Smooth density field of catalytic super-Brownian motion

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Abstract

Given an (ordinary) super-Brownian motion (SBM) ϱ on R^d of dimension $d = 2, 3$, we consider a (catalytic) SBM X^ϱ on R^d with "local branching rates" $\varrho_s(\mathrm{d}x)$.

We show that X_t^ϱ is absolutely continuous with a density function ξ_t^ϱ, say. Moreover, there exists a version of the map $(t, z) \mapsto \xi_t^\varrho(z)$ which is $\mathcal{C}^\infty$ and solves the heat equation off the catalyst ϱ, more precisely, off the (zero set of) closed support of the time-space measure $\mathrm{d}s\,\varrho_s(\mathrm{d}x)$.

Using self-similarity, we apply this result to answer the question of the long-term behavior of X^ϱ in dimension $d = 2$: If ϱ and X^ϱ start with a Lebesgue measure, then X_T^ϱ converges (persistently) as $T \to \infty$ towards a random multiple of Lebesgue measure.

Contents

1 Introduction

1.1 Motivation and sketch of results

Consider continuous super-Brownian motion $\varrho = (\varrho_t)_{t\geq 0}$ in R^d with a constant branching rate. Roughly speaking, the *catalytic super-Brownian motion* $X^\varrho = (X_t^\varrho)_{t\geq 0}$ is a continuous super-Brownian motion in R^d with local branching rate "proportional to" ϱ. A rigorous construction can be found in [DF97].

In [DF97] also the study of longtime behavior of X^ϱ was initiated, and then continued in [DF96] and [EF96]. From these papers it is known that if both initial states ϱ_0 and X_0^ϱ are Lebesgue measures ℓ_{c} and ℓ_{r}, respectively, then X^ϱ is *persistent* in all three dimensions $d \leq 3$ of its non-trivial existence. (In $d = 3$ the catalyst process ϱ was actually started from its steady state rather than from ℓ_{c} at time zero; this simplification is of course not possible in lower dimensions where ϱ clusters in the longtime limit, hence dies out locally.)

Here persistence means that all weak limit points of X_T^ϱ as $T \to \infty$ have the full intensity measure ℓ_{r} again. In dimensions one and three the stronger result of persistent *convergence* has been shown in ([DF97, DF96]). For dimension $d = 2$ persistence of X^ϱ was proved in ([EF96]). The approach of [EF96] was to show the relative compactness of the set of laws of random second moments by p.d.e. methods. However, uniqueness of the limit point, and hence convergence, remained open.

In dimension $d = 2$, the process X^ϱ has a self-similarity property that connects the long-term behavior of X^ϱ with local properties at a fixed time. Thus, as noted in [DF96, Remark 14], persistent convergence of X_T^ϱ as $T \to \infty$ is equivalent to the existence of the limit $\xi_1^\varrho(0)$ of $(2\varepsilon)^{-2} X_1^\varrho\left((-\varepsilon,\varepsilon)^2\right)$ as $\varepsilon \downarrow 0$, with full expectation, and hence to the absolute continuity of X_1^ϱ. Our main objective in this paper is to show that X_1^ϱ is absolutely continuous.

It is well-known that the (continuous) SBM with constant branching rate has *absolutely continuous states* only in dimension one. In $d = 1$ actually "every" catalytic SBM has densities ([DFR91]). [DF95] construct *higher-dimensional* catalytic SBM (with finite variance branching) with absolutely continuous states where the branching rate is given by a certain class of additive functionals of Brownian motion. This class includes catalysts concentrated on hyperplanes. They show absolute continuity via constructing fundamental solutions of the related cumulant equation.

Recently Delmas [Del96] considered a class of time-independent catalysts in R^d with carrying Hausdorff dimension greater than $d-2$. He shows that the reactant has a smooth density off the catalyst. His technique is a refinement of the *method of Brownian excursions,* introduced by [FL95] for a single point-catalytic model in $d = 1$. The procedure in those two papers is first to determine the (singular) occupation density measures λ, say, on the (time-independent) catalyst, and then to represent the SBM by means of Brownian excursion densities off the catalyst (supported by a Lebesgue zero set) starting with random masses according to λ. Clearly, excursion densities are smooth and satisfy the heat equation. At least at a heuristic level, this makes clear that in these cases a smooth density field exists.

Our strategy is to first show in $d = 2, 3$ that X^{ϱ} has densities in an L^2–sense on the complement of the support of ϱ. Next we use a modification of Delmas' representation of catalytic SBM "off the catalyst" on a local level to derive our *main result.* Namely, we show that off the catalyst, X^{ϱ} has a *smooth density field* ξ^{ϱ} that solves the heat equation (Theorem 1 at page 6).

Finally we use this result to answer the open question mentioned above: In two dimensions, if we start ϱ and X^{ϱ} in Lebesgue measures, then X^{ϱ}_T *converges* in law to the random multiple $\xi^{\varrho}_1(0)\,\ell$ of the (normed) Lebesgue measure ℓ (Corollary 2 (b)).

1.2 Informal description of the model

We consider a stochastic model for a *chemical (or biological) diffusion-reaction system* of two substances (or species) C and R, say. While C evolves independently of R, the *reaction* of R is *catalyzed* by C, that is takes place locally only in the presence of C but without affecting C.

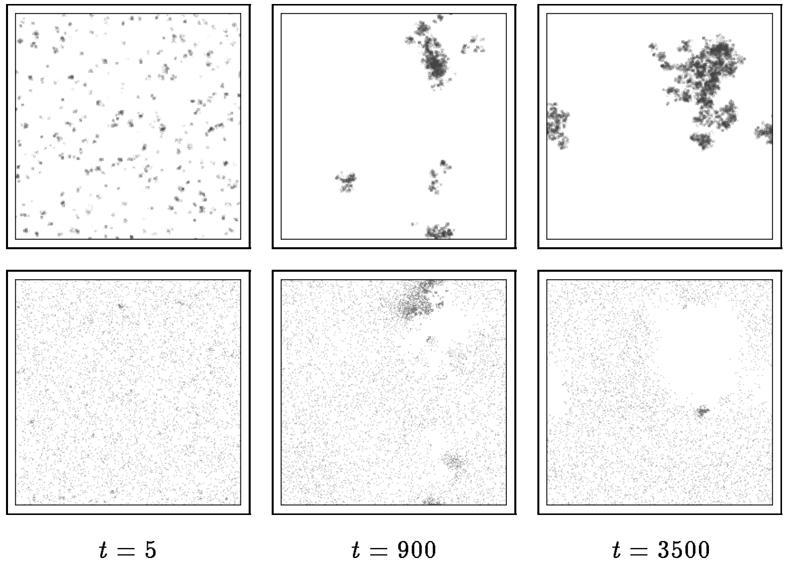
The mathematical model that we choose for the catalyst is the so-called *super-Brownian motion* (SBM) ϱ. It arises as the high density short lifetime limit of *branching Brownian motion.* The latter is an (infinite) particle system, where the particles move around in R^d according to independent Brownian motions. The catalyst particles die with a constant rate γ, say, and are replaced at the location of their death by zero or two offspring, each possibility occurring with probability $\frac{1}{2}$ (critical binary branching). The offspring continue to evolve in the same manner as their parent. Now assign the mass $\varepsilon > 0$ to each particle and replace the branching rate γ by γ/ε. Then (see, e.g., [Daw93, Section 4.4]) SBM ϱ is the limiting process if we let $\varepsilon \downarrow 0$ (provided that the initial states converge). Summarizing, the *catalyst* ϱ arises as a diffusion approximation to a critical binary branching Brownian motion with constant branching rate. For background on SBM we recommend [Daw93].

The mathematical model X^{ϱ} for the *reactant* is also SBM, however the branching rate of an "infinitesimal reactant particle" is the local concentration of catalytic matter. Consequently, the heuristic picture is the same except that the reactant particles die only when they are in contact with the catalyst. The catalyst itself varies in time and space and concentrates in some localized regions.

The model is interesting only in dimensions $d \leq 3$. Roughly speaking, the catalyst is a $(d \wedge 2)$–dimensional object in R^d, thus a reactant particle (which performs Brownian motion) cannot meet the catalyst if $d \geq 4$. Hence, in $d \geq 4$, the "reactant" X^{ϱ} is only the deterministic heat flow.

A mathematical approach to this *"one-way interaction"* model is possible by means of Dynkin's additive functional approach to superprocesses ([Dyn91]).

Simulation of the catalyst ϱ and the reactant X^{ϱ}



The first row shows a discrete version of ϱ (critical binary branching simple random walk) on a 250×250 grid with periodic boundaries, originally started from a "uniformly" distributed field. The number of particles per site is indicated by different grey scales. The "movie" clearly exhibits the well-known tendency of clustering in $d = 2$.

The second sequence of pictures shows a simulation of the analogous discrete version of X^{ϱ}, for the same realization of the branching rate ϱ. The figure illustrates that the reactant X^{ϱ} is uniformly spread out outside the catalytic clusters, except a few "hot spots" related to the catalyst, and that within the catalytic clumps mainly killing of the reactant happens.

In fact, given the medium ϱ, an intrinsic X^{ϱ}–particle (reactant) following a Brownian path W branches according to the clock given by the *collision local time*, $L_{[W,\varrho]}(\mathrm{d}s)$, of W with ϱ ([BEP91]). Somewhat more formally:

$$L_{[W,\varrho]}(\mathrm{d}s) \;=\; \mathrm{d}s \int \varrho_s(\mathrm{d}y)\, \delta_y(W_s). \tag{1}$$

For sufficiently nice initial states of ϱ, these collision local times $L_{[W,\varrho]}$ make sense non-trivially in dimensions $d \leq 3$ ([EP94]), although the measures $\varrho_s(\mathrm{d}y)$ are singular for $d \geq 2$ ([DH79]). For this reason, in dimensions $d \leq 3$ the *cat-*

alytic SBM X^{ϱ} could be constructed in [DF97] as a continuous measure-valued (time-inhomogeneous) Markov process $(X^{\varrho}, P^{\varrho}_{r,m})$, given the catalyst process ϱ (*quenched approach*). By standard notation, $P^{\varrho}_{r,m}$ denotes the law of the process X^{ϱ} (for ϱ fixed) if at time r we start X^{ϱ} in the measure m. The laws of the catalyst process ϱ will be denoted by $\mathbb{P}_{\mu}$ if $\varrho_0 = \mu$.

Averaging the random laws $P^{\varrho}_{0,m}$ by means of $\mathbb{P}_{\mu}$ gives the *annealed* distribution

$$\mathcal{P}_{\mu,m} := \mathbb{P}_{\mu} P^{\varrho}_{0,m} \tag{2}$$

of X^{ϱ}.

Of particular interest is the case

$$\mu = \ell_{\mathrm{c}} := i_{\mathrm{c}} \ell, \qquad m = \ell_{\mathrm{r}} := i_{\mathrm{r}} \ell \tag{3}$$

for some constants $i_{\mathrm{c}}, i_{\mathrm{r}} > 0$.

Consider for the moment the *critical dimension* $d = 2$ and initial states $(\varrho_0, X^{\varrho}_0) = (\ell_{\mathrm{c}}, \ell_{\mathrm{r}})$. Here the catalyst ϱ_T dies out locally in probability as $T \to \infty$. In the large regions without catalyst only the smoothing heat flow acts on the reactant X^{ϱ}. On the other hand, a finite window of observation will be visited by increasingly large catalytic clumps at arbitrarily large times (recall that the time averaged two-dimensional catalyst ϱ has a proper random limit despite local extinction, see, e.g., [FG86]). These clumps lead locally to a great variability of the concentration of reactant: In contact with the catalyst, reactant mass piles up in relatively small areas, whereas large areas become vacant. But according to [EF96, Theorem 1], the smoothing effect in the large catalyst free regions wins this competition with the *"turbulence"* at the catalyst, leading to persistence: The intensity measure ℓ_{r} of X^{ϱ}_T is preserved also for all accumulation points (in law) as $T \to \infty$.

A formal description of the pair (ϱ, X^{ϱ}) will be given in § 2.1.

1.3 Notation and regularity assumption

Let p denote the standard heat kernel:

$$\mathrm{p}_s(x) := (2\pi s)^{-d/2} \exp\Big[-\frac{|x|^2}{2s}\Big], \qquad s > 0, \quad x \in \mathsf{R}^d, \tag{4}$$

and set

$$\mu * \mathrm{p}_s(y) := \int \mu(\mathrm{d}x)\, \mathrm{p}_s(y - x). \tag{5}$$

Introduce the spatial *shift* operators θ_z defined on functions φ on R^d :

$$\theta_z \varphi(y) := \varphi(y - z), \qquad y, z \in \mathsf{R}^d, \tag{6}$$

and write $\langle \nu, \varphi \rangle$ for integral expressions as $\int \nu(\mathrm{d}y)\, \varphi(y)$.

The construction of our processes actually needs an integrability condition for the initial states μ and m. Namely, we will assume that $\mu, m \in \mathcal{M}_p$ for some $p > d$. Here $\mathcal{M}_p$ is the set of measures μ on R^d such that $\langle \mu, \phi_p \rangle < \infty$, where

$$\phi_p(x) := \frac{1}{(1+|x|^2)^{p/2}}, \qquad x \in \mathsf{R}^d. \tag{7}$$

$\mathcal{M}_p$ is endowed with the coarsest topology such that the map $\mu \mapsto \langle \mu, \varphi \rangle$ is continuous for $\varphi = \phi_p$ and for each φ in the cone $\mathcal{C}_+^{\text{comp}}$ of all non-negative continuous functions on R^d with compact support.

In addition we impose a hypothesis on the local structure of the initial states μ of the catalyst ϱ.

Notation *A measure $\mu \in \mathcal{M}_p$ is called* **strongly diffusive** *if there exists an $\eta \in (0, \frac{1}{4})$ such that the map*

$$(t, z) \mapsto \int_0^t \mathrm{d}s \; \mu * \mathrm{p}_s(z), \qquad (t, z) \in [0, \infty) \times \mathsf{R}^d, \tag{8}$$

is locally Hölder continuous of order η. $\diamond$

Example Some important examples for strongly diffusive measures are:

- $\mu = i_{\mathrm{c}} \ell$, for some constant $i_{\mathrm{c}} > 0$.
- μ absolutely continuous with a locally bounded density function.
- In $d \leq 3$, almost all samples of (ordinary) super Brownian motion at a (strictly) positive time (in particular almost all samples of a steady state in $d = 3$) are strongly diffusive (see Lemma 14 in the Appendix).

Note that in $d \geq 2$ a measure μ is *not* strongly diffusive if it has an atom. $\diamond$

1.4 Results

The *key* to our main result (Theorem 1) is the fact that in dimensions two and three the *closed support* S^ϱ of the locally finite measure $\mathrm{d}s\, \varrho_s(\mathrm{d}x)$ on $(0, \infty) \times \mathsf{R}^d$ is an $\ell^+ \times \ell$–zero set (Corollary 8). Here ℓ^+ denotes the (normed) Lebesgue measure on $(0, \infty)$. Write $\mathbf{Z}^\varrho \subset (0, \infty) \times \mathsf{R}^d$ for the *complement* of S^ϱ in $(0, \infty) \times \mathsf{R}^d$. In $\mathbf{Z}^\varrho$ only the heat flow acts on X^ϱ. This suggests that here X^ϱ has densities satisfying the heat equation.

Theorem 1 *Let $d = 2$ or 3, assume $r \geq 0$, $\mu, m \in \mathcal{M}_p$, and that μ is strongly diffusive. For $\mathbb{P}_\mu$–almost all ϱ the following statements hold.*

(a) (absolutely continuous states) *With $P^\varrho_{r,m}$–probability one, X^ϱ_t is absolutely continuous with respect to Lebesgue measure, for all $t > r$.*

(b) (smooth density field ξ^{ϱ}) *Denoting by* $\xi^{\varrho} = \{\xi^{\varrho}_t(z) : t > r,\ z \in \mathsf{R}^d\}$ *the density field of* X^{ϱ}*, there is a version of* ξ^{ϱ} *such that* $P^{\varrho}_{r,m}$*–a.s. the mapping* $(t,z) \mapsto \xi^{\varrho}_t(z)$*,* $(t,z) \in \mathbf{Z}^{\varrho}$*,* $t > r$*, is of class* C^{∞} *and solves the heat equation:*

$$\frac{\partial}{\partial t}\,\xi^{\varrho}_t(z) \;=\; \frac{1}{2}\,\Delta\,\xi^{\varrho}_t(z), \qquad (t,z) \in \mathbf{Z}^{\varrho}, \quad t > r. \tag{9}$$

(c) (moments) *The* $\xi^{\varrho}_t(z)$ *belong to* $L^2 = L^2(P^{\varrho}_{r,m})$*, have expectation*

$$P^{\varrho}_{r,m}\xi^{\varrho}_t(z) \;=\; m * \mathrm{p}_{t-r}(z) \tag{10}$$

and covariances

$$\begin{aligned} &\mathcal{C}\mathrm{ov}^{\varrho}_{r,m}\left[\xi^{\varrho}_{t_1}(z_1), \xi^{\varrho}_{t_2}(z_2)\right] \\ &\quad=\; 2\int_r^{t_1\wedge t_2} \mathrm{d}s\,\Big\langle \varrho_s\,,\ (m * \mathrm{p}_{s-r})\,(\theta_{z_1}\mathrm{p}_{t_1-s})\,(\theta_{z_2}\mathrm{p}_{t_2-s})\Big\rangle \;\ge\; 0, \end{aligned} \tag{11}$$

$(t_i\,, z_i) \in \mathbf{Z}^{\varrho},\ t_i > r,\ i = 1,2.$

(d) (local L^2–Lipschitz continuity) *The field* $\{\xi^{\varrho}_t(z) :\ (t,z) \in \mathbf{Z}^{\varrho},\ t > r\}$ *is locally* $L^2(P^{\varrho}_{r,m})$*–Lipschitz continuous: For every compact subset* C *of* $\mathbf{Z}^{\varrho} \cap \big((r,\infty) \times \mathsf{R}^d\big)$ *there is a constant* $c = c(\varrho, C)$ *such that*

$$\left\|\xi^{\varrho}_{t_1}(z_1) - \xi^{\varrho}_{t_2}(z_2)\right\|_2 \;\le\; c\,\left|(t_1, z_1) - (t_2\,, z_2)\right|, \tag{12}$$

$(t_1\,, z_1), (t_2\,, z_2) \in C.$

Note that in the case $m = \ell_{\mathrm{r}}$ the expectation and covariance formulas reduce to

$$P^{\varrho}_{r,\ell_{\mathrm{r}}}\,\xi^{\varrho}_t(z) \;\equiv\; i_{\mathrm{r}} \;>\; 0, \tag{13}$$

and

$$\begin{aligned} &\mathcal{C}\mathrm{ov}^{\varrho}_{r,\ell_{\mathrm{r}}}\left[\xi^{\varrho}_{t_1}(z_1), \xi^{\varrho}_{t_2}(z_2)\right] \\ &\quad=\; 2\,i_{\mathrm{r}}\int_r^{t_1\wedge t_2} \mathrm{d}s\,\Big\langle \varrho_s\,,\ (\theta_{z_1}\mathrm{p}_{t_1-s})\,(\theta_{z_2}\mathrm{p}_{t_2-s})\Big\rangle. \end{aligned} \tag{14}$$

Formula (11) has the following *genealogical interpretation.* The covariance measures the probability of two infinitesimal particles at $(t_1\,, z_1)$ and $(t_2\,, z_2)$ to have a common ancestor. On the other hand, the integrand at the r.h.s. is the "distribution" of the time-space location (s,x) of a possible latest common ancestor of these infinitesimal particles.

Remark Since a given point $(t,z) \in (r,\infty) \times \mathsf{R}^d$ belongs to $\mathbf{Z}^{\varrho}$ with $\mathbb{P}_{\ell_c}$–probability one, $\xi^{\varrho}_t(z)$ is a well-defined $P^{\varrho}_{r,m}$–random variable, $\mathbb{P}_{\mu}$–a.s. $\diamond$

Remark (annealed model) Statement (a) of Theorem 1 implies that also with respect to the *annealed* law $\mathcal{P}_{\mu,m}$ the catalytic SBM X^ϱ lives on the set of *absolutely continuous* measures. Clearly, (11) and (10) yield that the $\mathcal{P}_{\mu,m}$–*covariances* of ξ^ϱ are given by

$$\begin{aligned} &\mathcal{C}\mathrm{ov}_{\mu,m}\left[\xi^\varrho_{t_1}(z_1), \xi^\varrho_{t_2}(z_2)\right] \\ &\quad = 2\int_0^{t_1\wedge t_2} \mathrm{d}s \left\langle \ell,\ (\mu * \mathrm{p}_s)\,(m * \mathrm{p}_s)\,(\theta_{z_1}\mathrm{p}_{t_1-s})\,(\theta_{z_2}\mathrm{p}_{t_2-s})\right\rangle < \infty \end{aligned} \tag{15}$$

$(t_i\,, z_i) \in (0,\infty)\times \mathsf{R}^d$, $i = 1,2$, $(t_1\,, z_1) \neq (t_2\,, z_2)$. Hence (if $\mu, m \neq 0$), the covariance tends to infinity if $(t_2\,, z_2) \to (t_1\,, z_1)$. In particular,

$$\mathcal{V}\mathrm{ar}_{\mu,m}\,\xi^\varrho_t(z) \equiv \infty, \qquad (t,z) \in (0,\infty)\times \mathsf{R}^d. \qquad \diamond$$

Now we come back to the limiting behavior of X^ϱ_T as $T \uparrow \infty$ in $d = 2$ with $(\varrho_0\,, X^\varrho_0) = (\ell_{\mathrm{c}}\,, \ell_{\mathrm{r}})$. In this dimension, the long-term behavior of X^ϱ is connected to local properties (such as absolute continuity of states) by a *self-similarity* property. Proposition 13 in [DF96] states that

$$X^\varrho_T \overset{\mathcal{L}}{=} K^{-1} X^\varrho_{KT}(K^{1/2}\,\cdot\,), \qquad T, K > 0, \tag{16}$$

with respect to the random laws $P^\varrho_{0,\ell_{\mathrm{r}}}$. Here coincidence w.r.t. the random laws $P^\varrho_{0,\ell_{\mathrm{r}}}$ formally means that

$$\mathsf{P}_{\ell_{\mathrm{c}}}\left[P^\varrho_{0,\ell_{\mathrm{r}}}[X^\varrho_T \in (\cdot)] \in (\cdot)\right] = \mathsf{P}_{\ell_{\mathrm{c}}}\left[P^\varrho_{0,\ell_{\mathrm{r}}}\left[K^{-1}X^\varrho_{KT}(K^{1/2}\,\cdot\,) \in (\cdot)\right] \in (\cdot)\right]. \tag{17}$$

From this discussion the following corollary of Theorem 1 is immediate.

Corollary 2 *In dimension two, with respect to the random laws $P^\varrho_{0,\ell_{\mathrm{r}}}$ (with ϱ distributed according to $\mathsf{P}_{\ell_{\mathrm{c}}}$) the following two statements hold:*

(a) (self-similarity)

$$\xi^\varrho_T \overset{\mathcal{L}}{=} \xi^\varrho_{KT}(K^{1/2}\,\cdot\,), \qquad T, K > 0. \tag{18}$$

(b) (persistent convergence) X^ϱ_T *converges in distribution to a random multiple of Lebesgue measure:*

$$X^\varrho_T \underset{T\uparrow\infty}{\longrightarrow} \xi^\varrho_1(0)\,\ell. \tag{19}$$

Coincidence in law in statement (a) is understood in the same way as in (17). Similarly, the assertion in (b) has the following formal meaning. Given

ϱ, let Q_T^ϱ and Q_∞^ϱ denote the laws of the random measures X_T^ϱ and $\xi_1^\varrho(0)\,\ell$, respectively. Set

$$\mathbf{Q}_T := \mathbb{P}_{\ell_c}\left[Q_T^\varrho \in (\cdot)\right], \qquad \mathbf{Q}_\infty := \mathbb{P}_{\ell_c}\left[Q_\infty^\varrho \in (\cdot)\right]. \tag{20}$$

Then the formal expression for the claim in (b) is

$$\mathbf{Q}_T \text{ converges } \textit{weakly} \text{ to } \mathbf{Q}_\infty \text{ as } T \to \infty. \tag{21}$$

Note that for fixed medium ϱ one cannot expect convergence since ϱ_T itself does *not converge a.s.* as $T \to \infty$.

It is known from [DF97, Theorem 51] that in dimension *one*

$$X_T^\varrho \underset{T\uparrow\infty}{\longrightarrow} \ell_{\mathrm{r}}, \quad \text{in } P_{0,\ell_{\mathrm{r}}}^\varrho\text{–probability}, \quad \text{for } \mathbb{P}_{\ell_c}\text{–almost all } \varrho. \tag{22}$$

(It is still open whether this statement is true $P_{0,\ell_{\mathrm{r}}}^\varrho$–a.s.) The reason for this behavior is that in $d = 1$ the catalyst dies out locally *almost surely*. In contrast, in $d = 2$ the catalyst goes to local extinction only in $\mathbb{P}_{\ell_c}$–probability. Hence, the reactant meets the catalyst at arbitrarily large times. The randomness in the limit in (19) reflects the random medium as experienced by the reactant at large times. In particular, X^ϱ does not converge almost surely. The almost sure properties of Theorem 1 get lost on the way to Corollary 2 (b) by using the self-similarity that holds only in distribution.

Note that the two-dimensional reactant X^ϱ exhibits the following interesting phenomenon: Though started in a (spatially) ergodic state, the limit is *not* ergodic.

Remark 3 (annealed model) The self-similarity (16) holds also with respect to the annealed law $\mathcal{P}_{\ell_c,\ell_{\mathrm{r}}}$ ([DF96, Proposition 13]). Hence, (18) and (19) are true also w.r.t. the annealed law. In other words, we have the following weak convergence of averaged distributions:

$$\mathbb{P}_{\ell_c} Q_T^\varrho \underset{T\uparrow\infty}{\longrightarrow} \mathbb{P}_{\ell_c} Q_\infty^\varrho . \qquad \diamond$$

Remark 4 (lattice model) In the model of two-dimensional simple branching random walk in the simple branching random medium, one can show a statement analogous to Corollary 2 (b): Here the reactant converges to a mixed Poisson system (homogeneous Poisson point process) with random intensity ξ_1^ϱ ([GKW97, Theorem 1.3]). The proof of this statement is based on our Theorem 1. However, since there is no scaling property in the lattice model, things become rather complicated. $\diamond$

The rest of the *paper is laid out* as follows. In Section 2 we recall the formal characterization of the catalytic SBM X^ϱ. We establish the fact that around $\ell^+ \times \ell$–almost all time-space points (t, z) there is no catalytic mass. The key step in Section in 2 is to show that at those (t, z) an asymptotic spatial $L^2(P_{r,m}^\varrho)$–density $\xi_t^\varrho(z)$ of X_t^ϱ exists. Our theorem is proved in Section 3.

2 Preparations

2.1 Formal description of catalytic SBM

First we want to recall the formal characterization of the catalytic SBM X^{ϱ} in terms of its Laplace transition functional.

Recall that $p > d$ with d the dimension of R^d, and that ϕ_p is the reference function of (7). Write $\mathcal{B}^p$ for the set of all functions φ on R^d such that $|\varphi| \leq c_\varphi \, \phi_p$ for some (finite) constant c_φ, and $\mathcal{B}^p_+$ for the subset of its non-negative members.

Fix a constant $\gamma > 0$. By definition, the *catalyst process* $\varrho = (\varrho_t)_{t \geq 0}$ is a continuous (critical) SBM with branching rate γ. This is the continuous $\mathcal{M}_p$–valued time-homogeneous Markov process $(\varrho, \mathbb{P}_\mu)$ with Laplace transition functional

$$\mathbb{P}_\mu \exp \langle \varrho_t \, , -\varphi \rangle \;=\; \exp \langle \mu, -u(t) \rangle \, , \qquad t \geq 0, \quad \mu \in \mathcal{M}_p \, , \quad \varphi \in \mathcal{B}^p_+ \, . \tag{23}$$

Here $u = \{u(t) : t \geq 0\} = \{u(t,x) : \ t \geq 0, \ x \in \mathsf{R}^d\}$ is the unique non-negative solution to the *basic cumulant equation*

$$\frac{\partial}{\partial t} u \;=\; \frac{1}{2} \, \Delta u - \gamma \, u^2 \quad \text{on} \;\; (0,\infty) \times \mathsf{R}^d \tag{24}$$

with *initial* condition $u(0,x) = \varphi(x), \ \ x \in \mathsf{R}^d$. (Where needed, 'solution' has to be understood in a *mild* sense.)

The process ϱ serves as a random medium for a catalytic SBM X^{ϱ}. In order to characterize X^{ϱ}, roughly speaking, we have to replace the constant rate γ in (24) by the (randomly) varying rate $\varrho_t(x)$, where $\varrho_t(x)$ is the *generalized* derivative $\frac{\varrho_t(\mathrm{d}x)}{\mathrm{d}x}(x)$ of the measure $\varrho_t(\mathrm{d}x)$. Our aim is to define X^{ϱ} via its log-Laplace transition functionals v^{ϱ}_t that solve a certain cumulant equation. We do so by first making precise sense of this equation.

Because of time-inhomogeneity, it is convenient to write the formal cumulant equation in a *backward setting:*

$$-\frac{\partial}{\partial r} \, v^{\varrho}_t(r,x) \;=\; \frac{1}{2} \, \Delta v^{\varrho}_t(r,x) - \varrho_r(x) \, v^{\varrho}_t(r,x)^2, \tag{25}$$

$0 \leq r \leq t, \ \ x \in \mathsf{R}^d$. Note that the initial condition has become a *terminal* condition: $v^{\varrho}_t(t) = \varphi$. After a formal integration, we can rewrite (25) rigorously and probabilistically as

$$v^{\varrho}_t(r,x) \;=\; \Pi_{r,x} \left[\varphi(W_t) - \gamma \int_r^t L_{[W,\varrho]}(\mathrm{d}s) \, v^{\varrho}_t(s, W_s)^2 \right], \tag{26}$$

$0 \leq r \leq t, \ \ x \in \mathsf{R}^d$, where $\Pi_{r,x}$ is the law of (standard) Brownian motion W starting at time r from x, and $L_{[W,\varrho]}$ denotes the *collision local time* of W with ϱ, formally introduced in (1). For $d \leq 3$ and finite μ with some regularity,

[EP94] show that for $\mathbb{P}_\mu$–a.a. ϱ the collision local time $L_{[W,\varrho]}$ makes sense as a non-trivial additive functional of W. (In $d \geq 4$, actually $L_{[W,\varrho]} = 0$.) As pointed out in [DF97, Theorem 42], this is still true for strongly diffusive $\mu \in \mathcal{M}_p$ (recall the notation at page 6). Moreover ([DF97, Theorem 42 and Proposition 6]), for $\mathbb{P}_\mu$–almost all ϱ and for t, φ fixed, there is a unique non-negative solution v_t^ϱ to (26). Finally ([DF97, § 5.4]), for $\mathbb{P}_\mu$–a.a. ϱ, there exists a continuous $\mathcal{M}_p$–valued time-inhomogeneous Markov process $(X^\varrho, P_{r,m}^\varrho)$ with Laplace transition functional

$$P_{r,m}^\varrho \exp\langle X_t^\varrho, -\varphi\rangle \;=\; \exp\langle m, -v_t^\varrho(r)\rangle\,, \tag{27}$$

$0 \leq r \leq t,\ m \in \mathcal{M}_p\,,\ \varphi \in \mathcal{B}_+^p\,,$ and v_t^ϱ the solution to (26). This is the *catalytic SBM* X^ϱ with catalyst ϱ, which was intuitively introduced in § 1.2.

From now on we adopt the *convention,* that we consider only initial states $\varrho_0 = \mu \in \mathcal{M}_p$ strongly diffusive and those samples ϱ of the catalyst process such that the (continuous) catalytic SBM X^ϱ exists.

Since the branching mechanism is critical, X^ϱ has *expectation measure*

$$P_{r,m}^\varrho X_t^\varrho \;\equiv\; S_{t-r}m, \tag{28}$$

independent of the catalytic medium ϱ. Here

$$S = \{S_t : t \geq 0\} \tag{29}$$

is the semigroup of Brownian motion. In particular,

$$P_{r,\ell_{\mathrm{r}}}^\varrho\, X_t^\varrho \;\equiv\; \ell_{\mathrm{r}}\,, \tag{30}$$

independent of time. The *covariances* (given ϱ) related to (28) can be written as

$$\begin{aligned}&\mathrm{Cov}_{r,m}^\varrho\Big[\big\langle X_{t_1}^\varrho, \psi_1\big\rangle, \big\langle X_{t_2}^\varrho, \psi_2\big\rangle\Big]\\ &\quad=\; 2\int_r^{t_1\wedge t_2} \mathrm{d}s\,\Big\langle \varrho_s\,,\,(m * \mathrm{p}_{s-r})\,(S_{t_1-s}\psi)\,(S_{t_2-s}\psi_2)\Big\rangle,\end{aligned} \tag{31}$$

$0 \leq r \leq t_1, t_2\,,\ m \in \mathcal{M}_p\,,\ \psi_1, \psi_2 \in \mathcal{B}^p$; see [DF97, formula (95)]. In particular,

$$\begin{aligned}&\mathrm{Cov}_{r,\ell_{\mathrm{r}}}^\varrho\Big[\big\langle X_{t_1}^\varrho, \psi_1\big\rangle, \big\langle X_{t_2}^\varrho, \psi_2\big\rangle\Big]\\ &\quad=\; 2\,i_{\mathrm{r}}\int_r^{t_1\wedge t_2} \mathrm{d}s\,\Big\langle \varrho_s\,,(S_{t_1-s}\psi_1)\,(S_{t_2-s}\psi_2)\,\Big\rangle \;<\; \infty.\end{aligned} \tag{32}$$

2.2 Catalyst free regions

Denote by $B_\delta(z)$ the open ball in R^d of radius δ centered at z. The starting point for our development is the following observation.

Proposition 5 (catalyst free regions close to time-space points) *Consider $d \geq 2$. Assume that $\mu \in \mathcal{M}_p$. For $t > 0$, denote by $\mathbf{Z}_t^\varrho$ the open set of all those $z \in \mathsf{R}^d$ such that there exists a $\delta = \delta(\varrho, t, z) \in (0, t)$ with*

$$\sup_{s \in [t-\delta, t+\delta]} \varrho_s\left(B_\delta(z)\right) = 0. \tag{33}$$

(a) (full measure) *Then*

$$\sup_{t>0} \ell\left(\mathsf{R}^d \setminus \mathbf{Z}_t^\varrho\right) = 0, \quad \mathbb{P}_\mu\text{–}a.s. \tag{34}$$

(b) (absence at a given point) *In particular, for fixed $t > 0$ and $z \in \mathsf{R}^d$, there is a $\delta = \delta(\varrho, t, z)$ such that (33) holds for $\mathbb{P}_\mu$–almost all ϱ.*

Remark 6 (polarity of points) Statement (b) has been known in the case of finite initial measures μ ([Dyn93, Theorem 11.2]). In potential-theoretical language, Dynkin shows that (in $d \geq 2$) a given time-space *point* (t, z) *is polar for the graph* of SBM ϱ. $\diamond$

Proof of Proposition 5 1° (*reduction*) By a well-known scaling property of SBM (see, e.g., [FK94, Lemma 4.5.1]), w.l.o.g. we may take $\gamma = \frac{1}{2}$ for the catalyst's branching rate. It suffices to show that

$$\mathbb{P}_\mu\left(\ell\left((\mathbf{Z}_t^\varrho)^c \cap B_1(x)\right) = 0, \quad 0 < t \leq T\right) = 1, \tag{35}$$

for all $x \in \mathsf{R}^d$ and $T > 0$. Without loss of generality we will show this only for $x = 0$. We may reformulate (35) as

$$\mathbb{P}_\mu\left(\ell\Big(|z| < 1 : \sup_{s \in [t-\delta, t+\delta]} \varrho_s\left(B_\delta(z)\right) > 0 \ \forall \delta \in (0, t)\Big) > 0, \ \ 0 < t \leq T\right) = 0.$$

We want to distinguish between the contributions from different initial regions.

2° (*decomposition*) For $n, N \geq 1$, write

$$A^{n,N} := \Big\{x \in \mathsf{R}^d : N(n-1) \leq |x| < Nn\Big\}, \tag{36}$$

and $\mu^{n,N}$ for the 'restriction' $\mathbf{1}_{A^{n,N}}\mu$ of μ to the 'ring' $A^{n,N}$. Then, for N fixed, ϱ can be represented as the sum of independent SBM $\varrho^{n,N}$, $n \geq 1$, on R^d, where $\varrho^{n,N}$ starts from the finite measure $\mu^{n,N}$.

3° (*negligible contribution from outside*) First we will show that for $T, \eta > 0$,

$$\sum_{n \geq 2} \mathbb{P}_{\mu^{n,N}}\left(\sup_{s \leq T} \varrho_s\left(B_\eta(0)\right) > 0\right) \xrightarrow[N \uparrow \infty]{} 0. \tag{37}$$

For this purpose, we may assume that $\eta > 2\sqrt{T}$. Applying [DIP89, Theorem 3.3 (a)] (with R there replaced by η), we find a constant $c = c(d,T)$ such that for $n \geq 2$ and $N > (\eta + 2)$,

$$\begin{aligned} &\mathsf{P}_{\mu^{n,N}}\Big(\sup_{s\leq T} \varrho_s\big(B_\eta(0)\big) > 0\Big) \\ &\quad\leq\ c \int \mu^{n,N}(\mathrm{d}x)\,\big(|x|-(\eta+1)\big)^{d-2} \exp\Big[-\frac{\big(|x|-(\eta+1)\big)^2}{2T}\Big]. \end{aligned} \tag{38}$$

Hence, for the sum in (37) we get the upper bound

$$c \int_{|x|\geq N} \mu(\mathrm{d}x)\,\big(|x|-(\eta+1)\big)^{d-2} \exp\Big[-\frac{\big(|x|-(\eta+1)\big)^2}{2T}\Big] \underset{N\uparrow\infty}{\longrightarrow} 0,$$

proving (37).

4° (*main term*) By the previous step, it suffices to show that for $N \geq 1$ fixed, statement (34) and **(b)** hold with P_μ replaced by $\mathsf{P}_{\mu^{1,N}}$ (finite initial measure). Since **(b)** is simpler (and proved, e.g., in [Dyn93, Theorem 11.2]), we only show (34).

For $t > 0$, introduce the *closed support* $\mathrm{S}_t = \mathrm{S}_t^\varrho$ of the measure ϱ_t. From Corollary 1.3 in [Per89] we know that with $\mathsf{P}_{\mu^{1,N}}$-probability one, S_t is a Lebesgue *zero set* for all $t > 0$. Recall also that the process $t \mapsto \mathrm{S}_t$, $t > 0$, is *càdlàg* (with respect to the Hausdorff metric on compact sets) with $\mathsf{P}_{\mu^{1,N}}$-probability one; see [Per90, Theorem 1.4]. Moreover, by the same theorem, with probability one, the left-hand limits S_{t-} satisfy

$$\mathrm{S}_{t-} \supseteq \mathrm{S}_t \quad\text{and}\quad \mathrm{S}_{t-}\setminus \mathrm{S}_t \ \text{ is } \textit{empty or a singleton}, \text{ for all } t > 0. \tag{39}$$

(In each singleton, an isolated subpopulation of X^ϱ becomes extinct.) We claim that

$$\mathbf{Z}_t^\varrho \ =\ \mathrm{S}_{t-}^{\mathrm{c}}\,, \qquad t > 0, \quad \mathsf{P}_{\mu^{1,N}}\text{-a.s.} \tag{40}$$

In fact, first let t be a continuity point of the closed support process S. If $z \in \mathrm{S}_t^{\mathrm{c}}$, then by continuity a time-space box around (t,z) exists with no catalytic mass in the sense of (33), whereas for $z \in \mathrm{S}_t$ it does not. On the other hand, if t is a discontinuity point, then $z \in \mathbf{Z}_t^\varrho$ if and only if $z \in \mathrm{S}_{t-}^{\mathrm{c}}$, since S_{t-} is closed (together with S_t).

To finish the proof, now it remains to note that each S_{t-} is a Lebesgue zero set, by (39) since S_t is. ■

Remark 7 (dimension one) Properties as in Proposition 5 are *not* valid in $d = 1$ since there ϱ has a jointly continuous density field on $(0,\infty) \times \mathsf{R}$ (see, e.g., [KS88]). ◇

Recall that $\mathbf{Z}^\varrho \subset (0,\infty) \times \mathsf{R}^d$ denotes the complement in $(0,\infty) \times \mathsf{R}^d$ of the closed support S^ϱ of the measure $\mathrm{d}s\, \varrho_s(\mathrm{d}x)$. The following corollary is immediate from Proposition 5.

Corollary 8 (thin time-space support) *Let $d \geq 2$. We can write $\mathbf{Z}^\varrho$ in terms of the $\mathbf{Z}_t^\varrho$ from Proposition 5:*

$$\mathbf{Z}^\varrho = \Big\{(t,z) : \ t > 0, \ z \in \mathbf{Z}_t^\varrho\Big\}. \tag{41}$$

Hence,

$$\ell^+ \times \ell \left(\left((0,\infty) \times \mathsf{R}^d\right) \setminus \mathbf{Z}^\varrho\right) = 0, \quad \mathsf{P}_\mu\text{–}a.s. \tag{42}$$

2.3 Asymptotic L^2–densities of the reactant

Recall the definition of the reference function ϕ_p from (7). We state the following trivial heat kernel estimates without proof.

Lemma 9 (estimates for the heat kernel) *For $d \geq 1$, let $C \subset (0,\infty) \times \mathsf{R}^d$ be compact, and let $k, n \geq 1$. Choose $\delta > 0$ such that*

$$C^\delta := \bigcup_{(t,z) \in C} [t-\delta, t+\delta] \times B_\delta(z) \subset (0,\infty) \times \mathsf{R}^d. \tag{43}$$

Then there are constants $c_i = c_i(d, C, n, \delta)$, $i = 1,2,3$, such that for $(t,z) \in C$ and $(s,x) \in \left((0,\infty) \times \mathsf{R}^d\right) \setminus C^\delta$ with $s \leq t$, the following three statements hold:

$$\left| \frac{\partial}{\partial r} \theta_z\, \mathrm{p}_r(x) \Big|_{r = n(t-s)} \right| \leq c_1\, \theta_z\, \mathrm{p}_{2n(t-s)}(x), \tag{44}$$

$$\left| \frac{\partial}{\partial z} \theta_z\, \mathrm{p}_{n(t-s)}(x) \right| \leq c_2\, \theta_z\, \mathrm{p}_{2n(t-s)}(x), \tag{45}$$

$$\theta_z\, \mathrm{p}_{n(t-s)}(x) \leq c_3\, \phi_p^k(x). \tag{46}$$

We will also need the following estimate that is due to Theorem 42 of [DF97].

Lemma 10 *Assume that $\mu \in \mathcal{M}_p$ is strongly diffusive. Then, for P_μ–almost all ϱ, for every $t > 0$,*

$$\sup_{x \in \mathsf{R}^d} \frac{1}{\phi_p(x)} \int_0^t \mathrm{d}s \int \varrho_s(\mathrm{d}y)\, \mathrm{p}_s(y-x)\, \phi_p^2(y) < \infty. \tag{47}$$

The following L^2–result is the key of our development.

Proposition 11 (asymptotic L^2–densities at points in $\mathbf{Z}^\varrho$) *Let $d = 2$ or 3. Take $r \geq 0$, $\mu, m \in \mathcal{M}_p$ and assume that μ is strongly diffusive. For P_μ–almost all ϱ the following hold.*

(a) (existence on $\mathbf{Z}^\varrho$) *For each $(t,z) \in \mathbf{Z}^\varrho$, $t > r$, there is an element $\xi_t^\varrho(z) \geq 0$ in the Lebesgue space $L^2 = L^2(P_{r,m}^\varrho)$ such that the L^2–convergence*

$$X_t^\varrho * \mathrm{p}_\varepsilon(z) \underset{\varepsilon \downarrow 0}{\longrightarrow} \xi_t^\varrho(z) \tag{48}$$

takes place.

(b) (locally uniform convergence) *This convergence is uniform if (t,z) runs in a compact set $C = C(\varrho) \subset \mathbf{Z}^\varrho \cap \big((r,\infty) \times \mathsf{R}^d\big)$.*

(c) (moments) *$\xi_t^\varrho(z)$ has expectation $m * \mathrm{p}_{t-r}(z)$, and the covariances are given by (11).*

(d) (existence at a given point) *In particular, for $t > r$ and $z \in \mathsf{R}^d$ fixed, $\xi_t^\varrho(z)$ exists with those properties, for P_μ–almost all ϱ.*

Proof Since ϱ_r is strongly diffusive P_μ–a.s. (see Lemma 14 in the appendix), we may use the Markov property to conclude that we can assume that $r = 0$ without loss of generality. By the covariance formula (32),

$$\begin{aligned} &\Big\| X_t^\varrho * \mathrm{p}_{\varepsilon_1}(z) - X_t^\varrho * \mathrm{p}_{\varepsilon_2}(z) \Big\|_2^2 \\ &= \Big[m * \mathrm{p}_{t+\varepsilon_1}(z) - m * \mathrm{p}_{t+\varepsilon_2}(z) \Big]^2 \\ &\quad + 2 \int_0^t \mathrm{d}s \Big\langle \varrho_s \,,\, (m * \mathrm{p}_s) \big[\theta_z \mathrm{p}_{\varepsilon_1+t-s} - \theta_z \mathrm{p}_{\varepsilon_2+t-s} \big]^2 \Big\rangle, \end{aligned} \tag{49}$$

$t \geq 0$, $z \in \mathsf{R}^d$, $\varepsilon_1, \varepsilon_2 > 0$.

Fix a compact set $C \subset \mathbf{Z}^\varrho$ and $\delta > 0$ such that $C^\delta \subset \mathbf{Z}^\varrho$ (recall notation (43)). Further let $\tau := \sup\{(t : (t,z) \in C\}$: Clearly, the first summand on the r.h.s. of (49) goes to 0 as $\varepsilon_1, \varepsilon_2 \to 0$, uniformly in $(t,z) \in C$.

We use the convention $\mathrm{p}_t := 0$ if $t < 0$. By (46) in Lemma 9, there exists a constant $c_3 < \infty$ such that

$$\begin{aligned} &\int_0^\infty \mathrm{d}s \Big\langle \varrho_s \,,\, (m * \mathrm{p}_s) \sup_{\substack{(t_1,z_1),(t_2,z_2) \in C \\ 0 < \varepsilon_1, \varepsilon_2 < \delta/2}} \big(\theta_{z_1} \mathrm{p}_{\varepsilon_1+t_1-s}\big)\big(\theta_{z_2} \mathrm{p}_{\varepsilon_2+t_2-s}\big) \Big\rangle \\ &\leq c_3^2 \int_0^{\tau+\delta} \mathrm{d}s \Big\langle \varrho_s \,,\, (m * \mathrm{p}_s)\, \phi_p^2 \Big\rangle. \end{aligned} \tag{50}$$

By Lemma 10 the latter quantity is bounded by $c_4 \langle m, \phi_p \rangle < \infty$. Note that for all $(s,x) \in (C^\delta)^{\mathrm{c}}$,

$$\sup_{(t,z) \in C} \Big| \theta_z \left(\mathrm{p}_{\varepsilon_1+t-s} - \mathrm{p}_{\varepsilon_2+t-s} \right)(x) \Big| \to 0 \quad \text{as} \quad \varepsilon_1, \varepsilon_2 \downarrow 0. \tag{51}$$

If we combine (49), (50), and (51), the dominated convergence theorem yields that $\left(X_t^{\varrho} * \mathrm{p}_\varepsilon(z)\right)_{\varepsilon>0}$ is Cauchy in $L^2(P_{0,m}^{\varrho})$ as $\varepsilon \downarrow 0$, uniformly in $(t,z) \in C$. Hence, the L^2-limit $\xi_t^{\varrho}(z)$, say, exists, and

$$\left\| \sup_{(t,z)\in C} X_t^{\varrho} * \mathrm{p}_\varepsilon(z) - \xi_t^{\varrho}(z) \right\|_2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \tag{52}$$

This proves **(a)** and **(b)**.

Since L^2–convergence implies L^1–convergence, $P_{0,m}^{\varrho}\xi_t^{\varrho}(z) = m * \mathrm{p}_t(z)$ follows from (28). But then also the covariance formula (11) can be derived from (31) and domination according to (50).

Statement **(d)** is immediate (recall Proposition 5 (b)). ■

3 Proof of the theorem

In this section we prove Theorem 1. First we use Proposition 11 to show (a),(c), and (d). Next we proceed similarly as in [Del96] to get the smoothness of the density field. Delmas uses a representation of his catalytic SBM in terms of excursions started from his (time-independent) catalytic set. Our catalyst is *not* time-independent. However, it is not crucial in Delmas' argument to start the excursions from the catalyst. Our idea is to use a Delmas type representation of X^{ϱ} on a local level with an occupation density measure Γ^{ϱ} concentrated on a nice set outside the catalyst.

For notational simplicity we assume $r = 0$. This is no loss of generality (see Lemma 14 in the appendix).

(a) *Absolute continuity* of X^{ϱ} on $\mathbf{Z}^{\varrho}$ is immediate by the uniform convergence statement in Proposition 11. (An alternative argument for this fact will be given in the proof of part (b).) Since $\xi_t^{\varrho}(z)$ has expectation $m * \mathrm{p}_t(z)$ on $\mathbf{Z}_t^{\varrho}$, and $\ell\left((\mathbf{Z}_t^{\varrho})^{\mathrm{c}}\right) = 0$, we get the absolute continuity of X_t^{ϱ} on the whole space R^d by an exhaustion argument.

(c) (*moments*) This is Proposition 11 (c).

(d) (*local* $L^2(P_{0,m}^{\varrho})$*–Lipschitz continuity on* $\mathbf{Z}^{\varrho}$) We may assume that the compact set $C \subset \mathbf{Z}^{\varrho}$ is a closed box. Let $\delta > 0$ such that $C^{\delta} \subset \mathbf{Z}^{\varrho}$ (recall (43)). Set $\tau := \sup\{t : (t,z) \in C\}$, and let $(t_1,z_1),(t_2,z_2) \in C$ with $t_1 \le t_2$. From the moment formulas (10) and (11) we get

$$\left\| \xi_{t_1}^{\varrho}(z_1) - \xi_{t_2}^{\varrho}(z_2) \right\|_2^2 \; = \; I_1 + I_2 + I_3, \tag{53}$$

where

$$
\begin{aligned}
I_1 &:= \Big[m*\mathrm{p}_{t_1}(z_1) - m*\mathrm{p}_{t_2}(z_2)\Big]^2 \\
I_2 &:= 2\int_0^{t_1} \mathrm{d}s \Big\langle \varrho_s\,,\, (m*\mathrm{p}_s)\left[\theta_{z_1}\mathrm{p}_{t_1-s} - \theta_{z_2}\mathrm{p}_{t_2-s}\right]^2 \Big\rangle \qquad (54)\\
I_3 &:= 2\int_{t_1}^{t_2} \mathrm{d}s \Big\langle \varrho_s\,,\, (m*\mathrm{p}_s)\left[\theta_{z_2}\mathrm{p}_{t_2-s}\right]^2 \Big\rangle.
\end{aligned}
$$

We use the bound of the partial derivatives of $\theta_z\,\mathrm{p}_{t-s}$ in Lemma 9 to derive the existence of a constant c^1 (depending only on C and ϱ) such that

$$
I_2 \le \Big||t_1-t_2| + |z_1-z_2|\Big|^2 c^1 \int_0^{\mathcal{T}} \mathrm{d}s \Big\langle \varrho_s\,,\, (m*\mathrm{p}_s)\Big[\sup_{(t,z)\in C} \theta_z \mathrm{p}^2_{2(t-s)}\Big]\Big\rangle.
$$

By (46) and Lemma 10, this inequality can be continued with constants c^2, c^3 :

$$
\begin{aligned}
&\le \big|(t_1,z_1)-(t_2\,,z_2)\big|^2 c^2 \int_0^{\mathcal{T}} \mathrm{d}s \Big\langle \varrho_s\,,\, (m*\mathrm{p}_s)\,\phi_p^2\Big\rangle \\
&\le c^3\, \big|(t_1,z_1)-(t_2\,,z_2)\big|^2.
\end{aligned}
$$

Analogously we get the existence of c^4 such that

$$
I_1 \le c^4\, \big|(t_1,z_1)-(t_2\,,z_2)\big|^2.
$$

The estimate for I_3 is similar. Note that $\mathrm{p}_r(x) \le c^5 r^2$, $r>0$, for $|x|>\delta$ and a constant c^5 depending only on δ. Hence (again by (46) and Lemma 10) there exist c^6 and c^7 such that

$$
\begin{aligned}
I_3 &\le c^5\, |t_1-t_2|^2 \int_0^{t_2} \mathrm{d}s \Big\langle \varrho_s\,,\, (m*\mathrm{p}_s)\,\theta_{z_2}\mathrm{p}_{t_2-s}\Big\rangle \\
&\le c^6\, |t_1-t_2|^2 \int_0^{\mathcal{T}} \mathrm{d}s \Big\langle \varrho_s\,,\, (m*\mathrm{p}_s)\,\phi_p^2\Big\rangle \\
&\le c^7\, |t_1-t_2|^2.
\end{aligned}
$$

(b) (*smooth density field*) For the final part of proof we have the following strategy. We fix a cylinder $\mathcal{Z}$ contained in $\mathbf{Z}^\varrho$ and use Proposition 11 to construct the occupation density measure Γ^ϱ of X^ϱ on the lateral area $\mathcal{A}$ of the cylinder. Next we use Delmas' representation of catalytic SBM in terms of Brownian excursions starting from $\mathcal{A}$ to derive the smoothness of ξ^ϱ in $\mathcal{Z}$.

Let ϱ be such that the assertions in Proposition 11 (a) – (c) and in Corollary 8 hold. Recall the characterization of $\mathbf{Z}^\varrho$ from Corollary 8, and that $\mathbf{Z}^\varrho$ is open in $(0,\infty)\times\mathsf{R}^d$ and satisfies $\ell^+ \times \ell\left((\mathbf{Z}^\varrho)^{\mathrm{c}}\right) = 0$. Write $\overline{\ell}$ for the $(d-1)$–dimensional Lebesgue measure on the boundary $\partial B_r(z)$ of the open ball $B_r(z)$ around z of radius r.

Fix $(t,z) \in \mathbf{Z}^{\varrho}$, and a $\delta > 0$ such that

$$[t-\delta, t+\delta] \times B_{2\delta}(z) \subset \mathbf{Z}^{\varrho}. \tag{55}$$

Define a measure Γ^{ϱ} on $\mathcal{A} := [t-\delta, t+\delta] \times \partial B_{\delta}(z)$ by

$$\Gamma^{\varrho}(\mathrm{d}u, \mathrm{d}y) := \mathrm{d}u\ \xi^{\varrho}_u(y)\,\overline{\ell}(\mathrm{d}y), \qquad (u,y) \in \mathcal{A}, \tag{56}$$

with ξ^{ϱ}_u from Proposition 11 (a).

First we show that Γ^{ϱ} is the *occupation density measure (super-local time)* of X^{ϱ} on $\mathcal{A}$. For this purpose, we define random measures $\Gamma^{\varrho}_{\varepsilon}$, $\varepsilon > 0$, on the cylinder lateral area $\mathcal{A}$, via their density functions

$$(u,y) \mapsto X^{\varrho}_u * \mathrm{p}_{\varepsilon}(y), \qquad (u,y) \in \mathcal{A},$$

with respect to the measure $\ell^{+} \times \overline{\ell}$. The formal meaning of the statement that Γ^{ϱ} is the occupation density measure is that

$$\Gamma^{\varrho}_{\varepsilon} \quad \text{converges weakly to} \quad \Gamma^{\varrho} \quad \text{as } \varepsilon \downarrow 0, \quad P^{\varrho}_{0,m}\text{–a.s.} \tag{57}$$

To prove (57) it suffices to show that

$$\langle \Gamma^{\varrho}_{\varepsilon}, f \rangle \underset{\varepsilon \downarrow 0}{\longrightarrow} \langle \Gamma^{\varrho}, f \rangle, \quad P^{\varrho}_{0,m}\text{–a.s.}, \tag{58}$$

if f is a continuous function on $\mathcal{A}$. From the uniform convergence in Proposition 11 (b) we know that in $L^2(P^{\varrho}_{0,m})$,

$$\begin{aligned}
&\left\| \langle \Gamma^{\varrho}_{\varepsilon}, f \rangle - \langle \Gamma^{\varrho}, f \rangle \right\|_2 \\
&\quad \le \int_{t-\delta}^{t+\delta} \mathrm{d}u \int_{\partial B_{\delta}(z)} \mathrm{d}y\ |f(u,y)|\ \left\| X^{\varrho}_u * \mathrm{p}_{\varepsilon}(y) - \xi^{\varrho}_u(y) \right\|_2 \underset{\varepsilon \downarrow 0}{\longrightarrow} 0,
\end{aligned}$$

Hence, for every sequence $\varepsilon_n \downarrow 0$ as $n \uparrow \infty$, there exists a subsequence $\varepsilon_{n(k)}$ such that

$$\left\langle \Gamma^{\varrho}_{\varepsilon_{n(k)}}, f \right\rangle \underset{k \uparrow \infty}{\longrightarrow} \langle \Gamma^{\varrho}, f \rangle, \quad P^{\varrho}_{0,m}\text{–a.s.}$$

Since the mapping $\varepsilon \mapsto \langle \Gamma^{\varrho}_{\varepsilon}, f \rangle$, $\varepsilon > 0$, is continuous, we have shown (58), and hence (57).

The aim is now to use Γ^{ϱ} to get a representation of ξ^{ϱ} as in Proposition 7.1 of [Del96] (see also Theorem 2 of [FL95]). This is Proposition 12 below. From this representation it is easily shown that ξ^{ϱ} is $\mathcal{C}^{\infty}$ and solves the heat equation (see Theorem 8.1 of [Del96]).

We start by introducing the ingredients of the representation formula. Recall that $(W, \Pi_{r,x})$ denotes the Brownian motion on R^d. Define the exit time

$$\tau^{B} := \inf\{ s > 0 :\ W_s \notin B \} \tag{59}$$

of the open ball $B = B_\delta(z)$, and the *exit density*

$$\mathrm{q}^B \;=\; \left\{ \mathrm{q}_t^B(x,y): \quad t>0,\ x\in B,\ y\in\partial B \right\} \tag{60}$$

by

$$\Pi_{0,x}\, f\left(\tau^B, W_{\tau^B}\right) \;=\; \int_0^\infty \mathrm{d}t \int_{\partial B} \overline{\ell}(\mathrm{d}y)\, \mathrm{q}_t^B(x,y)\, f(t,y), \tag{61}$$

$f \in \mathcal{C}_b\left((0,\infty)\times\partial B\right)$, (that is f bounded and continuous). Clearly, for $y \in \partial B$ fixed, $(t,x) \mapsto \mathrm{q}_t^B(x,y)$ is of class $\mathcal{C}^\infty$ and solves the heat equation.

Fix $T > 0$ and a compact set $D \subset B$. By a simple induction argument we derive from Lemma 9 that the partial derivatives of all orders are bounded, uniformly in $x \in D,\ y \in \partial B,\ t \in (0,T]$. Hence, for every finite measure ν on $(0,\infty)\times\partial B$, also the mixture

$$\nu * \mathrm{q}_t^B(x) \;:=\; \int_0^t \int_{\partial B} \nu(\mathrm{d}u,\mathrm{d}y)\, \mathrm{q}_{t-u}^B(x,y), \qquad t>0, \quad x\in B, \tag{62}$$

is of class $\mathcal{C}^\infty$ and solves the heat equation in $(0,\infty)\times B$.

Define the transition density $\mathrm{p}^B \;=\; \left\{\mathrm{p}_t^B(x,x'): \ t>0,\ \ x\in B,\ x'\in \mathsf{R}^d\right\}$ of *Brownian motion killed on* B^{c} :

$$\Pi_{0,x}\, \mathbf{1}_{\tau>t}\, f\left(W_t\right) \;=\; \int_B \mathrm{d}x'\ \mathrm{p}_t^B(x,x')\, f(x'), \qquad f\in\mathcal{C}_b\left(\mathsf{R}^d\right). \tag{63}$$

As above, $(t,x) \mapsto \mathrm{p}_t^B(x,x')$ is $\mathcal{C}^\infty$ and solves the heat equation. Further, for $n \in \mathcal{M}_p$, also the mixture

$$n * \mathrm{p}_t^B(x) \;:=\; \int n(\mathrm{d}x')\, \mathrm{p}_t^B(x,x'), \qquad t>0, \quad x\in B, \tag{64}$$

is $\mathcal{C}^\infty$ and solves the heat equation.

Since $\nu * \mathrm{q}^B$ and $n * \mathrm{p}^B$ are $\mathcal{C}^\infty$ and solve the heat equation, the same is true for ξ^ϱ by the following proposition. Hence the proof of the theorem is complete. ■

Let $r \geq 0$ and $(t,z) \in \mathbf{Z}^\varrho,\ t > r$. Choose $\delta \in (0, t-r)$ such that (55) holds. Define the cylinder $\mathcal{Z} := (t-\delta, t+\delta)\times B_\delta(z)$.

Proposition 12 (representation by excursion densities) *For* $\mathbb{P}_\mu$*-a.a.* ϱ, *with* $P_{r,m}^\varrho$*-probability one,*

$$\xi_s^\varrho(x) \;=\; X_{t-\delta}^\varrho * \mathrm{p}_{s-(t-\delta)}^B(x) \,+\, \Gamma^\varrho * \mathrm{q}_{s-(t-\delta)}^B(x), \qquad (s,x)\in\mathcal{Z}. \tag{65}$$

Proof As above we may assume $r = 0$. We want to show that the difference of both sides of (65) vanishes in $L^2(P^{\varrho}_{0,m})$. Clearly,

$$P^{\varrho}_{0,m}\xi^{\varrho}_s(x) \;=\; m * \mathrm{p}_t(x) \;=\; P^{\varrho}_{0,m}\left[X^{\varrho}_{t-\delta} * \mathrm{p}^B_{s-(t-\delta)}(x) \;+\; \Gamma^{\varrho} * \mathrm{q}^B_{s-(t-\delta)}(x)\right].$$

Hence, it suffices to prove that the variance of the difference equals 0. We use the covariance formulas (11) and (32) to deduce that

$$\begin{aligned} &\mathcal{V}\mathrm{ar}^{\varrho}_{0,m}\left[\xi^{\varrho}_s(x) \;-\; X^{\varrho}_{t-\delta} * \mathrm{p}^B_{s-(t-\delta)}(x) \;-\; \Gamma^{\varrho} * \mathrm{q}^B_{s-(t-\delta)}(x)\right] \\ &= \int_0^s \mathrm{d}s' \int \varrho_{s'}(\mathrm{d}x')\, m * \mathrm{p}(x') \Big[\theta_x \mathrm{p}_{s-s'} \;-\; \mathbf{1}_{s'<t-\delta}\,\left(S_{t-\delta-s'}\,\mathrm{p}^B_{s-(t-\delta)}\right)(x') \\ &\qquad\qquad - \int_{s'\vee(t-\delta)}^{s} \mathrm{d}u \int_{\partial B} \overline{\ell}(\mathrm{d}y)\; \mathrm{p}_{u-s'}(y-x')\; \mathrm{q}^B_{s-u}(x,y)\Big]^2. \end{aligned}$$

However, the integrand vanishes if $(s,x) \in \mathcal{Z}$ and $(s',x') \in \mathcal{Z}^{\mathrm{c}}$. In fact, we distinguish between the two cases whether the backward Brownian motion path leaves the cylinder $\mathcal{Z}$ at the base $\{t-\delta\} \times B_\delta(z)$, or at the lateral area $\mathcal{A}$. This shows (65), hence the proof is complete. ■

Appendix

As mentioned at the end of § 1.3, almost all samples of SBM ϱ at positive time are strongly diffusive. This follows from [DF97, Theorem 32] via an expectation calculation. But for methodological reasons we give an independent proof.

We begin with a lemma that shows that mass is locally not "too concentrated":

Lemma 13 *Let $d \geq 2$. Fix $\zeta, \tau > 0$ and $\mu \in \mathcal{M}_p$. For $\mathbb{P}_\mu$-almost all ϱ_τ,*

$$\varepsilon^{\zeta-2}\,\varrho_\tau\left(B_\varepsilon(x)\right) \to 0 \quad as \quad \varepsilon \downarrow 0, \tag{66}$$

uniformly for x in compact subsets of R^d.

Proof For μ finite, this is implied by Corollary 4.8 of [BEP91]. To carry this over to general $\mu \in \mathcal{M}_p$, we use a decomposition as in step 2° of the proof of Proposition 5. ■

Lemma 14 *Let $d \leq 3$, and fix $\mu \in \mathcal{M}_p$ and $\tau > 0$. For $\mathbb{P}_\mu$-almost all ϱ, the map*

$$(t,x) \mapsto g(t,x) \;:=\; \int_0^t \mathrm{d}s\; \varrho_\tau * \mathrm{p}_s(x), \qquad (t,x) \in [0,\infty) \times \mathsf{R}^d, \tag{67}$$

is locally Hölder continuous of order η, for all $\eta \in \left(0, \frac{1}{4}\right)$. In particular, $\mathbb{P}_\mu$-almost all ϱ_τ are strongly diffusive.

Proof If $d=1$, then $\varrho_\mathcal{T}$ has a locally bounded density, hence we are done.

Let now $d=2,3$. Fix $\eta\in\left(0,\frac{1}{4}\right)$, choose $\alpha\in\left(\eta+\frac{1}{4},\frac{1}{2}\right)$ and ζ such that $0<\zeta<2\left[1-\alpha^{-1}(\eta+\frac{1}{4})\right]$, and fix $\varrho_\mathcal{T}$ such that the assertion in Lemma 13 holds. Take $K>0$ and $T>0$. We consider $g(t,x)$ for $(t,x)\in[0,T]\times B_K(0)$. By c we denote a constant that changes from place to place but depends only on $\varrho_\mathcal{T}$, K, and T.

First we show that

$$\left|g(t,x)-g(s,x)\right| \;\le\; c\,(t-s)^{2\eta}, \qquad 0\le s\le t\le T, \quad x\in B_K(0). \tag{68}$$

Note that for $y\in\mathsf{R}^d$ and $u\in[0,T]$,

$$\mathrm{p}_u(y)\,\mathbf{1}\left\{|y|\ge u^\alpha\right\} \;\le\; c\,\phi_p(y) \tag{69}$$

and

$$\mathrm{p}_u(y)\,\mathbf{1}\left\{|y|\le u^\alpha\right\} \;\le\; u^{-d/2}. \tag{70}$$

Hence,

$$\int_s^t \mathrm{d}u\;\varrho_\mathcal{T}*\mathrm{p}_u(x) \;\le\; c\,(t-s)\,\langle\varrho_\mathcal{T},\phi_p\rangle+\int_s^t\mathrm{d}u\;u^{-d/2}\,\varrho_\mathcal{T}\left(B_{u^\alpha}(x)\right). \tag{71}$$

By Lemma 13, the integral on the r.h.s. of (71) is dominated by

$$c\int_s^t\mathrm{d}u\;u^{-d/2}\,u^{\alpha\,(2-\zeta)} \;\le\; c\,(t-s)^{2\eta}. \tag{72}$$

Hence, we have shown (68).

Next we prove

$$\left|g(t,x)-g(t,y)\right| \;\le\; c\,|x-y|^\eta, \qquad x,y\in B_K(0), \quad t\in[0,T]. \tag{73}$$

Let $t' := t\wedge|x-y|^{1/2}$. Then by the triangular inequality,

$$\left|g(t,x)-g(t,y)\right| \;\le\; g(t',x)+g(t',y)+\int_{t'}^t\mathrm{d}s\int\mu(\mathrm{d}z)\,\left|\theta_z\mathrm{p}_s(x)-\theta_z\mathrm{p}_s(y)\right|.$$

Using (68) and the simple heat kernel estimate

$$\left|\theta_z\mathrm{p}_s(x)-\theta_z\mathrm{p}_s(y)\right| \le c\,\frac{|x-y|}{s}\,\mathrm{p}_{16s}(z+x), \qquad t'<s\le T,$$

we get

$$\begin{aligned}\left|g(t,x)-g(t,y)\right| \;&\le\; c\,|x-y|^\eta+c\,|x-y|^{1/2}\int_0^t\mathrm{d}s\int\mu(\mathrm{d}z)\,\mathrm{p}_{16s}(x+z)\\ &\le\; c\,|x-y|^\eta+c\,|x-y|^{1/2}\,g(16t,x).\end{aligned}$$

Since $g(16t,x)$ is bounded, this implies (73), and the proof is complete. ∎

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