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Abstract

This paper deals with controlled polyhedral sweeping processes under uncertainty and involving a terminal point constraint for the solution trajectory. More precisely, a cost-optimal (nominal) control is to be determined such that even after random perturbations of this control the terminal point of the resulting random trajectory falls into some target set with high probability. This leads to the well-known concept of chance constraints which have recently been successfully applied in the optimal control of partial or ordinary differential equations. Given the dynamics of sweeping processes – which are specific differential inclusions with unbounded and discontinuous right-hand side – a straightforward translation of these previous results is not possible. The focus of the paper lies on modeling of well-posed controlled polyhedral sweeping processes with randomly perturbed control and, as a main result, on the derivation of the existence of solutions to three types of such processes.

This work is deticated to the memory of Thomas Seidman.

1 Introduction

A sweeping process is a differential inclusion of the following type

$$-\dot{x}(t) \in N_{C(t)}(x(t)) \quad \text{a.e. } t \in [0,T]; \quad x(0) = x^0 \in C(0).$$

Here, $[0,T] \subseteq \mathbb{R}$ is a given time interval, $C : [0,T] \rightrightarrows \mathcal{H}$ is a convex-valued multifunction into some separable Hilbert space $\mathcal{H}$, $x^0 \in \mathcal{H}$ is a given initial point and N_A is the normal cone of convex analysis to a convex set $A \subseteq \mathcal{H}$:

$$N_A(x) := \left\{ w \in \mathcal{H} \;\middle|\; \forall y \in A : \; \langle w, y-x \rangle \leq 0 \quad \right\} \text{ if } x \in A \text{ and}$$
$$N_A(x) := \emptyset \text{ otherwise.}$$

Sweeping processes have been introduced by J.J. Moreau in [40, 41]. They have found numerous applications in problems of elasto-plasticity, robotics, traffic flow modeling, crowd motion control or electrical circuits. For introductions to the topic, we refer to [1, 37]. Traditionally, the moving set C has been given from the very beginning and the analysis focused on conditions for the moving set such that the existence and uniqueness of solutions to the resulting sweeping process could be guaranteed. First ideas to control sweeping processes considered additive perturbations of the normal cone to the moving set (e.g., [20]). The first paper to address the optimal control (position and shape) of a moving set itself in a sweeping process was [17]. A simpler special case arises, if the control consists in a translation of some fixed set, i.e., $C_u(t) = u(t) + A$. This model is closely related with the so-called

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'play operator' (e.g., [9, 34, 49]) which figures prominently in the modeling of hysteresis phenomena by so called 'hysteresis operators'. For instance, the control of the vectorial variant of the play operator is investigated in [8, 26, 44]. Considering a general formulation of the corresponding definition, it is shown in [43] that appropriate sweeping processes generate hysteresis operators. The second author would like to point out the interesting and fruitful joined work with Thomas Seidman on the paper [46] devoted to the analysis of models involving hysteresis operators.

As far as the control of moving sets also changing their shapes are concerned, most efforts so far have been spent to the polyhedral case as in (1) below ([17, 15, 16, 28, 47]). For the non-convex setting, we refer to, e.g., [2, 42]. The focus of these investigations has been on the existence of solutions (both to the sweeping process itself as well as to the optimal control problem), on the derivation of necessary optimality conditions (via discrete approximation [17, 16, 28]) or the formulation of a maximum principle [4, 14, 19]), and on algorithmic solution approaches. Terminal point constraints, i.e., conditions on the solution trajectory at the final time, are considered, for instance, in [19, 28, 42]. The optimal control of sweeping processes has numerous important applications, for example in elasto-plasticity [25], crowd motion dynamics [12], robotics [18], hysteresis [11], nano particle modelling [39], traffic flow models [18], obstacle problems [12, 39], or electrical circuits [7]. Important extensions of problem (1) concern the additional coupling of the sweeping process with an ODE [14] or the consideration of so-called implicit sweeping processes [3]. For a survey of some recent developments and applications of controlled sweeping processes including a comprehensive list of references, we refer to [38].

The consideration of stochastic versions of the (uncontrolled) sweeping process began almost in parallel with the deterministic model [13]. In a typical setting, the right-hand side of the sweeping process is perturbed by a stochastic process (e.g., reflected Brownian motion) [6, 5]. In the present paper, we want to make an attempt towards considering uncertainty within the optimal control of sweeping process much in a sense as in the risk-averse optimal control of uncertain PDEs [33]. More precisely, in a sweeping process where the moving set is controlled and additionally impacted by some random process, we want to minimize the cost of the control under the condition that the risk of the solution trajectory not reaching a given target at terminal time is kept small. Risk may be measured in different ways, for instance by using the conditional value at risk [32]. In engineering problems the use of *probabilistic* or *chance constraints* is very popular because the interpretation of risk in terms of probability arises in a very natural way. On the other hand, the use of chance constraints turns out to be more difficult than other risk measures. As examples for the successful use of chance constraints in the context of the optimal control of PDEs under uncertainty, we refer to [21, 23, 24, 45, 31, 22, 27, 29].

The paper is organized as follows: Section 2 develops a model for the optimal control of two polyhedral sweeping processes with randomly perturbed control under a probabilistic constraint on the terminal point of the trajectory. In Section 3, an existence result for this kind of control problems is derived. A numerical illustration for the presented models is provided in Section 4.

2 Model for the optimal control of special polyhedral sweeping processes with randomly perturbed control

The aim of this section is to present optimal control problems involving simple polyhedral sweeping processes whose nominal control (the one subject to optimization) is affected by posterior random perturbation. Accordingly, the solutions of the sweeping processes become random too. The goal is to find a cost-optimal (nominal) control such that the (random) terminal points of the solutions to the sweeping processes belong to some fixed target set with some minimum probability.

2.1 Existence of solutions to polyhedral sweeping processes

The sweeping process considered in [17] and in many subsequent papers was a polyhedral one, which means that the controlled moving set C is described by finitely many inequalities

$$\left(\mathcal{S}_{(v,b)}\right): \quad -\dot{x}(t) \in N_{C_{(v,b)}(t)}\left(x(t)\right) \quad \text{a.e. } t \in [0,T], \quad x(0) = x^0 \in C_{(v,b)}(0), \tag{1}$$

Here, the control is given by $u = (v,b) : [0,T] \to \mathcal{H}^m \times \mathbb{R}^m$, with the controlled set

$$C_{(v,b)}(t) := \{x \in \mathcal{H} \mid \langle v_i(t), x\rangle \leq b_i(t) \ i = 1,\ldots,m\}.$$

Note, that the inclusion $-\dot{x}(t) \in N_{C_{(v,b)}(t)}(x(t))$ in (1) induces the hidden constraint $x(t) \in C_{(v,b)}(t)$ because otherwise the normal cone $N_{C_{(v,b)}(t)}(x(t))$ is empty by definition.

We cite the following existence and uniqueness result [28, Theorem 3] relating to the sweeping process (1):

Theorem 2.1. *In* (1)*, let* $u = (v,b) \in W^{1,1}([0,T],\mathcal{H}^m \times \mathbb{R}^m)$ *be an absolutely continuous control satisfying the initial condition* $x^0 \in C_{(v,b)}(0)$*. Assume further that*

$$\forall t \in [0,T] \ \exists y \in \mathcal{H} : \langle v_i(t), y\rangle < b_i(t) \quad i = 1,\ldots,m \tag{2}$$

is satisfied, then, the sweeping process $(\mathcal{S}_{(v,b)})$ *admits a unique absolutely continuous solution* x*.*

The following simple lemma provides an equivalent condition for (2) in the special case of $m = 1$:

Lemma 2.2. *If* $m = 1$*, then* (2) *is equivalent with the condition*

$$\forall t \in [0,T]: \quad v(t) \neq 0 \quad \textit{or} \quad b(t) > 0. \tag{3}$$

Proof. If (2) holds true, then for each $t \in [0,T]$ the assumption $v(t) = 0$ implies that $b(t) > 0$ which is (3). Conversely, fix an arbitrary $t \in [0,T]$ and assume first that $v(t) \neq 0$. Define $y := \frac{b(t)-1}{\|v(t)\|^2} v(t)$ and verify that (2) holds true. If $b(t) > 0$ and $v(t) = 0$, then (2) holds trivially true (with arbitrary y). □

In the following we will consider stochastic polyhedral sweeping processes in finite dimensions and will put the focus on two special cases: as a first instance, we consider polyhedral sweeping processes with $m = 1$, i.e., the (deterministic) controlled moving set takes the form

$$C_{(v,b)}(t) := \{x \in \mathbb{R}^n \mid \langle v(t), x\rangle \leq b(t)\}.$$

Note that, in the typical situation of $v(t) \neq 0$, the moving set $C_{(v,b)}(t)$ represents some closed half space, whereas for $v(t) = 0$ it is the empty set if $b(t) < 0$ (in which case the corresponding sweeping process cannot have a solution) and it is the whole space $\mathbb{R}^n$ if $b(t) \geq 0$. The second instance will be given by the controlled translation of a rigid polyhedron $P \subseteq \mathbb{R}^n$: $\{u(t)\} + P$. If

$$P := \{x \in \mathbb{R}^n \mid \langle a_i, x\rangle \leq c_i \quad (i = 1,\ldots,m)\} \tag{4}$$

is a description of P by means of finitely many linear inequalities (and such description always exists), then it is easily seen that the moving rigid polyhedron can be described in the form of the general moving polyhedron above as follows:

$$\{u(t)\} + P = C_{(v,b)}(t) \quad \text{with} \quad v_i(t) := a_i, \ b_i(t) := c_i + \langle a_i, u(t)\rangle \tag{5}$$

for all $i = 1,\ldots,m$ and $t \in [0,T]$.

Thanks to Remark 2.6 below, the solution to the sweeping processes derived by considering the translation of a rigid polyhedron can be identified with the output of appropriate PLAY–*operators*, belonging to the class of hysteresis operators as considered, e.g., in [34, 49].

2.2 Special polyhedral sweeping processes with parametrically perturbed control

In the following, we consider two special sweeping processes in finite dimensions:

$$(\widetilde{\mathcal{S}}_{(v,b,z)}) : \begin{cases} -\dot{x}_z(t) \in N_{\widetilde{C}_{(v,b,z)}(t)}\,(x_z(t)) \quad \text{a.e. } \ t \in [0,T]\,; \\ x_z(0) = x^0 \in \widetilde{C}_{(v,b,z)}(0) \\ \widetilde{C}_{(v,b,z)}(t) := \{x \in \mathbb{R}^n \mid \langle v(t) + p(z,t), x\rangle \le b(t)\} \ \forall t \in [0,T]. \end{cases} \tag{6}$$

$$(\overline{\mathcal{S}}_{(u,z)}) : \begin{cases} -\dot{x}_z(t) \in N_{\overline{C}_{(u,z)}(t)}\,(x_z(t)) \qquad \text{a.e. } \ t \in [0,T]\,; \\ x_z(0) = x^0 \in \overline{C}_{(u,z)}(0) \\ \overline{C}_{(u,z)}(t) := \{u(t) + p(z,t)\} + P \quad \forall t \in [0,T]. \end{cases} \tag{7}$$

Evidently, the moving set in $(\widetilde{\mathcal{S}}_{(v,b,z)})$ is a closed half space in $\mathbb{R}^n$ (which could degenerate to the empty set or the whole space), whereas the moving set in $(\overline{\mathcal{S}}_{(u,z)})$ is the translation of a rigid polyhedron P defined by (4). In these processes, $(v,b) \in W^{1,1}([0,T],\mathbb{R}^{n+1})$ and $u \in W^{1,1}([0,T],\mathbb{R}^n)$, respectively, refer to some nominal controls of the polyhedra which will be subjected later on to optimization. In both processes, $z \in \mathbb{R}^k$ plays the role of an external parameter (later on interpreted as the realization of some random vector) determining a perturbation p of the nominal control. For this perturbation we shall assume the following structure:

$$p(z,t) := \sum_{i=1}^{k} z_i \alpha_i(t); \qquad (t \in [0,T];\ z \in \mathbb{R}^k), \tag{8}$$

where the α_i are given functions. Throughout the paper, we shall make the following assumption on these functions:

$$\alpha_i \in W^{1,1}([0,T],\mathbb{R}^n), \qquad \alpha_i(0) = 0 \qquad (i = 1,\dots,k). \tag{A1}$$

Remark 2.3. We point out, that here and in the rest of the paper, all pointwise relations on functions in $W^{1,1}([0,T])$ (as the one in (**A1**)) are to be understood with respect to their corresponding continuous representatives.

Remark 2.4. The perturbation in the process $\widetilde{S}_{(v,b,z)}$ affects the normal direction v only. One could add a perturbation term to the right-hand side control b as well and basically recover all of the subsequent results. This would require, however, an additional technical and notational effort which we prefer to avoid here for the sake of simplicity. Anyhow, if one would add a perturbation of b as well, one would need to request in (10) that there is also some lower bound m' for this perturbation such that $0 > m' > -m$ to ensure that the sum of b and its perturbation is bounded from below by $m - m' > 0$ such that one could adapt the proofs.

Figure 1 illustrates, as an instance of $(\overline{\mathcal{S}}_{(u,z)})$, the solution trajectory $x(t)$ (blue) for the control of a rigid square. The left part of the figure refers to a nominal (unperturbed) control of a square u pulling a particle (black dot) from an initial position x^0 (red) to some terminal position $x(T)$ (green). Here, the

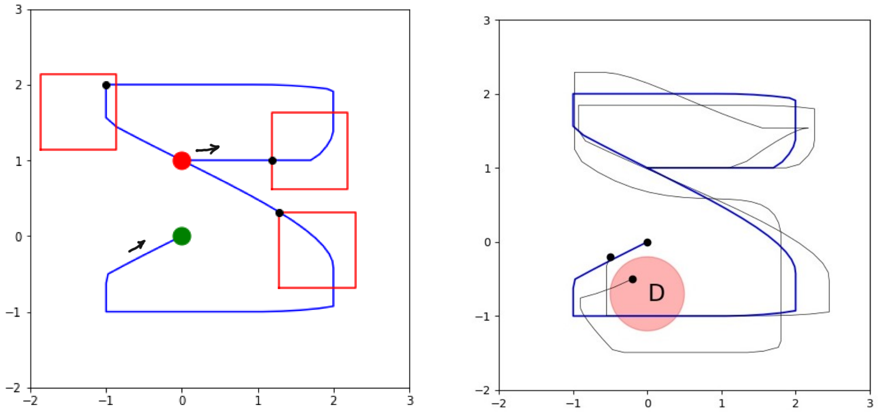

Figure 1: Illustration of the process $(\overline{\mathcal{S}}_{(u,z)})$ arising from the control of a rigid square. The effect of some nominal control is visualized in the left part of the figure, whereas the right part shows the solution trajectories under the nominal and two perturbed controls. For details, see text.

perturbation parameter z equals zero. The figure emphasizes three different situations at which the normal cone to the square at the position $x(t)$ of the particle is zero (particle at rest in the interior of the square), or one-dimensional (particle pulled along one edge of the square) or two-dimensional (particle pulled along a corner of the square). In the right part of the figure the trajectory of the particle under the nominal control (blue) is juxtaposed with the corresponding trajectories of two perturbed controls where $z \neq 0$ (black). As to be expected, with the same initial point given, the trajectories under the perturbed controls end up at different terminal points which influences the capacity of reaching some target set D. We shall get back later to this aspect.

From (6), (7), (8) and (**A1**) it follows immediately that

$$\widetilde{C}_{(v,b,z)}(0) = \{x \in \mathbb{R}^n \mid \langle v(0), x\rangle \leq b(0)\}; \quad \overline{C}_{(u,z)}(0) = \{u(0)\} + P \quad \forall z \in \mathbb{R}^k. \tag{9}$$

Lemma 2.5. *The sweeping process $\widetilde{\mathcal{S}}_{(v,b,z)}$ admits a unique, absolutely continuous solution for arbitrary $(v, b, z) \in W^{1,1}([0, T], \mathbb{R}^{n+1}) \times Z$ under the following alternative conditions:*

$$Z = \mathbb{R}^k, \quad b(t) \geq m > 0 \quad \forall t \in [0, T]; \quad \langle v(0), x^0\rangle \leq b(0) \tag{10}$$

or

$$Z = Z_*, \quad ||v(t)|| = 1 \quad \forall t \in [0, T]; \quad \langle v(0), x^0\rangle \leq b(0), \tag{11}$$

where $Z_* \subseteq \mathbb{R}^k$ is a compact neighborhood of $0_{\mathbb{R}^k}$ such that $||p(z, t)|| < 1$ for all $z \in Z_*$ and $t \in [0, T]$ (note that such a neighborhood clearly exists as a consequence of (8) *and (**A1**).)*

Similarly, the sweeping process $\overline{\mathcal{S}}_{(u,z)}$ admits a unique, absolutely continuous solution for arbitrary $(u, z) \in W^{1,1}([0, T], \mathbb{R}^n) \times \mathbb{R}^k$ under the following condition for the description (4) *of P*

$$\exists x^* \in \mathbb{R}^n : \langle a_i, x^*\rangle < c_i \quad (i = 1, \ldots, m) \tag{A2}$$

and on the control u:

$$\langle a_i, x^0\rangle \leq c_i + \langle a_i, u(0)\rangle \quad (i = 1, \ldots, m). \tag{12}$$

Proof. We apply Theorem 2.1 to the perturbed sweeping processes $(\widetilde{\mathcal{S}}_{(v,b,z)})$, $(\overline{\mathcal{S}}_{(u,z)})$, respectively. Observe first, that the initial conditions in both sweeping processes are guaranteed for all perturbation

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parameters z thanks to (9). In the second process, one has to take into account the equivalence

$$x^0 \in \overline{C}_{(u,z)}(0) = \{u(0)\} + P \iff x^0 \text{ satisfies (12)} \tag{13}$$

which follows from the second condition in $(\mathbf{A1})$ and from the description (4) of the polyhedron P. The absolute continuity of the perturbed controls in both processes is guaranteed by the absolute continuity of $v + p(z,\cdot), b, u + p(z,\cdot)$ as a consequence of our assumptions and of the first condition in $(\mathbf{A1})$. Second, condition (2), applied to the control $(v+p(z,\cdot),b)$, follows for the process $(\widetilde{\mathcal{S}}_{(v,b,z)})$ via Lemma 2.2 under the alternative assumptions made (note, in particular, that in the case of (11), $v(t) + p(z,t) \neq 0$ for all $t \in [0,T]$ thanks to the definition of Z_*).

As for the process $(\overline{\mathcal{S}}_{(u,z)})$, we refer to the description (4) of the polyhedron P and observe from (5) that the set $\{u(t) + p(z,t)\} + P$ can be described as a moving polyhedron $C_{(v,b)}(t)$ with

$$v_i(t) := a_i,\; b_i(t) := c_i + \langle a_i, u(t) + p(z,t)\rangle \quad (i = 1,\dots,m;\; t \in [0,T]). \tag{14}$$

Now, based on assumption (**A2**), we derive (2) for an arbitrarily fixed $t \in [0,T]$ by putting $y := x^* + u(t) + p(z,t)$:

$$\langle v_i(t), y\rangle = \langle a_i, x^*\rangle + \langle a_i, u(t) + p(z,t)\rangle < c_i + \langle a_i, u(t) + p(z,t)\rangle = b_i(t) \;\; (i = 1,\dots,m).$$

Altogether, this proves uniqueness and existence of absolutely continuous solutions to both (perturbed) sweeping processes. □

We note that the requirement of $||p(z,t)|| < 1$ for all $z \in Z_*$ and $t \in [0,T]$ in the definition Z_* could be equivalently expressed by the existence of some $\gamma < 1$ such that $||p(z,t)|| \le \gamma$ for all $z \in Z_*$ and $t \in [0,T]$. This follows from the compactness of Z_* and the continuity of p.

Similar to (**A1**), we shall tacitly assume in the following that (**A2**) is satisfied in all relevant results. Note, that this condition basically means, that the considered rigid polyhedron P has nonempty interior, which is a natural assumption. Observe that the constraints imposed on the controls in (10), (11), and (12) are simple control constraints which can be added to those already existing in some control problem so that one does not have to care about the existence of solutions to the perturbed sweeping processes. This leads us to introduce the following constraint sets:

$$Q^1 := \{(v,b) \in W^{1,1}([0,T],\mathbb{R}^{n+1}) \mid b(t) \ge m > 0 \;\forall t \in [0,T],\; \langle v(0), x^0\rangle \le b(0)\}, \tag{15}$$
$$Q^2 := \{(v,b) \in W^{1,1}([0,T],\mathbb{R}^{n+1}) \mid \|v(t)\| = 1 \;\forall t \in [0,T],\; \langle v(0), x^0\rangle \le b(0)\}, \tag{16}$$
$$Q^3 := \{u \in W^{1,1}([0,T],\mathbb{R}^n) \mid \langle a_i, x^0\rangle \le c_i + \langle u(0), x^0\rangle\}. \tag{17}$$

The conditions identified in Lemma 2.5 allow us to define parametric terminal point mappings

$$\widetilde{X}_1 : Q^1 \times \mathbb{R}^k \to \mathbb{R}^n; \quad \widetilde{X}_2 : Q^2 \times Z_* \to \mathbb{R}^n; \quad \overline{X} : Q^3 \times \mathbb{R}^k \to \mathbb{R}^n \tag{18}$$

(with $Q^1 \times \mathbb{R}^k$ and $Q^2 \times Z_*$ according to the case distinction in (10) and (11)) as being the terminal points $x_z(T)$ of the unique absolutely continuous solutions x_z to the sweeping processes $(\widetilde{\mathcal{S}}_{(v,b,z)})$, $(\widetilde{\mathcal{S}}_{(v,b,z)})$, and $(\overline{\mathcal{S}}_{(u,z)})$, respectively.

We conclude this section with two observations on the introduced polyhedral sweeping processes. The following remark establishes a relation between the sweeping process $(\overline{\mathcal{S}}_{(u,z)})$ and the theory of play operators.

Remark 2.6. Assume that (**A2**) and (12) are satisfied. Let $(u,z) \in W^{1,1}([0,T],\mathbb{R}^n) \times \mathbb{R}^k$ be given. Let $y : [0,T] \to \mathbb{R}^n$ be defined by $y(t) := u(t) + p(z,t) + x^*$ for all $t \in [0,T]$ and let $\psi_0 := y(0) - x^0$. Considering the *play operator* as defined in [34, Definition after Theorem I.1.9] or [35, Definition after Theorem 2.1] and starting from [36, Text after Theorem 3.4] or [43, Sec. 1, text after (11)], we can show that x_z is a solution to the sweeping process $(\overline{\mathcal{S}}_{(u,z)})$ if and only if x_z is the output of the play operator with characteristic set $\{x^*\} - P$, initial state ψ_0, and input function y.

The condition (10) implies a 'spiral-like' behavior of solutions to the sweeping process $\widetilde{\mathcal{S}}_{(v,b,z)}$, see Figure 2. This and an additional contrability result is formaly shown in the following lemma the proof of which is given in the Appendix, page 20.

Lemma 2.7. *We have the following result:*

1 *For each control $(v,b) \in W^{1,1}([0,T],\mathbb{R}^{n+1})$ satisfying (10), each $z \in \mathbb{R}^k$, and each initial value $x^0 \in \widetilde{C}_{(v,b,z)}(0)$ it holds for the solution x to $\widetilde{\mathcal{S}}_{(v,b,z)}$ (with initial value x^0):*

   1.1 *For all $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $x(t_1) \neq x(t_2)$ it holds $||x(t_1)|| > ||x(t_2)||$.*

   1.2 *If $x^0 \neq 0$ then it holds that $x(s) \neq 0$ for all $s \in [0,T]$.*

   1.3 *If $x^0 = 0$ then it holds that $x(s) = 0$ for all $s \in [0,T]$.*

2 *If $n > 1$ then it holds for each $x^0, x_1 \in \mathbb{R}^n$ with $0 < ||x_1|| < ||x^0||$ and each $t_1 \in (0,T]$: one can find a control $(v,b) \in W^{1,1}([0,T],\mathbb{R}^{n+1})$ satisfying (10) and $x^0 \in \widetilde{C}_{(v,b,z)}(0)$ such that it holds for the solution x to $\widetilde{\mathcal{S}}_{(v,b,0)}$ (with initial value x^0) that $x(t_1) = x_1$.*

Considering the situation as in Lemma 2.7.2 for $n = 1$, one can show the existence of a corresponding control if and only if x^0 and x_1 have the same sign.

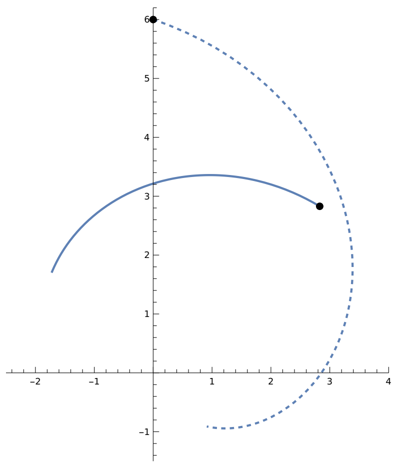

Figure 2: Two examples for solutions to the sweeping process $\widetilde{\mathcal{S}}_{(v,b,0)}$ for $n = 2$ with inital value marked by a point, as considered in the proof of Lemma 2.7.2, see page 21 in the appendix. These are parts of Archimedean spirals, see, e.g., [10].

2.3 Sweeping processes with randomly perturbed control

Next, we want to interpret the sweeping processes (6) and (7) with perturbed controls as random processes, where the perturbation parameter z is the realization of some random vector ξ_1 defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Accordingly, for some given random event $\omega \in \Omega$, we will refer to the induced random sweeping processes $\widetilde{S}_{(v,b,\xi_1(\omega))}$ and $\overline{S}_{(u,\xi_1(\omega))}$, respectively. Moreover, we will also consider the induced random sweeping processes $\widetilde{S}_{(v,b,\xi_2(\omega))}$ for a further appropriate random vector ξ_2. Considering Z_* as in Lemma 2.5, it is required that

$$\xi_1 : \Omega \to \mathbb{R}^k, \quad \xi_2 : \Omega \to Z_* \quad \text{are random variables on } (\Omega, \mathcal{A}, \mathbb{P}). \tag{A3}$$

A typical example would be continous random variables ξ_1 and ξ_2 such that

$$\xi_1 \sim \mathcal{N}(\mu, \Sigma), \quad \xi_2 \sim \mathcal{TN}(\mu, \Sigma, Z_*) \tag{19}$$

where $\mathcal{N}(\mu, \Sigma)$ refers to a k-dimensional Gaussian distribution with mean vector μ and covariance matrix Σ and $\mathcal{TN}(\mu, \Sigma, Z_*)$ means a truncation of such distribution to the set Z_* as introduced in (11). As an immediate consequence of Lemma 2.5, we obtain the following result, where the occurring sets Q^i are introduced in (15)-(17):

Lemma 2.8. *Assume* $(\mathbf{A3})$. *Let* $(v, b) \in Q^1$ *be arbitrary. Then,* $\mathbb{P}$*-almost surely, the sweeping process* $\widetilde{S}_{(v,b,\xi_1(\omega))}$ *admits a unique, absolutely continuous solution. The same holds true for* $(v, b) \in Q^2$ *and the sweeping process* $\widetilde{S}_{(v,b,\xi_2(\omega))}$. *Similarly, for arbitrary* $u \in Q^3$*, one has that* $\mathbb{P}$*-almost surely, the sweeping process* $\overline{S}_{(u,\xi_1(\omega))}$ *admits a unique, absolutely continuous solution.*

We emphasize that both constellations for the process $\widetilde{S}_{(v,b,\xi_i}(\omega))$ referred to in Lemma 2.8 involve constraints on the controls and/or on the distribution of their random perturbations. Indeed, without such constraints, the almost sure existence of unique, absolutely continuous solutions cannot be guaranteed as is illustrated by the following example:

Example 2.9. Consider the random sweeping process $\widetilde{S}_{(v,b,\xi(\omega))}$ with the data

$$\begin{gathered} n := k := T := 1,\ \xi \sim \mathcal{N}(0,1),\ x^0 := -1, \\ v(t) := 1,\ b(t) := -1,\ \alpha(t) := \sin(2\pi t)\ (t \in [0,1]). \end{gathered}$$

Then, in Lemma 2.8, we cannot consider the process $\widetilde{S}_{(v,b,\xi_2(\omega))}$ with $\xi_2 := \xi$ because the image set Z_* of ξ_2 is supposed to be compact which is not the case for ξ defined here. On the other hand, when considering the process $\widetilde{S}_{(v,b,\xi_1(\omega))}$ instead, we might put $\xi_1 := \xi$ consistently with $(\mathbf{A3})$ or (19). Then, however, the requirement $(v, b) \in Q^1$ would be violated due to b being negative. Therefore, none of the two alternative settings in Lemma 2.8 is valid. On the other hand, we will show that, with positive probability, the random sweeping process $\widetilde{S}_{(v,b,\xi(\omega))}$ has no solution, hence it cannot have a solution almost surely. To this aim, let $z \in \mathbb{R}$ with $z > 1$ be arbitrary. Then,

$$\begin{aligned} v(1/4) + p(z, 1/4) &= 1 + z\sin(\pi/2) = 1 + z > 0; \\ v(3/4) + p(z, 3/4) &= 1 + z\sin(3\pi/2) = 1 - z < 0. \end{aligned}$$

Consequently, for each $z > 1$, there is some $t_z \in (\frac{1}{4}, \frac{3}{4})$ such that $v(t_z) + p(z, t_z) = 0$. Moreover, with $'\sin'$ being strictly decreasing on $[\frac{\pi}{2}, \frac{3\pi}{2}]$, the function $v(t) + p(z,t) = 1 + z\sin(2\pi t)$ is strictly decreasing on $[\frac{1}{4}, \frac{3}{4}]$. It follows that $v(t) + p(z,t) > 0$ for $t \in [1/4, t_z)$. If the sweeping process

$\widetilde{S}_{(v,b,z)}$ had a solution x_z, then x_z would in particular have to be an absolutely continuous function, moreover satisfying the hidden constraint $x_z(t) \in C_{(v,b,z)}(t)$ a.e. $t \in [0,1]$ (see Introduction). In other words, $(v(t)+p(z,t))x_z(t) \leq b(t)$ a.e. $t \in [0,1]$. In particular,

$$x_z(t) \leq \frac{-1}{v(t)+p(z,t)} \text{ a.e. } t \in [1/4, t_z].$$

Therefore, we find a sequence $t_k \to_k t_z$ with $t_k < t_z$ and

$$x_z(t_k) \leq \frac{-1}{v(t_k)+p(z,t_k)} \quad \forall k.$$

Since $v(t_k)+p(z,t_k) > 0$ and $v(t_k)+p(z,t_k) \to_k v(t_z)+p(z,t_z) = 0$, it follows that $x_z(t_k) \to -\infty$ contradicting the fact that the continuous function x_z must be bounded on the interval $[0,1]$. Therefore, $\widetilde{S}_{(v,b,z)}$ cannot have a solution for arbitrary $z > 1$. Because of $\mathbb{P}(\xi > 1) > 0$ thanks to the assumed Gaussian distribution of ξ, the random sweeping process $\widetilde{S}_{(v,b,\xi(\omega))}$ has no solution with positive probability, hence it cannot have a solution almost surely.

Remark 2.10. We note that the previous counter-example would continue to work even if the solution concept for sweeping processes was weakened from absolute continuity to classes of functions which could be unbounded or have jumps. The established contradiction just relies on the hidden constraint of the differential inclusion, namely $x_z(t) \in C_{(v,b,z)}(t)$ for almost every $t \in [0,T]$.

As a consequence of Lemma 2.8, we may $\mathbb{P}$-almost surely define terminal-point events $\widetilde{X}_1(v,b,\xi_1(\omega))$, $\widetilde{X}_2(v,b,\xi_2(\omega))$, and $\overline{X}(u,\xi_1(\omega))$ via (18) in the cases of Lemma 2.8. Our aim is next to impose a constraint on the random terminal point, namely to fall into some closed target set $D \subseteq \mathbb{R}^n$. One way to do so would consist in formulating an almost-sure constraint:

$$\widetilde{X}_1(v,b,\xi_1(\omega)) \in D \text{ or } \widetilde{X}_2(v,b,\xi_2(\omega)) \in D \text{ or } \overline{X}(u,\xi_1(\omega)) \in D \quad \mathbb{P}-\text{almost surely}.$$

Such strong constraint may turn out not to be satisfiable at all or to shrink the feasible controls in a way that the associated costs in some optimization problem become prohibitively large. Therefore, we adopt a modeling approach which is very popular in engineering and requires only that the terminal point (in the respective model) belongs to the target set with sufficiently high probability $p \in (0,1]$:

$$\mathbb{P}(\widetilde{X}_1(v,b,\xi_1) \in D) \geq p \text{ or } \mathbb{P}(\widetilde{X}_2(v,b,\xi_2) \in D) \geq p \text{ or } \mathbb{P}(\overline{X}(u,\xi_1) \in D) \geq p. \tag{20}$$

Here, the probability level is typically chosen close to but different from one (the case $p = 1$ leading back to the almost sure constraint). Constraints of type (20) are called probabilistic or chance constraints. Formally, they are just additional inequality constraints imposed on the controls but the challenge to deal with them analytically and numerically relies on the fact that no explicit formula is available in general in order to deal with them directly. In the right part of Figure 1, the solution trajectories under both a nominal control and two perturbations thereof are visualized. These perturbations can be interpreted as the realizations of some random process. From the resulting two random terminal points of the trajectories only one falls into the target set D which corresponds to an empirical probability of $1/2$ for satisfying the terminal point constraint. This means, that the underlying (unperturbed) nominal control is feasible for the probabilistic constraint in the second part of (20), as long as $p \leq 1/2$ whereas it is infeasible for larger probabilities.

We note that the formulation of expressions involving probabilities such as (20) requires first the verification of Borel measurability of the terminal point mappings with respect to its last argument. We will provide a formal justification in Corollary 3.5 below in a slightly narrower context of $W^{1,2}$ spaces.

2.4 Optimal control problems

We are now in a position to formulate optimal control problems involving random sweeping processes $\widetilde{S}_{(v,b,\xi_1(\omega))}$, $\widetilde{S}_{(v,b,\xi_2(\omega))}$, $\overline{S}_{(u,\xi_1(\omega))}$ and respective probabilistic constraints (20) for the terminal point. Before doing so, we emphasize that we will place our controls no longer into the space $W^{1,1}$ but into the smaller space $W^{1,2}$ of square integrable functions with square integrable derivative. Therefore, all results formulated before remain valid in this smaller space. The benefit of doing so relies on the fact that $W^{1,2}$ is a reflexive space which allows in a more comfortable way to establish the main result of this paper, namely the existence of solutions to the optimal control problems formulated in the following.

We start with the first two problems relating to the processes $\widetilde{S}_{(v,b,\xi_1(\omega))}$ and $\widetilde{S}_{(v,b,\xi_2(\omega))}$ distinguished according to the alternative cases (10), (11):

$$\min_{(v,b)\in W^{1,2}([0,T],\mathbb{R}^{n+1})} f(v,b) \quad \text{subject to} \quad (v,b)\in U^i,\ \widetilde{\varphi}^i(v,b)\geq p \quad (\mathcal{P}^i) \quad (i=1,2), \tag{21}$$

where, with given constants $m, M>0$, the U^i are simple control constraints defined by

$$\begin{aligned} U^1 := \{(v,b)\in W^{1,2}([0,T],\mathbb{R}^{n+1}) \mid \|(v,b)\|_{W^{1,2}}\leq M,\ b(t)\geq m\ \forall t\in[0,T],\\ \langle v(0),x^0\rangle \leq b(0)\}, \end{aligned} \tag{22}$$

$$\begin{aligned} U^2 := \{(v,b)\in W^{1,2}([0,T],\mathbb{R}^{n+1}) \mid \|(v,b)\|_{W^{1,2}}\leq M,\ \|v(t)\|=1\ \forall t\in[0,T],\\ \langle v(0),x^0\rangle \leq b(0)\}, \end{aligned} \tag{23}$$

f is a cost function and $\widetilde{\varphi}^i : U^i\to\mathbb{R}$ $(i=1,2)$ are probability functions defined as

$$\widetilde{\varphi}^1(v,b) := \mathbb{P}(\widetilde{X}_1(v,b,\xi_1)\in D);$$
$$\widetilde{\varphi}^2(v,b) := \mathbb{P}(\widetilde{X}_2(v,b,\xi_2)\in D);$$

Observe that, by $W^{1,2}([0,T],\mathbb{R}^{n+1})\subseteq W^{1,1}([0,T],\mathbb{R}^{n+1})$ we have that $U^1\subseteq Q^1, U^2\subseteq Q^2$ for the sets Q^1, Q^2 introduced in (15), (16). Accordingly, the terminal point mappings $\widetilde{X}_1$ and $\widetilde{X}_2$ defined in (18) are well-defined on the sets $U^1\times\mathbb{R}^k$ and $U^2\times Z_*$, respectively. Consequently, taking into account Corollary 3.5 below, the probability functions $\widetilde{\varphi}^i$ are well-defined too on U^i for $i=1,2$.

In addition, we consider the third optimal control problem related with the random sweeping process $\overline{S}_{(u,\xi_1(\omega))}$:

$$\min_{u\in W^{1,2}([0,T],\mathbb{R}^n)} f(u) \quad \text{subject to} \quad u\in U^3,\ \overline{\varphi}(u)\geq p \quad (\mathcal{P}^3). \tag{24}$$

We assume that the polyhedron P in the sweeping process (7) is endowed with a description (4) satisfying the first condition of (12). Then, with given constant $M>0$, U^3 is defined as a simple control constraint

$$U^3 := \{u\in W^{1,2}([0,T],\mathbb{R}^n) \mid \|u\|_{W^{1,2}}\leq M,\ \langle a_i,x^0\rangle\leq c_i+\langle u(0),x^0\rangle\}. \tag{25}$$

Moreover, $\overline{\varphi}(u) : U^3\to\mathbb{R}$ is a probability function defined as

$$\overline{\varphi}(u) := \mathbb{P}(\overline{X}(u,\xi_1)\in D).$$

The fact that the terminal point mapping $\overline{X}$ is well-defined on $U^3\times\mathbb{R}^k$ and the probability function $\overline{\varphi}$ is well-defined on U^3 follows as before (now referring to (17)).

The meaning of these optimal control problems is as follows: we aim at finding a (nominal) control of some polyhedron such that the costs of the control are minimized and that, after a posterior random perturbation of this control, the terminal point of the corresponding sweeping process still lands in a region D with probability at least p. We understand the setting as a static 'here-and-now' decision which means that the optimized control does not depend on historic realizations of its random perturbation but is determined once and forever.

3 Existence of solutions for the optimal control problems

The aim of this section is to verify the existence of optimal solutions to the optimal control problems (21) and (24) introduced above. The main argument for proving such existence result turns out to be the weak sequential continuity of the terminal point mappings considered in these problems.

In the following we will use balls or norms indexed by '1,1', '1,2' or '∞', respectively, to refer to the balls in or the norms of the spaces $W^{1,1}, W^{1,2}$, or C, respectively. The pure index '1' will refer to the 1-norm in $\mathbb{R}^k$.

3.1 Preparatory results

In this section we prove several preparatory technical results which allow us in the next section to verify the weak sequential continuity of the terminal point mappings.

Proposition 3.1. *Under one of the alternative cases* (10) *or* (11)*, let a fixed nominal control* $(v, b) \in W^{1,1}([0,T], \mathbb{R}^{n+1})$ *and* $z \in Z$ *be arbitrarily given, with* $Z = \mathbb{R}^k$ *for the case* (10) *and* $Z = Z_*$ *for the case* (11)*. Denote by* x *the unique, absolutely continuous solution of the process* $\widetilde{S}_{(v,b,z)}$ *(see Lemma 2.5). Moreover, let some* $\widetilde{\rho} > 0$ *be given. Then, there exists* $\rho, K > 0$ *such that for all*

$$(v', b', z') \in \left[\mathbb{B}_{1,1}((0,0), \widetilde{\rho}) \cap \mathbb{B}_\infty((v,b),\rho)\right] \times \mathbb{B}_1(z,\rho)$$

with $\langle v'(0), x^0\rangle \leq b'(0)$ *the associated sweeping process* $\widetilde{\mathcal{S}}_{(v',b',z')}$ *admits a unique solution* x'*, and for this solutions it holds that*

$$\|x'(T) - x(T)\|^2 \leq K \left(\|(v'-v, b'-b)\|_\infty + \|z'-z\|_1\right). \tag{26}$$

Proof. For the given $\widetilde{\rho}$, let $\rho^* := \widetilde{\rho} + \max\{A,B\} + \|z\|_1 A$, where we define with respect to (8):

$$A := \max_{i=1,\ldots,k} \|\alpha_i\|_{1,1}; \quad B := \max_{i=1,\ldots,k} \|\alpha_i\|_\infty . \tag{27}$$

We want to apply Theorem 4.1 in the appendix to the given control pair $(v+p(z,\cdot), b)$ (corresponding to $(\bar{u}, \bar{b})$ in the statement of the Theorem). Recall from the proof of Lemma 2.5, that condition (2) required in Theorem 4.1 is enforced by our assumptions (10) or (11). From Theorem 4.1 we now derive the existence of $\rho', K' > 0$ such that for all $(v', b') \in W^{1,1}([0,T], \mathbb{R}^{n+1})$ and all $z' \in \mathbb{R}^k$ with

$$(v' + p(z', \cdot), b') \in \mathbb{B}_{1,1}\left((0,0), \rho^*\right) \cap \mathbb{B}_\infty\left((v+p(z,\cdot), b), \rho'\right)$$

and $\langle v'(0), x^0\rangle \leq b'(0)$ the associated sweeping process $\widetilde{\mathcal{S}}_{(v',b',z')}$ admits a unique solution x', and for this solution it holds that

$$\|x'(T) - x(T)\|^2 \leq K' \left\|\left(v' + p(z', \cdot) - (v + p(z,\cdot)), b' - b\right)\right\|_\infty . \tag{28}$$

From (8), we get that for all $(v', b', z') \in \mathbb{B}_{1,1}((0,0), \widetilde{\rho}) \times \mathbb{B}_1(z, \rho)$ with $\rho \leq 1$

$$\begin{aligned} &\|(v' + p(z', \cdot), b')\|_{1,1} \\ &\leq \|v', b'\|_{1,1} + (\|z' - z\|_1 + \|z\|_1) \max_{i=1,\ldots,k} \|\alpha_i\|_{1,1} \leq \widetilde{\rho} + (\rho + \|z\|_1)\, A \leq \rho^*. \end{aligned} \tag{29}$$

Similarly, it holds for all $(v', b', z') \in \mathbb{B}_\infty((v, b), \rho) \times \mathbb{B}_1(z, \rho)$ with $\rho \leq \rho'/(1 + B)$ that

$$\begin{aligned} \|(v' + p(z', \cdot) - (v + p(z, \cdot)), b' - b)\|_\infty &\leq \|(v' - v, b' - b)\|_\infty + B\|z' - z\|_1 \\ &\leq \rho(B + 1) \leq \rho'. \end{aligned} \tag{30}$$

Consequently, (28), (29), and (30) yield for all

$$(v', b', z') \in \left[\mathbb{B}_{1,1}((0,0), \widetilde{\rho}) \cap \mathbb{B}_\infty((v, b), \rho)\right] \times \mathbb{B}_1(z, \rho) \text{ with } \rho \leq \min\{1, \rho'/(1 + B)\}$$

and $\langle v'(0), x^0 \rangle \leq b'(0)$ that

$$\|x'(T) - x(T)\|^2 \leq K' \left(\|(v' - v, b' - b)\|_\infty + B\|z' - z\|_1\right).$$

Hence, (26) is satisfied for $K := K' \max\{1, B\}$. □

Proposition 3.2. *Under assumption (***A2***), let a fixed nominal control $u \in W^{1,1}([0,T], \mathbb{R}^n)$ with* (12) *and $z \in \mathbb{R}^k$ be arbitrarily given. Denote by x the unique, absolutely continuous solution of the process $\overline{S}_{(u,z)}$ (see Lemma 2.5). Moreover, let some $\widetilde{\rho} > 0$ be given. Then, there exists $\rho, K > 0$ such that for all*

$$(u', z') \in \left[\mathbb{B}_{1,1}(0, \widetilde{\rho}) \cap \mathbb{B}_\infty(u, \rho)\right] \times \mathbb{B}_1(z, \rho)$$

with $\langle a_i, x^0 \rangle \leq c_i + \langle a_i, u'(0) \rangle$ for $(i = 1, \ldots, m)$ the associated sweeping process $\overline{\mathcal{S}}_{(u',z')}$ admits a unique solution x', and for this solutions it holds that

$$\|x'(T) - x(T)\|^2 \leq K \left(\|u' - u\|_\infty + \|z' - z\|_1\right). \tag{31}$$

Proof. Let m, a_i, c_i be defined as in (4) and

$$\overline{A} := \sum_{i=1}^m ||a_i||, \quad \overline{C} := \sum_{i=1}^m |c_i|. \tag{32}$$

Let $v, b : [0,T] \to \mathbb{R}^m$ be defined by (14). Considering the argumentation at the end of the proof of Lemma 2.5, we see that x is also the solution to the sweeping process $\left(\mathcal{S}_{(v,b)}\right)$ from (1).

For the given $\widetilde{\rho}$ and with A as in (27), let

$$\rho^* := T(\overline{A} + \overline{C}) + \overline{A}\left(\widetilde{\rho} + (1 + \|z\|_1)\, A\right).$$

We now want to apply Theorem 4.1 from the appendix. In order to do so, we have to ensure condition (2) for the process $\left(\mathcal{S}_{(v,b)}\right)$. This, however, follows from our assumption (**A2**) as in the end of the proof of Lemma 2.5.

From Theorem 4.1, we derive that there exist $\rho', K' > 0$ such that for all

$$(v', b') \in \mathbb{B}_{1,1}((0,0), \rho^*) \cap \mathbb{B}_\infty\left((v, b), \rho'\right) \tag{33}$$

with $x^0 \in C_{(v',b')}(0)$ the sweeping processes $\left(\mathcal{S}_{(v',b')}\right)$ from (1) has a unique absolutely continuous solutions x', and that this solution satisfies the estimate

$$\|x'(T) - x(T)\|^2 \leq K' \|(v - v', b - b')\|_\infty . \tag{34}$$

Let $(u', z') \in \left[\mathbb{B}_{1,1}(0, \widetilde{\rho}) \cap \mathbb{B}_\infty(u, \rho)\right] \times \mathbb{B}_1(z, \rho)$ with $\langle a_i, x^0\rangle \leq c_i + \langle a_i, u'(0)\rangle$ for $(i = 1, \ldots, m)$ for $\rho := \min\left(1, \frac{\rho'}{\overline{A}(1+kB)}\right)$ be given. Hence, by (13), we see that $x^0 \in \overline{C}_{(u',z')}(0)$. As in the statement related with (14), we have that

$$\overline{C}_{(u',z')}(t) = \{u'(t) + p(z', t)\} + P = C_{(v',b')}(t) \quad \text{and} \quad x^0 \in C_{(v',b')}(0),$$

where the functions $v', b' : [0, T] \to \mathbb{R}^m$ are defined as

$$v_i'(t) := a_i, \; b_i'(t) := c_i + \langle a_i, u'(t) + p(z', t)\rangle \quad (i = 1, \ldots, m; \, t \in [0, T]).$$

Owing to (8) and (**A1**), we now get the estimates

$$\begin{aligned}
\|v'\|_{1,1} &\leq \sum_{i=1}^m ||a_i||T \leq T\overline{A}, \\
\|b'\|_{1,1} &\leq \sum_{i=1}^m \left(|c_i|T + ||a_i|| \left(\|u'\|_{1,1} + \|p(z', \cdot)\|_{1,1}\right)\right) \\
&\leq \overline{C}T + \overline{A}\left(\|u'\|_{1,1} + \sum_{j=1}^k |z_j'| \, \|\alpha_j\|_{1,1}\right) \\
&\leq \overline{C}T + \overline{A}\left(\|u'\|_{1,1} + (\|z' - z\|_1 + \|z\|_1)\, A\right) \leq \overline{C}T + \overline{A}\left(\widetilde{\rho} + (\rho + \|z\|_1)\, A\right).
\end{aligned}$$

Hence, we have $\|(v', b')\|_{1,1} \leq T(\overline{A} + \overline{C}) + \overline{A}\left(\widetilde{\rho} + (1 + \|z\|_1)\, A\right) = \rho^*$.

Moreover, we have $\|v - v'\|_\infty = 0$ and, with B as in (27),

$$\begin{aligned}
\|b - b'\|_\infty &\leq \max_{1\leq i\leq m} ||a_i||_{\mathbb{R}^n} \left(\|u - u'\|_\infty + \|p(z, \cdot) - p(z', \cdot)\|_\infty\right) \\
&\leq \overline{A}\left(\|u - u'\|_\infty + kB\|z - z'\|_\infty\right) \\
&\leq \overline{A}\rho(1 + kB) \leq \rho'.
\end{aligned} \tag{35}$$

Hence, we see that (33) is valid for (v', b'), yielding that the sweeping processes $\left(\mathcal{S}_{(v',b')}\right)$ from (1) has a unique absolutely continuous solutions x', and that this solution satisfies (34). Recalling again the argumentation at the end of the proof of Lemma 2.5, we deduce that x' is the unique absolutely continuous solutions to $\overline{\mathcal{S}}_{(u',z')}$. Combining (34), $v = v'$, and (35), we get (31) with $K := \overline{A}K' \max(1, B)$. □

Remark 3.3. Assume that (**A2**) and (12) are satisfied. Considering x and x' as in Proposition 3.2 and recalling Remark 2.6, we deduce that x and x' can be considered as outputs of a play operator with a characteristic set being a polyhedron and appropriate initial states ψ_0 and ψ_0' and input function y and y'. Recalling now [35, Theorem 7.1] and using an argumentation as in the proof of Proposition 3.2, we can find some $K > 0$ such that

$$\|x'(T) - x(T)\| \leq K\left(\|u' - u\|_\infty + \|z' - z\|_1\right).$$

for all $u, u' \in W^{1,1}([0, T], \mathbb{R}^n)$ and all $z, z' \in \mathbb{R}^k$.

3.2 Weak continuity of the terminal point mappings

Theorem 3.4. *The terminal point mappings $\widetilde{X}_1$ and $\widetilde{X}_2$ introduced in* (18) *are well defined and weakly sequentially continuous on the sets $U^1 \times \mathbb{R}^k$ and $U^2 \times Z_*$ (see* (22), (23) *and* (11)*), respectively. Similarly, the terminal point mapping $\overline{X}$ introduced in* (18) *is well defined and weakly sequentially continuous on the set $U^3 \times \mathbb{R}^k$ (see* (25)*).*

Proof. We start with the terminal point mappings $\widetilde{X}_1$ and $\widetilde{X}_2$. As stated in Section 2.4, $\widetilde{X}_1$ is well-defined on $U^1 \times \mathbb{R}^k$ and $\widetilde{X}_2$ on $U^2 \times Z_*$. We have to show that the weak convergence $(v_m, b_m, z_m) \rightharpoonup (v, b, z)$ in $U^1 \times \mathbb{R}^k$ implies the convergence $\widetilde{X}_1(v_m, b_m, z_m) \to \widetilde{X}_1(v, b, z)$ in $\mathbb{R}^n$ and that the analogous weak convergence in $U^2 \times Z_*$ implies the convergence $\widetilde{X}_2(v_m, b_m, z_m) \to \widetilde{X}_2(v, b, z)$ in $\mathbb{R}^n$. To see this, note that, by compact embedding, weak convergence in $W^{1,2}$ implies strong convergence in C (see, e.g., [30, eqs. (1.70) and (2.135)]). Hence, $(v_m, b_m) \to (v, b)$ strongly in $C([0, T]; \mathbb{R}^{n+1})$. Since weak and strong convergence coincide in $\mathbb{R}^k$, we also have that $z_m \to z$. As a consequence of weak convergence, the sequence $\{(v_m, b_m)\}$ is bounded in $W^{1,2}([0, T], \mathbb{R}^n)$. Since the latter space is continuously embedded into $W^{1,1}([0, T], \mathbb{R}^n)$, there exists some $\tilde{\rho} > 0$ with $(v_m, b_m) \in \mathbb{B}_{1,1}((0, 0), \tilde{\rho})$ for all $m \in \mathbb{N}$. This allows us to invoke Proposition 3.1 and to establish the existence of $\rho, K > 0$ such that the inequalities

$$\left\| \widetilde{X}_i(v_m, b_m, z_m) - \widetilde{X}_i(v, b, z) \right\|^2 \leq K \left(\|(v_m - v, b_m - b)\|_\infty + \|z_m - z\|_1 \right) \quad i = 1, 2$$

hold true for all m with $(v_m, b_m, z_m) \in \left[\mathbb{B}_{1,1}((0, 0), \widetilde{\rho}) \cap \mathbb{B}_\infty((v, b), \rho) \right] \times \mathbb{B}(z, \rho)$. Note that the assumption $\langle v_m(0), x^0 \rangle \leq b_m(0)$ for the application of that Proposition is satisfied thanks to (v_m, b_m) belonging to U^1 or U^2, respectively. From $(v_m, b_m) \to (v, b)$ strongly in $C([0, T]; \mathbb{R}^{n+1})$ and $z_m \to z$ it follows that the inequality above can be applied for m large enough and then yields the desired convergence $\widetilde{X}_i(v_m, b_m, z_m) \to \widetilde{X}_i(v, b, z)$ for $i = 1, 2$.

Recalling Section 2.4, we see that $\overline{X}$ is well-defined on $U^3 \times \mathbb{R}^k$. Now, we have to show that the weak convergence $(u_m, z_m) \rightharpoonup (u, z)$ in $U^3 \times \mathbb{R}^k$ yields the convergence $\overline{X}(u_m, z_m) \to \overline{X}(u, z)$ in $\mathbb{R}^n$. The same argument as in the first part of this proof yields that $u_m \to u$ strongly in $C([0, T]; \mathbb{R}^n)$ and $z_m \to z$. Now, Proposition 3.2 yields the desired result in the same way as Proposition 3.1 in the first part of the proof. □

As a corollary to the previous result, we may guarantee the measurability of the terminal point mappings in their last argument, thus justifying probability expressions in section 2.4.

Corollary 3.5. *Let $(v, b) \in U^1$ be arbitrary. Then, the mapping $\omega \mapsto \widetilde{X}_1(v, b, \xi_1(\omega)) \in \mathbb{R}^n$ is Borel measurable under $(\mathbf{A3})$. The same holds true for $(v, b) \in U^2$ and the mapping $\omega \mapsto \widetilde{X}_2(v, b, \xi_2(\omega)) \in \mathbb{R}^n$. Similarly, for arbitrary $u \in U^3$, the mapping $\omega \mapsto \overline{X}(u, \xi_1(\omega)) \in \mathbb{R}^n$ is Borel measurable under $(\mathbf{A3})$.*

Proof. Since ξ_1 and ξ_2 are $k-$dimensional random vectors, it is sufficient to verify the continuity of the mappings $\widetilde{X}_1(v, b, \cdot) : Z \to \mathbb{R}^n$, $\widetilde{X}_2(v, b, \cdot) : Z_* \to \mathbb{R}^n$, $\overline{X}(u, \cdot) : \mathbb{R}^k \to \mathbb{R}^n$ respectively. This, however, follows immediately from Theorem 3.4 because the weak continuity in the finite-dimensional $z-$argument coincides with strong continuity. □

3.3 Existence theorem

A further prerequisite for proving the desired existence result for solutions to the considered optimal control problems is the following property:

Lemma 3.6. *The simple control constraint sets U^1, U^2, and U^3 introduced in problems* (21) *and* (24) *are weakly sequentially compact.*

Proof. Since $W^{1,2}([0,T],\mathbb{R}^q)$ ($q=n$ or $q=n+1$) is a reflexive Banach space, it is sufficient to show that the U^i are bounded and weakly sequentially closed (see, e.g. [48, Theorem 2.10]). Boundedness follows in all three cases from $U^i \subseteq \mathbb{B}_{1,2}\left((0,0),M\right)$. To show the weak sequential closedness, observe that the sets U^i are intersections of the weakly sequentially closed ball $\mathbb{B}_{1,2}\left((0,0),M\right)$ in $W^{1,2}([0,T],\mathbb{R}^q)$ with respective sets which are strongly closed in $C([0,T],\mathbb{R}^q)$ since pointwise convergence is implied by uniform convergence. By the compact embedding of $W^{1,2}([0,T],\mathbb{R}^q)$ into $C([0,T],\mathbb{R}^q)$, strong closedness in $C([0,T],\mathbb{R}^q)$ implies weak sequential closedness in $W^{1,2}([0,T],\mathbb{R}^q)$. Hence, the U^i are weakly sequentially closed as intersections of two such sets. □

Now, we are in a position to prove an existence result for the optimal control problems introduced before:

Theorem 3.7. *In the optimal control problem* (21) *and* (24)*, let the objective f be weakly sequentially lower semicontinuous. Then, problem* (21) *has a solution (for $i=1,2$), if there exists some $(v,b)\in U^i$ ($i=1,2$) such that $\widetilde{\varphi}^i(v,b)\geq p$ ($i=1,2$). Similarly,* (24) *has a solution provided that there exists some $u\in U^3$ such that $\overline{\varphi}(u)\geq p$.*

Proof. By the classical Weierstrass Theorem for the existence of solutions to optimization problems, it is sufficient to verify that the respective constraint sets

$$\{(v,b)\in U^i \mid \widetilde{\varphi}^i(v,b)\geq p\}\quad (i=1,2)\quad\text{and}\quad \{u\in U^3 \mid \overline{\varphi}(u)\geq p\}$$

are weakly sequentially compact (note that these sets are nonempty by assumption). As for the U^i and $i=1,2,3$ themselves, their sequential weak compactness follows from Lemma 3.6. Hence, we are done, if we can show that the sets $\widetilde{\varphi}^i\geq p$ ($i=1,2$) and $\overline{\varphi}\geq p$ are sequentially weakly closed because then their intersections with the respective U^i, are sequentially weakly compact as desired. Now, since $\widetilde{\varphi}^i\geq p$ ($i=1,2$) and $\overline{\varphi}\geq p$ are upper level sets, their sequential weak closedness will follow from the weak sequential upper semicontinuity of these probability functions. In order to prove this property, we represent first the random terminal point constraints $\widetilde{X}_1(v,b,\xi_1(\omega))\in D$, $\widetilde{X}_2(v,b,\xi_2(\omega))\in D$, and $\overline{X}(u,\xi_1(\omega))\in D$ as random inequalities $\widetilde{g}_1(v,b,\xi_1(\omega))\leq 0$, $\widetilde{g}_2(v,b,\xi_2(\omega))\leq 0$, and $\overline{g}(u,\xi_1(\omega))\leq 0$ by defining $\widetilde{g}_1 := h\circ\widetilde{X}_1$, $\widetilde{g}_2 := h\circ\widetilde{X}_2$, and $\overline{g} := h\circ\overline{X}$, where $h := \operatorname{dist}_D$ is the distance function to the closed set D. Then, we may represent the probability functions as (recall that $\widetilde{\varphi}^i$ are defined on the different domains U^i for $i=1,2$)

$$\begin{aligned} &\widetilde{\varphi}^1(v,b)=\mathbb{P}(\widetilde{g}_1(v,b,\xi_1)\leq 0),\quad \widetilde{\varphi}^2(v,b)=\mathbb{P}(\widetilde{g}_2(v,b,\xi_2)\leq 0),\\ \text{and}\quad &\overline{\varphi}(u)=\mathbb{P}(\overline{g}(u,\xi_1)\leq 0).\end{aligned}$$

Owing to [21, Lemma 2], the desired weak sequential upper semicontinuity of the probability functions will follow from the weak sequential lower semicontinuity of the functions $\widetilde{g}_1$ and $\widetilde{g}_2$ in (v,b), and of the function $\overline{g}$ in u. Since the distance function h is continuous, the latter property is implied by the weak sequential continuity of the terminal point mappings $\widetilde{X}_1(v,b,z)$, $\widetilde{X}_2(v,b,z)$, and $\overline{X}(u,z)$ in their respective controls (v,b) and u. This, however was shown in Theorem 3.4. □

4 Numerical Illustration

As an illustration of the optimal control problems introduced in Section 2.4, we consider two special instances of problem $(\mathcal{P}^2)$ in (21). The first instance is given by

$$\min_{(v,b)\in W^{1,2}([0,T],\mathbb{R}^{n+1})} \|(\dot v,\dot b)\|^2_{L^2} \quad \text{subject to} \quad (v,b)\in U^2,\ \widetilde{\varphi}^2(v,b)\geq p, \tag{36}$$

where, with Z_* being a neighborhood of 0 introduced in Lemma 2.5,

$$\begin{aligned} U^2 &:= \{(v,b)\in W^{1,2}([0,T],\mathbb{R}^{n+1}) \mid \|(v,b)\|_{W^{1,2}}\leq M,\ \|v(t)\|=1\ \forall t\in[0,T],\\ &\qquad \langle v(0),x^0\rangle \leq b(0)\},\\ \widetilde{\varphi}^2(v,b) &:= \mathbb{P}(\widetilde{X}(v,b,\xi)\in D);\quad \xi\sim\mathcal{TN}(\mu,\Sigma,Z_*). \end{aligned}$$

The second instance is given by

$$\min_{(v,b,p)\in W^{1,2}([0,T],\mathbb{R}^{n+1})\times\mathbb{R}} -p \quad \text{subject to} \quad (v,b)\in U^2,\ \|(\dot v,\dot b)\|^2_{L^2}\leq c,\ \widetilde{\varphi}^2(v,b)\geq p \tag{37}$$

with U^2 and $\widetilde{\varphi}^2$ as above.

The objective in the first instance measures roughly speaking the kinetic effort in moving the controlled half space. In the second instance, the probability level p enters as an additional control variable and the objective along with the chance constraint aim at maximizing the probability level. We add an additional upper bound c for the kinetic effort here to ensure this problem has a solution (otherwise the probability could be driven to one). Indeed, (37) is equivalent to the problem

$$\max_{(v,b)\in W^{1,2}([0,T],\mathbb{R}^{n+1})} \widetilde{\varphi}^2(v,b) \quad \text{subject to} \quad (v,b)\in U^2,\ \|(\dot v,\dot b)\|^2_{L^2}\leq c$$

of maximizing the probability directly over the remaining constraints. This problem finds, irrespective of any costs, the most robust control for reaching the target set D with the terminal point of the trajectory. As illustration is our main purpose here, we consider just a bivariate truncated Gaussian random vector $\xi_2\sim\mathcal{TN}(\mu,\Sigma,Z_*)$ according to the second case in (19) with parameters

$$\mu=(0,0);\quad \Sigma:=\begin{pmatrix}0.09 & 0\\ 0 & 0.09\end{pmatrix};\quad Z_*:=\{z\in\mathbb{R}^2 \mid \|z\|_1\leq 1/2\},$$

with time horizon $T:=1$ and base functions for $k:=2$ from (8)

$$\alpha_1(t):=\begin{pmatrix}\sin(2.5\pi t)\\ \sin(3.5\pi t)\end{pmatrix};\quad \alpha_2(t):=\begin{pmatrix}\sin(4.5\pi t)\\ \sin(5.5\pi t)\end{pmatrix}.$$

Note that the choices of Z_*, α_1, α_2, and k yield thanks to (8) that $||p(z,t)||\leq 1/\sqrt{2}$ for all $z\in Z_*$ and all $t\in[0,T]$, so they are coherent with the requirement made in Lemma 2.5. The target set, initial point and values for c,p according to the problem considered, were chosen as

$$D:=\{x\in\mathbb{R}^2 \mid x_1\geq 0, x_2\leq 0\};\quad x^0:=(0,1);\quad c:=0.2;\quad p:=0.6.$$

The optimization was carried out by a derivative free nonlinear optimization solver with the required probabilities (values of $\widetilde{\varphi}^2$) approximated by Monte Carlo simulation with 20.000 samples. Of course, in more realistic settings, also derivatives of $\widetilde{\varphi}^2$ would be needed in order to employ more efficient solvers such as SQP methods. An efficient way to deal with derivatives of probability functions under elliptically

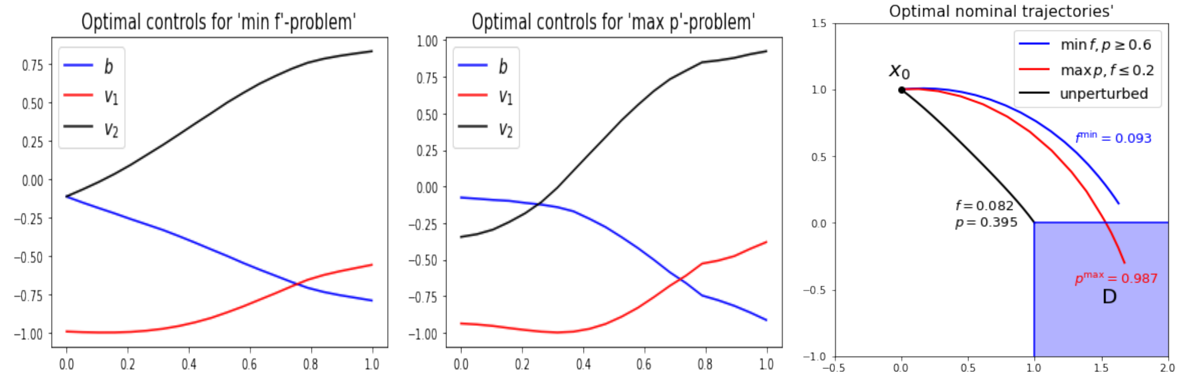

Figure 3: Optimal nominal controls for problems (36) (left) and (37) (middle) and associated trajectories (right).

symmetric distributions (such as Gaussian) is based on the so-called *spherical-radial decomposition* and has found numerous recent applications in optimal control of PDEs under chance constraints, e.g., [22, 29]. Figure 3 shows the optimal (nominal) controls for problems (36) (left) and (37)Unfortunately (middle) as well as the associated (nominal) trajectories (right). For the sake of comparison, we have added the solution of the 'unoerturbed' problem, when ignoring stochastic perturbations of the control (i.e., problem (36) without the chance constraint $\widetilde{\varphi}^2(v,b) \geq p$). Not surprisingly, the solution of this latter problem consists in driving the solution trajectory along the shortest path to the target set D. This results in the smallest kinetic effort $f = 0.082$. When this trivial control is faced with the given random perturbation, the probability that the terminal point of the trajectory reaches the target set is as small as $p = 0.395$. After imposing a chance constraint of reaching the target set with probability at least $p = 0.6$, the required kinetic effort increases only slightly to $f = 0.093$. When constraining the kinetic effort to $f \leq 0.2$, then the maximum probability of reaching the target set increases to $p = 0.987$. Note from Figure 3 (right diagram) that the nominal trajectory for the probability maximization problem reaches the target set while it does not in the problem minimizing the kinetic effort under a chance constraint. This latter observation does not contradict the fact that 60% of the trajectories associated with randomly perturbed controls reach the target set.

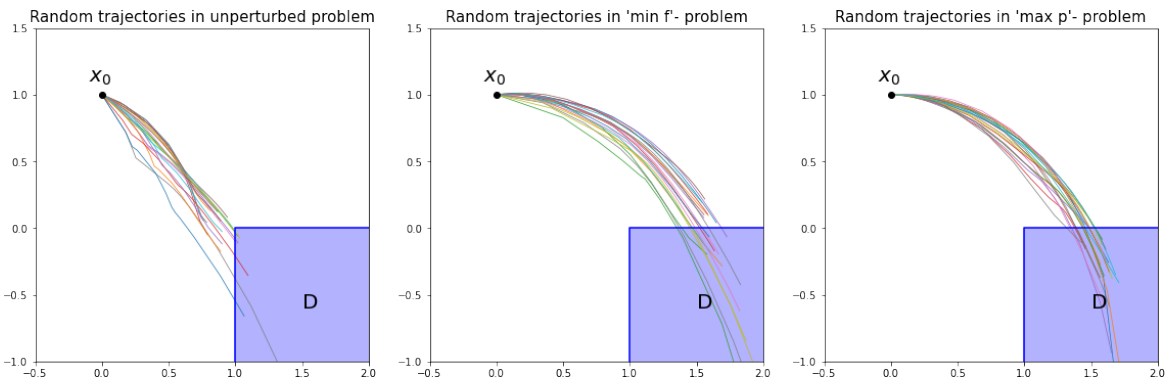

Figure 4: Trajectories associated with the randomly perturbed nominal controls of problem (36) without chance constraint (left), problem (36) (middle) and problem (37) (right).

Figure 4 shows the trajectories associated with 20 randomly perturbed nominal controls of the three problems. For better visibility, Figure 5 plots just the terminal points of 100 simulated such perturbations. Empirically, the probabilities of reaching the target set are in good coincidence with the theoret-

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ical probabilities indicated in Figure 3 (right diagram).

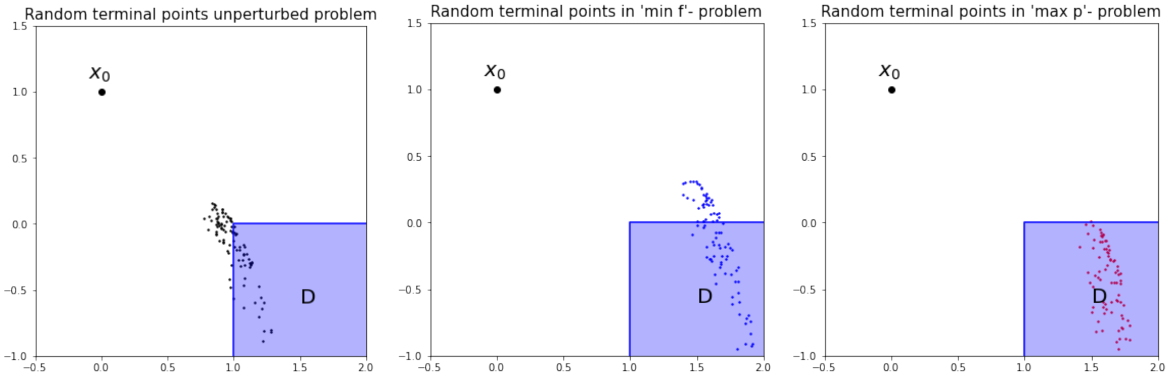

Figure 5: Terminal points associated with the randomly perturbed nominal controls of problem (36) without chance constraint (left), problem (36) (middle) and problem (37) (right).

Appendix

The following theorem which is related with the general polyhedral sweeping process $(\mathcal{S}_{(v,b)})$ introduced in (1), is the basis (via (3.1)) for the proof of the existence result in Section 3. It is an improved version of [28, Theorem 4,Unfortunately Lemma 3]. Since the proofs of these results are rather lengthy and the necessary modifications for our modification only occasional, we do not provide here more or less a repetition of the existing proofs but rather identify precisely at which positions the necessary modifications have to be made there.

Theorem 4.1. *Let $\mathcal{H}$ be a separable Hilbert space, and let the condition* (2) *hold for a given control pair $(\overline{u}, \overline{b}) \in W^{1,1}([0,T], \mathcal{H}^m) \times W^{1,1}([0,T], \mathbb{R}^m)$. Let $\widetilde{\rho} > 0$ be given. Then there exist a number $\rho > 0$ and a continuous function $K : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ such that for all control pairs*

$$(u, b), (u', b') \in \mathbb{B}_{1,1}((0,0), \widetilde{\rho}) \cap \mathbb{B}_\infty \left((\overline{u}, \overline{b}), \rho\right) \tag{38}$$

and for all initial values $x^0 \in C_{(u,b)}(0)$, $(x')^0 \in C_{(u',b')}(0)$, the sweeping processes $\left(\mathcal{S}_{(u,b)}\right)$ (with initial value x^0) and $\left(\mathcal{S}_{(u',b')}\right)$ (with initial value $(x')^0$), respectively, from (1) *have unique absolutely continuous solutions x, x', respectively, and these solutions satisfy the estimate*

$$\|x - x'\|_\infty^2 \le \left\|x^0 - (x')^0\right\|^2 + K(x^0, (x')^0) \, \|(u - u', b - b')\|_\infty.$$

Proof. Thanks to (2) for $(\overline{u}, \overline{b})$ and [28, Proposition 1] is holds that there is some $\varepsilon > 0$ such that

$$\forall t \in [0,T] \ \exists y \in \mathcal{H} : \langle \overline{u}_i(t), y \rangle \le \overline{b}_i(t) - \varepsilon \quad i = 1, \ldots, m.$$

Let $\widetilde{\rho} > 0$ be given. We fix an arbitrary $\delta \in (0, \varepsilon)$. According to [28, Lemma 1, eq. (2.14)], there exists some $\hat{x} \in \mathcal{C}([0,T], \mathcal{H})$ such that

$$\hat{x}(t) \in C^{(\delta)}_{(u,b)}(t) \quad \forall t \in [0,T] \, \forall (u,b) \in \mathbb{B}_\infty \left((\bar{u}, \bar{b}), \frac{\varepsilon - \delta}{3(1 + \|\hat{x}\|_\infty)} \right), \tag{39}$$

where $C^{(\delta)}_{(u,b)}(t) := \{x \in \mathcal{H} \mid \langle u_i(t), x\rangle \le b_i(t) - \delta$ for $i = 1, \dots, m\}$. Furthermore, thanks to [28, Corollary 1] we may associate with this $\hat{x}$ the unique solution y_δ of the sweeping process

$$-\dot{x}(t) \in N(C^{(\delta)}_{(\bar{u},\bar{b})}(t), x(t)) \quad \text{a.e. } t \in [0,T], \quad x(0) = \hat{x}^0 \in C^{(\delta)}_{(\bar{u},\bar{b})}(0).$$

In particular, $y_\delta(t) \in C^{(\delta)}_{(\bar{u},\bar{b})}(t)$ for all $t \in [0,T]$. We now set ρ in (38) as:

$$\rho := \min\left\{\frac{\delta}{1 + \|y_\delta\|_\infty}, \frac{\varepsilon - \delta}{3(1 + \|\hat{x}\|_\infty)}\right\} > 0.$$

Let (u,b) satisfying (38) and an initial value $x^0 \in C_{(u,b)}(0)$ be arbitrarily fixed. Then, the sweeping processes $\left(\mathcal{S}_{(u,b)}\right)$ (with initial value x^0) has a unique absolutely continuous solution x. This follows from Theorem 2.1 upon observing that, thanks to (39), and to the definition of $C^{(\delta)}_{(u,b)}$, $\hat{x}(t)$ is a uniform Slater point in the sense of (2) for all $t \in [0,T]$. From $y_\delta(t) \in C^{(\delta)}_{(\bar{u},\bar{b})}(t)$ for all $t \in [0,T]$ it follows that one has that

$$\begin{aligned}
&\langle u_i(t), y_\delta(t)\rangle - b_i(t) \le \langle u_i(t) - \bar{u}_i(t), y_\delta(t)\rangle + \bar{b}_i(t) - b_i(t) - \delta \\
&\le \|u - \bar{u}\|_\infty \|y_\delta\|_\infty + \|b - \bar{b}\|_\infty - \delta \\
&\le \|(u,b) - (\bar{u},\bar{b})\|_\infty (1 + \|y_\delta\|_\infty) - \delta \le 0 \quad \forall t \in [0,T]\ \forall i = 1, \dots, m.
\end{aligned}$$

It follows that

$$y_\delta(t) \in C_{(u,b)}(t) \quad \text{for all } t \in [0,T]. \tag{40}$$

We now want to follow the proof of [28, Lemma 3]. In its original form, this lemma used the condition

$$(u,b) \in \mathbb{B}_{1,1}\left((\bar{u},\bar{b}), \frac{\delta}{1 + \|y_\delta\|_\infty}\right) \tag{41}$$

for proving (40). We have shown (40) here under the condition (38) instead. Unfortunately, a second constraint

$$(u,b) \in \mathbb{B}_\infty\left((\bar{u},\bar{b}), \frac{\varepsilon - \delta}{3(1 + \|\hat{x}\|_\infty)}\right).$$

on the control has been erroneously omitted in the statement of [28, Lemma 3] although its necessity is clear from the proof (by appealing to [28, eq. (2.16)]). This is now corrected by adding the second term in the definition of ρ above. As a consequence, we may follow the proof of [28, Lemma 3] until [28, (4.31) on page 424]. Thanks to the derivation of (40) above, we can now follow line by line the proof from the top of page 425 until the end in order to dervie the estimate [28, eq. (4.28)].

Now, we may pass to the proof of [28, Theorem 4], which we will read with its original assumption [28, eq. (4.33)] replaced by our new assumption (38). We may follow the proof of [28, Theorem 4] until the estimates [28, eq. (4.39), eq.(4.40)]. Adapting theses estimates, we get for some arbitrary, fixed $\delta \in (0, \varepsilon)$ and constants ρ, C, C' as in [28, eq. (4.35), eq. (4.38)]

$$\begin{aligned}
&\int_0^t \|\dot{x}(s)\| ds \le \delta^{-1} C \left\|(u,b)\right\|_{1,1} \le \delta^{-1} C \tilde{\rho}, \\
&\int_0^t \|\dot{x}'(s)\| ds \le \delta^{-1} C' \left\|(u',b')\right\|_{1,1} \le \delta^{-1} C' \tilde{\rho}
\end{aligned} \tag{42}$$

for all $t \in [0,T]$ and all $(u,b),(u',b')$ satisfying (38). We now pick up the proof of [28, Theorem 4] until the inequality

$$\begin{aligned}&\|x(t)-x'(t)\|^2-\|x(0)-x'(0)\|^2\\&\le \delta^{-1}(C+C')\,\|(u-u',b-b')\|_\infty\int_0^t(\|\dot{x}(s)\|+\|\dot{x}'(s)\|)\,ds \quad \forall t\in[0,T]\end{aligned}$$

which by virtue of (42) can be continued as

$$\|x(t)-x'(t)\|^2\le\|x(0)-x'(0)\|^2+\underbrace{\delta^{-2}(C+C')^2\,\widetilde{\rho}}_{K(x^0,(x')^0)}\|(u-u',b-b')\|_\infty \quad \forall t\in[0,T].$$

This proves the assertion of our theorem (the continuity of the function K follows with the same argument as in the end of the proof of [28, Theorem 4]).

□

Proof of Lemma 2.7: Let $m>0$ as in (10) be given.

1 1.1 Let $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $x(t_1)\neq x(t_2)$ be given. Thanks to (6), it holds for a.e. $t\in[0,T]$ that $-\dot{x}(t)\in N_{\widetilde{C}_{(v,b,z)}(t)}(x(t))$. We will consider such t in the following:

If $\dot{x}(t)\neq 0$**:** Then it follows that $x(t)\in\partial\widetilde{C}_{(v,b,z)}(t)$. The definition of the set $\widetilde{C}_{(v,b,z)}(t)$ in (6) and the definition of the normal cone now yield that $\langle v(t)+p(z,t),x(t)\rangle=b(t)$ and that there is some $\lambda(t)>0$ such that $-\dot{x}(t)=\lambda(t)\,(v(t)+p(z,t))$. Hence, we have

$$\langle\dot{x}(t),x(t)\rangle=-\lambda(t)\langle v(t)+p(z,t),x(t)\rangle=-\lambda(t)b(t)\le-\lambda(t)m.$$

If $\dot{x}(t)=0$ Defining now $\lambda(t):=0$, we have

$$-\dot{x}(t)=\lambda(t)\,(v(t)+p(z,t))\,,\ \langle\dot{x}(t),x(t)\rangle=-\lambda(t)b(t)\le-\lambda(t)m. \tag{43}$$

Overall, we see that there is a function $\lambda:[0,T]\to[0,\infty)$ such that (43) holds for a.e. $t\in[0,T]$. Since b is continuous on $[0,T]$ and uniformly bounded from below by $m>0$ thanks to (10), we see that $1/b$ is a continuous function on $[0,T]$. Since (43) yields that $\lambda(t)=-\frac{\langle\dot{x}(t),x(t)\rangle}{b(t)}$ for a.e. $t\in[0,T]$, we see that λ is measurable. With $x(t_1)\neq x(t_2)$, we get from (43)

$$0\neq\int_{t_1}^{t_2}\dot{x}(t)dt=-\int_{t_1}^{t_2}\lambda(t)\,(v(t)+p(z,t))\,dt. \tag{44}$$

Hence, we get from (44) and $\lambda\ge0$ that $0<\int_{t_1}^{t_2}\lambda(t)dt$, since otherwise we would have $\lambda(t)=0$ for a.e. $t\in[t_1,t_2]$ and therefore a contradiction to (44). Recalling now (43), we get

$$\|x(t_2)\|^2-\|x(t_1)\|^2=2\int_{t_1}^{t_2}\langle\dot{x}(t),x(t)\rangle dt\le-2\int_{t_1}^{t_2}\lambda(t)m\,dt<0.$$

Hence, we have $\|x(t_2)\|<\|x(t_1)\|$.

1.2 Assume that $x_0 \neq 0$. To prove that $x(s) \neq 0$ for all $s \in [0,T]$ by contradiction, we assume that there is some $s \in [0,T]$ with $x(s) = 0 \neq x_0 = x(0)$. Hence, we see that $s > 0$. Recalling (6), we can find an increasing sequence $(t_k)_{n\in\mathbb{N}}$ in $(0,s)$ such that $\lim_{k\to\infty} x(t_k) = 0$, $\dot{x}(t_k) \neq 0$, and $-\dot{x}(t_k) \in N_{\widetilde{C}_{(v,b,z)}(t_k)}(x(t_k))$. The definition of the normal cone yields that $x(t_k) \in \partial\widetilde{C}_{(v,b,z)}(t_k)$. The definition of this set in (6) yields that $\langle v(t_k) + p(z,t_k), x(t_k)\rangle = b(t_k)$. Thanks to (10), $v \in W^{1,1}([0,T],\mathbb{R}^n)$, (8), (**A1**), we get the contradiction

$$\begin{aligned} 0 < m &\leq \langle v(t_k) + p(z,t_k), x(t_k)\rangle \leq \|v(t_k) + p(z,t_k)\|\|x(t_k)\| \\ &\leq \|v + p(z,\cdot)\|_\infty \|x(t_k)\| \to_{k\to\infty} 0. \end{aligned}$$

1.3 Defining $\widetilde{x} : [0,T] \to \mathbb{R}^n$ by $\widetilde{x}(t) = 0$, we get by (6), that $\widetilde{x}(t) = 0 \in \widetilde{C}(v,b,z)(t)$ for all $t \in [0,T]$ and therefore, by the definiton of the normal cone, $-\dot{\widetilde{x}}(t) = 0 \in N_{\widetilde{C}_{(v,b,z)}(t)}(\widetilde{x}(t))$ for all $t \in [0,T]$. Thanks to (6), we see that $\widetilde{x}$ is the solution to $\widetilde{\mathcal{S}}_{(v,b,z)}$ (with initial value $0 = x^0$) and therefore $x = \widetilde{x}$.

2 Let $x^0, x_1 \in \mathbb{R}^n$ with $0 < ||x_1|| < ||x^0||$ and $n > 1$ be given. Let $t_1 \in (0,T]$ be given. We can find orthonormal unit vectors $e_a, e_b \in \mathbb{R}^n$ and numbers $\lambda_{1,a}, \lambda_{1,b}$ such that $x^0 = \|x^0\| e_a$ and $x_1 = \lambda_{1,a} e_a + \lambda_{1,b} e_b$. Since $n > 1$, we can find such e_a, e_b also if x^0 and x_1 are linear dependent. Let $\omega_1 \in]-\pi,\pi]$ be given such that $\lambda_{1,a} = \|x_1\|\cos(\omega_1)$ and $\lambda_{1,b} = \|x_1\|\sin(\omega_1)$. Defining $x : [0,T] \to \mathbb{R}^n$ by

$$x(t) = \left(\left(1 - \frac{t}{t_1}\right)\|x^0\| + \frac{t}{t_1}\|x_1\|\right)\left(\cos\left(\frac{t}{t_1}\omega_1\right)e_a + \sin\left(\frac{t}{t_1}\omega_1\right)e_b\right),$$

for all $t \in [0,t_1]$ and $x(t) = x(t_1)$ for all $t \in]t_1,T]$, we see that $x(0) = x^0$, $x(t_1) = x_1$ and

$$\begin{aligned} &\dot{x}(t) \\ &= \frac{1}{t_1}\left(\|x_1\| - \|x^0\|\right)\left(\cos\left(\frac{t}{t_1}\omega_1\right)e_a + \sin\left(\frac{t}{t_1}\omega_1\right)e_b\right) \\ &+ \left(\left(1 - \frac{t}{t_1}\right)\|x^0\| + \frac{t}{t_1}\|x_1\|\right)\frac{\omega_1}{t_1}\left(-\sin\left(\frac{t}{t_1}\omega_1\right)e_a + \cos\left(\frac{t}{t_1}\omega_1\right)e_b\right), \end{aligned} \tag{45}$$

for all $t \in [0,t_1]$. Therefore, we have $x \in W^{1,\infty}([0,T],\mathbb{R}^n)$.

Hence, we have, thanks to the orthonormality of e_a and e_b:

$$\begin{aligned} \mu(t) := \langle \dot{x}(t), x(t)\rangle = &\left(\left(1 - \frac{t}{t_1}\right)\|x^0\| + \frac{t}{t_1}\|x_1\|\right)\frac{1}{t_1}\left(\|x_1\| - \|x^0\|\right) \\ &\leq \|x_1\|\frac{1}{t_1}\left(\|x_1\| - \|x^0\|\right) < 0, \quad \forall t \in [0,t_1]. \end{aligned}$$

Recalling now (45) and the definition of m in (10), we see that the function $v : [0,T] \to \mathbb{R}^n$ with $v(t) := m\dot{x}(t)/\mu(t)$ for $t \in [0,t_1]$ and $v(t') = v(t_1)$ for all $t' \in]t_1,t]$ belongs to $W^{1,1}([0,T],\mathbb{R}^n)$. Since (8) yields that $p(0,t) = 0$ for all $t \in [0,T]$, we have

$$\begin{aligned} -\dot{x}(t) = \frac{-\mu(t)}{m}v(t) &= \frac{-\mu(t)}{m}\left(v(t) + p(0,t)\right), \quad -\dot{x}(t') = 0, \\ \langle v(t) + p(0,t), x(t)\rangle &= \frac{m}{\mu(t)}\langle \dot{x}(t), x(t)\rangle = m, \\ \langle v(t') + p(0,t'), x(t')\rangle &= \langle v(t_1), x(t_1)\rangle = m, \end{aligned}$$

for all $t \in [0, t_1]$ and all $t' \in]t_1, T]$. Hence, defining $b : [0, T] \to \mathbb{R}$ by $b \equiv m$ and recalling (6), we have $x(t) \in \widetilde{C}_{(v,b,0)}(t)$ and $-\dot{x}(t) \in N_{\widetilde{C}_{(v,b,0)}(t)}$ for all $t \in [0, T]$. Hence, we see that x is a solution to $(\widetilde{\mathcal{S}}_{(v,b,0)})$.

□

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