epsilon^2}{K}}$.

On the other hand, to bound $Z^c_{x,\beta,\delta\zeta}$, we proceed as in [BG2] and first note that using the Theorem 4.1 of [BG6], we have if $\frac{\alpha}{\gamma\ell}$ and ϵ are small enough

$$\sup_{x\in\Delta_R}\|m_\ell(x,\sigma)\|_2^2 \leq \sup_{x\in\Delta_R}\|A(x)\| \leq (1+\sqrt{\frac{\alpha}{\gamma\ell}})^2 + \epsilon \leq 2 \tag{3.78}$$

with $I\!P$-probability $\geq 1 - KR(\gamma\ell)^{-1}e^{-\frac{\ell\epsilon^2}{K}}$. Next, we introduce the lattice $\mathcal{W}_{\ell,M}$ with spacing $1/\sqrt{\ell}$ in $I\!R^M$ and we denote by $\mathcal{W}_{\ell,M}(2)$ the intersection of this lattice with the ball of radius 2 in $I\!R^M$. We have

$$|\mathcal{W}_{\ell,M}(2)| \leq \exp\left(M\ln\left(\frac{2\ell}{M}\right)\right) \tag{3.79}$$

Now, we can cover the ball of radius 2 in $I\!R^M$ with balls of radii $\hat\rho \equiv \sqrt{M/\ell}$ centered at the points of $\mathcal{W}_{\ell,M}(2)$. Assuming that $\delta\zeta > \hat\rho$ this yields,

$$\begin{aligned} Z^c_{x,\beta,\delta\zeta} &\leq \sum_{m\in\mathcal{W}_{\ell,M}(2)} 1\!\!1_{\{m\in\mathcal{B}^c_{\delta\zeta-\hat\rho}\}} Z_{x,\beta,\hat\rho}(m)[\omega] \\ &\leq \sum_{m\in\mathcal{W}_{\ell,M}(2)} 1\!\!1_{\{m\in\mathcal{B}^c_{\delta\zeta-\hat\rho}\}} \exp\left(-\beta\ell\left(\Phi_{x,\beta}(m)[\omega] - \frac{1}{2}\hat\rho^2\right)\right) \end{aligned} \tag{3.80}$$

It is rather tedious to check that it is possible to modify the Theorem 1 [BG3] (or Theorem 6.1 of [BG4]) by changing almost all the constants and making different choices each time it is necessary to get that for all $\epsilon > 0$, with $I\!P$-probability greater than $1 - Ke^{-\frac{\ell\epsilon^2}{K}}$, if $\beta > 1$, there exists a strictly positive $c(\beta,\epsilon)$, with $c(\beta,\epsilon) \geq c(\beta)(1-g(\epsilon,\beta))$ for some

$g(\epsilon,\beta)\downarrow 0$, where $\epsilon\downarrow 0$ and $c(\beta)$ is defined in the Theorem 1 of [BG3], such that if $\delta\zeta-\hat\rho > c\sqrt{\frac{\alpha}{a(\beta)}}$ for some positive constant c then

$$\Phi_{x,\beta}(m)[\omega]-\Phi_\beta(a(\beta))\geq c(\beta,\epsilon)(\delta\zeta-\hat\rho)^2 \tag{3.81}$$

Therefore, with $I\!P$-probability greater that $1-R(\gamma\ell)^{-1}Ke^{-\frac{\ell\epsilon^2}{K}}$ we get

$$\begin{aligned}&\sup_{x\in\Delta_R} Z^c_{x,\beta,\delta\zeta}\\ &\leq \exp\left(-\beta\ell\left(\phi(a(\beta))+c(\beta,\epsilon)(\delta\zeta-\hat\rho)^2-\tfrac{1}{2}\hat\rho^2-\tfrac{M}{\beta\ell}\ln\left(\tfrac{2\ell}{M}\right)\right)\right)\end{aligned} \tag{3.82}$$

Therefore, using (3.75) and (3.76), we get

$$\begin{aligned}&\sup_{x\in\Delta_R}\frac{Z^c_{x,\beta,\delta\zeta}}{Z_{x,\beta,\rho}(a(\beta)e^1)}\\ &\leq \exp\left(-\beta\ell\left[c(\beta,\epsilon)\left(\delta\zeta-\hat\rho\right)^2-c\tfrac{1}{2}\hat\rho^2-\tfrac{M}{\beta\ell}\ln\left(\tfrac{2\ell}{M}\right)\right]\right)\left(1-e^{-\beta\ell\frac{\rho^2}{4}}\right)^{-1}\end{aligned} \tag{3.83}$$

with $I\!P$-probability greater that $1-R(\gamma\ell)^{-1}Ke^{-\frac{\ell\epsilon^2}{K}}$. Thus the product $T^{(1)}_S T^{(2)}_S$ defined in (3.54) is bounded by

$$\begin{aligned}&T^{(1)}_S T^{(2)}_S\\ &\leq \exp\left(-\beta\gamma^{-1}|S|\left[c(\beta,\epsilon)\left(\delta\zeta-\hat\rho\right)^2-c\tfrac{1}{2}\hat\rho^2-\tfrac{M}{\beta\ell}\ln\left(\tfrac{2\ell}{M}\right)\right]\right)\\ &e^{\beta\gamma^{-1}c\left[\frac{\alpha}{\gamma\ell}(1-\log\epsilon)\right]}\left(1-e^{-\beta\ell\frac{\rho^2}{4}}\right)^{-(\gamma\ell)^{-1}}\end{aligned} \tag{3.84}$$

with $I\!P$-probability greater that $1-R(\gamma\ell)^{-1}Ke^{-\frac{\ell\epsilon^2}{K}}$.

Hence calling

$$\left(1-e^{-\beta\ell\frac{\rho^2}{4}}\right)^{-(\gamma\ell)^{-1}|\Delta_R|}\equiv \Xi^{-(\gamma\ell)^{-1}|\Delta_R|} \tag{3.85}$$

we get

$$\begin{aligned}
&\prod_{u\in\Delta_R}\sum_{r\in u}\sum_{S\in\mathcal{V}_{\delta,\zeta}(r)} T_S^{(1)}T_S^{(2)}\\
&\leq \Xi^{-(\gamma\ell)^{-1}|\Delta_R|}\prod_{u\in\Delta_R}\\
&\sum_{r\in u}\sum_{S\in\mathcal{V}_{\delta,\zeta}(r)} \exp\left(-\beta\gamma^{-1}|S|\left[c(\beta,\epsilon)\left(\delta\zeta-\hat\rho\right)^2 - c\tfrac{1}{2}\hat\rho^2-\tfrac{M}{\beta\ell}\ln\left(\tfrac{2\ell}{M}\right)\right]\right)\\
&\leq e^{\left(-\beta\gamma^{-1}|\Delta_R|\left[\gamma L\zeta c(\beta,\epsilon)(\delta\zeta-\hat\rho)^2\right]\right)}\\
&\times e^{\left(-\beta\gamma^{-1}|\Delta_R|\left[-c\frac{1}{2}\hat\rho^2-\frac{M}{\beta\ell}\ln\left(\frac{2\ell}{M}\right)-\gamma|\ln(\gamma L)|-\gamma L\frac{\ln 2}{\ell}-\frac{1}{\beta\ell}\ln\Xi\right]\right)}
\end{aligned} \tag{3.86}$$

with $I\!P$-probability greater that $1-R(\gamma\ell)^{-1}Ke^{-\frac{\ell\epsilon^2}{K}}$. Now, if

$$\beta\ell\frac{\rho^2}{4} > \log 2 \tag{3.87}$$

we get

$$\left(1-e^{-\beta\ell\frac{\rho^2}{4}}\right)^{-(\gamma\ell)^{-1}|\Delta_R|} \leq \exp\left[2\gamma^{-1}|\Delta_R|\frac{1}{\ell}\exp\left(-\beta\ell\frac{\rho^2}{4}\right)\right] \tag{3.88}$$

therefore the right hand side of (3.86) is bounded from above by

$$e^{\left(-\beta\gamma^{-1}R\left[\gamma L\zeta c(\beta,\epsilon)(\delta\zeta-\hat\rho)^2-c\frac{1}{2}\hat\rho^2-\frac{M}{\beta\ell}\ln\left(\frac{2\ell}{M}\right)-\gamma|\ln(\gamma L)|-\gamma L\frac{\ln 2}{\ell}-\frac{1}{\beta\ell}e^{-\beta\ell\frac{\rho^2}{4}}\right]\right)} \tag{3.89}$$

We collect all the constraints on the various parameters we have introduced, recalling that we have imposed (3.76) and (3.63). We first assume that the various parameters are chosen in such a way that

$$\begin{aligned}
&c\tfrac{1}{2}\hat\rho^2-\tfrac{M}{\beta\ell}\ln\left(\tfrac{2\ell}{M}\right)-\gamma\ln(\gamma L)^{-1}+\gamma L\frac{\ln 2}{\ell}+\frac{1}{\beta\ell}e^{-\beta\ell\frac{\rho^2}{4}}\\
&\leq \frac{1}{3}\gamma L\zeta c(\beta,\epsilon)\left(\delta\zeta-\hat\rho\right)^2
\end{aligned} \tag{3.90}$$

Noticing that in (3.46) there is an exponential prefactor $e^{\beta\gamma^{-1}|\Delta_R|(16\gamma\ell+4\gamma M+4\rho^2)}$, we assume also that

$$16\gamma\ell+4\gamma M+4\rho^2 \leq \frac{1}{3}\gamma L\zeta c(\beta,\epsilon)\left(\delta\zeta-\hat\rho\right)^2 \tag{3.91}$$

Let us note that it is precisely here that it is crucial to have two different scales ℓ and L with $L >> \ell$ to be able to satisfy this inequality (for fixed ζ, δ arbitrarily small). This was already observed in [COP].

Note that (3.91) implies that ρ has to satisfy $\rho^2 << \gamma L$. Assuming that

$$\frac{1}{2}\delta\zeta > \hat{\rho} = \sqrt{\frac{M}{\ell}} \tag{3.92}$$

and since from the Lemma 3.6, we need $\delta < \zeta$, we can take $\delta = \zeta/2$ and replace $\delta\zeta - \hat{\rho}$ by $\frac{\zeta^2}{4}$.

Therefore a simple way to satisfy (3.91) is to impose

$$16\gamma\ell + 4\gamma M \leq \frac{1}{24}\gamma L\zeta^5 c(\beta, \epsilon) \tag{3.93}$$

and

$$\rho^2 \leq \frac{1}{48}\gamma L\zeta^5 c(\beta, \epsilon) \tag{3.94}$$

Since we have to satisfy also (3.76) and (3.63) which implies $\frac{\alpha}{\gamma\ell}(1 - \log\epsilon) \leq c\epsilon^2 \log\epsilon$ if we satisfy, (3.63) and

$$c\epsilon \leq \frac{\rho^2}{4} \leq \frac{1}{48}\gamma L\zeta^5 c(\beta, \epsilon) \tag{3.95}$$

for some positive constant c, we get (3.76). Therefore if we assume (3.63), (3.95), and

$$16\gamma\ell + 4\gamma M \leq c\epsilon \tag{3.96}$$

we get (3.93).

To satisfy (3.90), we first note that since

$$\hat{\rho}^2 = \frac{M}{\ell} << -\frac{M}{\beta\ell}\ln\left(\frac{2\ell}{M}\right) \tag{3.97}$$

we can ignore the first term $c\frac{1}{2}\hat{\rho}^2$ in (3.90). Cutting the condition (3.90) in three, and recalling (3.95) we impose, for some positive constant c

$$\tfrac{M}{\beta\ell}\ln\left(\tfrac{2\ell}{M}\right) = \tfrac{\alpha}{\beta\gamma\ell}\ln\left(\tfrac{2\gamma\ell}{\alpha}\right) \leq c\epsilon \tag{3.98}$$

$$\gamma \ln(\gamma L)^{-1} + \gamma L \frac{\ln 2}{\ell} \leq c\epsilon \tag{3.99}$$

and

$$\frac{1}{\beta\ell} e^{-\beta\ell\frac{\rho^2}{4}} \leq c\epsilon \tag{3.100}$$

Taking into account of (3.63), since $c^2\epsilon^2 \log \frac{2}{c^2\epsilon} \leq c'\epsilon$ if ϵ is small enough, (3.95) implies (3.98). we consider (3.99). Now $L \leq \gamma^{-1}$ which is assumed already, implies $\gamma \ln(\gamma L)^{-1} \leq 2\gamma \log \frac{1}{\gamma}$ and (3.63) implies

$$\gamma L \frac{\ln 2}{\ell} \leq \frac{c}{\ell} \leq c^2\epsilon^2 \frac{\gamma}{\alpha} \leq c^2\epsilon^2 \tag{3.101}$$

since $\frac{\gamma}{\alpha} \leq 1$, therefore (3.95) and

$$\gamma \log \frac{1}{\gamma} \leq c\epsilon \tag{3.102}$$

imply (3.99). Concerning (3.100), it is an immediate consequence of $\ell^{-1} \leq c^2\epsilon^2$ and (3.95). Therefore we remain with the conditions (3.63), (3.95), (3.96) and (3.102). Note that to satisfy (3.96), we need $16\gamma\ell \leq c\epsilon$ and to satisfy (3.63), we need $\gamma\ell \geq \alpha c^{-2}\epsilon^{-2}$. This impose

$$\alpha \leq c\epsilon^3 \tag{3.103}$$

and we get that ℓ has to be chosen in such a way that

$$\alpha\gamma^{-1}\epsilon^{-2} \leq \ell \leq c\epsilon\gamma^{-1} \tag{3.104}$$

Note that $\ell\rho^2 > c\epsilon^{-1}$ and therefore taking ϵ small enough we get (3.87). It is certainly better to choose $\ell = c\epsilon\gamma^{-1}$ to have $\ell\rho^2 \geq \epsilon^3\gamma^{-1}$, note that the choice $\ell = \alpha\gamma^{-1}\epsilon^{-2}$ gives $\ell\rho^2 \geq \frac{\alpha}{\gamma\epsilon}$ which is not very good if $M = \alpha\gamma^{-1}$ is bounded. Assuming (3.103), (3.96) is a consequence of

$$16\gamma\ell \leq c\epsilon \tag{3.105}$$

Collecting, if (3.63), (3.95),(3.105)and (3.102) are satisfied then

$$\mathcal{G}_{\beta,\gamma,\Delta_R}[\omega] \ \left(A^+(R)\right) \leq \exp\left(-\beta\gamma^{-1}R\left[\gamma L\frac{1}{3}c(\beta,\epsilon)\zeta^3\delta^2\right]\right) \tag{3.106}$$

with $I\!P$-probability greater than $1 - R(\gamma\ell)^{-1}Ke^{-\frac{\ell\epsilon^2}{K}}$.

Remark: We have made explicit all the various constraints on the parameters ϵ ℓ, L ζ δ since depending on the kind of results we want they can be chosen in various ways. Let us notice that we have insisted to choose first ϵ and for this choice, ℓ, L ζ, δ and ρ are choosen in that order. This has been done to have always an uniformity on volume of order $\exp(c(\epsilon)\gamma^{-1})$ for all values of M that are such that $\gamma M \downarrow 0$. A possible procedure is the following: For a given ϵ small enough, take γ small enough such that $\alpha < \epsilon^3$ and $\gamma \log \frac{1}{\gamma} < \epsilon$. Then choose ℓ such that $\ell = c\epsilon\gamma^{-1}$. To satisfy (3.95), take for example $L = c\gamma^{-1}\epsilon \log \frac{1}{\epsilon}$ and $\zeta^5 = \log \frac{1}{\epsilon}$ and everything work perfectly.

4. Self averaging properties of the free energy

In this chapter we study the self averaging properties of the free energy of the Hopfield-Kac model with *mesoscopic* boundary conditions. This chapter is crucial to understand the volume restriction we will impose. It is here that the main restrictions will come.

We denote the partition function on the volume Δ with boundary condition $s^- a(\beta) e^{\mu^-}$ on the left of Δ and $s^+ a(\beta) e^{\mu^+}$ on the right of Δ by

$$
\begin{aligned}
&Z_\Delta^{(\mu^\pm, s^\pm)} \\
&\equiv I\!\!E_{\sigma_\Delta}\left[e^{-\beta\left(H_{\gamma,\Delta}(\sigma) + W_{\gamma,\Delta,\partial^-\Delta}(\sigma_\Delta | m^{(\mu^-,s^-)}) + W_{\gamma,\Delta,\partial^+\Delta}(\sigma_\Delta) | m^{(\mu^+,s^+)})\right)}\right]
\end{aligned}
\tag{4.1}
$$

and the corresponding free energy

$$
f_\Delta^{(\mu^\pm, s^\pm)} \equiv f_\Delta = -\frac{\gamma}{\beta|\Delta|} \ln Z_\Delta^{(\mu^\pm, s^\pm)} \tag{4.2}
$$

To include the case of free boundary conditions, we set $m^{(0,0)} \equiv 0$.

We are interested in the behavior of the fluctuations of $f_\Delta^{(\mu^\pm, s^\pm)}$ around it mean value. We will use the Theorem 6.6 of Talagrand [T] that we state for the convenience of the reader. We denote by $I\!\!MX$ a median of the random variable X. Recall that a number x is called the median of a random variable X if both $I\!\!P[X \geq x] \geq \frac{1}{2}$ and $I\!\!P[X \leq x] \geq \frac{1}{2}$.

Theorem 4.1. *[T] Consider a real valued function f defined on $[-1,+1]^N$. We assume that, for each real number a the set $\{f \leq a\}$ is convex. Consider a convex set $B \subset [-1,+1]^N$, and assume that for all $x, y \in B$, $|f(x) - f(y)| \leq k\|x - y\|_2$ for some positive k. Let X denote a random vector with i.i.d. components $\{X_i\}_{1 \leq i \leq N}$ taking values in $[-1,+1]$. Then for all $t > 0$,*

$$
I\!\!P\left[|f(X) - I\!\!Mf(X)| \geq t\right] \leq 4b + \frac{4}{1-2b} \exp\left(-\frac{t^2}{16k^2}\right) \tag{4.3}
$$

where $b \equiv I\!\!P\left[X \notin B\right]$ and we assume that $b < \frac{1}{2}$.

The first result of this chapter is the following proposition:

Proposition 4.2. . *If $\gamma\ell$, M/ℓ and γM are small enough, then for all*

$t > 0$, *there exists a universal numerical constant* K *such that*

$$
\begin{aligned}
&I\!P\left[\left|f_\Delta^{(\mu^\pm,s^\pm)} - I\!\!E f_\Delta^{(\mu^\pm,s^\pm)}\right| \geq t + K\left(\sqrt{\gamma^{-1}|\Delta|}\right)^{-1}\right] \\
&\quad\leq K \exp\left(-\frac{\gamma^{-1}}{4}|\Delta|(\sqrt{1+\frac{t^2}{8}}-1)\right)
\end{aligned}
\tag{4.4}
$$

Remark: Note that in this proposition we have fixed Δ, $(\mu^\pm, s^\pm)$.
Proof. Note first that the set $\{f_\Delta \leq a\}$ is convex. This follows from the fact that the Hamiltonian $H_{\gamma,\Delta}$ is a convex function of the variable ξ. The main difficulty that remains is to establish that f_Δ is a Lipshitz function of the independent random variables ξ with a constant k that is small with large probability. To prove the Lipshitz continuity of f_Δ it is obviously enough to prove the corresponding bounds for $H_{\gamma,\Delta}(\sigma)$ and $W_{\gamma,\Delta,\partial^\pm\Delta}(\sigma_\Delta|m^{(\mu^\pm,s^\pm)})$.

Let us first prove that $H_{\gamma,\Delta}(\sigma)$ is Lipshitz in the random variable ξ. Let us write $\xi \equiv \xi[\omega]$ and $\hat{\xi} \equiv \xi[\omega']$. Denoting by $\xi^\mu\sigma$ the coordinate-wise product of the two vectors ξ^μ and σ and $J_\gamma(i-j)$ the symmetric $\gamma^{-1}|\Delta| \times \gamma^{-1}|\Delta|$ matrix with i,j entries, we have

$$
|H_{\gamma,\Delta}[\omega](\sigma) - H_{\gamma,\Delta}[\omega'](\sigma)| = \left|\sum_{\mu=1}^{M}\left(\left[\xi^\mu\sigma - \hat{\xi}^\mu\sigma\right], J_\gamma\left[\xi^\mu\sigma + \hat{\xi}^\mu\sigma\right]\right)\right|
\tag{4.5}
$$

Since J_γ is a symmetric and positive definite matrix, its square root $J_\gamma^{1/2}$ exists. Thus using the Schwarz inequality we may write

$$
\begin{aligned}
&\left|\sum_{\mu=1}^{M}\left([\xi^\mu\sigma - \hat{\xi}^\mu\sigma], J_\gamma[\xi^\mu\sigma + \hat{\xi}^\mu\sigma]\right)\right| \leq \\
&\sum_{\mu=1} \|J_\gamma^{1/2}[\xi^\mu\sigma - \hat{\xi}^\mu\sigma]\|_2 \|J_\gamma^{1/2}[\xi^\mu\sigma + \hat{\xi}^\mu\sigma]\|_2 \\
&\leq \mathcal{J}^+\mathcal{J}^-
\end{aligned}
\tag{4.6}
$$

where

$$
\mathcal{J}^+ \equiv \left(\sum_{\mu=1}^{M}([\xi^\mu\sigma + \hat{\xi}^\mu\sigma], J_\gamma[\xi^\mu\sigma + \hat{\xi}^\mu\sigma])\right)^{1/2}
\tag{4.7}
$$

and

$$\mathcal{J}^- \equiv \left(\sum_{\mu=1}^{M} ([\xi^\mu \sigma - \hat{\xi}^\mu \sigma], J_\gamma [\xi^\mu \sigma - \hat{\xi}^\mu \sigma]) \right)^{1/2} \leq \|\xi - \hat{\xi}\|_2 \tag{4.8}$$

The last inequality in (4.8) follows since $\|J_\gamma\| \leq 1$.

On the other hand, by convexity

$$\begin{aligned} \left(\mathcal{J}^+\right)^2 &\leq 2 \sum_{\mu=1}^{M} (\xi^\mu \sigma J_\gamma \xi^\mu \sigma) + 2 \sum_{\mu=1}^{M} (\hat{\xi}^\mu \sigma J_\gamma \hat{\xi}^\mu \sigma) \\ &= 2 H_{\gamma,\Delta}[\omega](\sigma) + 2 H_{\gamma,\Delta}[\omega'](\sigma) \end{aligned} \tag{4.9}$$

Collecting, we get

$$|H_{\gamma,\Delta}[\omega](\sigma) - H_{\gamma,\Delta}[\omega'](\sigma)| \leq \sqrt{2} \|\xi - \hat{\xi}\|_2 \left(H_{\gamma,\Delta}[\omega](\sigma) + H_{\gamma,\Delta}[\omega'](\sigma) \right)^{1/2} \tag{4.10}$$

This means that as in [T], we are in a situation where the upper bound for the Lipshitz norm of $H_{\gamma,\Delta}[\omega](\sigma)$ is not uniformly bounded. However the estimates of Section 2, allow us to give reasonable estimates on the probability distribution of this Lipshitz norm. Recalling (2.5) we have

$$I\!P \left[\sup_{\sigma \in \mathcal{S}_\Delta} |\Delta H_{\gamma,\Delta}(\sigma)| \geq \gamma^{-1} |\Delta| (16(1+c)\gamma\ell + 4\gamma M) \right] \leq 16 e^{-c\gamma^{-1}|\Delta|} \tag{4.11}$$

Therefore, using (2.1) we get

$$\begin{aligned} &I\!P \left[\sup_{\sigma \in \mathcal{S}_\Delta} |H_{\gamma,\Delta}(\sigma)| \geq \gamma^{-1} |\Delta| (C + (16(1+c))\gamma\ell + 4\gamma M) \right] \\ &\leq 16 e^{-c\gamma^{-1}|\Delta|} + I\!P \left[\sup_{\sigma \in \mathcal{S}_\Delta} |\gamma^{-1} E^\ell_{\gamma,\Delta}(m_\ell(\sigma))| \geq C \gamma^{-1} \Delta \right] \end{aligned} \tag{4.12}$$

To estimate this last probability, we notice that by convexity

$$2(m_\ell(x,\sigma), m_\ell(y,\sigma)) \leq \|m_\ell(x,\sigma)\|_2^2 + \|m_\ell(y,\sigma)\|_2^2 \tag{4.13}$$

Therefore

$$
\begin{aligned}
|\gamma^{-1}E^{\ell}_{\gamma,\Delta}(m_\ell(\sigma))| &= 1/2 \left| \sum_{x,y\in\Delta} J_{\gamma\ell}(x-y)(m_\ell(x,\sigma),m_\ell(y,\sigma)) \right| \\
&\le \ell/2 \sum_{x\in\Delta} \|m_\ell(x,\sigma)\|_2^2
\end{aligned}
\tag{4.14}
$$

Now we have

$$
\begin{aligned}
& I\!P\left[\sup_{\sigma\in\mathcal{S}_\Delta} \ell \sum_{x\in\Delta} \|m_\ell(x,\sigma)\|_2^2 \ge 2C\gamma^{-1}|\Delta|\right] \\
& \le 2^{\gamma^{-1}|\Delta|} I\!P\left[\ell \sum_{x\in\Delta} \|m_\ell(x,\sigma)\|_2^2 \ge 2C\gamma^{-1}|\Delta|\right] \\
& \le 2^{\gamma^{-1}|\Delta|} \inf_{0\le t<1/2} e^{-2C\gamma^{-1}|\Delta|t} \prod_{x\in\Delta}\prod_{\mu=1}^{M} I\!E e^{t\ell\left(\frac{1}{\ell}\sum_{i\in x}\xi_i^\mu\sigma_i\right)^2}
\end{aligned}
\tag{4.15}
$$

Using the well known inequality [BG1]

$$
I\!E \exp\left(t\ell\left(\frac{1}{\ell}\sum_{i\in x}\xi_i^\mu\sigma_i\right)^2\right) \le \frac{1}{\sqrt{1-2t}} \tag{4.16}
$$

and choosing $t=1/4$, the r.h.s of (4.15) is bounded from above by

$$
\exp\left(-\gamma^{-1}|\Delta|\left(\frac{C}{2}-(1+M/2\ell)\ln 2\right)\right) \tag{4.17}
$$

Collecting, we get

$$
\begin{aligned}
& I\!P\left[\sup_{\sigma\in\mathcal{S}_\Delta} \ell \sum_{x\in\Delta} \|m_\ell(x,\sigma)\|_2^2 \ge \gamma^{-1}|\Delta| 2\left(2C+(1+M/2\ell)\ln 2\right)\right] \\
& \le e^{-C\gamma^{-1}|\Delta|}
\end{aligned}
\tag{4.18}
$$

which implies, if $\gamma\ell$, γM and M/ℓ are small enough, that

$$
I\!P\left[\sup_{\sigma\in\mathcal{S}_\Delta} |H_{\gamma,\Delta}(\sigma)| \ge \gamma^{-1}|\Delta|(4c+1)\right] \le 17e^{-c\gamma^{-1}|\Delta|} \tag{4.19}
$$

which is the estimate we wanted.

To treat the boundary terms let us call $W^-_{\gamma,\Delta}[\omega]$ (respectively $W^+_{\gamma,\Delta}[\omega]$) the terms corresponding to interactions with the left (respectively right) part of the boundary $\partial\Delta$. We estimate first the Lipshitz norm of $W^-_{\gamma,\Delta}[\omega]$, the one of $W^+_{\gamma,\Delta}[\omega]$ being completely identical.

$$\begin{aligned}
&|W^-_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^-,s^-)}) - W^-_{\gamma,\Delta}[\omega'](\sigma_\Delta, m^{(\mu^-,s^-)})| \\
&\le a(\beta)\left|\sum_{i\in\Delta}\sigma_i(\xi_i^{\mu^-} - \hat{\xi}_i^{\mu^-})\left(\sum_{j\in\partial^-\Delta} J_\gamma(i-j)\right)\right| \\
&\le a(\beta)\left(\sum_{i\in\Delta}(\xi_i^{\mu^-} - \hat{\xi}_i^{\mu^-})^2\right)^{1/2}\left(\sum_{i\in\Delta}\left(\sum_{j\in\delta^-\Delta} J_\gamma(i-j)\right)^2\right)^{1/2} \\
&\le \gamma^{1/2}a(\beta)\|\xi-\hat{\xi}\|_2^2 \\
&\le \gamma^{1/2}\|\xi-\hat{\xi}\|_2^2
\end{aligned} \tag{4.20}$$

where we have used the Schwarz inequality and

$$\sum_{i\in\Delta}\left(\sum_{j\in\partial^-\Delta} J_\gamma(i-j)\right)^2 \le \gamma^{-1} \tag{4.21}$$

Therefore if we denote by

$$\Omega_B \equiv \left\{\xi\in[-1,+1]^{\gamma^{-1}\Delta M};\ \sup_{\sigma\in\mathcal{S}_\Delta}|H_{\gamma,\Delta}(\sigma)| \le \gamma^{-1}|\Delta|(4c+1)\right\} \tag{4.22}$$

Using (4.3), (4.19), (4.20) and some easy computations, we get

$$I\!P\left[\left|f_\Delta^{(\mu^\pm,s^\pm)} - I\!M f_\Delta^{(\mu^\pm,s^\pm)}\right| \ge t\right] \le 68e^{-c\gamma^{-1}|\Delta|} + 68e^{-\frac{t^2}{32(4c+2)}\gamma^{-1}|\Delta|} \tag{4.23}$$

Choosing c such that $c = \frac{t^2}{32(4c+2)}$ we get

$$\begin{aligned}
&I\!P\left[\left|f_\Delta^{(\mu^\pm,s^\pm)} - I\!M f_\Delta^{(\mu^\pm,s^\pm)}\right| \ge t\right] \\
&\le 136\exp\left(-\frac{\gamma^{-1}}{4}|\Delta|(\sqrt{1+\frac{t^2}{8}}-1)\right)
\end{aligned} \tag{4.24}$$

Finally, the simple fact:

$$\left| I\!M f_\Delta^{(\mu^\pm,s^\pm)} - I\!E f_\Delta^{(\mu^\pm,s^\pm)} \right| \le I\!E \left(\left| f_\Delta^{(\mu^\pm,s^\pm)} - I\!M f_\Delta^{(\mu^\pm,s^\pm)} \right| \right)$$
$$= \int_0^\infty I\!P \left[\left| f_\Delta^{(\mu^\pm,s^\pm)} - I\!M f_\Delta^{(\mu^\pm,s^\pm)} \right| \ge t \right] dt \tag{4.25}$$

and easy estimates show that (4.24) implies that

$$|I\!M f_\Delta^{(\mu^\pm,s^\pm)} - I\!E f_\Delta^{(\mu^\pm,s^\pm)}| \le 26 \left(\sqrt{\gamma^{-1}|\Delta|} \right)^{-1} \tag{4.26}$$

and this implies the claim of Proposition 5.2.

The next step is to control the uniformity with respect the possible boundary conditions, and the uniformity with respect to the possible volumes that could occur in the problem. To be more explicit, since we want to analyze the various Gibbs measures that appears in (3.16) and (3.17) and those are related to the base of the cylindrical function F we consider, we want to find the largest volume centered at the origin where we have a good estimate of the deviation from the mean of the free energy uniformly with respect to all the various mesoscopic boundary conditions and all the possible subvolumes included in this fixed volume.

Proposition 5.3. *Given $\epsilon > 0$, $\delta > 0$ and $\Lambda_{\max}$ a macroscopic volume centered at the origin such that*

$$|\Lambda_{\max}| < \frac{\epsilon^2}{64\gamma(2\log M + (3+\delta)\log\gamma^{-1})} \tag{4.27}$$

then if γ is small enough, with $I\!P$-probability greater than $1 - 4\gamma^{1+\delta}$

$$\sup_{\mu^\pm,s^\pm} \sup_{\Delta \subset \Lambda_{\max}} \left(\left| \log Z_\Delta^{(\mu^\pm,s^\pm)} - I\!E \log Z_\Delta^{(\mu^\pm,s^\pm)} \right| \right) \le \epsilon\gamma^{-1} \tag{4.28}$$

Remark: Note that the previous estimate for the $I\!P$-probability allows us to use the first Borel-Cantelli Lemma to get an almost sure result in the case $\gamma = 1/n$ and $n \uparrow \infty$. The numerical constant 64 in (4.27) is not relevant and is linked to the 16 in Talagrand's result. The only relevant fact is the scale $(\gamma \log \gamma^{-1})^{-1}$ in (4.27) where we could expect that the almost sure fluctuations of the free energy around it mean value are of order γ^{-1}.

Proof. We simply write:

$$\mathbb{P}\left[\sup_{\mu^\pm,s^\pm}\sup_{\Delta\in\Lambda_{\max}}\left(\left|\log Z_\Delta^{(\mu^\pm,s^\pm)}-\mathbb{E}\log Z_\Delta^{(\mu^\pm,s^\pm)}\right|\right)\geq\epsilon\gamma^{-1}\right]$$
$$\leq 4M^2\sum_{\kappa=1}^{|\Lambda_{\max}|}||\Lambda_{\max}|-k|\,\mathbb{P}\left[\left|\log Z_{\Delta_k}^{(1,1)}-\mathbb{E}\log Z_{\Delta_k}^{(1,1)}\right|\geq\epsilon\gamma^{-1}\right] \tag{4.29}$$

by fixing the length k of the subvolumes Δ_k, and using the fact that the number of different volumes of fixed length k in $\Lambda_{\max}$ is just $||\Lambda_{\max}|-k|$. Using (4.4) we have, if $|\Delta|=k$

$$\mathbb{P}\left[\left|\log Z_\Delta^{(\mu^\pm,s^\pm)}-\mathbb{E}\log Z_\Delta^{(\mu^\pm,s^\pm)}\right|\geq tk\gamma^{-1}+26(\sqrt{\gamma^{-1}k})\right]$$
$$\leq K\exp\left\{-\frac{\gamma^{-1}k}{4}\left(\sqrt{1+\frac{t^2}{8}}-1\right)\right\} \tag{4.30}$$

we choose $t=t(k)$ such that

$$t\gamma^{-1}k+26(\sqrt{\gamma^{-1}k})=\epsilon\gamma^{-1} \tag{4.31}$$

that is $t=\epsilon k^{-1}(1-26\epsilon^{-1}\sqrt{\gamma k})$. Using the fact that $\gamma k\leq c\epsilon^2(\log\gamma^{-1})^{-1}$ we get, for all $0<\eta<1$, if γ is small enough, $t\geq\epsilon k^{-1}(1-\eta)$. On the other hand since for all $x\geq 0$, $\sqrt{1+x}-1\geq x/2(1-x/2)$ we get immediately, for all $\epsilon>0$ and γ small enough:

$$\mathbb{P}\left[\left|\log Z_\Delta^{(\mu^\pm,s^\pm)}-\mathbb{E}\log Z_\Delta^{(\mu^\pm,s^\pm)}\right|\geq\epsilon\gamma^{-1}\right]\leq\exp-\left\{\frac{\epsilon^2(1-\eta)^2}{64\gamma k}\right\} \tag{4.32}$$

It remains to estimate the sum:

$$\Sigma\equiv 4M^2\sum_{\kappa=1}^{|\Lambda_{\max}|}||\Lambda_{\max}|-k|\exp-\left\{\frac{\epsilon^2(1-\eta)^2}{64\gamma k}\right\} \tag{4.33}$$

since the term into the bracket in the previous exponential is an increasing function of k it is easy to check that the previous sum is bounded from above by

$$4M^2|\Lambda_{\max}|^2\exp\{-\frac{(1-\eta)^2\epsilon^2}{64\gamma|\Lambda_{\max}|}\} \tag{4.34}$$

Therefore if

$$|\Lambda_{\max}| \leq \frac{(1-\eta)^2\epsilon^2}{64\gamma(2\log M + (3+\delta)\log\gamma^{-1})} \tag{4.35}$$

we get $\Sigma \leq 4\gamma^{1+\delta}$.

5. Typical profiles under the local Gibbs measures

We consider here the Gibbs measure with free boundary conditions, in a *macroscopic* volume $\Lambda \equiv [v_-, v_+]$ included in the volume $\Lambda_{\max}$ centered around the origin of length

$$|\Lambda_{\max}| \equiv \frac{\epsilon^2}{64\gamma(2\log M + (3+\delta)\log\gamma^{-1})} \tag{5.1}$$

As it is clear from the last chapter, this is a volume where the random fluctuations of the difference between the free energy and its mean are bounded by $\epsilon\gamma^{-1}$, uniformly in all possible volumes involved and boundaries conditions. On a larger scale we expect that these random fluctuations will become of order $c\gamma^{-1}$ and will govern the typical configurations of the Gibbs measure. Note that the fundamental fact that allows [COP] to work in the infinite volume in the use of the symmetry of the system on the global spin flip. In random system such a symmetry does not exist. However, taking average over the disorder restore this symmetry. Therefore, as far as we are in volume where it is possible to replace the involved quantities by their averages, we can expect to have similar behavior as in a tranlations invariant system.

Our main result is about the typical configurations:

Theorem 5.1. *Given $\epsilon > 0$, assume that $\Lambda \subset \Lambda_{\max}$, $\beta > 1$ and $\gamma M(\gamma) \downarrow 0$. Then we can find $\gamma^{-1} \gg \hat{L} \gg 1$ and $\hat{\zeta} \downarrow 0$, such that on a subset $\Omega_\epsilon \subset \Omega$ with $I\!P(\Omega_\epsilon^c) \le \gamma^{1+\delta}$ we have that for all $\omega \in \Omega_\Lambda$*

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\exists_{u\in\Lambda}\eta_{\hat{\zeta},\hat{L}}(u,\sigma) = 0\right) \le e^{-\hat{L}h(\hat{\zeta})} \tag{5.2}$$

where $h(\zeta) = c(\beta,\epsilon)\beta\zeta^3$, and

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\exists_{u\in\Lambda}\eta_{\hat{\zeta},\hat{L}}(u,\sigma) \neq \eta_{\hat{\zeta},\hat{L}}(u+1,\sigma)\right) \le e^{-c\gamma^{-1}} \tag{5.3}$$

for some positive constant $c \ge \frac{1}{8}(1-2\gamma\hat{L})^2(a(\beta)^2 - 2\hat{\zeta})$

Remark: In the ferromagnetic case, the event $\exists_{u\in\Lambda}\eta_{\hat{\zeta},\hat{L}}(u,\sigma) \neq \eta_{\hat{\zeta},\hat{L}}(u+1,\sigma)$ occurs with Gibbs probability 1, on a scale which is of the order $e^{c\gamma^{-1}}$. Here we expect that such a result is true on macroscopic volume which is roughly speaking of the order γ^{-1} with some $\log\gamma^{-1}$ and/or $\log\log\gamma^{-1}$ corrections.

The proof of this theorem makes use of large deviation type estimates, that we will state now. We will consider events F that are measurable with respect to the sigma-algebra generated by the variables $\{m_\ell(\sigma,x)\}_{x\in I}$ with $I=[u_-,u_+]\subset\Lambda$, where $|I|\ll 1/(\gamma\ell)$ is very small compared to Λ. We call this sigma-algebra, the cylinder sigma-algebra and I will be called the basis of the cylinder. Note that the cylinder sigma-algebra generated by σ_i with $i\in I$ will never be used and there is no ambiguity. Let us define the functions $U_\Delta^{s^\pm,\mu^\pm}$ and $\mathcal{F}_{\Delta,\beta,\rho}^{s^\pm,\mu^\pm}$ by

$$
\begin{aligned}
U_\Delta^{s^\pm,\mu^\pm}(m_\ell) \equiv \gamma\ell \sum_{x,y\in\Delta} J_{\gamma\ell}(x-y)\frac{\|m_\ell(x)-m_\ell(y)\|_2^2}{4} \\
+\gamma\ell \sum_{x\in\Delta,y\in\partial\Delta} J_{\gamma\ell}(x-y)\frac{\|m_\ell(x)-m^{(\mu^\pm,s^\pm)}\|_2^2}{2}
\end{aligned}
\tag{5.4}
$$

and

$$
\mathcal{F}_{\Delta,\beta,\rho}^{(\mu^\pm,s^\pm)}(m_\ell)\equiv U_\Delta^{(\mu^\pm,s^\pm)}(m_\ell)+\gamma\ell\sum_{x\in\Delta} f_{x,\beta,\rho}(m_\ell(x)) \tag{5.5}
$$

where

$$
f_{x,\beta,\rho}(m_\ell(x))\equiv -\frac{1}{\beta\ell}\ln \mathbb{E}_\sigma e^{\frac{\beta\ell}{2}\|m_\ell(\sigma,x)\|_2^2}\mathbb{1}_{\{\|m_\ell(\sigma,x)-m_\ell(x)\|_2\le\rho\}} \tag{5.6}
$$

For any given $\delta>0$ define the δ-covering F_δ of F as $F_\delta\equiv\{\sigma|\exists_{\sigma'\in F}: \forall_{x\in I}\|m_\ell(\sigma,x)-m_\ell(\sigma',x)\|_2<\delta\}$.

With these notations we have the following large deviation estimates:

Theorem 5.2. *Let F and F_δ be as defined above. Assume that $\Lambda\subset\Lambda_{\max}$ Then there exist ℓ,L,ζ,R all depending on γ and a set $\Omega_\Lambda\subset\Omega$ with $\mathbb{P}[\Omega_\Lambda^c]\le\gamma^{1+\delta}$ such that for all $\omega\in\Omega_\Lambda$*

$$
\begin{aligned}
&-\frac{\gamma}{\beta}\ln\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F)\ge\\
&\inf_{\mu^\pm,s^\pm,\pm(w_\pm-u_\pm)\le R}\left[\inf_{m_\ell\in F}\mathcal{F}_{[w_-,w_+],\beta,\gamma}^{(\mu^\pm,s^\pm)}(m_\ell)-\inf_{m_\ell}\mathcal{F}_{[w_-,w_+],\beta,\gamma}^{(1,1,1,1)}(m_\ell)\right]\\
&-er(\ell,L,M,\zeta,R)
\end{aligned}
\tag{5.7}
$$

and for any $\delta > 0$, for γ small enough

$$
\begin{aligned}
&-\frac{\gamma}{\beta}\ln \mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F_\delta) \leq \\
&\inf_{\mu^\pm,s^\pm,\pm(w_\pm-u_p m)\leq R}\left[\inf_{m_\ell\in F}\mathcal{F}^{(\mu^\pm,s^\pm)}_{[w_-,w_+],\beta,\gamma}(m_\ell)+\inf_{m_\ell}\mathcal{F}^{(1,1,1,1)}_{[w_-,w_+],\beta,\gamma}(m_\ell)\right] \\
&+er(\ell,L,M,\zeta,R)
\end{aligned}
\tag{5.8}
$$

where $er(\ell,L,M,\zeta,R)$ is a function of $\alpha \equiv \gamma M$ that tends to zero as $\alpha \downarrow 0$.

Proof. Relative to the interval I, the base of the cylinder corresponding to F, we introduce again the partition of the spin configuration space S from Section 3. While we will use again the fondamental estimate (3.15), we treat the terms corresponding to S_R somewhat differently. Let us introduce the constrained partition functions

$$
Z_{\beta,\gamma,\Lambda}[\omega](F) \equiv \mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F) Z_{\beta,\gamma,\Lambda}[\omega] \tag{5.9}
$$

Just as in Proposition 4.1, for given $\epsilon > 0$, $\zeta > 0$ and L, calling $\hat\zeta \equiv \hat\zeta(\epsilon) \equiv \zeta(1+\sqrt{\gamma M})(1+\epsilon)$, we have that

$$
\begin{aligned}
&Z_{\beta,\gamma,\Lambda}(F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm)) \leq Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\}) \\
&\times Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F) Z_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma)=s^+e^{\mu^+}\}) \\
&\times e^{8\gamma^{-1}(\hat\zeta+2\gamma L)}
\end{aligned}
\tag{5.10}
$$

and

$$
\begin{aligned}
&Z_{\beta,\gamma,\Lambda}(F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm)) \geq Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\}) \\
&\times Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F) Z_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma)=s^+e^{\mu^+}\}) \\
&\times e^{-8\gamma^{-1}(\hat\zeta+2\gamma L)}
\end{aligned}
\tag{5.11}
$$

where $\Delta = [w_- + \frac{1}{2}, w_+ - \frac{1}{2}]$ and $\Lambda^\pm$ are the two connected components of the complement of Δ in Λ. Using the trivial observation that

$$
Z_{\beta,\gamma,\Lambda} \geq Z_{\beta,\gamma,\Lambda}(\mathcal{A}(\mu^\pm = 1, s^\pm = 1, w_\pm)) \tag{5.12}
$$

this combines to

$$
\begin{aligned}
&\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap\mathcal{A}(\mu^{\pm},s^{\pm},w_{\pm}))\\
&\le \frac{Z^{(\mu^{\pm},s^{\pm})}_{\beta,\gamma,\Delta}(F)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Delta}}\\
&\times \frac{Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\})}{Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=e^1\})}\frac{Z_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma)=s^+e^{\mu^+}\})}{Z_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,s)=e^1\})}\\
&\times e^{16\gamma^{-1}(\hat{\zeta}+2\gamma L)}
\end{aligned}
\tag{5.13}
$$

It is precisely at this step that [COP] used the symmetry of the ferromagnetic system to simply replace the ratio of partition functions on $\Lambda^{\pm}$ by one.

Here this is clearly impossible, the idea is to use the self averaging property proved in the previous section. We have in fact an approximate symmetry on volume that are not too large.

Lemma 5.3: *Given $\epsilon>0$, $\delta>0$, let $\Lambda=[\lambda^-,\lambda^+]\subset\Lambda_{\max}$, let $w_-,w_+\in\Lambda$, and $\Lambda^-=[\lambda^-,w_-+\frac{1}{2}]$, $\Lambda^+=[w_+-\frac{1}{2},\lambda^+]$. Then, uniformly with respect to $s^{\pm},\mu^{\pm}$ and $w_{\pm}\in\Lambda_{\max}$*

$$
\begin{aligned}
&\left|\ln Z_{\beta,\gamma,\Lambda^{\pm}}(\{\eta(w_{\pm},\sigma)=s^{\pm}e^{\mu^{\pm}}\})-\ln Z_{\beta,\gamma,\Lambda^{\pm}}(\{\eta(w_{\pm},\sigma)=e^1\})\right|\le\\
&\beta\gamma^{-1}\left[c\hat{\zeta}+cL^{-1}e^{-\beta L(\zeta^2-\frac{\alpha}{\gamma L}(1-\log\epsilon))}+16\gamma L+4\gamma M\frac{\alpha}{\gamma L}(1-\log\epsilon)\right]
\end{aligned}
\tag{5.14}
$$

with probability greater than $1-\gamma^{(1+\delta)}$.

Proof. We consider the case where $\Lambda^{\pm}=\Lambda^-$, the other one being similar. Introducing a carefully chosen zero and using the triangle inequality, we see that

$$
\begin{aligned}
&\left|\ln Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\})-\ln Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=e^1\})\right|\\
&\le\Big|\ln Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\})-\ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda^-\backslash w_-}\\
&+\ln Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda^-\backslash w_-}-\ln Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=e^1\})\Big|\\
&+\left|\ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda^-\backslash w_-}-I\!E\ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda^-\backslash w_-}\right|\\
&+\left|I\!E\ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda^-\backslash w_-}-I\!E\ln Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda^-\backslash w_-}\right|\\
&+\left|I\!E\ln Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda^-\backslash w_-}-\ln Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda^-\backslash w_-}\right|
\end{aligned}
\tag{5.15}
$$

The third term on the right hand side of (5.15) is zero by symmetry, while the second and fourth are bounded by the Proposition 4.2 by $\gamma^{-1}\epsilon$ with probability at least $1-\gamma^{(1+\delta)}$. To bound the first term we proceed as in the proof of Proposition 4.1, part i, that is we use the same decomposition as in (3.24) and (3.26). Calling

$$
\begin{aligned}
&D_{\Lambda,w_-,s^-,\mu^-}\\
&\equiv \ln Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\})\\
&-\ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda^-\setminus w_-}-\ln Z_{w_-,\beta,\gamma}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\})
\end{aligned}
\tag{5.16}
$$

this gives that, with $I\!P$-probability greater than $1-8KM^2R^2e^{-\frac{\epsilon^2\gamma^{-1}}{K}}$,

$$
\sup_{s^-,\mu^-}\sup_{w_-\in\Lambda_{\max}},\left|D_{w_-,s^-,\mu^-}\right|\le 4\gamma^{-1}\left(\hat\zeta+2\gamma L+\gamma M\right)
\tag{5.17}
$$

The constrained partition function on the block w_- is easily dealt with. First, we note that by (2.5) with probability greater than $1-R^2\exp(-c\gamma^{-1})$ we can replace the Hamiltonian by its blocked version on the scale L with an error of order $\gamma^{-1}(16\gamma L+\gamma M)$. Then we can repeat what was done on the scale ℓ in (3.75) but here on the scale L to get

$$
\begin{aligned}
&\inf_{s^-,\mu^-}\inf_{w_-\in\Lambda_{\max}}\ln Z_{w_-,\beta,\gamma}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\})\ge\\
&-\beta\gamma^{-1}\big[\phi(a(\beta))+\zeta^2+cL^{-1}e^{-\beta L(\zeta^2-\delta(\epsilon,L)-\frac{\alpha}{\gamma L}(1-\log\epsilon))}\\
&+16\gamma L+4\gamma M\big]
\end{aligned}
\tag{5.18}
$$

with $I\!P$-probability greater than $1-4M^2\gamma^{-2}e^{-\frac{\epsilon^2L}{K}}$. To get an upper bound we simply use (3.65) to get

$$
\begin{aligned}
&\sup_{s^-,\mu^-}\sup_{w_-\in\Lambda_{\max}}\ln Z_{w_-,\beta,\gamma}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\})\\
&\le-\beta\gamma^{-1}\left[\phi(a(\beta))-c\delta(\epsilon,L)-\frac{\alpha}{\gamma L}(1-\log\epsilon)\right]
\end{aligned}
\tag{5.19}
$$

with $I\!P$-probability at least $1-4M^2\gamma^{-2}e^{-\frac{\epsilon^2L}{K}}$.

Therefore, we get an upper bound

$$
\beta\gamma^{-1}\left[c\hat\zeta+cL^{-1}e^{-\beta L(\zeta^2-\frac{\alpha}{\gamma L}(1-\log\epsilon))}+16\gamma L+4\gamma M\frac{\alpha}{\gamma L}(1-\log\epsilon)\right]
\tag{5.20}
$$

for the first term on the right of (5.14). Putting all things together, and noticing that the worst probability is $1-\gamma^{1+\delta}$, we arrive at the assertion of the lemma.

Lemma 5.3 asserts that to leading order, only the first ratio of partition functions is relevant in (5.13). On the other hand, using Proposition 4.1, part (ii), we see that by choosing R large enough, we only need to consider the case $|\Delta| \leq R$. We use the block approximation on the scale ℓ for those, committing an error at most of order $\beta\gamma^{-1}(R\gamma\ell)$. We will make this precise in the next lemma.

Lemma 5.4: *For any given $(s^\pm, \mu^\pm, w_\pm)$ and $I \subset \Delta \subset \Lambda$ and any F that is measurable with respect to the sigma algebra generated by $\{m_\ell(\sigma, x)\}_{x\in I}$ we have*

$$\frac{\gamma}{\beta} \ln \frac{Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Delta}} \leq - \inf_{m_\ell \in F} \mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell) + \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{\Delta,\beta,\rho}(m_\ell) \\ + c' \left(|\Delta|\gamma\ell + |\Delta|\gamma M |\ln \tfrac{2\ell}{M}| + |\Delta| \tfrac{M}{\ell}\right) \tag{5.21}$$

and $\forall \delta > 0$, for sufficiently small γ

$$\frac{\gamma}{\beta} \ln \frac{Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F_\delta)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Delta}} \geq - \inf_{m_\ell \in F} \mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell) + \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{\Delta,\beta,\rho}(m_\ell) \\ + c' \left(|\Delta|\gamma\ell + |\Delta|\gamma M |\ln \tfrac{2\ell}{M}| + |\Delta| \tfrac{M}{\ell}\right) \tag{5.22}$$

with probability greater than $1 - e^{-c\ell\epsilon^2}$.

Proof. The first step is use the block approximation on the scale ℓ:

Using Lemma 2.1,i) with $\delta = 1$, we see that for given F, $\mu^\pm, s^\pm$ and Δ, with $I\!P$-probability greater than $1 - 16e^{-|\Delta|\gamma^{-1}}$

$$Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F) \\ \leq I\!E_\sigma \mathbb{1}_{\{m_\ell(\sigma)\in F\}} e^{-\beta\gamma^{-1}\left[E^\ell_{\gamma,\Delta}(m_\ell(\sigma)) + E^{\ell,L}_{\gamma,\Delta}\left(m_\ell(\sigma_\Delta), m^{(\mu^\pm,s^\pm)}\right)\right]} \\ \times e^{\beta\gamma^{-1}40|\Delta|(\gamma\ell+\gamma M)} \tag{5.23}$$

and

$$Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F) \\ \geq I\!E_\sigma \mathbb{1}_{\{m_\ell(\sigma)\in F\}} e^{-\beta\gamma^{-1}\left[E^\ell_{\gamma,\Delta}(m_\ell(\sigma)) + E^{\ell,L}_{\gamma,\Delta}\left(m_\ell(\sigma_\Delta), m^{(\mu^\pm,s^\pm)}\right)\right]} \\ \times e^{-\beta\gamma^{-1}40|\Delta|(\gamma\ell+\gamma M)} \tag{5.24}$$

It is not difficult to check that

$$\begin{aligned}
&E^{\ell}_{\Delta}\left(m_\ell(\sigma_\Delta)\right)+E^{\ell,L}_{\Delta,\partial\Delta}\left(m_\ell(\sigma_\Delta)|m^{(\mu^\pm,s^\pm)}\right)\\
&+\gamma\ell\sum_{x\in\Delta}\frac{\|m_\ell(\sigma,x)\|_2^2}{2}+\gamma\ell\sum_{x\in\partial\Delta}\frac{[a(\beta)]^2}{2}\\
&=\gamma\ell\sum_{x,y\in\Delta}J_{\gamma\ell}(x-y)\frac{\|m_\ell(\sigma,x)-m_\ell(\sigma,y)\|_2^2}{4}\\
&+\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\frac{\|m_\ell(\sigma,x)-m^{(\mu^\pm,s^\pm)}\|_2^2}{2}\\
&-\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\frac{1}{2}[a(\beta)]^2\\
&\equiv U^{\mu^\pm,s^\pm}_{\Delta}\left(m_\ell(\sigma_\Delta)\right)-C(|\Delta|,\beta)
\end{aligned}\tag{5.25}$$

where $C(|\Delta|,\beta)$ is an irrelevant σ-independent constant that will drop out of all relevant formulas and may henceforth be ignored. For given ρ, to be chosen later, we introduce a lattice $\mathcal{W}_{M,\rho}$ in $\mathbb{R}^M$ with spacing $\rho/\sqrt{M}$. Then for any domain $D\subset\mathbb{R}^M$, the balls of radius ρ centered at the points of $\mathcal{W}_{M,\rho}\cap D$ cover D. We choose $\rho=2\sqrt{\frac{M}{\ell}}$. With probability greater than $1-\exp(-c\ell)$, $f_{x,\beta,\rho}(m_\ell(x))=\infty$ if $\|m\|_2^2>2$, while the number of lattice points within the ball of radius 2 are bounded by $\exp\left(M\ln\frac{2\ell}{M}\right)$. But this implies that

$$\begin{aligned}
&\ln\left(\mathbb{E}_{\sigma_\Delta}\mathbb{I}_{\{m_\ell(\sigma)\in F\}}e^{-\beta\gamma^{-1}\left[E^{\ell}_{\Delta}(m_\ell(\sigma_\Delta)+E^{\ell,L}_{\Delta,\partial\Delta}\left(m_\ell(\sigma_\Delta)|m^{(\mu^\pm,s^\pm)}\right)\right]}\right)\\
&\leq-\gamma^{-1}\beta\inf_{m_\ell\in F}\left[\mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell)-C(|\Delta|,\beta)\right]+|\Delta|\left(M|\ln\tfrac{2\ell}{M}|+2\frac{M}{\ell}\right)
\end{aligned}\tag{5.26}$$

and also, if $\delta>2\sqrt{\frac{M}{\ell}}$,

$$\begin{aligned}
&\ln\left(\mathbb{E}_{\sigma_\Delta}\mathbb{I}_{\{m_\ell(\sigma)\in F_\delta\}}e^{-\beta\gamma^{-1}\left[E^{\ell}_{\Delta}(m_\ell(\sigma_\Delta)+E^{\ell,L}_{\Delta,\partial\Delta}\left(m_\ell(\sigma_\Delta)|m^{(\mu^\pm,s^\pm)}\right)\right]}\right)\\
&\geq-\gamma^{-1}\beta\inf_{m_\ell\in F}\left[\mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell)-C(|\Delta|,\beta)\right]-|\Delta|2\frac{M}{\ell}
\end{aligned}\tag{5.27}$$

Treating the denominator in the first line of (5.13) in the same way and putting everything together concludes the proof of the lemma.

An immediate corollary of Lemmata 5.3 and 5.4 is

Lemma 5.5: *For any $\Lambda \subset \Lambda_{\max}$ and any F that is measurable with respect to the sigma algebra generated by $\{m_\ell(\sigma,x)\}_{x\in I}$,*

$$
\begin{aligned}
&\sup_{s^\pm,\mu^\pm,w_\pm} \Big[\gamma\beta \ln \mathcal{G}_{\beta,\gamma,\Lambda}(F\cap \tilde{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)) \\
&+ \inf_{m_\ell\in F} \mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell) - \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{\Delta,\beta,\rho}(m_\ell)\Big] \\
&\le c\left(\gamma L + \epsilon + \zeta + |\Delta|\gamma\ell + |\Delta|\gamma M|\ln \tfrac{2\ell}{M}| + |\Delta|\tfrac{M}{\ell}\right)
\end{aligned}
\tag{5.28}
$$

with probability greater than $1-\gamma^{1+\delta}$ for some finite positive numerical constants c.

We are now set to prove the upper bound in Theorem 5.2. Using the notation of Section 3 we have that

$$
\begin{aligned}
&\ln \mathcal{G}_{\beta,\gamma,\Lambda}(F) \\
&\ln\left(\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R) + \mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R^c)\right) \\
&\le \ln 2 + \max\left(\ln \mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R); \ln \mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R^c)\right) \\
&\le \ln(8M^2 2R) \\
&+ \max\Big[\sup_{\mu^\pm,s^\pm,\pm(w_\pm-u_\pm)\le R} \ln \mathcal{G}_{\beta,\gamma,\Lambda}(F\cap \mathcal{A}(\mu^\pm,s^\pm,w_\pm)); \\
&\left(-c(\beta,\epsilon)\beta L R\zeta^3\right)\Big]
\end{aligned}
\tag{5.29}
$$

where we used (3.15) at the last step. It is clear that for a given F, L, ζ we can always choose R in such a way that the previous maximum is realized with the first term. This impose that

$$
R \gg \frac{1}{\gamma L \zeta^3}
\tag{5.30}
$$

On the other hand, in order for the error terms in (5.21) to go to zero, we must assure that (note that $|\Delta| = |I| + 2R$ is of order R) $R(\gamma\ell + \frac{M}{\ell})$ tends to zero. With $\alpha \equiv \gamma M$, this means

$$
R\left(\gamma\ell + \tfrac{\alpha}{\gamma\ell} + \alpha \ln \frac{\gamma\ell}{\alpha}\right) \downarrow 0
\tag{5.31}
$$

We want to find the smallest possible R such that this true. Since the minimum of the term into parenthesis occurs for $\gamma\ell \sim \sqrt{\alpha}$ if α is small enough, R must satisfies $R(\sqrt{\alpha} + \alpha\ln\alpha) \downarrow 0$, that is $R\sqrt{\alpha} \downarrow 0$.

(5.30) and (5.31) impose conditions on L and ζ, namely that

$$\tfrac{\sqrt{\alpha}}{\gamma L \zeta^3} \downarrow 0 \tag{5.32}$$

Of course we also need that $\zeta \downarrow 0$ and $\gamma L \downarrow 0$, but clearly these constraints can be satisfied provided that $\alpha \downarrow 0$ as $\gamma \downarrow 0$. Thus the upper bound of Theorem 5.2 follows.

To prove the lower bound, we will actually need to make use of the upper bound. To do so, we need more explicit control of the functional $\mathcal{F}$, i.e. we have to use the explicit bounds on $f_{x,\beta,\rho}(m_\ell(x))$ in terms of the function Φ.

Lemma 5.6: *The functional $\mathcal{F}$ defined in (5.5) satisfies*

$$\mathcal{F}_{\Delta,\beta,\rho}^{(\mu^\pm,s^\pm)}(m_\ell) \geq U_\Delta^{(\mu^\pm,s^\pm)}(m_\ell) + \gamma\ell \sum_{x\in\Delta} \Phi_{x,\beta}(m_\ell(x)) - \frac{1}{2}|\Delta|\rho^2 \tag{5.33}$$

and

$$\inf_{m_\ell} \mathcal{F}_{\Delta,\beta,\rho}^{(1,1,1,1)}(m_\ell) \leq |\Delta|\phi_\beta(a(\beta)) + |\Delta|\frac{\ln 2}{\ell\beta} \tag{5.34}$$

where $\phi_\beta(a) \equiv \frac{a^2}{2} - \beta^{-1}\ln\cosh(\beta a)$.

Proof.

To get (5.34), just note that U is non-negative and is equal to zero for any constant m_ℓ, while from Lemma 3.1 it follows that

$$\begin{aligned}
\inf_{m_\ell(x)} f_{x,\beta,\rho}(m_\ell(x)) &\leq \inf_{m_\ell(x)} \Phi_{x,\beta}(m_\ell(x)) + \frac{\ln 2}{\ell\beta} \\
&\leq \Phi_{x,\beta}(m^{(1,1)}) + \frac{\ln 2}{\ell\beta} \\
&= \phi_\beta(a(\beta)) + \frac{\ln 2}{\ell\beta}
\end{aligned} \tag{5.35}$$

This concludes the derivation of the upper bound. We now turn to the corresponding lower bound. What is needed for this is an upper bound on the partition function that would be comparable to the lower bound (5.12). Now

$$
\begin{aligned}
&Z_{\beta,\gamma,\Lambda}\\
&= \sum_{(\mu^\pm,s^\pm)} \mathbb{E}_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} \mathbb{1}_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}}\\
&\frac{Z_{\beta,\gamma,\Lambda}}{\sum_{(\mu^\pm,s^\pm)} \mathbb{E}_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} \mathbb{1}_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}}}\\
&= \sum_{(\mu^\pm,s^\pm)} \mathbb{E}_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} \mathbb{1}_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}}\\
&\frac{Z_{\beta,\gamma,\Lambda}}{\mathbb{E}_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} \left(1-\mathbb{1}_{\{\eta(w_\pm,\sigma)=0\}}\right)}\\
&= \sum_{(\mu^\pm,s^\pm)} \mathbb{E}_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} \mathbb{1}_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}}\\
&\left[1-\mathcal{G}_{\beta,\gamma,\Lambda}\left(\{\eta(w_\pm,\sigma)=0\}\right)\right]^{-1}
\end{aligned}
\tag{5.36}
$$

This is almost the same form as the one we want, except for the last factor. The point is now that we want to use our upper bound from Theorem 5.2 to show that $\mathcal{G}_{\beta,\gamma,\Lambda}\left(\{\eta(w_\pm,\sigma)=0\}\right)$ is small, e.g. smaller than $1/2$. so that this entire factor is negligible on our scale. Remembering our estimate (3.15), one may expect an estimate of the order $\exp(-c_2\beta L\zeta\epsilon(\zeta))$, up to the usual errors. Unfortunately, these errors are of order $\exp(\pm\beta\gamma^{-1}(\zeta+\gamma L))$ and thus may offset completely the principal term. A way out of this apparent dilemma is given by our remaining freedom of choice in the parameters ζ and L; that is to say, to obtain the lower bound, we will use a $\hat\zeta$ and a $\hat L$ in such that first they still satisfy the requirement (5.32) while second $c_2\hat L\hat\zeta(\epsilon(\hat\zeta)\gg\gamma^{-1}\zeta+L$. This is clearly possible. With this in mind we get

Lemma 5.7: *With the same probability as in Lemma 5.5,*

$$
\begin{aligned}
&\frac{\gamma}{\beta}\ln\mathcal{G}_{\beta,\gamma,\Lambda}\left(\{\eta_{\hat\zeta,\hat L}(w_\pm,\sigma)=0\}\right)\\
&\leq -\gamma\hat L\hat\zeta\frac{1-\delta}{2-\delta\hat\zeta}\epsilon(\delta\hat\zeta)+c'\left(\gamma L+\epsilon+\zeta+R\gamma\ell+R\gamma M|\ln\tfrac{2\ell}{M}|+R\tfrac{M}{\ell}\right)
\end{aligned}
\tag{5.37}
$$

Proof. The proof of this Lemma is very similar to the proof of (ii) of Proposition 4.1, except that in addition we use the upper bound of Lemma 5.5 to reduce the error terms. We will skip the details of the proof.

Choosing $\hat{L}$ and $\hat{\zeta}$ appropriately, we can thus achieve that $\left[1-\mathcal{G}_{\beta,\gamma,\Lambda}\left(\{\eta(w_\pm,\sigma)=0\}\right)\right]^{-1}\leq 2$ so that

$$
\begin{aligned}
&Z_{\beta,\gamma,\Lambda}\\
&\leq 2\sum_{(\mu^\pm,s^\pm)} I\!\!E_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} \mathbb{I}_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}}\\
&\leq 2(2M)^2 e^{+8\gamma^{-1}\beta(\hat{\zeta}+2\gamma\hat{L})}\\
&\sup_{\mu^\pm,s^\pm}\left[Z_{\beta,\gamma,\Lambda_-}(\{\eta(w_-\sigma)=s^-e^{\mu^-}\})\right.\\
&\left.Z_{\beta,\gamma,\Delta}^{(\mu^\pm,s^\pm)} Z_{\beta,\gamma,\Lambda_+}(\{\eta(w_+\sigma)=s^+e^{\mu^+}\})\right]
\end{aligned}
\tag{5.38}
$$

(we will drop henceforth the distinction between $\hat{L}$ and L and $\hat{\zeta}$ and ζ). The first and third factor in the last line are, by Lemma 5.3, independent of $\mu^\pm, s^\pm$, up to the usual errors. The second partition function is maximal for $(\mu^+,s^+)=(\mu^-,s^-)$, (this will be shown later). Thus on a set of probability greater than $1-\gamma^{1+\delta}$, which is uniform with respect to $\mu^\pm, s^\pm, w_\pm$ we have

$$
\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm))\geq \frac{Z_{\beta,\gamma,\Delta}^{(\mu^\pm,s^\pm)}(F)}{Z_{\beta,\gamma,\Delta}^{(1,1,1,1)}} e^{-c'\beta\gamma^{-1}(\zeta+\gamma L+\epsilon)}
\tag{5.39}
$$

for some numerical constant c, c'. Using the second assertion of Lemma 6.4 allows us to conclude the poof of Theorem 5.2.

We are now ready to prove Theorem 5.1:

Proof of Theorem 5.1: Notice first that the first assertion (5.2) follows immediately from Lemma 6.7. Just note that

$$
\begin{aligned}
&\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\exists_{u\in\Lambda}\eta_{\hat{\zeta},\hat{L}}(u,\sigma)=0\right)\\
&\leq\sum_{u\in\Lambda}\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\{\eta_{\hat{\zeta},\hat{L}}(u,s)=0\}\right)\\
&\leq|\Lambda|e^{-c\beta\hat{L}\hat{\zeta}^3}
\end{aligned}
\tag{5.40}
$$

for suitably chosen $\hat{L}, \hat{z}$. To prove (5.3), note that we need only consider the case where both $\eta(u,\sigma)$ and $\eta(u+1,\sigma)$ are non-zero. This follows then simply from the upper bound of Theorem 6.2 and the lower bound

$$U_{\Delta}^{(\mu^{\pm},s^{\pm})}(m_\ell) \geq \tfrac{1}{4}\gamma\ell \sum_{x\in u}\sum_{y\in u+1} J_{\gamma\ell}(x-y)\|m_\ell(x)-m_\ell(y)\|_2^2 \tag{5.41}$$

Using convexity, we see that

$$\begin{aligned}
&\gamma\ell \sum_{x\in u}\sum_{y\in u+1} J_{\gamma\ell}(x-y)\|m_\ell(x)-m_\ell(y)\|_2^2\\
&\geq (\gamma\ell)^2 \sum_{\substack{r\in u, s\in u+1\\ |r-s|\leq(\gamma\hat{L})^{-1}-2}} \sum_{x\in r}\sum_{y\in s}\|m_\ell(x)-m_\ell(y)\|_2^2\\
&\geq (\gamma\hat{L})^2 \sum_{\substack{r\in u, s\in u+1\\ |r-s|\leq(\gamma\hat{L})^{-1}-2}} \left\| \tfrac{\ell}{\hat{L}}\sum_{x\in r} m_\ell(x) - \tfrac{\ell}{\hat{L}}\sum_{y\in s} m_\ell(y)\right\|_2^2\\
&= (\gamma\hat{L})^2 \sum_{\substack{r\in u, s\in u+1\\ |r-s|\leq(\gamma\hat{L})^{-1}-2}} \left\|m_{\hat{L}}(r)-m_{\hat{L}}(s)\right\|_2^2
\end{aligned} \tag{5.42}$$

Therefore

$$\begin{aligned}
&\inf_{\mu^{\pm},s^{\pm}} \inf_{m_\ell:\eta(u,m_\ell)\neq\eta(u+1,m_\ell)\neq 0} U_{\Delta}^{(\mu^{\pm},s^{\pm})}(m_\ell)\\
&\geq \tfrac{1}{4} \sum_{\substack{r\in u, s\in u+1\\ |r-s|\leq(\gamma\hat{L})^{-1}-2}} \left((a(\beta))^2 - 2\hat{\zeta}\right)\\
&\geq \tfrac{1}{8}(1-2\gamma\hat{L})^2\left((a(\beta))^2-2\hat{\zeta}\right)
\end{aligned} \tag{5.43}$$

From here the proof of (5.3) is obvious.

This concludes our analysis of the Gibbs measure with free boundary condition in volumes of order $(\gamma(2\log M+(3+\delta)\log\gamma^{-1}))^{-1}$.

The next step is to consider the case of symmetric boundary conditions, that is when the boundary conditions are the same on both side of the volume Λ. We consider only the case where the volume Λ is smaller than $(\gamma(2\log M+(3+\delta)\log\gamma^{-1}))^{-1}$. Since the random fluctuations are negligeable here, the typical profile will be the constant one, compatible with the boundary condition.

Theorem 5.8. *Given ϵ,δ, assume that $\Lambda\subset\Lambda_{\max}$ and $\gamma M\downarrow 0$. Then there exist ℓ, L, ζ, R all depending on γ and a set $\Omega_\Lambda\subset\Omega$ with $I\!P[\Omega_\Lambda^c]\leq$*

$\gamma^{1+\delta}$ *such that for all* $\omega \in \Omega_\Lambda$

$$
\begin{aligned}
&-\frac{\gamma}{\beta} \ln \mathcal{G}^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}[\omega](F) \\
&\geq \inf_{\pm(w_\pm - u_\pm)\leq R} \left[\inf_{m_\ell \in F} \mathcal{F}^{(\mu,s,\mu,s)}_{[w_-,w_+],\beta,\gamma}(m_\ell) - \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{[w_-,w_+],\beta,\gamma}(m_\ell) \right] \\
&- er(\ell, L, M, \zeta, R)
\end{aligned}
\tag{5.44}
$$

and for any $\delta > 0$*, for* γ *small enough*

$$
\begin{aligned}
&-\frac{\gamma}{\beta} \ln \mathcal{G}^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}[\omega](F_\delta) \\
&\leq \inf_{\pm(w_\pm - u_\pm)\leq R} \left[\inf_{m_\ell \in F} \mathcal{F}^{(\mu,s,\mu,s)}_{[w_-,w_+],\beta,\gamma}(m_\ell) - \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{[w_-,w_+],\beta,\gamma}(m_\ell) \right] \\
&+ er(\ell, L, M, \zeta, R)
\end{aligned}
\tag{5.45}
$$

where $er(\ell, L, M, \zeta, R)$ *is a function of* $\alpha \equiv \gamma M$ *that tends to zero as* $\alpha \downarrow 0$.

An immediate corollary of Theorem 5.8 is

Theorem 5.9. *Given* ϵ, δ*, assume that* $\Lambda \subset \Lambda_{\max}$ *and* $\gamma M \downarrow 0$*. Then there exist* ℓ, L, ζ, R *all depending on* γ *and a set* $\Omega_\Lambda \subset \Omega$ *with* $I\!P[\Omega^c_\Lambda] \leq \gamma^{1+\delta}$ *such that for all* $\omega \in \Omega_\Lambda$

$$
\mathcal{G}^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}[\omega] \left(\exists_{u\in\Lambda} \eta_{\hat\zeta,\hat L}(u,\sigma) \neq s e^\mu \right) \leq e^{-\hat L g(\hat\zeta)} \tag{5.46}
$$

where $h(\zeta) = c(\beta, \epsilon)\zeta^3$.

Remark: Eq. (5.46) implies that with $I\!P$-probability one

$$
\lim_{\gamma\downarrow 0} \mathcal{G}^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}[\omega] \left(\forall_{u\in\Lambda} \eta_{\hat\zeta,\hat L}(u,\sigma) = s e^\mu \right) = 1 \tag{5.47}
$$

For the proof of these two theorems see [BGP4].

At last we consider the case of asymmetric boundary conditions. In this case the typical profile will have to make a jump somewhere in the volume Λ, to be compatible with the boundary conditions. This comes from the interaction part of the potential.

Theorem 5.10. *Given* ϵ, δ*, assume that* $\Lambda \subset \Lambda_{\max}$ *and* $\gamma M \downarrow 0$*. Then there exist* ℓ, L, ζ, R *all depending on* γ *and a set* $\Omega_\Lambda \subset \Omega$ *with*

$I\!P[\Omega_\Lambda^c] \le \gamma^{1+\delta}$ *such that for all* $\omega \in \Omega_\Lambda$

$$
\begin{aligned}
&-\frac{\gamma}{\beta}\ln \mathcal{G}_{\beta,\gamma,\Lambda}^{(\tilde\mu,\tilde s,\mu,s)}[\omega](F)\\
&\ge \inf_{\pm(w_\pm-u_\pm)\le R}\left[\inf_{m_\ell\in F}\mathcal{F}_{[w_-,w_+]}^{(\tilde\mu,\tilde s,\mu,s)}(m_\ell)-\inf_{m_\ell}\mathcal{F}_{[w_-,w_+]}^{(1,1,2,1)}(m_\ell)\right]\\
&+er(\ell,L,M,\zeta,R)
\end{aligned}
\tag{5.48}
$$

and for any $\delta>0$*, for* γ *small enough,*

$$
\begin{aligned}
&-\frac{\gamma}{\beta}\ln \mathcal{G}_{\beta,\gamma,\Lambda}^{(\tilde\mu,\tilde s,\mu,s)}[\omega](F_\delta)\\
&\le \inf_{\pm(w_\pm-u_\pm)\le R}\left[\inf_{m_\ell\in F}\mathcal{F}_{[w_-,w_+]}^{(\tilde\mu,\tilde s,\mu,s)}(m_\ell)-\inf_{m_\ell}\mathcal{F}_{[w_-,w_+]}^{(1,1,2,1)}(m_\ell)\right]\\
&+er(\ell,L,M,\zeta,R)
\end{aligned}
\tag{5.49}
$$

where $er(\ell,L,M,\hat\zeta,R)$ *is a function of* $\alpha\equiv\gamma M$ *that tends to zero as* $\alpha\downarrow 0$*.*

Finally, we want to give a characterization of the typical profile in the case of asymmetric boundary conditions.

Let us define the following subset of spin configurations

$$
\begin{aligned}
&E_{1,\Lambda}^{(\mu,s,\mu',s')}\equiv\\
&\left\{\exists_{\substack{u_0\le u_1\in\Lambda\\ u_1-u_0\le 2R}}\forall_{\lambda_-\le u<U_0}\eta_{\hat\zeta,\hat L}(u,\sigma)=se^{\mu},\forall_{u_0<v\le\lambda_+}\eta_{\hat\zeta,\hat L}(u,\sigma)=s'e^{\mu'}\right\}
\end{aligned}
\tag{5.50}
$$

Theorem 5.11. *Given* ϵ,δ*, assume that* $\Lambda\subset\Lambda_{\max}$ *and* $\gamma M\downarrow 0$*. Then there exist* ℓ,L,ζ,R *all depending on* γ *and a set* $\Omega_\Lambda\subset\Omega$ *with* $I\!P[\Omega_\Lambda^c]\le\gamma^{1+\delta}$ *such that for all* $\omega\in\Omega_\Lambda$

$$
\mathcal{G}_{\beta,\gamma,\Lambda}^{(\mu,s,\mu',s')}[\omega]\left(E_{1,\Lambda}^{(\mu,s,\mu',s')}\right)\ge 1-2Re^{-\hat L c(\hat\zeta)}
\tag{5.51}
$$

the proof of this theorem can be found in [BGP4].

Remark: This theorem implies that for any volume Λ such that $\Lambda\subset\Lambda_{\max}$, we have $I\!P$-almost surely,

$$
\lim_{\gamma\downarrow 0}\mathcal{G}_{\beta,\gamma,\Lambda}^{(\mu,s,\mu',s')}[\omega]\left(E_{1,\Lambda}^{(\mu,s,\mu',s')}\right)=1
\tag{5.52}
$$

(Here one may, to avoid complications with the "almost sure" statement due to the uncountability of the number of possible sequences γ_n, assume for simplicity that $\lim_{\gamma\downarrow 0}$ is understood to be taken along some fixed discrete sequence, e.g. $\gamma_n = 1/n$. To show that the convergence holds also with probability one for *all* sequences tending to zero, one can use a continuity result as given in Lemma 2.3 of [BGP2]).

We are now ready to state a precise version of the main result of this paper: We define the events

$$E_{0,\Lambda}^{(\mu,s)} \equiv \left\{ \sigma \,\middle|\, \forall_{u\in\Lambda} \eta_{\hat{\zeta},\hat{L}}(u,\sigma) = se^{\mu} \right\} \tag{5.53}$$

and set

$$E_{0,\Lambda} \equiv \cup_{(\mu,s)} E_{0,\Lambda}^{(\mu,s)} \tag{5.54}$$

$$E_{1,\Lambda} \equiv \cup_{(\mu,s)\neq(\mu',s')} E_{1,\Lambda}^{(\mu,s,\mu',s')} \tag{5.55}$$

This this notation we have

Theorem 5.12. *For any macroscopic box V such that $|V| \leq (\gamma(2\log M + (3+\delta)\log\gamma^{-1}))^{-1}$, $I\!\!P$-almost surely,*

$$\lim_{\gamma\downarrow 0} \lim_{\Lambda\uparrow \boldsymbol{Z}} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega] \left(E_{0,V} \cup E_{1,V}\right) = 1 \tag{5.56}$$

The proof is immediate and can be found in [BGP4].

References

[ABCP] G. Alberti, G. Bellettini, M. Cassandro, and E. Presutti, "Surface tension in Ising systems with Kac potentials", J. Stat. Phys. **82**, 743-796 (1996). Ann. Phys. **173**, 30-67 (1987).

[B] T. Bodineau, "Interface for one-dimensional random Kac model", to appear Ann. Inst. H. Poincaré.

[BCP] G. Bellettini, M. Cassandro, and E. Presutti, "Constrained minima of non local free energy functionals", J. Stat. Phys. **84**, 1337-1349 (1996).

[BG1] A. Bovier and V. Gayrard, "Rigorous results on the thermodynamics of the dilute Hopfield model", J. Stat. Phys. **69**, 597-627 (1993).

[BG2] A. Bovier and V. Gayrard, "An almost sure large deviation principle for the Hopfield model", Ann. Probab. **24**, 1444-1475 (1996).

[BG3] A. Bovier and V. Gayrard, "The retrieval phase of the Hopfield model, A rigorous analysis of the overlap distribution", Probab. Theor. Rel. Fields **107**, 61-98 (1996).

[BG4] A. Bovier and V. Gayrard, "Hopfield models as generalized random mean field models", this volume.

[BGP1] A. Bovier, V. Gayrard, and P. Picco, "Gibbs states of the Hopfield model in the regime of perfect memory", Prob. Theor. Rel. Fields **100**, 329-363 (1994).

[BGP2] A. Bovier, V. Gayrard, and P. Picco, "Large deviation principles for the Hopfield model and the Kac-Hopfield model", Prob. Theor. Rel. Fields **101**, 511-546 (1995).

[BGP3] A. Bovier, V. Gayrard, and P. Picco, "Gibbs states of the Hopfield model with extensively many patterns", J. Stat. Phys. **79**, 395-414 (1995).

[BGP4] A. Bovier, V. Gayrard and P. Picco, "Distribution of profiles for the Kac-Hopfield model", to appear in Comm. Math. Phys. (1997)

[BS] M. Brokate and J. Sprekels, "Hysteresis and Phase Transitions", Applied Mathematical Sciences, vol. 121, Springer-Verlag, New York, (1996).

[BZ] A. Bovier and M. Zahradník, "The low-temperature phase of Kac-Ising models" to appear in J. Stat. Phys. **87** (1997)

[BPRS] L. Bertini, E. Presutti, B. Rüdiger, and E. Saada, "Dynamical critical fluctuations and convergence to a stochastic non linear PDE in one dimension", Theor. Prob. Appl. **38**, 689-741 (1993).

[CMP] M. Cassandro, R. Marra, and E. Presutti, "Corrections to the critical temperature in 2d Ising systems with Kac potentials", J. Stat. Phys. **78**, 1131-1138 (1995).

[COPi] M. Cassandro, E. Olivieri, and P. Picco, "Small random perturbations of infinite dimensional dynamical systems and nucleation theory", Ann. IHP. **44**, 343 (1986).

[COP] M. Cassandro, E. Orlandi, and E. Presutti, "Interfaces and typical Gibbs configurations for one-dimensional Kac potentials", Prob. Theor. Rel. Fields **96**, 57-96 (1993).

[CP] M. Cassandro and E. Presutti, "Phase transitions in Ising systems with long but finite range interactions", Markov Proc. Rel.Fields **2**, 241-262 (1996).

[CT] Y.S. Chow and M. Teicher, "Probability Theory", Springer, Berlin, (1978)

[DGP] A. De Masi, T. Gobron, and E. Presutti, "Travelling fronts in non local evolution equations", Arch. Rat. Mech. Anal. **132**, 143-205 (1995).

[DOPT1] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, "Motion by curvature by scaling non local evolution equations", J. Stat. Phys. **73**, 543-570 (1993).

[DOPT2] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, "Glauber evolution with Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics", Nonlinearity **7**, 1-67 (1994).

[DOPT3] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, "Stability of the interface in a model of phase separation", Proc. Royal Soc. Edinburgh **124A**, 1013-102 (1994).

[DOPT4] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, "Uniqueness and global stability of the instanton in non local evolution equations", Rendiconti di Matematica **14**, 693-723 (1994).

[DOPT5] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, "Glauber evolution with Kac potentials. III. Spinodal decomposition", Nonlinearity **9**, 53-114 (1996).

[DOPT6] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, "Glauber evolution with Kac potentials. II. Fluctuation", Nonlinearity **9**, 27-51 (1996).

[FP] L.A. Pastur and A.L. Figotin, "Infinite range limit for a class of disordered spin systems", Theor. Math. Phys. **51**, 564-569 (1982).

[Ho] J.J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities", Proc. Natl. Acad. Sci. USA **79**, 2554-2558 (1982).

[HL] P.C. Hemmer and J.L. Lebowitz, "Systems with weak long-range potentials", in "Phase transition and critical phenomena" vol **5 b** Domb and Green ed. (1976).

[K] H. Koch, "A free energy bound for the Hopfield model", J. Phys. **A 26**, L353-L355 (1993).

[KUH] M. Kac, G. Uhlenbeck, and P.C. Hemmer, “On the van der Waals theory of vapour-liquid equilibrium. I. Discussion of a one-dimensional model”, J. Math. Phys. **4**, 216-228 (1963); “II. Discussion of the distribution functions”, J. Math. Phys. **4**, 229-247 (1963); “III. Discussion of the critical region”, J. Math. Phys. **5**, 60-74 (1964).

[LP] J. Lebowitz and O. Penrose, “Rigorous treatment of the Van der Waals Maxwell theory of the liquid-vapour transition”, J. Math. Phys. **7**, 98-113 (1966).

[LOP] J.L.Lebowitz, E. Orlandi, and E. Presutti, “ A particle model for spinodal decomposition”, J. Stat. Phys. **63**, 933-974 (1991).

[P1] E. Presutti,“Phase separation and interfaces for spin systems with Kac potentials”, in “ Phase transitions: mathematics, physics, biology”, World Scientific (Singapore) ed: R. Kotecký (1993).

[P2] E. Presutti, “Critical fluctuations in a spin system”, Fields Institute Communications.**6**, 189-202 (1996) Publ. Math. I.H.E.S. **81**, 73-205 (1995).

[T] M. Talagrand, “A new look at independence”, Ann. Prob. **24**, 1-37 (1996).