Stability Analysis of Quadrature Methods for Two-Dimensional Singular Integral Equations

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Abstract

In this paper we apply a quadrature method based on the tensor product trapezoidal rule to the solution of a singular integral equation over the two-dimensional torus. We prove that this method is stable if and only if a certain numerical symbol does not vanish. For a special kernel function, we present a plot of numerically computed symbol values and, for symmetric kernels (Mikhlin-Giraud kernels), we show that the symbol is different from zero if the singular integral operator is invertible. Finally, we prove the convergence of our method and present numerical tests.

1 Introduction

In the last two decades a lot of problems in elasticity, fluid mechanics, acoustics, optics, electrostatics, and other fields of engineering have been tackled by boundary element methods (cf. e.g. the overview articles by Mazya [15] and Wendland [36]). These methods include the analysis of strongly singular boundary integral equations

$$\tilde{A}\tilde{u} \;=\; (\tilde{a}I + \tilde{K})\tilde{u} \;=\; \tilde{f}, \tag{1.1}$$

where $\tilde{a}I$ stands for the multiplication operator

$$(\tilde{a}I)\tilde{u}(s) :=\; \tilde{a}(s)\tilde{u}(s) \tag{1.2}$$

multiplying by a real valued function $\tilde{a}$ and $\tilde{K}$ for the integral operator

$$(\tilde{K}\tilde{u})(s) :=\; \int_S \tilde{k}(s,t)\tilde{u}(t)d_tS \tag{1.3}$$

over the boundary manifold S. We suppose that the kernel $k(s,t)$ is strongly singular (cf. Section 2). This means that the integral in (1.3) is to be understood in the sense of a Cauchy principal value. In order to get the unknown function $\tilde{u}$ we solve (1.1) numerically. Originally, in the boundary element method this was done by a finite element discretization of (1.1). However, nowadays p- and h-p-methods, collocation, and quadrature schemes are popular as well. Several monographs are devoted to the study of Equation (1.1) and its numerical solution. Let us mention here e.g. the books written by Mikhlin, Prößdorf [16], Mikhlin, Morozov, Paukshto [17], Muskhelishvili [19], Prößdorf, Silbermann [28], and Parton, Perlin [20].

The main objective of this paper is to analyze quadrature methods for the numerical solution of singular integral equations over two-dimensional boundary manifolds and to prove convergence results similar to those known for collocation. Note that using the concept of strong ellipticity (cf. Stephan and Wendland [35]), the analysis of Galerkin methods for strongly elliptic singular integral equations is easy. The realization of these Galerkin schemes, however requires the computation of two-fold integrals over the boundary and, thus, is very time consuming. To reduce these efforts, collocation methods are applied. In contrast to their successful implementation, the convergence analysis is done for very special situations, only (cf. Prößdorf and Schneider [26]). Moreover, the collocation still requires the computation of singular integrals, which is accomplished by using quadratures. The advantage of quadrature schemes in comparison to Galerkin and collocation methods is that all the integrals are discretized within one discretization step, i.e., quadrature methods are so-called fully discrete schemes. The corresponding number of quadrature knots and therewith, the computation time is much less than for other discretization schemes. The draw back of the quadrature methods is the larger discretization error. Hence a quadrature method could be a good choice if the convergence of Galerkin and collocation schemes is slow due to the lack of smoothness of the right-hand side and the underlying manifold. Moreover, low order quadrature methods can be considered as a starting point for an analysis of higher order fully discrete methods with minimal numbers of quadrature points. We expect that the optimal methods are slight modifications of our quadrature methods.

The theory of one-dimensional spline collocation has been established by Prößdorf and Schmidt [24, 25], Arnold and Wendland [1, 2], Saranen and Wendland [31], and Schmidt [32, 33]. In the end of the 80-ies Hsiao, Prößdorf, and Schneider started to generalize these results to the case of multi-dimensional pseudo-differential equations. Unfortunately, the technique of Arnold and Wendland [1] could be generalized only by a difficult technical modification (cf. Hsiao and Prößdorf [13]). The techniques of Fourier analysis (or circulant techniques) take over to the multi-dimensional case if the underlying manifold is a torus or an open subset of the plane (cf. Prößdorf and Schneider [26, 27]). Note that the restriction to the artificial torus manifold means the following: The stability of collocation is a local property. Collocation is stable if and only if it is locally stable in the neighborhood of any point of the underlying manifold. The problem of local stability, however, is solved only for points where the mesh is regular, i.e., close to a rectangular mesh over a torus. E.g., if we consider a sphere and take a partition along the lines of constant longitude and latitude, then the resulting grid is regular at any point except the two poles. In other words, the local stability problem is solved at any point of the sphere but the poles. The local stability near the poles is not solved yet. Further investigations for collocation methods are due to Costabel and McLean [8], Dahmen, Prößdorf, and Schneider [9], and Hagen, Roch, and Silbermann [12]. Note that the authors of [9, 12] have even dealt with wavelet collocation methods.

Similar to the analysis of collocation, we have to restrict our consideration for quadrature methods to the special case that the underlying manifold is diffeomorphic to the torus. Suppose

$$\phi \;:\; [0,1]\times[0,1] \;\to\; S$$

is a parametrization of S which is 1-periodic in each argument. Then Equation (1.1) takes the form

$$Au(s) \;:=\; a(s)u(s) \;+\; \int_{[0,1]\times[0,1]} k(s,t)u(t)dt \;=\; g(s), \quad s\in[0,1]\times[0,1], \tag{1.4}$$

where

$$\begin{aligned} a(s) &= \tilde{a}\left(\phi(s)\right), \quad g(s) = \tilde{g}\left(\phi(s)\right), \quad u(s) = \tilde{u}\left(\phi(s)\right), \\ k(s,t) &= \tilde{k}\left(\phi(s),\phi(t)\right)|\phi^{'}(t)|. \end{aligned} \tag{1.5}$$

Discretizing (1.4) with the help of the trapezoidal rule

$$\int_{[0,1]\times[0,1]} \varphi(t)dt \;\sim \frac{1}{n^2}\sum_{i,j=0}^{n-1}\varphi(\frac{i}{n},\frac{j}{n}), \tag{1.6}$$

we arrive at the quadrature method

$$a\left((\frac{i}{n},\frac{j}{n})\right)u_n\left((\frac{i}{n},\frac{j}{n})\right)+\frac{1}{n^2}\sum_{\substack{k,l=0\\(k,l)\neq(i,j)}}^{n-1}k\left((\frac{i}{n},\frac{j}{n}),(\frac{k}{n},\frac{l}{n})\right)u_n(\frac{k}{n},\frac{l}{n})$$
$$=g\left((\frac{i}{n},\frac{j}{n})\right),\quad i,j=0,\dots,n-1. \tag{1.7}$$

Note that this method can be derived using the so-called singularity subtraction technique if the kernel function satisfies certain symmetry conditions (cf. Section 2). Only if this symmetry assumption is fulfilled, then (1.7) can be convergent. If the assumption is violated, then (1.7) needs to be modified.

For the quadrature method (1.7), we first investigate the stability, i.e., we check if the discretized integral operator is invertible and if the inverse discretized operator is uniformly bounded for sufficiently small mesh size $1/n$. Note that stability of the quadrature method implies that the linear systems arising after discretization are well conditioned and that the convergence order of the approximate solution is the same as that of the quadrature rules. In analogy to the collocation, it turns out that stability is a local property. This means, the quadrature method (1.7) for A in (1.4) is stable if and only if it is locally stable (cf. Sections 3 and 4) at any point of the boundary S. The local stability at a given point t on S, however, is the same as the stability of the quadrature method to a convolution operator defined over the tangent plane, if this convolution operator coincides with A in the neighborhood of t. In other words, it is sufficient to consider the stability of quadrature methods applied to singular convolution equations over the plane. The discretized convolution operator turns out to be a discrete convolution matrix. Its stability is determined by the generating symbol which is called numerical symbol of the quadrature method. As the first main result of this paper we prove that the numerical symbol is bounded (cf. Proposition 3.2). It is invertible (cf. Theorem 3.1) if a simple symmetry assumption for the kernel $k(s,t)$ is fulfilled. Thus we derive a sufficient condition for the local stability. As the second main result we show that quadrature method (1.7) is stable if and only if it is locally stable (cf. Theorem 4.1), i.e., if and only if the numerical symbol does not vanish over S. Unfortunately, the values of the numerical symbol are given in form of an infinite sum and can be computed by numerical methods, only. We give one example for such a numerical computation (cf. Subsection 3.2). However, for the special case of the integral operator corresponding to the oblique derivative boundary value problem (cf. Section 6 and [18]), the sufficient condition for the local stability is fulfilled and global stability can be proved. The third main result (cf. Theorem 5.1) concerns the convergence of the quadrature method (1.7). Using the just established stability, we prove that, for any Lipschitz continuous right-hand side f, the solution u_n of (1.7) tends to the exact solution u of (1.4) in L_2, i.e.,

$$\frac{1}{n}\sqrt{\sum_{i,j=0}^{n-1}\left|u_n\left((\frac{i}{n},\frac{j}{n})\right)-u\left((\frac{i}{n},\frac{j}{n})\right)\right|^2}\to 0.$$

To confirm the theoretically obtained results, we present some numerical tests in Section 6. We consider the singular integral equation corresponding to the oblique derivative boundary value problem for Laplace's equation on an unbounded domain with a boundary manifold diffeomorphic to the torus. For this equation, we present the approximation errors of the quadrature method (1.7).

2 The Quadrature Method over the Torus

Collocation methods and Galerkin methods are so-called semi-discrete schemes. In fact, to compute the integrals contained in the definition of the matrix entries one has to apply analytic formulas or quadrature schemes. We now like to give the quadratures in an optimal way (minimal number of quadrature knots) and to perform the stability analysis for the quadrature algorithm simultaneously. This can be done by considering quadrature discretization schemes right from the start. In the case of one-dimensional singular integral equations this is done by Belotserkovski, Lifanov, Prößdorf, Rathsfeld, Sloan, Silbermann [3, 23, 29, 28]. We shall try to generalize these results to two dimensions.

Let us consider the singular integral equation (with a classical pseudo-differential operator of order zero corresponding to the symbol function $\sigma_A(x,\xi) \in S^0$, cf. e.g. [7])

$$Au(x) = a(x)u(x) + \int_{\mathbb{T}^2} k(x,y)u(y)d_y\mathbb{T}^2 = g(x), \quad x \in \mathbb{T}^2 \tag{2.1}$$

over the torus $\mathbb{T}^2 := \mathbf{R}^2/\mathbb{Z}^2$, where

$$k(x,y) = k_S(x, x-y) + k_R(x,y). \tag{2.2}$$

Here $k_S(x, x-y)$ is defined by

$$k_S(x,z) = \int_{\mathbf{R}^2} \sigma_A(x,\xi)e^{iz\cdot\xi}d\xi. \tag{2.3}$$

We may suppose that σ_A is a positive homogeneous function in ξ of degree zero with $\sigma_A \in C^\infty(\mathbb{T}^2 \times \mathbf{R}^2\backslash\{0\})$ and that the kernel k_S satisfies the following conditions :

a) $k_S(x,z) \in C^\infty(\mathbb{T}^2 \times \mathbf{R}^2\backslash\{0\})$.

b) $k_S(x,tz) = t^{-2}k(x,z)$, $t > 0$, $x \in \mathbb{T}^2$, $z \in \mathbf{R}^2\backslash\{0\}$.

c) $\int_{S^1} k_S(x,z)d\sigma(z) = \int_0^{2\pi} k(x, e^{i\eta})d\eta = 0$, $x \in \mathbb{T}^2$.

The additional kernel $k_R(x,y)$ is supposed to be continuous and to generate a compact operator. (Note that, for a general classical pseudo-differential operator of order zero, the kernel k_R is weakly singular only. The corresponding operators and discrete operators should be treated in a similar manner as the singular operators. For the sake of simplicity, however, we suppose k_R to be continuous). The integral in (2.1) is to be understood as

$$\begin{aligned}\int_{\mathbb{T}^2} k(x,y)u(y)d_y\mathbb{T}^2 &= \int_{x_1-\frac{1}{2}}^{x_1+\frac{1}{2}} \int_{x_2-\frac{1}{2}}^{x_2+\frac{1}{2}} k(x,y)u(y)dy_1dy_2 \\ &= \lim_{\varepsilon\longrightarrow 0} \int_{y:\,\varepsilon\le|x-y|} k(x,y)u(y)dy.\end{aligned}$$

For the computation of an integral over the square, we choose the tensor product trapezoidal rule. Setting $N = n^2$ with n even, $m = (m_1, m_2)$, and $t^N_{m_1,m_2} = (\frac{m_1}{n}, \frac{m_2}{n})$ and assuming $x = (x_1, x_2) = t^N_k = t^N_{k_1,k_2}$, we write

$$\int_{x_1-\frac{1}{2}}^{x_1+\frac{1}{2}} \int_{x_2-\frac{1}{2}}^{x_2+\frac{1}{2}} h(y) dy_1 dy_2 \sim \frac{1}{n^2} {\sum_l}' h(t^N_l) := \frac{1}{n^2} \left\{ \sum_{l_1=k_1-\frac{n}{2}}^{k_1+\frac{n}{2}} \sum_{l_2=k_2-\frac{n}{2}}^{k_2+\frac{n}{2}} h(t^N_{l_1,l_2}) \omega_{l_1,l_2} \right\},$$

$$\omega_{l_1,l_2} := \begin{cases} 1 & \text{if } |l_1-k_1| < n/2,\ |l_2-k_2| < n/2 \\ 1/4 & \text{if } |l_1-k_1| = |l_2-k_2| = n/2 \\ 1/2 & \text{else.} \end{cases} \tag{2.4}$$

Note that $\mathbb{T}^2$ is the tensor product of the periodic interval [0,1] by itself. In this sense we get $t^N_{l_1\pm n, l_2} = t^N_{l_1,l_2} = t^N_{l_1,l_2\pm n}$. To set up a quadrature method for solving (2.1) numerically, we consider (2.1) at x from the set of collocation points $\{t^N_{k_1,k_2}\}$ and replace the integration by the corresponding quadrature rule (2.4). Since the value $k(x,x)$ is infinite, we have to modify the quadrature. We do this by dropping the term in the quadrature sum containing $k(x,x)$. This way we arrive at the quadrature method

$$a(t^N_k)u_N(t^N_k) + \frac{1}{n^2} {\sum_{l:\, l\neq k}}' k(t^N_k, t^N_l) u_N(t^N_l) = g(t^N_k), \quad k_1, k_2 = 0, \ldots, n-1. \tag{2.5}$$

Unfortunately, the method (2.5) is not convergent in the general case. Namely, if usual quadrature rules are applied to a singular integral, convergence cannot be expected. The remedy for this is the so-called singularity subtraction technique. Suppose we can compute (cf. (2.2))

$$b(t^N_k) = \int_{\mathbb{T}^2} k_S(t^N_k, t^N_k - y) d_y \mathbb{T}^2 = \int_{\frac{k_1}{n}-\frac{1}{2}}^{\frac{k_1}{n}+\frac{1}{2}} \int_{\frac{k_2}{n}-\frac{1}{2}}^{\frac{k_2}{n}+\frac{1}{2}} k_S(t^N_k, t^N_k - y) dy_1 dy_2$$

(analytically or numerically with finer quadrature procedures). Then, we write

$$\int_{\mathbb{T}^2} k_S(t^N_k, t^N_k - y) u(y) d_y \mathbb{T}^2 = b(t^N_k) u(t^N_k) + \int_{\mathbb{T}^2} k_S(t^N_k, t^N_k - y)[u(y) - u(t^N_k)] d_y \mathbb{T}^2.$$

The last integral is weakly singular only and the usual quadratures converge for this weakly singular integral. Applying this step to (2.1), we arrive at the quadrature method

$$[a(t^N_k) + b(t^N_k)] u_N(t^N_k) + \frac{1}{n^2} {\sum_{l:\, l\neq k}}' k_S(t^N_k, t^N_k - t^N_l)[u_N(t^N_l) - u_N(t^N_k)]$$
$$+ \frac{1}{n^2} {\sum_{l:\, l\neq k}}' k_R(t^N_k, t^N_l) u_N(t^N_l) = g(t^N_k), \quad k_1, k_2 = 0, \ldots, n-1, \tag{2.6}$$

which is equivalent to

$$\left[a(t^N_k) + b(t^N_k) - \frac{1}{n^2} {\sum_{l:\, l\neq k}}' k_S(t^N_k, t^N_k - t^N_l) \right] u_N(t^N_k)$$
$$+ \frac{1}{n^2} {\sum_{l:\, l\neq k}}' k(t^N_k, t^N_l) u_N(t^N_l) = g(t^N_k), \quad k_1, k_2 = 0, \ldots, n-1. \tag{2.7}$$

E.g., if the kernel $k_S(x, x-y)$ is odd with respect to the second variable $z = x - y$ (i.e. if it is a Mikhlin-Giraud kernel), then we get $b(t_k^N) = 0$ and

$$\frac{1}{n^2} \sum_{l:\, l \neq k}{}' k_S(t_k^N, t_k^N - t_l^N) = 0. \tag{2.8}$$

Note that (2.8) is true also if instead of

$$k_S(x, (z_1, z_2)) \;=\; -k_S(x, (-z_1, -z_2)) \tag{2.9}$$

one of the following symmetry properties is satisfied for the kernel:

$$\begin{aligned} k_S(x, (z_1, z_2)) &= -k_S(x, (-z_1, z_2)), && (2.10)\\ k_S(x, (z_1, z_2)) &= -k_S(x, (z_2, z_1)). && (2.11) \end{aligned}$$

Consequently,

$$\left[b(t_k^N) - \frac{1}{n^2} \sum_{l:\, l \neq k}{}' k_S(t_k^N, t_k^N - t_l^N) \right] = 0 \tag{2.12}$$

and the method (2.7) is equivalent to (2.5). Hence, the quadrature method (2.5) is useful if $k(x, x-y)$ is odd with respect to the second variable (cf. (2.9)) or if (2.10) or (2.11) is satisfied.

In the quadrature methods (2.5) and (2.7), the unknown solution u_N is a sequence of point values $\{u_N(t_{k_1,k_2}^N), k_1, k_2 = 0, \ldots, n-1\}$. We denote the matrix in the linear system (2.5) and (2.7) by A_N. However, we shall identify u_N with a piecewise constant function and A_N with an operator acting in the space of piecewise constant functions. To this reason, we introduce the characteristic function

$$\chi_{l_1,l_2}^N(x) = \begin{cases} 1 & \text{if} \quad l_j/n \leq x_j < (l_j+1)/n, \quad j = 1, 2 \\ 0 & \text{else} \end{cases}$$

and denote the space of piecewise constant functions by $\mathcal{S}^N$, i.e.,

$$\mathcal{S}^N = span\{\chi_{l_1,l_2}^N : l_1, l_2 = 0, \ldots, n-1\}.$$

Then we identify $\{u_N(t_{l_1,l_2}^N) : l_1, l_2 = 0, \ldots, n-1\}$ with the piecewise constant interpolation

$$u_N = \sum_{l_1,l_2=0}^{n-1} u_N(t_{l_1,l_2}^N) \chi_{l_1,l_2}^N$$

and the matrix A_N with the operator in $\mathcal{L}(\mathcal{S}^N)$ whose matrix with respect to the basis $\{\chi_{l_1,l_2}^N : l_1, l_2 = 0, \ldots, n-1\}$ is just A_N.

We call the quadrature method stable if the operators A_N are invertible for sufficiently large N and if the inverse operators $A_N^{-1} \in \mathcal{L}(\mathcal{S}^N)$ are uniformly bounded with respect to N (i.e. the norms of $A_N^{-1} \in \mathcal{L}(\mathcal{S}^N)$ induced by the L_2 norm are uniformly bounded). The quadrature method is called convergent if, for any right-hand side g such that

$$\left\| \sum_{l_1,l_2=0}^{n-1} g(t^N_{l_1,l_2})\chi^N_{l_1,l_2} - g \right\|_{L_2(\mathbb{T}^2)} \to 0,$$

there exist unique solutions $\{u_N(t^N_{l_1,l_2})\}$ of the quadrature equations (2.5) or (2.7) with

$$u_N = \sum_{l_1,l_2=0}^{n-1} u_N(t^N_{l_1,l_2})\chi^N_{l_1,l_2}$$

tending in the L_2-norm to the exact solution u.

The Sections 3, 4, and 5 are devoted to the stability and convergence analysis of method (2.5). Method (2.7) can be treated with slight modifications.

3 Localized Operators and Localized Quadrature Method on the Plane

Stability is a local property. Therefore it is necessary to introduce the quadrature scheme for the localized singular integral operator over the plane and to investigate the stability by analyzing the corresponding numerical symbol of the method. For singular kernels with a natural symmetry property, we shall prove the local stability.

3.1 The Operators and the Numerical Scheme over the Plane

In this subsection we introduce simple local problems over the plane which later will turn out to be the quadrature methods applied to the singular integral operators with frozen symbols. We consider the singular integral operator

$$\begin{aligned} Au(x) &= au(x) + (Ku)(x), \quad x \in \mathbf{R}^2, & (3.1)\\ (Ku)(x) &= \int_{\mathbf{R}^2} k(x-y)u(y)dy. & (3.2)\end{aligned}$$

with a real constant $a > 0$ and the convolution kernel

$$k(x-y) = \frac{f(\theta)}{r^2} \ , \quad r = |x-y|, \quad \theta = \frac{x-y}{|x-y|}.$$

Moreover, we suppose f to be a Lipschitz function and

$$\int_{S^1} f(z)d\sigma(z) = 0. \tag{3.3}$$

To define the quadrature method for singular integral equation (3.1), we rewrite (3.1) in the form

$$Au(x) \;=\; au(x) + \int_{\mathbf{R}^2} k(x-y)\Big[u(y) - u(x)\Big]dy + \int_{\mathbf{R}^2} k(x-y)dy \; u(x). \tag{3.4}$$

Since $\int_{S^1} f(\theta)d\theta = 0$ (cf. (3.3)), we have $\int k(x-y)dy = 0$ and get

$$Au(x) \quad = \quad au(x) + \int_{\mathbf{R}^2} \frac{f(\theta)}{|x-y|^2}\Big[u(y) - u(x)\Big]dy \ . \tag{3.5}$$

In order to evaluate the integral in Equation (3.5), we use the quadrature rule

$$\int_{\mathbf{R}^2} h(t)dt \sim \sum_{j\in\mathbb{Z}^2} h(t_j)\frac{1}{n^2}\ , \quad t_j = (\frac{j_1}{n}, \frac{j_2}{n}). \tag{3.6}$$

Applying this to (3.5) and neglecting the term corresponding to $j = k$, we obtain

$$\begin{aligned} Au(t_k) \quad &\sim \quad au(t_k) + \sum_{\substack{j\in\mathbb{Z}^2\\ j\neq k}} \frac{f\Big(\theta(t_k,t_j)\Big)}{|t_j-t_k|^2}\Big[u(t_j) - u(t_k)\Big]\frac{1}{n^2} \\ &\sim \quad au(t_k) + \sum_{\substack{j\in\mathbb{Z}^2\\ j\neq k}} \frac{f\Big(\theta(t_k,t_j)\Big)}{|t_j-t_k|^2}u(t_j)\frac{1}{n^2} - \Big[\sum_{\substack{j\in\mathbb{Z}^2\\ j\neq k}} \frac{f\Big(\theta(t_k,t_j)\Big)}{|t_j-t_k|^2}\frac{1}{n^2}\Big]u(t_k), \\ &\qquad\qquad \theta(t_k,t_j) = \frac{t_k-t_j}{|t_k-t_j|}. \end{aligned} \tag{3.7}$$

Now we shall show that the last sum vanishes under an additional assumption. To this end, we suppose that f can be split into $f(\theta) = f_1(\theta) + f_2(\theta) + f_3(\theta)$, where

$$f_1\Big((\cos\varphi, \sin\varphi)\Big) \quad = \quad -f_1\Big((-\cos\varphi, -\sin\varphi)\Big), \tag{3.8}$$
$$f_2\Big((\cos\varphi, \sin\varphi)\Big) \quad = \quad -f_2\Big((-\cos\varphi, \sin\varphi)\Big), \tag{3.9}$$
$$f_3\Big((\cos\varphi, \sin\varphi)\Big) \quad = \quad -f_3\Big((\sin\varphi, \cos\varphi)\Big). \tag{3.10}$$

Similarly to (2.12), we obtain

$$\sum_{\substack{j\in\mathbb{Z}^2\\ j\neq k}} \frac{f\Big(\theta(t_k,t_j)\Big)}{|t_j-t_k|^2}\frac{1}{n^2} = 0. \tag{3.11}$$

Equation (3.7) takes the form

$$Au(t_k) \sim au(t_k) + \sum_{\substack{j\in\mathbb{Z}^2\\ j\neq k}} \frac{f\Big(\theta(t_k,t_j)\Big)}{|t_j-t_k|^2}u(t_j)\frac{1}{n^2}.$$

Hence, the quadrature method over the plane is defined by

$$au_N(t_k) + \sum_{\substack{j\in\mathbb{Z}^2\\ j\neq k}} \frac{f\Big(\theta(t_k,t_j)\Big)}{|t_j-t_k|^2}u_N(t_j)\frac{1}{n^2} = g(t_k), \quad k\in\mathbb{Z}^2. \tag{3.12}$$

Though this method (3.12) could be used as a numerical scheme for the plane equation, the application of (3.12) would require a further step of reduction to a finite linear system of equations. However, we are not interested in solving the plane equation. The method (3.12) serves us only as a tool in the stability analysis of the corresponding method over the torus.

3.2 Stability of the Quadrature Method over the Plane

The matrix of the system (3.12) is

$$A_N = (a_{k,j})_{k,j\in \mathbb{Z}^2}, \quad a_{k,j} = \begin{cases} \dfrac{f(\theta(t_k,t_j))}{|t_k-t_j|^2}\dfrac{1}{n^2} = \dfrac{f\Big(\dfrac{k-j}{|k-j|}\Big)}{|j-k|^2} & \text{if } j\neq k \\ a & \text{if } j=k\,. \end{cases} \tag{3.13}$$

Thus the entries of A_N are independent of $N=n^2$ and we get $A_N = A_1$. Moreover, the entries of A_1 depend only on the difference $k-j$.

$$a_{k,j} = a_{k-j}, \quad a_m = \begin{cases} f\Big(\frac{m}{|m|}\Big)|m|^{-2} & \text{if } m\neq 0 \\ a & \text{if } m=0\,. \end{cases} \tag{3.14}$$

We identify A_N with the operator acting in the space of piecewise constant functions

$$\mathcal{S}^N(\mathbf{R}^2) = span\{\chi^N_{k_1,k_2} : k_1,k_2\in \mathbb{Z}\}$$

whose matrix with respect to the basis $\{\chi^N_k, k\in\mathbb{Z}^2\}$ is A_N. Since

$$\Big\|\sum_{k\in\mathbb{Z}^2}\xi_k\chi^N_k\Big\|_{L_2(\mathbf{R}^2)} = \frac{1}{n}\sqrt{\sum_{k\in\mathbb{Z}^2}|\xi_k|^2}, \tag{3.15}$$

the operator norm of A_N induced by the L_2 space is equivalent to the matrix norm of the space

$$l_2(\mathbb{Z}^2) := \Big\{\xi = (\xi_j)_{j\in\mathbb{Z}^2} : \sqrt{\sum_{k\in\mathbb{Z}^2}|\xi_k|^2} < \infty\Big\}.$$

It is a well-known fact that each discrete convolution operator can be represented as (cf. e.g. [4]) $A_N = F^{-1}MF$, where the unitary operators $F:\ l_2(\mathbb{Z}^2)\to L_2(\mathbb{T}^2)$ and $F^{-1}:\ L_2(\mathbb{T}^2)\to l_2(\mathbb{Z}^2)$ are defined by

$$F:\{\xi_j\}_{j\in\mathbb{Z}^2}\mapsto \sum_{j\in\mathbb{Z}^2}\xi_j\, e^{i2\pi j\cdot t}, \quad F^{-1}: f(t)\mapsto\{\xi_j\}_{j\in\mathbb{Z}^2},$$

$$\xi_j := \int_0^1\int_0^1 f(e^{i2\pi s_1}, e^{i2\pi s_2})e^{-i2\pi s_1 j_1}\, e^{-i2\pi s_2 j_2}ds_1\, ds_2.$$

The operator M mapping $L_2(\mathbb{T}^2)$ into $L_2(\mathbb{T}^2)$, takes the form $M\, f(t) = \sigma(t)\, f(t)$ with the continuous function $\sigma:\mathbb{T}^2\to\mathbf{R}$ given by (cf. (3.14))

$$\sigma\,(t) = \sum_{k\in\mathbb{Z}^2} a_k\, e^{i2\pi k\cdot t}. \tag{3.16}$$

Obviously, the inverse operator M^{-1} mapping $L_2(\mathbb{T}^2)$ to $L_2(\mathbb{T}^2)$ is of the form $M^{-1}\, f(t) = \sigma^{-1}(t)\, f(t)$.

Proposition 3.1 *1) There holds*

$$\|A_1\| = \|M\|_{\mathcal{L}(L_2(\mathbb{T}^2))} = ess\ \sup_{t\in\mathbb{T}^2}|\sigma(t)|, \quad \|M^{-1}\|_{\mathcal{L}(L_2(\mathbb{T}^2))} = ess\ \sup_{t\in\mathbb{T}^2}|\sigma^{-1}(t)|.$$

2) Operator A_1 is invertible if and only if

$$ess \inf_{t\in\mathbb{T}^2} |\sigma(t)| > 0. \tag{3.17}$$

The function σ is called the symbol of the discrete convolution operator and the numerical symbol of the method (3.12). Now let us show that the sequence A_N is uniformly bounded. In view of $A_N = A_1$ and of Proposition 3.1, we have to prove that $\sigma : \mathbb{T}^2 \to \mathbf{R}$ defined by the formula (3.16) is bounded.

Proposition 3.2 *For the function σ, we get* $\sup |\sigma(t)| < \infty$.

Proof. We shall utilize the Galerkin method with piecewise constant trial functions. Let Q_n be the orthogonal projection onto the span of the system $\{\chi_j : j \in \mathbb{Z}^2\}$, where $\chi_j := \chi_j^1$.

$$Q_n : \quad L_2(\mathbf{R}^2) \to span\{\chi_j : i \in \mathbb{Z}^2\}, \quad Q_n f \; = \; \sum_{j\in\mathbb{Z}^2} (f, \chi_j)\chi_j.$$

With respect to the basis $\{\chi_j\}_{j\in\mathbb{Z}^2}$ of $im\, Q_n$ the matrix of $A_n^G = Q_n A|_{im\, Q_n}$ is bounded (because $\|Q_n\| = 1$ and A is bounded). The matrix of A_n^G with respect to the basis $\{\chi_j\}_{j\in\mathbb{Z}^2}$ is defined by

$$A_n^G \; = \; \left(a_{k,j}^G\right)_{k,j\in\mathbb{Z}^2}, \quad a_{k,j}^G \; = \; \left(A\chi_j, \chi_k\right) \; = \; \left(A\chi_0, \chi_{k-j}\right) \; = \; a_{k-j}^G.$$

Then $A_n^G = \left(a_{k,j}^G\right)_{k,j}$ is a discrete convolution operator. Since A_n^G is a bounded operator in $l_2(\mathbb{Z}^2)$, there exist a bounded $m_G : \mathbb{T}^2 \to \mathbf{R}$,

$$m_G(t) = \sum_{k\in\mathbb{Z}^2} a_k^G \; e^{i2\pi k\cdot t}$$

such that $A_n^G = F^{-1} M_G F$, that M_G is the operator of multiplication by m_G, and that M_G is bounded (cf. Proposition 3.1). Now let $\left(a_{k-j}\right)_{k,j}$ denote the matrix of the quadrature method and $m = \sigma$ the corresponding symbol. We write

$$m(t) \;\; = \;\; [m(t) - m_G(t)] \; + m_G(t) \; = \; \sum_{k\in\mathbb{Z}^2} (a_k - a_k^G) e^{i2\pi k\cdot t} + \sum_{k\in\mathbb{Z}^2} a_k^G \; e^{i2\pi k\cdot t}.$$

In order to prove that m is bounded, it is sufficient to prove that $(m - m_G)$ is bounded. We prove this by showing

$$\sum_{k\in\mathbb{Z}^2} |a_k - a_k^G| < \infty. \tag{3.18}$$

Since $a_k = a_{k,0}$, we get

$$\begin{aligned}
a_k^G \;\; &= \;\; \left(A\chi_0, \chi_k\right) = \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \left(A\chi_0\right)(t) dt \\
&= \;\; \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \int_0^1 \int_0^1 \frac{f\left(\dfrac{t-s}{|t-s|}\right)}{|t-s|^2} ds_2 ds_1 dt_2 dt_1,
\end{aligned}$$

$$a_k^G - a_k \;\; = \;\; \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \int_0^1 \int_0^1 \left[\frac{f\left(\dfrac{t-s}{|t-s|}\right)}{|t-s|^2} - \frac{f\left(\dfrac{k}{|k|}\right)}{|k|^2} \right] ds_2 ds_1 dt_2 dt_1. \tag{3.19}$$

For the integrand, we get

$$\left| \frac{f\Big(\frac{t-s}{|t-s|}\Big)}{|t-s|^2} - \frac{f\Big(\frac{k}{|k|}\Big)}{|k|^2} \right| \le f\Big(\frac{k}{|k|}\Big) \left| \frac{1}{|t-s|^2} - \frac{1}{|k|^2} \right| + \frac{1}{|t-s|^2} \left| f\Big(\frac{t-s}{|t-s|}\Big) - f\Big(\frac{k}{|k|}\Big) \right|. \quad (3.20)$$

Estimating the first term on the right-hand side, we easily conclude

$$\left| f\Big(\frac{k}{|k|}\Big) \Big| \frac{1}{|t-s|^2} - \frac{1}{|k|^2} \Big| \right| \ \le \ C \frac{1}{|k|^3}. \quad (3.21)$$

To estimate the second term in (3.20), we observe that f is Lipschitz by assumption. Hence

$$\frac{1}{|t-s|^2} \Big| f\Big(\frac{t-s}{|t-s|}\Big) - f\Big(\frac{k}{|k|}\Big) \Big| \ \le \ C \frac{1}{|t-s|^2} \Big| \frac{t-s}{|t-s|} - \frac{k}{|k|} \Big| \ \le \ C \frac{1}{|k|^3}.$$

We arrive at

$$\sum_{k \in \mathbb{Z}^2} |a_k^G - a_k| \ < \ C \sum_{\substack{k \in \mathbb{Z}^2 \\ k \neq (0,0)}} |k|^{-3} \ < \ \infty. \qquad \diamond$$

Remark 3.1 *It is not hard to see that σ is continuous on $[-\frac{1}{2}, \frac{1}{2}]^2 \backslash \{(0,0)\}$. At $(0,0)$ the function σ has limits along all rays starting at $(0,0)$.*

Next we turn to the stability of A_N. Since $A_N = A_1$, we only have to prove the invertibility, i.e. (3.17). Unfortunately, we cannot prove stability for the general case or for the case of strongly elliptic singular integral equation either. Instead we prove stability for the special case of singular integral equation with Mikhlin-Giraud kernels and present a numerical stability proof for singular kernels with an operator for which the constant a is a complex number.

Theorem 3.1 *Suppose the integral equation to which we apply (3.12) is given by (3.1) with constant $a > 0$ and a convolution kernel $k(x,y) = f(\theta) r^{-2}$ such that $f(-\theta) = -f(\theta)$. Then the quadrature method (3.12) is stable.*

Proof. We only have to show (3.17). Recall that (cf. (3.14) and (3.16))

$$\sigma(t) = a + \sigma^{\#}(t), \quad \sigma^{\#}(t) = \sum_{\substack{k \in \mathbb{Z}^2 \\ k \neq (0,0)}} a_k e^{i2\pi k \cdot t}, \quad a_k = f\Big(\frac{k}{|k|}\Big) |k|^{-2}.$$

Since f is an odd function, we get $a_{-k} = -a_k$ as well as

$$\overline{\sigma^{\#}(t)} \ = \ \sum_{\substack{k \in \mathbb{Z}^2 \\ k \neq (0,0)}} a_k \ e^{-i2\pi k \cdot t} \ = \ - \sum_{\substack{k \in \mathbb{Z}^2 \\ k \neq (0,0)}} a_{-k} \ e^{-i2\pi k \cdot t} \ = \ -\sigma^{\#}(t).$$

Hence, $\sigma^{\#}(t)$ is purely imaginary and

$$|\sigma(t)| = \sqrt{a^2 + [\frac{\sigma^{\#}(t)}{i}]^2} \ge a. \quad \diamond$$

Finally, let us suppose there exist real constants α, β with $\alpha^2+\beta^2=1$ and

$$f(\theta) \quad = \quad \frac{1}{2\pi}\{\alpha \sin\varphi + \beta\cos\varphi\}, \quad \theta = e^{i\varphi}. \tag{3.22}$$

The symbol of the corresponding singular operator is

$$\sigma_A(x,\xi) \quad = \quad a + i\{\alpha\sin\varphi + \beta\cos\varphi\}, \quad \xi = e^{i\varphi}. \tag{3.23}$$

In this case we get the numerical symbol

$$\sigma(t) = a + i\{\alpha\sigma_1(t) + \beta\sigma_2(t)\},$$

where the numerical symbols σ_1 and σ_2 are real and correspond to the characteristics $\frac{1}{i2\pi}\sin\varphi$ and $\frac{1}{i2\pi}\cos\varphi$, respectively. Numerical computations of $\sigma_1^2+\sigma_2^2$ confirm (cf. Figure 1) that $\sigma_1^2+\sigma_2^2 \le 1$. Hence, $-1 \le [\alpha\sigma_1 + \beta\sigma_2] \le 1$ and we obtain: If f is given by (3.22) with real numbers α, β such that $\sqrt{\alpha^2+\beta^2} = 1$ and if $a \in \mathbb{C} \setminus \{z \in \mathbb{C} : -1 \le \operatorname{Im} z \le 1\}$, then the quadrature method (3.12) is stable. Note that the condition $a \in \mathbb{C} \setminus \{z \in \mathbb{C} : -1 \le \operatorname{Im} z \le 1\}$, is equivalent to the fact that A defined by (3.23) is strongly elliptic at least after multiplication by a suitable constant.

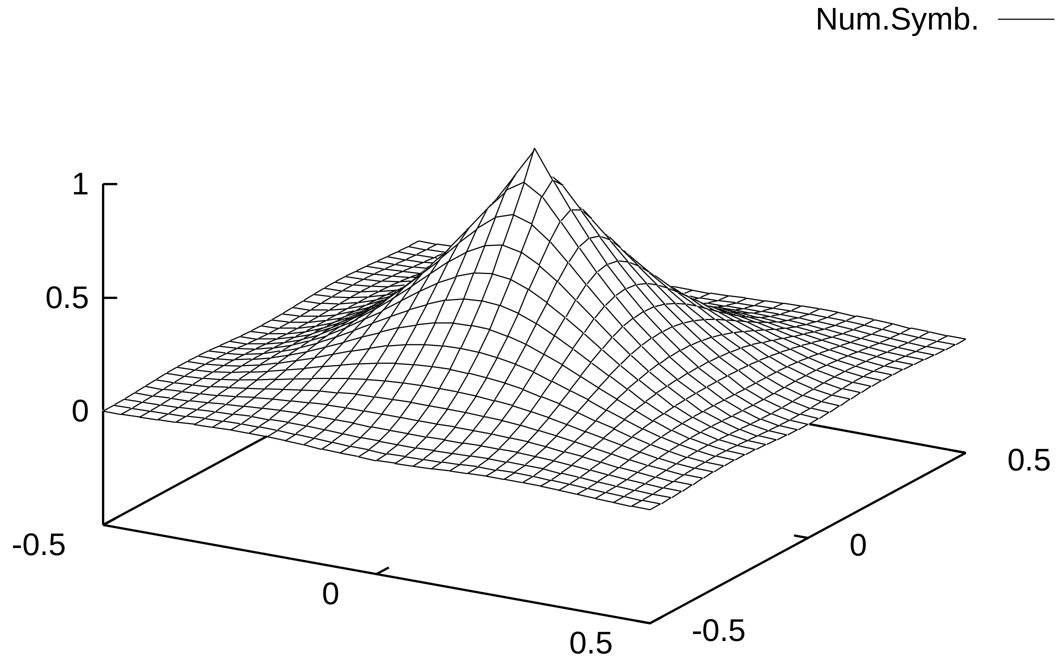

Figure 1: The numerical symbol $\sigma_1^2+\sigma_2^2$.

4 Localization Principle

4.1 The Theorem

Let us start with a few historical remarks. Localization techniques (principle of freezing the coefficients) have been known and applied for a long time to the analysis of partial differential operators or pseudo-differential operators. Later on these techniques have been reformulated in an algebra language which has turned out to be useful in the analysis of several kind of operator classes (cf. Simonenko [34], Gohberg, Krupnik [11], and Douglas [10]). The first one to apply these techniques to numerical methods was Kozak [14]. His ideas have been generalized and developed into a very nice abstract scheme by the school of Silbermann (for details cf. the corresponding chapters of [28]). Parallel to this, an abstract setting for the application to spline methods is due to Prößdorf [21].

We shall use the same localization techniques. However, instead of using the abstract schemes of e.g. Silbermann, we perform the corresponding steps of proof directly. This is possible because the local principle in our situation is not very complicated. To get a better feeling for the localization, we recommend the reader to study the corresponding sections of [11, 28].

Let us consider the quadrature method (2.5) applied to the singular integral equation (2.1) over the torus and suppose (2.9) is satisfied. To the corresponding singular integral operator and to this quadrature method, we introduce a localized singular integral operator and a localized quadrature method at any point $\tau \in \mathbb{T}^2$. Thus let us fix a $\tau \in \mathbb{T}^2$. The localized operator is the singular integral operator over the tangent plane with the same values of the kernel function $k_S(x, x-y)$ at $x = \tau$. To get an operator over the plane, we freeze the local variable x and consider the convolution kernel $k_S(\tau, x-y)$. In other words the localized singular integral operator A_τ at τ is the singular convolution operator over the plane $\mathbf{R}^2$ with the kernel function

$$k_\tau(x-y) = k_S(\tau, x-y) \quad ,$$

and with the multiplication operator $a(x)$ replaced by the constant $a_\tau = a(\tau)$. Thus the localized equation corresponding to (2.1) is

$$a_\tau u(x) + \int_{\mathbf{R}^2} k_\tau(x-y)u(y)dy = g(x). \tag{4.1}$$

To this we apply the quadrature method (3.12). The resulting scheme is the localized quadrature method of (2.5). We denote the matrix (or the discretized operator of the quadrature method) by $(A_\tau)_N \in \mathcal{L}\Big(\mathcal{S}^N(\mathbf{R}^2)\Big)$. With this notation the localization principle for the quadrature method can be formulated as follows:

Theorem 4.1 *Let us consider the quadrature method (2.5) applied to the singular integral equation (2.1) including the invertible operator A which is supposed to be a pseudo-differential operator of order zero and to posses a symbol from the class S^0. Suppose the local operators A_τ are defined by the left-hand side of (4.1) and consider their quadrature approximation $(A_\tau)_N$ of the form (3.12). Then the method (2.5) is stable if and only if it is locally stable, i.e., if for any $\tau \in \mathbb{T}^2$, the quadrature operators $(A_\tau)_N$ are stable.*

The stability of the quadrature methods $(A_\tau)_N$ has been investigated in Section 3.

4.2 Sufficiency of Local Stability

In this subsection, we prove the sufficiency of the local stability. We retain the notation $\mathcal{S}^N$ for the space of piecewise constant functions (cf. Section 2) and denote the orthogonal projection onto $\mathcal{S}^N$ by L_N. For the stability of the sequence of operators A_N it is sufficient to prove a representation

$$A_N B_N = I_N + D_N + L_N T C_N \quad , \tag{4.2}$$

where $I_N \in \mathcal{L}(\mathcal{S}^N)$ is the identity, $\|D_N\|_{\mathcal{L}(\mathcal{S}^N)} \leq \frac{1}{2}$, the operators $C_N, B_N \in \mathcal{L}(\mathcal{S}^N)$ are uniformly bounded with respect to N, and $T \in \mathcal{L}(L_2(\mathbb{T}^2))$ is compact. Indeed, from (4.2), we get

$$A_N\Big[B_N\big(I_N + D_N\big)^{-1}\Big] = I_N + L_N T C_N \big(I_N + D_N\big)^{-1}. \tag{4.3}$$

and the stability of A_N follows from the following lemma and the strong convergence $A_N L_N \to A$ which will be proved in Section 5.

Lemma 4.1 *(cf. e.g. [22]) Suppose $A \in \mathcal{L}(L_2(\mathbb{T}^2))$ is invertible and $A_N L_N \to A$ for $A_N \in \mathcal{L}(\mathcal{S}^N)$. Moreover, suppose E_N, $F_N \in \mathcal{L}(\mathcal{S}^N)$ are sequences of uniformly bounded operators and $T \in \mathcal{L}(L_2(\mathbb{T}^2))$ is compact. Then*

$$A_N E_N = I_N + L_N T F_N$$

implies that A_N is stable. The same conclusion holds if there exist more than one term of the form $L_N T F_N$ on the right-hand side.

Let us derive (4.2). To get B_N, we introduce a finite set of points $\tau_k \in \mathbb{T}^2$, $k = 1, \ldots, M$. We choose cut off functions $\psi_k, \psi_k' \in C^\infty(\mathbb{T}^2)$ in the neighborhood of τ_k such that

i) The values of ψ_k, ψ_k' belong to $[0, 1]$.

ii) There holds:

$$\tau_k \in supp\ \psi_k \subseteq \{t \in \mathbb{T}^2 : \psi_k'(t) \equiv 1\} \subseteq supp\ \psi_k', \quad \psi_k \psi_k' = \psi_k.$$

iii) Let $f = \sum_{k=1}^M \psi_k$. Then we suppose that f is a positive function with values less than 4. Moreover, we suppose that, for any $t_0 \in \mathbb{T}^2$, there exist at most four functions ψ_k' not vanishing at t_0.

We introduce the piecewise constant interpolation projector by

$$K_N h = \sum_{l_1, l_2 = 0}^{n-1} h(t^N_{l_1,l_2}) \chi^N_{l_1,l_2}.$$

For a function g on $\mathbb{T}^2$, we set $g_N := K_N g|_{\mathcal{S}^N}$. In other words, the matrix of g_N with respect to the basis $\{\chi_l^N\}$ is

$$g_N = \Big(g(t_i^N)\delta_{i,j}\Big)_{i,j=0}^{n-1},$$

and we get $(\psi_k')_N (\psi_k)_N = (\psi_k' \psi_k)_N = (\psi_k)_N$. Using all these definitions, we choose the matrix operator B_N for (4.2) as

$$B_N = \sum_{k=1}^{M} (\psi_k)_N (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N,$$

where the operator B_N^k is defined as $B_N^k = (A_\tau)_N$ and $(A_\tau)_N$ is the localized quadrature operator of Subsection 4.1 defined for a fixed $\tau \in$ *supp* ψ_k. To explain the expression $(\psi_k)_N$ $(B_N^k)^{-1}$ $(\psi_k')_N$, we note that, for fixed $\tau_k = (\tau_{k,1}, \tau_{k,2}) \in \mathbb{T}^2$, the torus $\mathbb{T}^2$ can be identified with the periodic square

$$[\tau_{k,1} - \frac{1}{2}, \tau_{k,1} + \frac{1}{2}] \times [\tau_{k,2} - \frac{1}{2}, \tau_{k,2} + \frac{1}{2}]$$

and can be embedded into $\mathbf{R}^2$. The functions ψ_k, ψ_k' with

$$\mathit{supp}\, \psi_k,\ \mathit{supp}\, \psi_k' \ \subseteq \ \left(\tau_{k,1} - \frac{1}{2}, \tau_{k,1} + \frac{1}{2}\right) \times \left(\tau_{k,2} - \frac{1}{2}, \tau_{k,2} + \frac{1}{2}\right)$$

can be considered as functions over $\mathbf{R}^2$. If K_N stands for the interpolation projection onto $\mathcal{S}^N(\mathbf{R}^2)$ (We use the same symbol as for the corresponding operator on $\mathbb{T}^2$.), then we can set $h_N = K_N h|_{\mathcal{S}^N(\mathbf{R}^2)}$ for any function h over $\mathbf{R}^2$. In particular, we arrive at a second definition for $(\psi_k)_N$ and $(\psi_k')_N$. These different operators, one over $\mathbb{T}^2$ and the other over $\mathbf{R}^2$, however, can be identified since for each piecewise constant basis function $\chi^N_{l_1,l_2}$ over $\mathbb{T}^2$ with *supp* $\chi^N_{l_1,l_2} \cap$ *supp* $\psi_k \neq \emptyset$ there exists a unique basis function $\chi^N_{l_1',l_2'}$ over $\mathbf{R}^2$ with $\chi^N_{l_1',l_2'} = \chi^N_{l_1,l_2}$ over $(\tau_{k,1} - \frac{1}{2}, \tau_{k,1} + \frac{1}{2}) \times (\tau_{k,2} - \frac{1}{2}, \tau_{k,2} + \frac{1}{2})$. Identifying these basis functions, we can identify the two operators. In this sense the operator $(B_N^k)^{-1}$ over $\mathbf{R}^2$ multiplied by $(\psi_k)_N$ and $(\psi_k')_N$ over $\mathbf{R}^2$ can be considered as an operator $(\psi_k)_N$ $(B_N^k)^{-1}$ $(\psi_k')_N$ over the torus.

We conclude

$$\begin{aligned}
A_N B_N &= A_N \sum_{k=1}^{M} (\psi_k)_N (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N \\
&= \sum_{k=1}^{M} \Big[A_N (\psi_k')_N - (\psi_k')_N A_N \Big] (\psi_k)_N (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N \\
&\quad + \sum_{k=1}^{M} \Big[(\psi_k')_N A_N (\psi_k)_N - (\psi_k')_N B_N^k (\psi_k)_N \Big] (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N \\
&\quad + \sum_{k=1}^{M} (\psi_k')_N \Big[B_N^k (\psi_k)_N - (\psi_k)_N B_N^k \Big] (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N \\
&\quad + \sum_{k=1}^{M} (\psi_k')_N (\psi_k)_N B_N^k (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N \\
&= \sum_{k=1}^{M} \Big[A_N (\psi_k')_N - (\psi_k')_N A_N \Big] (\psi_k)_N (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N + \tilde{T}_N \\
&\quad + \sum_{k=1}^{M} (\psi_k')_N \Big[B_N^k (\psi_k)_N - (\psi_k)_N B_N^k \Big] (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N + I_N, \qquad (4.4)
\end{aligned}$$

where

$$\tilde{T}_N = \sum_{k=1}^{M} \Big[(\psi_k')_N A_N (\psi_k)_N - (\psi_k')_N B_N^k (\psi_k)_N \Big] (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N. \qquad (4.5)$$

The representation (4.4) will imply (4.2) if we can show :

a) The operator $\Big[A_N(\psi_k)_N - (\psi_k)_N A_N\Big]$ is the sum of an operator $L_N T C_N$ with T compact and C_N uniformly bounded plus an operator D_N tending to zero in the operator norm.

b) The operator $(\psi'_k)_N\Big[B_N^k(\psi_k)_N - (\psi_k)_N B_N^k\Big]$ is the sum of an operator $L_N T C_N$ with T compact and C_N uniformly bounded plus an operator D_N tending to zero in the operator norm.

c) The operator $\tilde{T}_N$ of (4.5) has a norm less than any prescribed $\epsilon > 0$ if the τ_k, ψ_k, ψ'_k are chosen suitably.

It remains to prove a), b), and c). We start with a). Let us consider the kernel

$$\tilde{k}(x,y) = k(x,y)\Big[\psi_k(x) - \psi_k(y)\Big], \tag{4.6}$$

which is the weakly singular kernel of a compact integral operator T and which satisfies

$$|\tilde{k}(x,y)| \leq C|x-y|^{-1}. \tag{4.7}$$

It is not hard to see that

$$A_N(\psi_k)_N - (\psi_k)_N A_N = T_N = (\tilde{k}(t_j^N, t_k^N)\frac{1}{n^2})_{j,k}.$$

Consequently, it remains to prove that

$$\|T_N - L_N T|_{S^N}\| \to 0. \tag{4.8}$$

We put $\tilde{k} = \tilde{k}^1 + \tilde{k}^2$,

$$\tilde{k}^1(x,y) \;=\; \tilde{k}(x,y)\chi^*(|x-y|), \qquad \tilde{k}^2(x,y) \;=\; \tilde{k}(x,y)\Big[1-\chi^*(|x-y|)\Big],$$

where $\chi^* \in C^\infty$ is chosen such that *supp* $\chi^* \subseteq (-\epsilon,\epsilon)$ and $\chi^* \equiv 1$ on $(-\epsilon/2,\epsilon/2)$ for a prescribed $\epsilon > 0$. According to the splitting of the kernel, we get the splitting

$$T = T^1 + T^2.$$

Operator T^2 has a smooth kernel. For (4.8) it remains to prove that

$$\Big\|(T^2)_N - L_N T^2|_{S^N}\Big\| \;\to\; 0, \tag{4.9}$$

$$\|L_N T^1|_{S^N}\| \;\leq\; C\epsilon, \tag{4.10}$$

$$\Big\|(T^1)_N\Big\| \;\leq\; C\epsilon, \tag{4.11}$$

where the constant C is independent of ϵ and χ^*. Let us prove (4.9). Since $T^2 : L_2 \to C$ is compact, since $L_N, K_N : C \to L_2$ are uniformly bounded, and since $(K_N - L_N)$ tends to zero strongly, the operator $(K_N - L_N)T^2$ tends to zero in operator norm. On the other hand, for the quadrature discretization

$$T_N^2 = \Big(\tilde{k}^2(t_j^N, t_k^N)\frac{1}{n^2}\Big)_{j,k},$$

we obtain

$$K_N T^2|_{imL_N} - (T^2)_N = \left(b^2_{j,k}\right)_{j,k}.$$

For the difference of the entries, we conclude

$$\begin{aligned}
\left|b^2_{j,k}\right| &= \left|\int \tilde{k}^2(t^N_j, y)\chi^N_k(y)dy - \tilde{k}^2(t^N_j, t^N_k)\frac{1}{n^2}\right| \\
&= \left|\int \left[\tilde{k}^2(t^N_j, y) - \tilde{k}^2(t^N_j, t^N_k)\right]\chi^N_k(y)dy\right| \\
&\leq \int \left|\chi^N_k(y)\right|dy \cdot \sup_{y\in supp\,\chi^N_k} \left|\tilde{k}^2(t^N_j, y) - \tilde{k}^2(t^N_j, t^N_k)\right|.
\end{aligned}$$

Since

$$\left|\tilde{k}^2(t^N_j, y) - \tilde{k}^2(t^N_j, t^N_k)\right| \leq C_\epsilon\left|y - t^N_k\right| \leq C_\epsilon\frac{1}{n},$$

we continue

$$\begin{aligned}
\left|b^2_{j,k}\right| &\leq C_\epsilon\frac{1}{n}\int\left|\chi^N_k(y)\right|dy \;\leq\; C_\epsilon\frac{1}{n}\cdot\frac{1}{n^2}, \\
\left\|(b^2_{j,k})_{j,k}\right\| &\leq \left\|(\frac{1}{n}\cdot\frac{1}{n^2})_{j,k}\right\| \;\leq\; C_\epsilon\sum_{k:\,|k|\leq n}\frac{1}{n}\cdot\frac{1}{n^2} \;\leq\; C_\epsilon\frac{1}{n}.
\end{aligned} \tag{4.12}$$

This implies $\left\|K_N T^2|_{S^N} - (T^2)_N\right\| \to 0$ for any fixed $\epsilon > 0$. And, together with $\left\|(K_N - L_N)T^2\right\| \to 0$, we obtain (4.9).

Let us turn to (4.11) and estimate the entries $b^1_{j,k} = \tilde{k}^1(t^N_j, t^N_k)/n^2$.

$$\left|b^1_{j,k}\right| \;\leq\; C\begin{cases} \dfrac{1}{\left|t^N_j - t^N_k\right|\cdot n^2} = \dfrac{1}{|j-k|}\dfrac{1}{n} & \text{if} \quad |j-k| \leq C\epsilon\cdot n \\ 0 & \text{otherwise}\end{cases} \tag{4.13}$$

Here ϵ is the number used for $supp\,\chi^* \subseteq (-\epsilon, \epsilon)$ in the splitting of $\tilde{k}$. By Young's inequality we conclude

$$\left\|(b^1_{j,k})_{j,k}\right\| \;\leq\; C\sum_{\substack{j\neq 0\\ |j|\leq C\epsilon\cdot n}}\frac{1}{|j|}\frac{1}{n} \;\leq\; C\epsilon. \tag{4.14}$$

Hence, (4.11) is proved. Relation (4.10) follows analogously if instead of the entry of the discretized operator the kernel function of the integral operator T^1 is considered. The proof of (4.9), (4.10), and (4.11) finishes the proof of assertion a).

Let us turn to the proof of b). This proof, however, is completely analogous to that of a). Indeed, instead of (4.6) we get

$$\tilde{k}(x,y) = \psi'_k(x)k(x,y)[\psi_k(y) - \psi_k(x)] \tag{4.15}$$

which satisfies (cf. (4.7))

$$|\tilde{k}(x,y)| \leq \begin{cases} C_\epsilon|x-y|^{-1} & \text{if} \quad |y| \leq \epsilon^{-1} \\ 0 & \text{if } x \notin supp\;\psi'_k \\ C|x-y|^{-2} & \text{else}\end{cases}$$

for sufficiently small $\epsilon > 0$. Since the support of ψ_k' is compact, the integral operator with kernel function (4.15) is compact. Using the function χ^*, we split T into T^1 and T^2, and, analogously to (4.12) we arrive at

$$
\begin{aligned}
|b^2_{j,k}| &\leq \begin{cases} C_\epsilon n^{-3} & \text{if } |k| \leq \epsilon^{-1} n \\ C|k|^{-2} & \text{if } |k| \geq \epsilon^{-1} n, \end{cases} \\
\left\| (b^2_{j,k})_{j,k} \right\| &\leq C_\epsilon n^{-1} + C \sqrt{\sum_{j,k:\ |k| \geq \epsilon^{-1} n,\ j \leq 2n} |k|^{-4}} \leq C_\epsilon n^{-1} + C\epsilon.
\end{aligned}
$$

Thus we obtain $\|(T^2)_N - L_N T^2|_{S^N(\mathbf{R}^2)}\| \leq C\epsilon$ for sufficiently large n. Similarly to (4.13) and (4.14), we get

$$
\begin{aligned}
|b^1_{j,k}| &\leq \begin{cases} C\dfrac{1}{|j-k|}\dfrac{1}{n} & \text{if } |j-k| \leq C\epsilon \cdot n,\ |k| \leq \epsilon^{-1} n \\ C|k|^{-2} & \text{if } |k| \geq \epsilon^{-1} n, \\ 0 & \text{if } |j-k| \geq C\epsilon \cdot n, \end{cases} \\
\left\| (b^1_{j,k})_{j,k} \right\| &\leq C\epsilon + C \sqrt{\sum_{j,k:\ |k| \geq \epsilon^{-1} n,\ j \leq 2n} |k|^{-4}} \leq C\epsilon + C\epsilon.
\end{aligned}
$$

This means $\|L_N T^1|_{S^N(\mathbf{R}^2)}\| \leq C\epsilon$ and all these facts together prove that b) is valid.

Now let us prove assertion c). We consider a vector $\xi = (\xi_j)_{j=0}^{n-1}$ and arbitrary matrices F_N^k. Then we get

$$
\begin{aligned}
\Big\| \sum_{k=1}^M (\psi_k')_N F_N^k (\psi_k')_N \xi \Big\|^2 &\leq C \sum_{k=1}^M \Big\| (\psi_k')_N F_N^k (\psi_k')_N \xi \Big\|^2 \\
&\leq C \sum_{k=1}^M \Big\| (\psi_k')_N F_N^k (\psi_k')_N \Big\|^2 \Big\| (\chi_k')_N \xi \Big\|^2 \\
&\leq C \sup_{k=1,\dots,M} \Big\| (\psi_k')_N F_N^k (\psi_k')_N \Big\|^2 \sum_{k=1}^M \Big\| (\chi_k')_N \xi \Big\|^2 \\
&\leq C \sup_{k=1,\dots,M} \Big\| (\psi_k')_N F_N^k (\psi_k')_N \Big\|^2 \Big\| \xi \Big\|^2. \qquad (4.16)
\end{aligned}
$$

Here χ_k' denotes the characteristic function of the support of ψ_k' and satisfies the relation $(\psi_k')_N (\chi_k')_N = (\psi_k' \chi_k')_N = (\psi_k')_N$. Moreover, the estimates corresponding to the second and last line of (4.16) are correct since, for each $j = (j_1, j_2)$ with $0 \leq j_1, j_2 \leq n-1$, there exist at most four vectors $(\psi_k')_N F_N^k (\psi_k')_N \xi$ and at most four $(\chi_k')_N \xi$ such that the j-th component does not vanish (cf. condition iii) for the definition of the ψ_k'). Hence,

$$
\Big\| \sum_{k=1}^M (\psi_k')_N F_N^k (\psi_k')_N \Big\| \leq C \sup_{k=1,\dots,M} \Big\| (\psi_k')_N F_N^k (\psi_k')_N \Big\|
$$

and, choosing

$$
F_N^k = \Big[(\chi_k')_N A_N (\psi_k)_N - (\chi_k')_N B_N^k (\psi_k)_N \Big] (B_N^k)^{-1} (f^{-1})_N,
$$

we arrive at

$$
\begin{aligned}
\Big\| \tilde{T}_N \Big\| &\leq C \sup_{k=1,\dots,M} \Big\| \Big[(\psi_k')_N A_N (\psi_k)_N - (\psi_k')_N B_N^k (\psi_k)_N \Big] (B_N^k)^{-1} (\psi_k')_N (f^{-1})_N \Big\| \\
&\leq C \sup_{k=1,\dots,M} \Big\| (\psi_k')_N A_N (\psi_k)_N - (\psi_k')_N B_N^k (\psi_k)_N \Big\|.
\end{aligned}
$$

It remains to prove that $\left[(\psi_k')_N A_N(\psi_k)_N - (\psi_k')_N B_N^k(\psi_k)_N\right]$ is small provided that the supports of ψ_k and ψ_k' have a small diameter.

First we consider the case that A is a multiplication operator. We get

$$\begin{aligned}\left[(\psi_k')_N A_N(\psi_k)_N - (\psi_k')_N B_N^k(\psi_k)_N\right] &= \left[(\psi_k')_N a_N(\psi_k)_N - (\psi_k')_N (a(\tau))_N(\psi_k)_N\right] \\ &= \left(\psi_k(t_{j_1,j_2}^N)[a(t_{j_1,j_2}^N) - a(\tau)]\delta_{i,j}\right)_{i,j}, \\ \left\|\left[(\psi_k')_N A_N(\psi_k)_N - (\psi_k')_N B_N^k(\psi_k)_N\right]\right\| &\leq C \sup_{t\in supp\ \psi_k} |a(t) - a(\tau)|.\end{aligned}$$

Since τ is taken from $supp\ \psi_k$ too, we obtain that $\left[(\psi_k')_N A_N(\psi_k)_N - (\psi_k')_N B_N^k(\psi_k)_N\right]$ is small for ψ_k with sufficiently small support $supp\ \psi_k$.

Now, in the second case, suppose that operator A is an integral operator with bounded kernel function k_R. For this A, the localized operator A_τ is zero. Thus $B_N^k = 0$ and we have to prove that $(\psi_k')_N A_N(\psi_k)_N$ is small provided the functions ψ_k, ψ_k' have supports with sufficiently small diameter. However, due to the quadrature weight n^{-2}, each entry of $(\psi_k')_N A_N(\psi_k)_N$ is less than Cn^{-2}. The dimension of the non-zero part of $(\psi_k')_N A_N(\psi_k)_N$ is less than $[\delta n]^2$ if the diameter of the supports $supp\ \psi_k$ and $supp\ \psi_k'$ is less than δ. Consequently, Young's inequality implies

$$\left\|(\psi_k')_N A_N(\psi_k)_N\right\| \leq \sum_{l\in\mathbb{Z}^2:|l|\leq\delta n} Cn^{-2} \leq C\delta^2$$

and $\left[(\psi_k')_N A_N(\psi_k)_N - (\psi_k')_N B_N^k(\psi_k)_N\right]$ is small for a small diameter δ of $supp\ \psi_k$ and $supp\ \psi_k'$.

In the third and last case we suppose that A is the singular integral operator with kernel k_S. Moreover, we may assume that

$$k_S(x, x-y) = b(x)f(\frac{x-y}{|x-y|})|x-y|^{-2}. \tag{4.17}$$

Indeed, the characteristic $f(x, x-y) = |x-y|^2 k_S(x, x-y)$ is a smooth function for a pseudo-differential operator with a symbol from the class S^0. We can approximate f in the Lipschitz norm by the truncated trigonometric series with respect to the second variable $z = x - y$. The singular integral operator and its quadrature discretization corresponding to the approximated characteristic are close to the original singular operator and its quadrature discretization (cf. [5, 6] and Lemma 5.1 for the discretized operators). Hence, we can replace A by the operator corresponding to the truncated trigonometric series of its characteristic and can treat each term of the sum separately. This way we arrive at kernels of the form (4.17). However, operators with kernel (4.17) are products of a multiplication operator (multiplication by b) and a convolution operator G with kernel

$$f(\frac{x-y}{|x-y|})|x-y|^{-2}. \tag{4.18}$$

Similarly, A_N is the product of the diagonal matrix b_N (discretized multiplication operator) and the discretized convolution operator G_N, and B_N^k the product of $b(\tau)I_N$ and G_N^k (discretized convolution operator over $\mathbf{R}^2$). We conclude

$$(\psi_k')_N A_N(\psi_k)_N - (\psi_k')_N B_N^k(\psi_k)_N$$

$$\begin{aligned}
&= (\psi_k')_N b_N G_N(\psi_k)_N - (\psi_k')_N b(\tau) I_N G_N^k(\psi_k)_N \\
&= \Big[(\psi_k')_N b_N(\chi_k')_N - (\psi_k')_N b(\tau) I_N(\chi_k')_N\Big] G_N(\psi_k)_N \\
&\quad + (\psi_k')_N b(\tau) I_N \Big[(\chi_k')_N G_N(\psi_k)_N - (\chi_k')_N G_N^K(\psi_k)_N\Big]. \qquad (4.19)
\end{aligned}$$

The last bracket is zero since the kernel of the frozen operator with kernel (4.18) is the same as (4.18). The first bracket on the right-hand side of (4.19) is small by the proof for the case when A is a multiplication operator. This completes the proof of assertion c).

4.3 Necessity of Local Stability

Suppose $\{A_N\}$ is stable and fix a $\tau \in \mathbb{T}^2$. We have to show that $(A_\tau)_1$, i.e., the quadrature operator $(A_\tau)_N$ for $N=1$ is invertible. We shall show that $\{A_N\}$ can be considered as a stable and convergent approximation method for operator $(A_\tau)_1$ which implies that $(A_\tau)_1$ is invertible. In order to simplify the notation we suppose $\tau = (0,0)$.

In the previous subsections we have identified the operator $(A_\tau)_N \in \mathcal{L}\big(\mathcal{S}^N(\mathbf{R}^2)\big)$ with its matrix. Now we consider $(A_\tau)_N = (A_\tau)_1$ to be the fixed matrix operator acting in $l_2(\mathbb{Z}^2)$. For the identification of $A_N \in \mathcal{L}(\mathcal{S}^N)$ with its matrix, we introduce the isomorphism of $\mathcal{S}^N$ and the finite l_2-space explicitly. We consider the set

$$\mathbb{Z}_N^2 = \{l \in \mathbb{Z}^2 : -\frac{1}{2} \le \frac{l_j}{n} < \frac{1}{2},\ j = 1,2\} \subseteq \mathbb{Z}^2$$

and introduce $E_N : l_2(\mathbb{Z}_N^2) \to \mathcal{S}^N$ by

$$E_N(\xi_l)_{l\in\mathbb{Z}_N^2} = \sum_{l\in\mathbb{Z}_N^2} \xi_l\, \chi_l^N.$$

Clearly, E_N is invertible. To each operator $B_N \in \mathcal{L}(\mathcal{S}^N)$ there corresponds the matrix operator $\tilde{B}^N := E_N^{-1} B_N E_N$, i.e., $\tilde{B}^N$ is the matrix of B_N with respect to the basis $\{\chi_l^N : l \in \mathbb{Z}_N^2\}$. Moreover

$$\big\|B_N\big\|_{\mathcal{L}(\mathcal{S}^N)} = \big\|\tilde{B}_N\big\|_{\mathcal{L}(l_2(\mathbb{Z}_N^2))}.$$

Now $l_2(\mathbb{Z}_N^2)$ can be embedded into $l_2(\mathbb{Z}^2)$ by identifying $l_2(\mathbb{Z}_N^2)$ with

$$\{(\xi_l)_{l\in\mathbb{Z}^2} \in l_2(\mathbb{Z}^2) : \xi_l = 0 \ \mathit{for}\ l \in \mathbb{Z}^2 \backslash \mathbb{Z}_N^2\}.$$

We denote the orthogonal projection from $l_2(\mathbb{Z}^2)$ to $l_2(\mathbb{Z}_N^2)$ by P_N. Clearly, P_N tends strongly to the identity operator in $l^2(\mathbb{Z}^2)$. Thus we can consider the operator $\tilde{A}_N \in \mathcal{L}(imP_N)$ corresponding to our quadrature operator A_N as an approximate operator for $(A_\tau)_1 \in \mathcal{L}(l_2(\mathbb{Z}^2))$. We shall prove that

$$\tilde{A}_N P_N \ \to\ (A_\tau)_1, \quad \tilde{A}_N^* P_N \ \to\ (A_\tau)_1^* \qquad (4.20)$$

is true in strong operator topology. If this is done, then we conclude from the stability $\|A_N^{-1}\| \le C$ (which means also $\|\tilde{A}_N^{-1}\| \le C$) that

$$\begin{aligned}
\big\|(A_\tau)_1\xi\big\| &= \lim_{N\to\infty} \big\|\tilde{A}_N P_N \xi\big\| \ \ge\ \lim_{N\to\infty} C^{-1}\big\|P_N\xi\big\| \ \ge\ C^{-1}\|\xi\|, \qquad (4.21)\\
\big\|(A_\tau)_1^*\xi\big\| &\ge C^{-1}\|\xi\| \qquad (4.22)
\end{aligned}$$

holds for any $\xi \in l_2$. Relation (4.21) implies that $(A_\tau)_1$ has a trivial null space and that the image space of $(A_\tau)_1$ is closed. The inequality (4.22) proves that the kernel of $(A_\tau)_1^*$ is trivial, i.e., the cokernel of $(A_\tau)_1$ is trivial, too. Hence, $(A_\tau)_1$ is invertible. It remains to show (4.20).

To prove the strong convergence we use the Banach-Steinhaus theorem. The uniform boundedness of the operators A_N (and hence also of the $\tilde{A}_N$) will be proved in Lemma 5.1. Thus it remains to prove that, for any fixed $e_m = (\delta_{j,m})_{j\in\mathbb{Z}^2}$,

$$\tilde{A}_N P_N e_m \;\to\; (A_\tau)_1 e_m\,, \quad \tilde{A}_N^* P_N e_m \;\to\; (A_\tau)_1^* e_m. \tag{4.23}$$

Moreover, the adjoint matrices $\tilde{A}_N^*$, $(A_\tau)_1^*$ are of the same structure as $\tilde{A}_N$, $(A_\tau)_1$ since they correspond to the adjoint singular integral operators. In other words, we only prove the first part of (4.23). We observe that, for any cut off function ψ which is equal to one in a small neighborhood of $\tau = 0$, there holds

$$\tilde{\psi}_N e_m \;=\; \left(\psi(t_j^N)\delta_{i,j}\right) e_m \;=\; \psi(t_m^N) e_m = e_m$$

for sufficiently large N. We introduce a cut off function ψ' such that

$$supp\ \psi \subseteq \{t \in \mathbb{T}^2 : \psi' \equiv 1\}$$

and write (Recall that the matrix $(A_\tau)_N$ is independent of N.)

$$\begin{aligned}\tilde{A}_N P_N e_m \;&=\; (A_\tau)_1 e_m + \left(\tilde{\psi}'_N - \tilde{I}_N\right)(A_\tau)_N \tilde{\psi}_N e_m \\ &\quad + \left[\tilde{\psi}'_N \tilde{A}_N \tilde{\psi}_N - \tilde{\psi}'_N (A_\tau)_N \tilde{\psi}_N\right]\tilde{\psi}_N e_m \;+\; \left(\tilde{I}_N - \tilde{\psi}'_N\right)\tilde{A}_N \tilde{\psi}_N e_m.\end{aligned}$$

The third term on the right-hand side is small if ψ and ψ' are suitably chosen. Indeed, the corresponding operators without the tilde have been shown to be small in the proof to assertion c) in Subsection 4.2. The smallness of the second and of the last term follows from the next lemma. In other words, for any $\epsilon > 0$, we can choose appropriate ψ and ψ' such that

$$\left\| \tilde{A}_N P_N e_m - (A_\tau)_1 e_m \right\|_{l_2} < \epsilon$$

for N sufficiently large. Thus $\tilde{A}_N P_N \to (A_\tau)_1$ and the necessity is proved.

Lemma 4.2 *Suppose that*

$$supp\ \psi \;\subseteq\; \{x \in \mathbb{Z}^2 : |x| < \delta_1\} \;\subseteq\; \{x \in \mathbb{Z}^2 : |x| < \delta_2\} \;\subseteq\; \{x \in \mathbb{Z}^2 : \psi'(x) = 1\},$$

where $0 < \delta_1 < \delta_2$ *and* δ_2 *is much larger than* δ_1. *Then we get*

$$\left\|(\psi'_N - I_N)(A_\tau)_N \psi_N\right\| \;\le\; C\frac{\delta_1}{\delta_2}, \quad \left\|(\psi'_N - I_N)A_N \psi_N\right\| \;\le\; C\frac{\delta_1}{\delta_2}. \tag{4.24}$$

Proof. Let us consider the matrices of $(\psi_k)_N$, $(\psi'_k)_N$, $(A_\tau)_N$ with respect to the basis $\{\chi_l^N : l \in \mathbb{Z}^2\}$. We get

$$(A_\tau)_N \;=\; \left(b_{i,j}\right)_{i,j\in\mathbb{Z}^2}, \quad (\psi_k)_N \;=\; \left(c_i\delta_{ij}\right)_{i,j\in\mathbb{Z}^2}, \quad I_N - (\psi'_k)_N \;=\; \left(d_i\delta_{ij}\right)_{i,j\in\mathbb{Z}^2},$$

where obviously

$$
\begin{aligned}
\left|b_{ij}\right| &\le C\left|t_i^N - t_j^N\right|^{-2}\cdot\frac{1}{n^2} \le C\left|i-j\right|^{-2},\\
|c_i| &\le \begin{cases} 1 & \text{if } t_i^N \in supp\ \psi_k \\ 0 & \text{if } t_i^N \notin supp\ \psi_k \end{cases} \le \begin{cases} 1 & \text{if } |i/n| \le \delta_1 \\ 0 & \text{else ,} \end{cases}\\
|d_i| &\le \begin{cases} 0 & \text{if } |i/n| \le \delta_2 \\ 1 & \text{else .} \end{cases}
\end{aligned}
$$

Consequently, the norm in the first part of (4.24), i.e., the l_2 matrix norm of the corresponding matrix with respect to the basis $\{\chi_l^N : l \in \mathbb{Z}^2\}$ is less than the norm of the matrix $E_N F_N$, where

$$
\begin{aligned}
F_N &= (f_i\delta_{ij}) \in \mathbb{Z}^2, \quad f_i = \begin{cases} 1 & \text{if } i \le \delta_1 n \\ 0 & \text{else ,} \end{cases}\\
E_N &= (e_{i,j})_{i,j\in\mathbb{Z}^2}, \quad e_{i,j} = e_{i-j} = \begin{cases} C|i-j|^{-2} & \text{if } |i-j| \ge (\delta_2-\delta_1)n \\ 0 & \text{else.} \end{cases}
\end{aligned}
$$

Applying E_N to a vector $\xi = (\xi_l)_{l\in\mathbb{Z}^2}$, we get from Young's inequality

$$
\left\|E_N\xi\right\|_{l_2} \le \sqrt{\sum_{|i|\ge(\delta_2-\delta_1)n} C^2|i|^{-4}}\cdot\|\xi\|_{l_1} \le C\Big[(\delta_2-\delta_1)n\Big]^{-1}\cdot\|\xi\|_{l_1}.
$$

Now we use the Cauchy-Schwarz inequality to get

$$
\begin{aligned}
\left\|E_N F_N\xi\right\|_{l_2} &\le C\Big[(\delta_2-\delta_1)n\Big]^{-1}\left\|F_N\xi\right\|_{l_1} \le C\Big[(\delta_2-\delta_1)n\Big]^{-1}\sum_{|i|\le\delta_1 n}|\xi_i|\\
&\le C\Big[(\delta_2-\delta_1)n\Big]^{-1}\sqrt{\sum_{|i|\le\delta_1 n} 1}\sqrt{\sum_{|i|\le\delta_1 n}|\xi_i|^2}\,.
\end{aligned}
$$

Consequently,

$$
\left\|E_N F_N\xi\right\|_{l_2} \le C\frac{\delta_1}{\delta_2-\delta_1}\|\xi\|_{l_2}, \quad \left\|E_N F_N\right\| \le C\frac{\delta_1}{\delta_2-\delta_1} \le C\frac{\delta_1}{\delta_2}.
$$

The second estimate of (4.24) follows analogously. $\diamond$

5 The Convergence of the Quadrature Method

This section is devoted to the convergence of the quadrature method. We shall show that the discretized operator A_N is uniformly bounded with respect to N. Using a Banach-Steinhaus argument, we shall prove the strong convergence of the discretized operator $A_N L_N$ to the singular integral operator A. This together with the stability implies the convergence of the quadrature method.

Theorem 5.1 *(cf. e.g. [28]) Suppose the quadrature method (2.5) applied to (2.1) is stable and that the discretized operator $A_N L_N$ corresponding to (2.5) converges strongly to the operator on the left-hand side of (2.1). Then the method (2.5) is convergent, i.e. for any right-hand side g such that*

$$\| \sum_{j_1,j_2=0}^{n-1} g(t_{j_1,j_2}^N)\chi_{j_1,j_2}^N - g\|_{L_2} \to 0,$$

the equation (2.5) has a unique solution u_N if N is sufficiently large, and u_N tends in L_2 to the exact solution u of (2.1).

Now let us turn to the boundedness of the discretized operator A_N defined in Section 3.3.

Lemma 5.1 *There exists a constant C independent of N and of the operator A defined on the left-hand side of (2.1) such that the L_2-operator norm of A_N (or equivalently the l_2-matrix norm of A_N) is bounded as*

$$\|A_N\| \le C\Big\{\|A\|_{\mathcal{L}(L_2(\mathbb{T}^2))} + \|a\|_{L_\infty(\mathbb{T}^2)} + \|f\|_{Lip} + \|k_R\|_{L_\infty(\mathbb{T}^2\times\mathbb{T}^2)}\Big\}.$$

Here the Lipschitz norm $\|f\|_{Lip}$ of the characteristic of kernel k_S is defined by

$$\|f\|_{Lip} = \|f\|_{L_\infty} + \sup_{\substack{x,\,x'\in\mathbb{T}^2\\ x'\neq x\\ \theta\in S^1}} \frac{|f(x,\theta)-f(x',\theta)|}{|x-x'|} + \sup_{\substack{\theta,\theta'\in S^1\\ \theta'\neq\theta\\ x\in\mathbb{T}^2}} \frac{|f(x,\theta)-f(x,\theta')|}{|\theta-\theta'|}.$$

Proof. Let us consider the Galerkin method where the trial space is spanned by the orthonormal basis $\{n\chi_k^N : k_1,k_2 = 0,\dots,n-1\}$. For the entries $a_{j,k}^G$ of the Galerkin matrix A_N^G we get

$$a_{j,k}^G = \langle A[n\chi_k^N],[n\chi_j^N]\rangle = \delta_{j,k}n^2\int_{(j-1)/n}^{j/n} a(x)dx \,+\, n^2\int_{(j-1)/n}^{j/n}\int_{(k-1)/n}^{k/n} k(x,x-y)dxdy.$$

We denote the corresponding entries of the matrix A_N for the quadrature method by $a_{j,k}$. Since

$$\|A_N^G\| = \|L_N A|_{imL_N}\| \le \|A\|,$$

we only have to show

$$\Big\|(a_{j,k}-a_{j,k}^G)_{j,k}\Big\|_{\mathcal{L}(l^2)} \le C\Big\{\|f\|_{Lip} + \|a\|_{L_\infty} + \|k_R\|_{L_\infty}\Big\} \tag{5.1}$$

Moreover, since the boundedness proofs for the multiplication operator and for the integral operator with bounded kernel function k_R are straight forward, we suppose $a\equiv 0$ and $k_R\equiv 0$. We shall estimate $(a_{j,k}-a_{j,k}^G)_{j,k}$ in two steps. First we shall derive a bound for the matrix with all entries corresponding to the indices i,j such that $|i-j|>2$ ("off diagonal" entries) and later we consider the matrix with the entries such that $|i-j|\le 2$ ("almost diagonal" entries). Let us estimate the "off diagonal" entries.

$$\Big|a_{j,k} - a^G_{j,k}\Big| = \Big|k\Big(\frac{j}{n}, \frac{j}{n} - \frac{k}{n}\Big)\frac{1}{n^2} - n^2\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{k}{n}}^{\frac{k+1}{n}} k(x, x-y)dxdy\Big|$$
$$= \Big|\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{k}{n}}^{\frac{k+1}{n}} n^2\Big[k\Big(\frac{j}{n}, \frac{j}{n} - \frac{k}{n}\Big) - k(x, x-y)\Big]dxdy\Big|.$$

If we put $x = \frac{j}{n} + \theta$, $y = \frac{k}{n} + \nu$, $\theta = (\theta_1, \theta_2)$, and $\nu = (\nu_1, \nu_2)$, then

$$\Big|a_{j,k} - a^G_{j,k}\Big| = \Big|\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} n^2\Big[k\Big(\frac{j}{n}, \frac{j}{n} - \frac{k}{n}\Big) - k\Big(\frac{j}{n} + \theta, \frac{j}{n} + \theta - (\frac{k}{n} + \nu)\Big)\Big]d\theta d\nu\Big|.$$

Putting $l = j - k$, and $\lambda = \theta - \nu$ we get

$$\begin{aligned}
\Big|a_{j,k} - a^G_{j,k}\Big| &= \Big|\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\int_{\theta_2 - \frac{1}{n}}^{\theta_2}\int_{\theta_1 - \frac{1}{n}}^{\theta_1} n^2\Big[k\Big(\frac{l+k}{n}, \frac{l}{n}\Big) \\
&\qquad - k\Big(\frac{l+k}{n} + \theta + \lambda, \frac{l}{n} + \lambda\Big)\Big]d\lambda d\theta\Big| \\
&\leq \int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\int_{\theta_2 - \frac{1}{n}}^{\theta_2}\int_{\theta_1 - \frac{1}{n}}^{\theta_1} n^2\Big[T_1 + T_2\Big]d\lambda d\theta, \qquad (5.2) \\
T_1 &= \Big|k\Big(\frac{l+k}{n} + \theta + \lambda, \frac{l}{n}\Big) - k\Big(\frac{l+k}{n} + \theta + \lambda, \frac{l}{n} + \lambda\Big)\Big|, \\
T_2 &= \Big|k\Big(\frac{l+k}{n}, \frac{l}{n}\Big) - k\Big(\frac{l+k}{n} + \theta + \lambda, \frac{l}{n}\Big)\Big|.
\end{aligned}$$

For T_1 we get

$$\begin{aligned}
&\Big|k\Big(\frac{l+k}{n} + \theta + \lambda, \frac{l}{n}\Big) - k\Big(\frac{l+k}{n} + \theta + \lambda, \frac{l}{n} + \lambda\Big)\Big| \\
&\quad\leq \Big|f\Big(\frac{l+k}{n} + \lambda + \theta, \frac{l}{n}\Big)\frac{1}{|l/n|^2} - f\Big(\frac{l+k}{n} + \lambda + \theta, \frac{l}{n} + \lambda\Big)\frac{1}{|l/n + \lambda|^2}\Big| \\
&\quad\leq \Big|f\Big(\frac{l+k}{n} + \lambda + \theta, \frac{l}{n} + \lambda\Big)\Big|\Big|\frac{1}{|l/n|^2} - \frac{1}{|l/n + \lambda|^2}\Big| \\
&\qquad + \frac{1}{|l/n|^2}\cdot\Big|f\Big(\frac{l+k}{n} + \theta + \lambda, \frac{l}{n}\Big) - f\Big(\frac{l+k}{n} + \theta + \lambda, \frac{l}{n} + \lambda\Big)\Big|. \qquad (5.3)
\end{aligned}$$

The function f is bounded and

$$\Big|\frac{1}{|l/n|^2} - \frac{1}{|l/n + \lambda|^2}\Big| \leq C\frac{n^2}{|l|^3}. \qquad (5.4)$$

Since f satisfies a Lipschitz condition with respect to the second variable, we find for the second term on the right-hand side of (5.3)

$$\Big|f\Big(\frac{l+k}{n} + \lambda + \theta, \frac{l}{n}\Big) - f\Big(\frac{l+k}{n} + \lambda + \theta, \frac{l}{n} + \lambda\Big)\Big| \leq C\Big|\frac{l/n}{|l/n|} - \frac{l/n + \lambda}{|l/n + \lambda|}\Big| \leq \frac{C}{|l|}. \quad (5.5)$$

Substitution of (5.4) and (5.5) into (5.3) provides us with $|T_1| \le Cn^2|l|^{-3}$. For T_2, we arrive at

$$\begin{aligned}
\Big|k\Big(\frac{l+k}{n},\frac{l}{n}\Big) - k\Big(\frac{l+k}{n}+\theta+\lambda,\frac{l}{n}\Big)\Big| &= \frac{1}{|l/n|^2}\Big|f\Big(\frac{l+k}{n},\frac{l}{n}\Big) - f\Big(\frac{l+k}{n}+\theta+\lambda,\frac{l}{n}\Big)\Big| \\
&\le C\frac{1}{n}\frac{1}{|l/n|^2} \le C\frac{n^2}{|l|^3}. \qquad (5.6)
\end{aligned}$$

Substituting the estimates for T_1 and T_2 into (5.2), we obtain

$$\Big|a_{j,k} - a^G_{j,k}\Big| \le C\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\int_{\theta_1-\frac{1}{n}}^{\theta_1}\int_{\theta_2-\frac{1}{n}}^{\theta_1}\frac{n^4}{|l|^3}d\theta d\lambda \le C\frac{1}{|l|^3}.$$

Young's inequality implies for the "off diagonal" part of $A_N - A_N^G$ that

$$\Big\|(a_{j,k}-a^G_{j,k})_{j,k}\Big\|_{\mathcal{L}(l_2(\mathbb{Z}^2))} \le C\sum_{l\in\mathbb{Z}^2}\frac{1}{\{1+|l|\}^3} \le C.$$

On the other hand, let us turn to the "almost diagonal" entries. For the Galerkin matrix A_N^G we conclude

$$|a^G_{j,k}| \le \Big\|(a^G_{j,k})_{j,k}\Big\|_{\mathcal{L}(l_2(\mathbb{Z}^2))} = \|A_N^G\| = \|L_nAL_n\| \le C\|A\| \le C.$$

For the "almost diagonal" entries of the quadrature method, we get

$$a_{j,k} = \begin{cases} \frac{1}{n^2}k_S\Big(\frac{j}{n},\frac{j}{n}-\frac{k}{n}\Big) & \text{if } |j-k|>0 \\ 0 & \text{if } j=k. \end{cases}$$

If $l = j-k \neq 0$, then we obtain

$$a_{j,k} = \frac{1}{n^2}f(\frac{j}{n},\frac{l}{n})\left|\frac{l}{n}\right|^{-2}, \quad |a_{j,k}| \le \left|f(\frac{j}{n},\frac{l}{n})\right| \le C.$$

Hence, each "almost diagonal" entry $[a_{j,k} - a^G_{j,k}]$ is bounded. Consequently, the "almost diagonal" part of $A_N - A_N^G$ is bounded, too. $\diamond$

Lemma 5.2 *Suppose that the operator A given by the left-hand side of (2.1) is a pseudo-differential operator of order zero with a symbol from S^0. Moreover, let A_N stand for the discretized quadrature operator of (2.5). We suppose that (2.9) is satisfied. Then A_NL_Nu tends to u in the L_2-norm for any $u \in L_2(\mathbb{T}^2)$.*

Proof. In Lemma 5.1 we have shown that A_N is uniformly bounded. Hence, in view of the Banach-Steinhaus theorem, we may suppose that f is smooth and have to prove $\|A_NL_Nf - Af\| \to 0$ for any smooth f. Since f is smooth, K_NAf tends to Af if K_N is the piecewise linear interpolation projector. It remains to prove $\|A_NL_Nf - K_NAf\| \to 0$. Moreover, since $K_Nf \to f$ and since A_N is bounded, we conclude $A_NL_Nf - A_NK_Nf \to 0$. It remains to prove $\|A_NK_Nf - K_NAf\| \to 0$. This, however, is a consequence of $\|A_NK_Nf - K_NAf\|_{L_\infty} \to 0$ which is equivalent to

$$\sup_i\Big|A_NK_Nf(t_i^N) - Af(t_i^N)\Big| \to 0. \qquad (5.7)$$

Now we study the difference $Af(t_i^N) - A_NK_Nf(t_i^N)$ in three cases :

1) A is a multiplication operator

2) A is an integral operator with a continuous and bounded kernel k_R

3) A is the singular integral operator with kernel k_S.

Case 1) is very simple since $Af(t_i^N) = A_N K_N f(t_i^N)$ holds for multiplication operators. The assertion for Case 2) is well known, too. Indeed, the quadrature rule used for $A_N K_N f(t_i^N)$ has non-negative quadrature weights. Hence, it converges on continuous functions and even uniformly over the compact set of functions $y \mapsto k_R(x,y)f(y)$. It remains to consider Case 3).

The difference $Af(t_i^N) - A_N K_N f(t_i^N)$ takes the form (cf. Section 2)

$$\begin{aligned} Af(t_i^N) - A_N K_N f(t_i^N) &= \int_{\mathbb{T}^2} k_S(t_i^N, t_i^N - y)[f(y) - f(t_i^N)] d_y \mathbb{T}^2 && (5.8) \\ &\quad - \sum_{l:\, l \neq i} k_S(t_i^N, t_i^N - t_l^N)[f(t_l^N) - f(t_i^N)] \frac{1}{n^2} = T^1 + T^2, \\ T^1 &= \int_{\mathbb{T}^2 \setminus B(t_i^N, \epsilon)} k_S(t_i^N, t_i^N - y)[f(y) - f(t_i^N)] d_y \mathbb{T}^2 && (5.9) \\ &\quad - \sum_{l:\, |t_i^N - t_l^N| > \epsilon} k_S(t_i^N, t_i^N - t_l^N)[f(t_l^N) - f(t_i^N)] \frac{1}{n^2}, \\ T^2 &= \int_{B(t_i^N, \epsilon)} k_S(t_i^N, t_i^N - y)[f(y) - f(t_i^N)] d_y \mathbb{T}^2 && (5.10) \\ &\quad - \sum_{l:\, l \neq i,\, |t_i^N - t_l^N| \leq \epsilon} k_S(t_i^N, t_i^N - t_l^N)[f(t_l^N) - f(t_i^N)] \frac{1}{n^2}, \end{aligned}$$

where the number ϵ stands for a fixed positive real, and $B(t_i^N, \epsilon) \subseteq \mathbb{T}^2$ is the ball with center t_i^N and radius ϵ. In a minute we will prove that $T^2 \to 0$ for $\epsilon \to 0$. On the other hand, the integral in T^1 is regular for fixed $\epsilon > 0$. Thus the same arguments as for Case 2) imply $T^1 \to 0$ for $N \to \infty$. We conclude $T^1, T^2 \to 0$ for $N \to 0$, and, using (5.8), we get (5.7). It remains to show $T^2 \to 0$ for $\epsilon \to 0$.

We estimate the two terms in (5.9) separately. For the integral, we get

$$\Big| \int_{B(t_i^N, \epsilon)} k_S(t_i^N, t_i^N - y)[f(y) - f(t_i^N)] d_y \mathbb{T}^2 \Big| \leq \int_{B(t_i^N, \epsilon)} C |t_i^N - y|^{-1} d_y \mathbb{T}^2 \leq C\epsilon.$$

The quadrature sum can be estimated as

$$\left| \sum_{l:\, l \neq i,\, |t_i^N - t_l^N| \leq \epsilon} k_S(t_i^N, t_i^N - t_l^N)[f(t_l^N) - f(t_i^N)] \frac{1}{n^2} \right| \leq \frac{1}{n} \sum_{l:\, l \neq i,\, |l - i| \leq \epsilon n} |i - l|^{-1} \leq C\epsilon.$$

Hence $|T^2| \leq C\epsilon$ and $T^2 \to 0$ for $\epsilon \to 0$ is proved. $\diamond$

6 Numerical Tests

In order to check the convergence properties of our quadrature method, we consider the following oblique derivative problem. We define the two-dimensional surface S by the parametrization

$$
\begin{aligned}
S &= \{\phi(s,t), 0 \le s,t \le 1\}, \qquad (6.1)\\
\phi(s,t) &= \Big([2+\cos(2\pi s)]\cos(2\pi t)\ ,\ \sin(2\pi s)\ ,\ [2+\cos(2\pi s)]\sin(2\pi t) \Big).
\end{aligned}
$$

Clearly, S is homeomorphic to the torus. The space $\mathbf{R}^2 \backslash S$ is the union of the bounded ring shaped domain Ω_- and the unbounded exterior domain Ω. For this domain Ω, we solve the oblique derivative boundary value problem (cf. [18])

$$\triangle V = 0 \quad in \quad \Omega, \qquad (6.2)$$

$$\frac{\partial}{\partial f} V = g \ \ on \ \ S = \partial\Omega, \quad f:\ S = \partial\Omega \ \to \ \mathbf{R}^3. \qquad (6.3)$$

The oblique direction vector $f(P)$ is defined as

$$f(P) = n(P) + \frac{1}{2}(0,0,1), \qquad (6.4)$$

where $n(P)$ is the normal vector of unit length at $P \in S$ pointing into Ω_-. We represent the unknown potential V in the form of a Newton potential

$$Vx(P) = \frac{1}{4\pi} \int_S \frac{x(Q)}{|P-Q|}\, d_Q S, \qquad (6.5)$$

where $x(Q)$ denotes an unknown single layer surface density. We apply the boundary operator of oblique derivative, and, with the well-known jump relations for the Newton potential, we obtain the boundary integral equation

$$
\begin{aligned}
g(P) &= \frac{\partial}{\partial f(P)} (Vx)(P)\\
&= -\frac{1}{2}\Big\langle f(P), n(P)\Big\rangle\, x(P) - \frac{1}{4\pi} \int_S \frac{f(P)\cdot(Q-P)}{|P-Q|^3}\, x(Q)\, d_Q S\ . \qquad (6.6)
\end{aligned}
$$

This is a strongly singular integral equation of the second kind for the unknown function $x(Q)$. Using the parametrization ϕ, we transform (6.6) into (1.4), where the kernel takes the form

$$k(t,s) = \frac{f\,(\phi(t))\cdot(\phi(s)-\phi(t))}{|\phi(s)-\phi(t)|^3}\, |\phi'(s)|, \qquad (6.7)$$

and where

$$|\phi'(s)| = |\partial_{s_1}\phi(s) \ \times\ \partial_{s_2}\phi(s)|$$

is the density of the surface measure. Note that operator A is strongly elliptic since $\langle f, n\rangle > 0$. Moreover, the singular part k_S of the kernel is a Mikhlin-Giraud kernel, i.e., it satisfies (2.9). This equation (1.4) is solved numerically by the quadrature method (2.5).

Before we solve the linear equations, we check whether the quadrature approximation of the singular integral operator converges. For this purpose, we consider the singular integral

$$
\begin{aligned}
v(P) &= -\frac{1}{2}\Big\langle f(P), n(P)\Big\rangle w(P) + \frac{1}{4\pi}\int_S \frac{\langle f(P), P-Q\rangle}{|P-Q|^3} w(Q) d_Q S,\\
w\,(\phi(t)) &= \sin(2\pi t_1)\sin(2\pi t_2), \quad t=(t_1,t_2)\in[0,1]^2
\end{aligned}
$$

together with its approximation v_N given at the grid points $t_j = t_{j_1,j_2}$ by

$$v_N\left(\phi(t_j)\right) := \sum_{k_1,k_2=0}^{n-1} a_{j,k} w\left(\phi(t_k)\right),$$

where $A_N = (a_{j,k})_{j,k}$ is the matrix of the quadrature method. For several $n = n_l = 2^l$ and $N_l = n_l^2$, we compute the L_2-Norm error

$$\|v_{N_l} - v_{N_{l+1}}\| := \frac{1}{n_l}\sqrt{\sum_{j_1,j_2=0}^{n_l-1} |v_{N_l}(\phi(t_j)) - v_{N_{l+1}}(\phi(t_j))|^2}$$

and the approximate convergence order

$$\alpha_{N_l} := \frac{\log\|v_{N_l} - v_{N_{l+1}}\| - \log\|v_{N_{l-1}} - v_{N_l}\|}{\log 2}.$$

The results are presented in Table 1. It turns out that the approximate operator A_N converges with order 1.

n_l	Degrees of Freedom: N_l	$\|v_{N_l} - v_{N_{l+1}}\|$	α_{N_l}
4	16	$5.35 \cdot 10^{-2}$	
8	64	$2.00 \cdot 10^{-2}$	1.42
16	256	$8.48 \cdot 10^{-3}$	1.24
32	1024	$4.16 \cdot 10^{-3}$	1.03
64	4096	$2.08 \cdot 10^{-3}$	1.00
128	16384	$1.04 \cdot 10^{-3}$	1.00

Table 1: Approximation order of the quadratures

The discretized operators are stable by the Theorems 3.1 and 4.1. Stability means that the matrices A_N together with their inverses A_N^{-1} are uniformly bounded with respect to N. Though we have not computed the Euclidean matrix norms of A_N and A_N^{-1}, we have an indicator for the uniform boundedness. Normally, for bounded norms $\|A_N\|$ and $\|A_N^{-1}\|$, the iterative solution of the matrix equation requires a number of iteration steps which is bounded independently of N. In Table 2 we present the number of GMRES iterations (cf. [30]) necessary to achieve an error less than 10^{-12}. Indeed, these numbers seem to grow very slowly.

n_l	N_l	Number of GMRES iterations
2	4	4
4	16	12
8	64	22
16	256	25
32	1024	28
64	4096	32

Table 2: Numbers of GMRES iterations

Next we compute an approximate solution from solving (2.5). After determining the solution u_N of the quadrature method at the grid points t_{j_1,j_2}, $j_1, j_2 = 0, \ldots, n-1$, we compute an approximate solution U_N for the Laplace equation by discretizing the single layer representation (6.5).

$$U(x) \approx U_N(x) := \frac{1}{4\pi}\frac{1}{n^2}\sum_{j_1,j_2=0}^{n-1}\frac{u_N(\phi(t_{j_1,j_2}))}{|\phi(t_{j_1,j_2})-x|}|\phi'(t_{j_1,j_2})|. \tag{6.8}$$

In our first example, we take a known solution of (6.2), (6.3) given by

$$U(P) = |P-(2,0,0)|^{-1}. \tag{6.9}$$

The oblique derivative is given by

$$g(P) = \frac{\partial}{\partial f}U(P) = \frac{f(P)\cdot((2,0,0)-P)}{|(2,0,0)-P|^3}. \tag{6.10}$$

For this right-hand side g, we have solved the quadrature equations (2.5) and computed the L_2 errors

$$\|u_{N_l}-u_{N_{l+1}}\| := \frac{1}{n_l}\sqrt{\sum_{j_1,j_2=0}^{n_l-1}|u_{N_l}(\phi(t_j))-u_{N_{l+1}}(\phi(t_j))|^2}$$

and the approximate convergence orders

$$\beta_{N_l} := \frac{\log\|u_{N_l}-u_{N_{l+1}}\|-\log\|u_{N_{l-1}}-u_{N_l}\|}{\log 2}.$$

Moreover, we have computed the approximate values $U_N(P)$ for $P=(1,0,0)$ and $P=(0.3,0.2,0.1)$, the relative errors $|U_N(P)-U(P)|/|U(P)|$ with $U(P)$ from (6.9), and the approximate convergence orders

$$\gamma_{N_l} := \frac{\log|U_{N_l}(P)-U(P)|-\log|U_{N_{l-1}}(P)-U(P)|}{\log 2}.$$

The numerical results are presented in the Table 3. They show that our quadrature solutions converge to the exact solutions. The convergence orders are close to one.

n_l	N_l	$\|u_{N_l}-u_{N_{l+1}}\|$	β_{N_l}	$\frac{\|U_{N_l}(P)-U(P)\|}{\|U(P)\|}$ $P=(1,0,0)$	γ_{N_l}	$\frac{\|U_{N_l}(P)-U(P)\|}{\|U(P)\|}$ $P=(0.3,0.2,0.1)$	γ_{N_l}
2	4			1.49		1.95	
4	16	0.87		0.0032	8.88	0.79	1.32
8	64	0.13	2.69	0.22	-6.12	0.13	2.62
16	256	0.04	1.76	0.16	0.49	0.028	2.24
32	1024	0.019	1.09	0.08	0.98	0.013	1.12
64	4096	0.01	0.81	0.04	1.00	0.0063	1.00

Table 3: Convergence of the quadrature method for $g(Q) = \frac{\partial}{\partial f(Q)}|Q-(2,0,0)|^{-1}$

In a second example we consider an oblique derivative g for which the exact solution is unknown. Since our quadrature method is a low order method, we choose g with a low degree of smoothness. In particular, we have taken

$$g_1(\phi(s,t)) = \begin{cases} 1 & \text{if } s<\frac{1}{2} \\ 0 & \text{else}, \end{cases}$$

Note that $g_1 \in H^\sigma(s)$ for $\sigma < 1$. Instead of the error $|U_{N_l}(P) - U(P)|/|U(P)|$ we now compute the error $|U_{N_l}(P) - U_{N_{l+1}}(P)|$ and the corresponding convergence rates

$$\delta_{N_l} = \frac{\log|U_{N_l}(P) - U_{N_{l+1}}(P)| - \log|U_{N_{l-1}}(P) - U_{N_l}(P)|}{\log 2}.$$

The numerical results are presented in Table 4. They show that our quadrature method converges with order one even for solutions with low degree of smoothness.

n_l	N_l	$\lvert U_{N_l}(P) - U_{N_{l+1}}(P)\rvert$	δ_{N_l}
2	4		
4	16	0.53	
8	64	0.59	-0.14
16	256	0.31	0.93
32	1024	0.13	1.35
64	4096	0.049	1.30

Table 4: Convergence of the quadrature method for g_1 and $\epsilon = 0.5$ at P=(1,0,0)

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