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Chance-constrained linear complementarity problems

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ABSTRACT. We study linear complementarity problems (LCPs) under uncertainty, which we model using chance constraints. Since the complementarity condition of the LCP is an equality constraint, it is required to consider relaxations, which naturally leads to optimization problems in which the relaxation parameters are minimized for given probability levels. We focus on these optimization problems and first study the continuity of the related probability functions and the compactness of the feasible sets. This leads to existence results for both types of models: one with a joint chance constraint and one with separate chance constraints for both uncertainty-affected conditions of the LCP. For both, we prove the differentiability of all probability functions and derive respective gradient formulae. For the separate case, we prove convexity of the respective optimization problem and use the gradient formulae to derive necessary and sufficient optimality conditions. In a small case study regarding a Cournot oligopoly among energy producers, we finally illustrate the applicability of our theoretical findings.

1. INTRODUCTION

Linear complementarity problems (LCP) are an important tool in many fields of mathematics such as optimization or game theory as well as in many applications like contact mechanics, optimal stopping, or energy markets (Cottle et al. 2009; Gabriel et al. 2012). While the deterministic setting, in which all data describing the instance at hand is known, is very mature with a rich theory and and many effective algorithms, the study of LCPs with uncertain data is still much less developed. Nevertheless, application areas such as the modeling of energy markets using complementarity problems very often require the treatment of uncertain data. In mathematical optimization, two fields emerged to deal with uncertainties: stochastic (Birge and Louveaux 2011; Kall and Wallace 1994) and robust optimization (Ben-Tal et al. 2009; Bertsimas et al. 2011). While the latter usually does not need to make assumptions about the distributions of the uncertain parameters, it is often criticized due to its conservatism. This is why more sophisticated models of robustness have been developed in the past such as Γ-robustness by Sim (2004) and Bertsimas and Sim (2004) or adjustable robustness; see, e.g., Ben-Tal et al. (2009), Ben-Tal et al. (2004), and Yanikoglu et al. (2019). On the contrary, stochastic optimization explicitly uses distributional information and considers the optimization of stochastic measures such as the expected value or the (conditional) value-at-risk, to just name a few prominent ones.

Both paradigms have also been applied to LCPs. In the stochastic setting, most of the papers consider the minimization of the so-called expected residual gap function of the LCP (Chen et al. 2009; Chen and Fukushima 2005; Chen et al. 2012; Lin and Fukushima 2006). The robust treatment started with the paper by Wu et al. (2011), where the authors apply the concept of strict robustness to LCPs. The same path is followed by Xie and Shanbhag (2014) and Xie and Shanbhag (2016), where different uncertainty sets such as box and ellipsoidal uncertainties are considered and where a special focus is on the tractability of the resulting robust counterparts. The results of the two latter papers are then applied to Cournot–Bertrand equilibria on power networks by Mather and Munsing (2017), whereas Γ-robust LCP models for Nash–Cournot and perfect competition equilibria are studied by Çelebi et al. (2023) and Kramer et al. (2021). For Γ-robust LCPs, the basic theory has been developed by Krebs et al. (2022) and Krebs and Schmidt (2022), covering again different uncertainty sets such ℓ_1- and box-uncertainty sets as well as ellipsoids. Finally, the robust treatment of LCPs has also been studied in adjustable settings by Biefel et al. (2022) and Biefel and Schmidt (2024).

Another prominent way to tackle uncertainty in optimization is the use of chance constraints. Here, a decision is declared to be feasible if the probability of satisfying some random inequality constraints exceeds a given safety level. A modern introduction to this topic is contained, for instance, in the monograph by van Ackooij and de Oliveira (2025). Chance constraints have proven to provide an

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efficient tool for risk-averse decisions over the last decades, first of all in engineering problems and problems of energy management. While their focus was originally on finite-dimensional conventional optimization problems, attention has recently shifted towards optimal control for ODEs, PDEs, or sweeping processes (Farshbaf-Shaker et al. 2018; Henrion et al. 2025) and hierarchical optimization problems (Heitsch et al. 2022). Chance constraints have also been used in game theory; see, e.g., Singh et al. (2016) for a primer. For instance, Singh et al. (2017) consider random bimatrix games using chance constraints. They relate the solutions of such a random game to an appropriately chosen nonlinear complementarity problem. A similar approach has been followed by Riccardi et al. (2023) for an empirical analysis of the Italian electricity market. However, the complementarity problems themselves are not directly tackled using chance constraints.

The contribution of this paper is to directly apply chance-constrained modeling to LCPs. The use of chance constraints in connection with complementarity problems has recently been initiated in the engineering literature, namely in robotics, by Drnach et al. (2022) and Shirai et al. (2023). The approach chosen there relies on a complete individualization of chance constraints, which drastically simplifies the analysis and numerics, but provides quite conservative approximations of the probability control for the entire random LCP system. In the present paper, we address random LCPs by chance constraints that are joint with respect to the entire system. As an alternative, we also consider a model separating the complementarity equality from the remaining affine inequality constraints. It is emphasized that the latter are still kept as joint chance constraints, in contrast to the complete individualization mentioned before. We provide a detailed structural analysis along with a numerical illustration for both models.

In the most general case, both the matrix M and the vector q that define the LCP are considered to be uncertain. While some of our results apply to this very general setting, our core contributions are related to the case in which the LCP's matrix M is known and only the vector q is subject to uncertainty. The complementarity condition itself is an equality constraint that would lead to ill-posed chance-constrained models. Hence, we relax this equality and consider optimization models for the uncertain LCP in which, for given probability levels, the relaxation parameter is minimized. Our core contributions are the following.

(i) We formalize chance-constrained LCP models, motivate the required relaxations, and present their optimization counterparts; see Section 2.
(ii) We prove the continuity of the respective probability functions and the compactness of the feasible sets under mild conditions, leading to existence results for both optimization problems—one having joint and the other one having separate chance constraints for the two uncertainty-affected LCP conditions. These results are provided in Section 3.
(iii) For the model with a separate chance constraint for the complementarity condition, we show the convexity of the respective optimization model under the assumption of a Gaussian distribution for the vector q; see Section 4.
(iv) We prove the differentiability of all studied probability functions and derive gradient formulae (Section 5) that we then use (Section 6) to develop optimality conditions.
(v) By relying on all theoretical results derived so far, we present a small case study using chance-constrained LCPs to model a simple Cournot oligopoly among energy producers; see Section 7. Here, we compare the results between the joint and the separated chance-constrained model as well as the expected-value model.

2. Problem Statement

The linear complementarity problem (LCP) is the problem to find a vector $z \in \mathbb{R}^n$ that satisfies the conditions

$$z \geq 0, \tag{1a}$$

$$q + Mz \geq 0, \tag{1b}$$

$$z^\top (q + Mz) = 0 \tag{1c}$$

or to show that no such vector exists. In the deterministic setting of (1), $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$ are a given vector and a given matrix in appropriate dimensions. We also abbreviate the problem as LCP(q, M). For more details on this problem we refer to the seminal textbook by Cottle et al. (2009).

2.1. Chance-Constrained LCPs.

In many applications of LCPs, the given data, i.e., q and M are not known with certainty but are subject to some uncertainty. To model this, we replace q and M with random variables q^ξ and M^ξ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, having a known joint distribution $(q^\xi, M^\xi) \sim \mathcal{D}$. Then, for a concrete realization of the random parameter, (1) becomes the random system

$$z \geq 0, \quad q^\xi(\omega) + M^\xi(\omega) z \geq 0, \quad z^\top (q^\xi(\omega) + M^\xi(\omega) z) = 0. \tag{2}$$

For a proper interpretation of this system, one has to make a choice on the chronology of searching z and of observing the realizations of the random parameter. If z is searched after observing randomness ("wait-and-see"), then z would become a random variable itself and one could investigate its distribution. We shall rather consider a decision on z before randomness reveals itself ("here-and-now"). Then, however, (2) does not make sense due to the presence of unknown data. A remedy is to make z only depend on the distribution $\mathcal{D}$ of (q^ξ, M^ξ) rather than on its realizations. A popular way to do so is to formulate a so-called chance constraint, which requires that the random constraints in (2) are satisfied with a given minimum probability $p \in (0, 1]$:

$$z \geq 0, \quad \mathbb{P}\left(q^\xi + M^\xi z \geq 0,\, z^\top (q^\xi + M^\xi z) = 0\right) \geq p. \tag{3}$$

This model relates to the probability of simultaneously satisfying both random conditions. In many applications, both conditions might have different meanings so that it might make sense to consider them separately in the probabilistic model using different probabilities:

$$z \geq 0, \quad \mathbb{P}(q^\xi + M^\xi z \geq 0) \geq p_1, \quad \mathbb{P}(z^\top (q^\xi + M^\xi z) = 0) \geq p_2. \tag{4}$$

Note that the first chance constraint of (4) is still of joint type in that it does not transform the single random constraints $q_i^\xi + M_i^\xi z \geq 0$ into individual chance constraints.

The logical inconsistency in both models (3) and (4) relies on the fact that, under continuous distributions $\mathcal{D}$, the probability of satisfying an equation reduces to zero in general, so that no levels $p, p_2 > 0$ can ever be realized. Trivially, the equation inside the probability of (3) can be replaced by an inequality of type "≤ 0" thanks to the first inequality and to $z \geq 0$. This, however, is just a reformulation of the original problem. While it looks like the model would be fine then because both expressions inside the probability are inequalities, the implicit presence of an equality is not removed, and so the entire probability would still reduce to zero in general. However, passing to an inequality in the second expression suggests the way to get rid of the problem, namely by relaxing this inequality with a positive parameter $\rho \geq 0$:

$$z \geq 0, \quad \mathbb{P}(q^\xi + M^\xi z \geq 0,\, z^\top (q^\xi + M^\xi z) \leq \rho) \geq p. \tag{5}$$

The need of relaxing the complementarity condition in the context of chance constraints has already been observed in Drnach et al. (2022) and Shirai et al. (2023) albeit not with respect to the entire scalar product but with respect to each of its summands.

With regard to the separate model (4), the argumentation is slightly different. When replacing the equation inside the second probability by an inequality of type "≤ 0", one would arrive at two reasonable probability expressions—none of which would imply the other to reduce to zero. So, unlike the joint model discussed before, there seems to be no reason to relax the second inequality. However, considering the constraint system

$$z \geq 0, \quad \mathbb{P}(q^\xi + M^\xi z \geq 0) \geq p_1, \quad \mathbb{P}(z^\top (q^\xi + M^\xi z) \leq 0) \geq p_2$$

would drastically reduce the choice of probabilities p_1, p_2 to unreasonably small values as it is shown in the next example.

Example 1. Let $n = 1$, $M^\xi \equiv 1$, and $q^\xi \sim \mathcal{N}(\mu, \sigma)$. Then, it is easy to check that

$$\min\left\{\mathbb{P}(q^\xi + M^\xi z \geq 0), \mathbb{P}(z^\top (q^\xi + M^\xi z) \leq 0)\right\} \leq 0.5 \quad \forall z \in \mathbb{R}.$$

This implies that the set of feasible z is empty, whenever both probability levels p_1, p_2 are larger than 0.5.

However, when relaxing the second inequality as in the joint model before, the feasible set will turn out to be non-empty for all probability levels smaller than one provided that the relaxation parameter is sufficiently large; see Proposition 11 below. This leads us to formulate the following relaxed version of the separate model (4):

$$z \geq 0, \quad \mathbb{P}(q^\xi + M^\xi z \geq 0) \geq p_1, \quad \mathbb{P}(z^\top (q^\xi + M^\xi z) \leq \rho) \geq p_2. \tag{6}$$

Remark 2. It is evident from the definitions, that every (z, ρ) that is feasible for (5), is also feasible for (6) provided that $p \geq \max\{p_1, p_2\}$.

Remark 3. Although the setting in (5) with joint chance constraints seems to be the adequate one by considering the probability of the entire uncertain LCP, Model (6) has certain structural advantages over (5) in that it will lead to convex optimization problems. Moreover, it can be related with (5) in a certain way; see Remark 4.

Remark 4. The relaxation of the complementarity constraint (1c) just by a positive right-hand side ρ might leave the impression that the negative deviation from zero remains uncontrolled. However, thanks to the nonnegativity of z, it can be immediately seen that (5) is equivalent to

$$z \geq 0, \quad \mathbb{P}(q^\xi + M^\xi z \geq 0,\, z^\top (q^\xi + M^\xi z) \in [0, \rho]) \geq p.$$

Moreover, the probabilities p_1, p_2 in the separate model (6) can be adjusted to guarantee feasible points of the joint model (5): A feasible point of (6) is also feasible for (5), whenever $p_1 + p_2 \geq 1 + p$. This can always be achieved by choosing $p_1 := p_2 := (1 + p)/2$. The statement follows immediately from the general relation $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$ for arbitrary events.

For the remainder of the paper, we make the following assumption.

Assumption 1. *The LCP's matrix is known, i.e., not subject to uncertainty: $M^\xi \equiv M$.*

While the last assumption is of course restrictive, the case of only q being affected by uncertainty still has many applications; see, e.g., the Cournot oligopoly problem considered in Section 7.

2.2. **Optimization Versions of the Problems.** It is clear that larger values of ρ also allow for larger values of p or p_2 and vice versa. Thus, we can also state related optimization problems. To this end, we always assume the probability levels (p or (p_1, p_2)) to be given and try to minimize the relaxation parameter ρ. This leads to the problem

$$\min_{z,\rho} \quad \rho \quad \text{s.t.} \quad (5),\ \rho \geq 0 \tag{7}$$

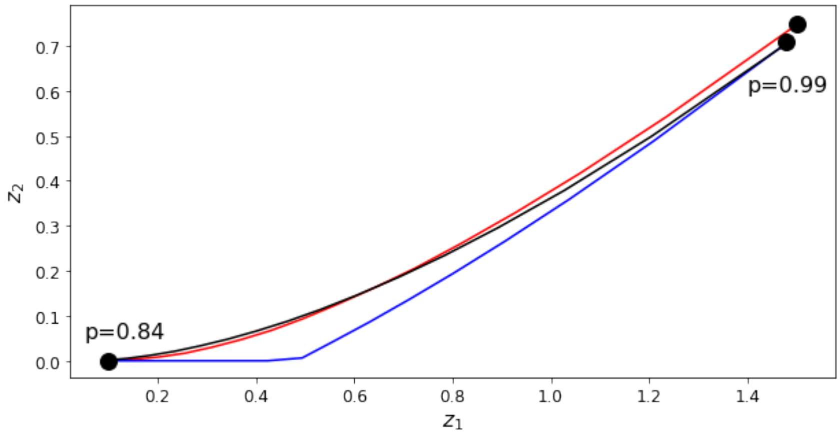

FIGURE 1. Plot of the $z = (z_1, z_2)$-part of the solution to Problem (7) for different values of p ranging from 0.84 to 0.99 (red). For the same variation of p, the corresponding solutions to Problem (8) are plotted for the settings $p_1 = p_2 = p$ (blue) and $p_1 = p$, $p_2 = 0.99$ (black) are also plotted.

for the jointly chance-constrained case as well as to

$$\min_{z,\rho} \quad \rho \quad \text{s.t.} \quad (6), \ \rho \geq 0. \tag{8}$$

for the other case.

We note that both problems are nonlinear optimization problems for which the probabilistic parts of the constraints are given by inequalities $\varphi(z,\rho) \geq p$ as well as $\varphi_1(z) \geq p_1, \varphi_2(z,\rho) \geq p_2$, respectively, with

$$\varphi(z,\rho) := \mathbb{P}(q^\xi + Mz \geq 0, \ z^\top(q^\xi + Mz) \leq \rho), \tag{9}$$

$$\varphi_1(z,\rho) := \mathbb{P}(q^\xi + Mz \geq 0), \tag{10}$$

$$\varphi_2(z,\rho) := \mathbb{P}(z^\top(q^\xi + Mz) \leq \rho). \tag{11}$$

Example 5. For the sake of illustration, we consider a two-dimensional instance for both optimization problems (7) and (8), where we choose M as the identity matrix and the distribution

$$q^\xi \sim \mathcal{N}(\mu, \Sigma), \quad \mu := (1,2)^\top, \quad \Sigma = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

for the random parameter. Figure 1 shows the $z = (z_1, z_2)$-part of the solution of Problem (7) in red for different values of p ranging from 0.84 to 0.99. Further, the corresponding results for the solutions of problem (8) are displayed for the settings $p_1 = p_2 = p$ (blue) and $p_1 = p, p_2 = 0.99$ (black). Figure 2 displays the minimum ρ-values associated with the p-values in the same setting as in Figure 1. Overall, the two models produce quite similar solutions, in particular if in the separate model one of the probabilities is high and the other one corresponds to the probability in the joint model; see the red and black curves in Figure 1. It is also clear that, for given p the minimal ρ is larger in the joint model than in the separate model with all equal probabilities because then the feasible set of (5) is than contained in that of (6); see Figure 2.

3. Continuity of the Probability Functions and Existence of Solutions

The following example shows that even for a nice joint distribution as the Gaussian one, continuity of the probability functions cannot be taken for granted in all cases without further assumptions. We

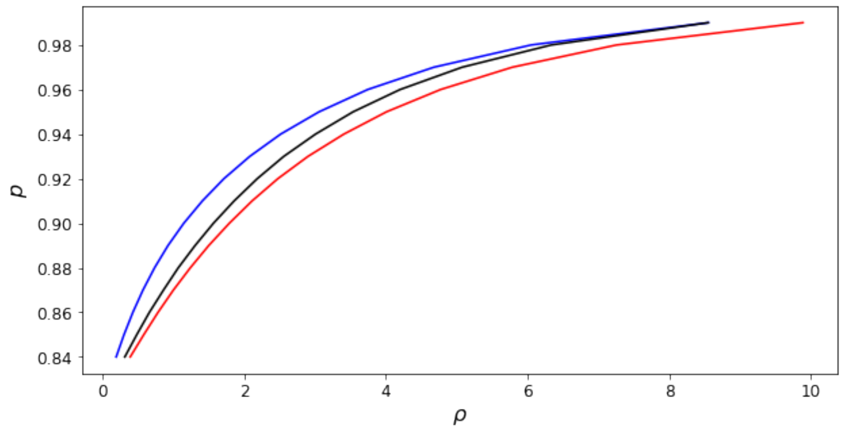

FIGURE 2. p/ρ-diagrams for the solutions to problems (7) and (8); same setting as in Figure 1.

shall make use of the notation $X \sim \mathcal{N}(\mu, \Sigma)$ to indicate that a random variable X follows a (regular) Gaussian distribution with mean μ and (positive definite) covariance matrix Σ.

Example 6. In dimension $n = 1$, let the two-dimensional random vector (q^ξ, M^ξ) have a bivariate standard normal distribution, i.e., $(q^\xi, M^\xi) \sim \mathcal{N}(0_2, I_2)$. Then, $\varphi_2(0, 0) = 1$, whereas, for any $\varepsilon > 0$, one has that $\varepsilon q^\xi + \varepsilon^2 M^\xi \sim \mathcal{N}(0, \varepsilon^2 + \varepsilon^4)$, whence

$$\varphi_2(\varepsilon, 0) = \mathbb{P}(\varepsilon q^\xi + \varepsilon^2 M^\xi \le 0) = 0.5 \quad \forall \varepsilon > 0.$$

It follows that φ_2 fails to be continuous, and in particular to be lower semicontinuous, at $(0, 0)$.

However, the example above evokes an exceptional setting and continuity of all three probability functions may be expected to hold true under mild assumptions. For our first result, we can easily consider the case of uncertain LCP matrix M so that we state it in full generality here although we will mainly focus on the case of deterministic M later on; see Assumption 1.

Proposition 7. *Consider the probability functions φ, φ_1, and φ_2 defined in* (9)–(11) *and restrict them to arguments $z \in \mathbb{R}^n_{\ge 0}$ and $\rho \in \mathbb{R}_{\ge 0}$. Then, all of them are upper semicontinuous. If the joint distribution of (q^ξ, M^ξ) has a density, then φ_1 is lower semicontinuous (and hence continuous), whereas φ and φ_2 are lower semicontinuous (and hence continuous) except possibly at $(z, \rho) = (0, 0)$.*

Proof. We apply well-known results for the semicontinuity of probability functions. To this aim, we define functions $g_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ by

$$g_i(z, \rho, y, A) := (y + Az)_i \quad (i = 1, \dots, n); \quad g_{n+1}(z, \rho, y, A) := \rho - z^\top (y + Az)$$

and observe that with

$$g := \min_{i=1,\dots,n+1} g_i; \quad g^{(1)} := \min_{i=1,\dots,n} g_i; \quad g^{(2)} := g_{n+1},$$

we get the representations (with compressed alternative cases)

$$[\varphi/\varphi_1/\varphi_2](z, \rho) = \mathbb{P}([g/g^{(1)}/g^{(2)}](z, \rho, q^\xi, M^\xi) \ge 0).$$

Clearly, all g_i are continuous and so are g, $g^{(1)}$, and $g^{(2)}$. This immediately implies the upper semicontinuity of the three probability functions without further assumptions on the random vectors; see, e.g., Lemma 2 by Farshbaf-Shaker et al. (2018).

The complementary lower semicontinuity property of the probability functions at an arbitrarily fixed $(z, \rho) \in \mathbb{R}^n_{\ge 0} \times \mathbb{R}_{\ge 0}$ will follow under the additional conditions

(12) $$\mathbb{P}([g/g^{(1)}/g^{(2)}](z, \rho, q^\xi, M^\xi) = 0) = 0;$$

see again Lemma 2 by Farshbaf-Shaker et al. (2018). Clearly all three conditions would follow from the stronger condition

$$\mathbb{P}(g_i(z,\rho,q^\xi,M^\xi)=0)=0 \quad (i=1,\dots,n+1), \tag{13}$$

where only the first n equations are relevant for $g^{(1)}$. Under our additional assumption (for the lower semicontinuity part) that the joint distribution of (q^ξ, M^ξ) has a density, it is sufficient to verify that the sets

$$H_i := \{(y,A) \mid g_i(z,\rho,y,A)=0\} \quad (i=1,\dots,n+1)$$

have Lebesgue measure zero. This is evident for $i=1,\dots,n$ because the corresponding H_i are hyperplanes in the space of variables (y,A). Hence, the first n equations in (13) are satisfied. This implies that (12) is satisfied for $g^{(1)}$ and, hence, φ_1 is lower semicontinuous at (z,ρ). As for H_{n+1}, it is an affine hyperplane in the space of variables (y,A) in the case of $z\neq 0$. If $z=0$ and $\rho>0$, then $H_{n+1}=\emptyset$ has trivially Lebesgue measure zero. Hence, φ and φ_2 are lower semicontinuous except possibly at $(z,\rho)=(0,0)$. This completes the proof. □

As an immediate consequence of the upper semicontinuity results above, and, without any further assumptions we have the following.

Corollary 8. *The feasible sets of the optimization problems* (7) *and* (8) *are closed.*

Directing our attention now to the existence of solutions for the optimization problems (7) and (8), a key argument will be related to the boundedness of certain feasible sets. Note that no evident boundedness is enforced in those problems by additional simple constraints on z such as box constraints. We just require the non-negativity condition $z\geq 0$ in the LCP.

Proposition 9. *Suppose Assumption 1 holds and let M be positive definite. Moreover, let $\rho\geq 0$ and $p\in(0,1]$ be given. Then, the set*

$$Z := \left\{z\in\mathbb{R}^n \mid \mathbb{P}(z^\top(q^\xi+Mz)\leq\rho)\geq p\right\} \tag{14}$$

is bounded.

Proof. Assume to the contrary that the set (14) fails to be bounded. Thus, there exists a sequence $\{z^k\}_k\subseteq Z$ with $\|z^k\|\to\infty$ for $k\to\infty$. We define the norm induced by the positive definite matrix M by $\|z^k\|_M := \sqrt{z^\top Mz}$ for all $z\in\mathbb{R}^n$. It follows from norm equivalence that

$$(z^k)^\top Mz^k = \|z^k\|_M^2 \to_k \infty.$$

Define next the sequence of half spaces

$$H_k := \left\{y\in\mathbb{R}^n \mid (z^k)^\top y\leq \rho-\|z^k\|_M^2\right\}, \quad k\in\mathbb{N}.$$

We shall denote by H_k^c the complements of these half spaces. Let us also consider a centered ball $\mathbb{B}(0,\ell)$ of arbitrary but fixed radius $\ell\in\mathbb{N}$. We claim that there exists a $k=k(\ell)$ such that

$$\mathbb{B}(0,\ell)\subseteq H_{k'}^c \quad \text{holds for all} \quad k'\geq k. \tag{15}$$

To this aim, let $y\in\mathbb{B}(0,\ell)$ be chosen arbitrarily. Define $k:=k(\ell)$ such that

$$-\ell\|z^{k'}\| > \rho-\|z^{k'}\|_M^2 \quad \text{for all} \quad k'\geq k$$

(by equivalence of norms). Therefore,

$$(z^{k'})^\top y \geq -\|z^{k'}\|\|y\| \geq -\ell\|z^{k'}\| > \rho-\|z^{k'}\|_M^2 \quad \text{for all} \quad k'\geq k.$$

Hence, $y\in H_{k'}^c$ for all $k'\geq k$, which proves (15). Since $\ell\in\mathbb{N}$ was arbitrary, we can now find a sequence k_ℓ such that $\mathbb{B}(0,\ell)\subseteq H_{k'}^c$ holds for all $k'\geq m_\ell := \max\{k_\ell,\ell\}$, where $m_\ell\to_{\ell\to\infty}\infty$. Consequently, we get that $\mathbb{B}(0,\ell)\subseteq H_{m_\ell}^c$ for all $\ell\in\mathbb{N}$. This implies that $\mathbb{P}(q^\xi\in H_{m_\ell}^c)\to_{\ell\to\infty} 1$ and, hence, $\mathbb{P}(q^\xi \in H_{m_\ell}) \to_{\ell\to\infty} 0$. Since $p > 0$ by assumption, this means that there exists an index $\bar{\ell}$ so that $\mathbb{P}(q^\xi \in H_{m_{\bar{\ell}}}) < p$. In other words, $z^{m_{\bar{\ell}}} \notin Z$, which is a contradiction to our assumption. □

We note that a similar boundedness result does not apply to the chance constraint $\mathbb{P}(q^\xi + Mz \geq 0) \geq p_1$. We are now in a position to derive existence of solutions for our optimization problems.

Theorem 10. *Suppose Assumption 1 holds and that M is positive definite. Moreover, let $p \in (0, 1)$ be arbitrary. Then, Problem* (7) *has a solution.*

Proof. Recall, that (7) is a minimization problem. Consequently, we may prove the existence of solutions by showing that some lower level set of the objective function intersected with the feasible set is non-empty and compact. Since the objective function f of (7) is $f(z, \rho) = \rho$, this amounts to finding some $\alpha \in \mathbb{R}$ such that the set

$$X_\alpha := \left\{(z, \rho) \in \mathbb{R}^{n+1} \mid z \geq 0,\ \rho \in [0, \alpha],\ \mathbb{P}(q^\xi + Mz \geq 0,\ z^\top(q^\xi + Mz) \leq \rho) \geq p\right\}$$

is non-empty and compact.

We recall the fact that, as a positive definite matrix, M disposes of some $z^* \geq 0$ such that $(Mz^*)_i > 0$ for $i = 1, \ldots, n$; see Lemma 3.1.3 by Cottle et al. (2009). Define $s := \min\{(Mz^*)_i \mid i = 1, \ldots, n\} > 0$ and the sequence $z_k := kz^*$. In particular, $z_k \geq 0$ for all $k \in \mathbb{N}$. Then,

$$H_k := \{y \in \mathbb{R}^n \mid y_i \geq -ks \quad (i = 1, \ldots, n)\} \quad (k \in \mathbb{N})$$

is a monotonically increasing sequence of sets with $\cup_{k=1}^\infty H_k = \mathbb{R}^n$. It follows that

$$\mathbb{P}(q^\xi \in H_k) \to 1 \quad \text{for} \quad k \to \infty.$$

We clearly have the relation

$$H_k \subseteq \{y \in \mathbb{R}^n \mid y + Mz_k \geq 0\} \quad (k \in \mathbb{N}),$$

which entails that

$$\mathbb{P}(q^\xi + Mz_k \geq 0) \to 1 \quad \text{for} \quad k \to \infty.$$

By our assumption $p < 1$, we may choose $k_1 \in \mathbb{N}$ large enough such that

(16) $$\mathbb{P}(q^\xi + Mz_{k_1} \geq 0) \geq (1 + p)/2.$$

Next, we define another increasing sequence of sets

$$W_\ell := \left\{y \in \mathbb{R}^n \mid z_{k_1}^\top(y + Mz_{k_1}) \leq \ell\right\} \quad (\ell \in \mathbb{N})$$

with $\cup_{\ell=1}^\infty W_\ell = \mathbb{R}^n$. It follows that

$$\mathbb{P}(q^\xi \in W_\ell) = \mathbb{P}(z_{k_1}^\top(q^\xi + Mz_{k_1}) \leq \ell) \to 1 \quad \text{for} \quad \ell \to \infty.$$

Again, we may choose ℓ_1 large enough such that

(17) $$\mathbb{P}(z_{k_1}^\top(q^\xi + Mz_{k_1}) \leq \ell_1) \geq (1 + p)/2.$$

Exploiting the general relation $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$ for arbitrary events, we derive from (16) and (17) that

$$\mathbb{P}(q^\xi + Mz_{k_1} \geq 0,\ z_{k_1}^\top(q^\xi + Mz_{k_1}) \leq \ell_1) \geq p.$$

In other words, $(z_{k_1}, \ell_1) \in X_{\ell_1} \neq \emptyset$ and it remains to show that X_{ℓ_1} is compact. Since we have that $X_{\ell_1} = Q \cap (\mathbb{R}^n \times [0, \ell_1])$, where Q is the feasible set of (7) and, since Q is closed by Corollary 8, it follows that X_{ℓ_1} is closed. Finally, observe that

$$\begin{aligned}\mathbb{P}(z^\top(q^\xi + Mz) \leq \ell_1) &\geq \mathbb{P}(z^\top(q^\xi + Mz) \leq \rho) \\ &\geq \mathbb{P}(q^\xi + Mz \geq 0,\ z^\top(q^\xi + Mz) \leq \rho) \quad \forall z \in \mathbb{R}^n\ \forall \rho \in [0, \ell_1].\end{aligned}$$

Therefore,

$$X_{\ell_1} \subseteq \left\{(z,\rho) \in \mathbb{R}^{n+1} \mid z \geq 0,\ \rho \in [0,\ell_1],\ \mathbb{P}(z^\top(q^\xi + Mz) \leq \ell_1) \geq p\right\}$$
$$\subseteq \left\{z \in \mathbb{R}^n \mid \mathbb{P}(z^\top(q^\xi + Mz) \leq \ell_1) \geq p\right\} \times [0,\ell_1].$$

The first factor in the last expression is bounded in $\mathbb{R}^n$ as a consequence of Proposition 9. Therefore, X_{ℓ_1} is bounded as was to be shown. □

Theorem 11. *Suppose that Assumption 1 holds, that M is positive definite, and that $p_1, p_2 \in (0,1)$. Assume further that the support of q^ξ is the entire space. Then, Problem* (8) *has a solution.*

Proof. We follow the same approach as in the proof of Theorem 10 and, accordingly, have to show the existence of some $\alpha \in \mathbb{R}$ such that this time the set

$$X^*_\alpha := \{(z,\rho) \in \mathbb{R}^{n+1} \mid z \geq 0,\ \rho \in [0,\alpha],\ \mathbb{P}(q^\xi + Mz \geq 0) \geq p_1,$$
$$\mathbb{P}(z^\top(q^\xi + Mz) \leq \rho) \geq p_2\}$$

is non-empty and compact. We apply Theorem 10 with $p := \max\{p_1, p_2\} \in (0,1)$ and recall that we have constructed in the proof of that proposition some $\ell_1 \in \mathbb{N}$ such that $X_{\ell_1} \neq \emptyset$ with X_α defined there. Since $X_{\ell_1} \subseteq X^*_{\ell_1}$ as a consequence of Remark 2, it follows that $X^*_{\ell_1}$ is non-empty as well.

Since we have that $X^*_{\ell_1} = Q \cap (\mathbb{R}^n \times [0,\ell_1])$, where Q is the feasible set of (6) and, since Q is closed by Corollary 8, it follows that $X^*_{\ell_1}$ is closed. Finally, from

$$\mathbb{P}(z^\top(q^\xi + Mz) \leq \ell_1) \geq \mathbb{P}(z^\top(q^\xi + Mz) \leq \rho) \quad \forall z \in \mathbb{R}^n\ \forall \rho \in [0,\ell_1]$$

it follows that

$$X^*_{\ell_1} \subseteq \left\{(z,\rho) \in \mathbb{R}^{n+1} \mid z \geq 0,\ \rho \in [0,\ell_1],\ \mathbb{P}(z^\top(q^\xi + Mz) \leq \ell_1) \geq p_2\right\}$$
$$\subseteq \left\{z \in \mathbb{R}^n \mid \mathbb{P}(z^\top(q^\xi + Mz) \leq \ell_1) \geq p_2\right\} \times [0,\ell_1].$$

The first factor in the last expression is bounded in $\mathbb{R}^n$ as a consequence of Proposition 9. Therefore, $X^*_{\ell_1}$ is bounded and altogether compact, as was to be shown. □

The following one-dimensional example shows why it is essential for the existence results of Theorem 10 and 11 that M is positive definite and not just positive semi-definite.

Example 12. Let $n = 1$, $M := 0$, and let q^ξ have a one-dimensional standard Gaussian distribution. Then, for all $z \in \mathbb{R}$, $\mathbb{P}(q^\xi + Mz \geq 0) = \mathbb{P}(q^\xi \geq 0) = 0.5$. This implies that the feasible set in (3) is empty whenever $p > 0.5$. Similarly, the feasible set in (4) is empty whenever $p_1 > 0.5$. As a consequence, both problems (7) and (8) have no solutions unless the probability levels are restricted to rather low levels.

4. Convexity

We now turn to convexity properties of the formulated optimization problems. While it is not clear, whether on the basis of existing theory the feasible set of (7) is convex or not, we will be able to discern the convexity of the feasible set of (8) in the separate model. This will lay the foundations for deriving necessary and sufficient optimality conditions later on. The result will require a further refinement of our distribution assumptions towards Gaussian ones.

Theorem 13. *Suppose that Assumption 1 holds and let M be positive semi-definite. Let furthermore q^ξ have a Gaussian distribution according to $q^\xi \sim \mathcal{N}(\mu,\Sigma)$ and assume that $p_2 \geq 0.5$. Then,* (8) *is a convex optimization problem in its equivalent description*

(18) $$\min_{z,\rho} \quad \rho \quad \textit{s.t.} \quad \log p_1 - \log \varphi_1(z,\rho) \leq 0,\ \ \alpha(z,\rho) \leq 0,\ \ \rho \geq 0,\ \ z \geq 0,$$

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where

$$\alpha(z,\rho) := \Phi^{-1}(p_2)\sqrt{z^\top \Sigma z} + z^\top \mu + z^\top M z - \rho$$

and Φ denotes the cumulative distribution function of the one-dimensional standard Gaussian distribution.

Proof. Taking into account that the objective function and the non-probabilistic constraints of Problem (8) are linear, it remains to prove the convexity of the chance constraints in (6), which, by referring to (10) and (11), may compactly be written as $\varphi_1(z,\rho) \geq p_1$ and $\varphi_2(z,\rho) \geq p_2$, respectively. Unfortunately, these inequalities as they are do not yet provide a convex description of the feasible set because the probability functions φ_1, φ_2 are never concave. However, a certain equivalent reformulation of the same feasible set is possible. As far as φ_1 is concerned, the log-concavity of this probability function is an immediate consequence of the log-concavity of the Gaussian density function of q^ξ; see, e.g., Theorem 10.2.1 by Prékopa (1995). Moreover, thanks to the Gaussian density being positive on the whole space, we infer that $\varphi_1(z,\rho) > 0$ for all (z,ρ). Thus, the log is well-defined (finite-valued) at all arguments and the original chance constraint $\varphi_1(z,\rho) \geq p_1$ can be equivalently reformulated as the convex inequality $\log p_1 - \log \varphi_1(z,\rho) \leq 0$. Coming to φ_2 now, we recall that $q^\xi \sim \mathcal{N}(\mu,\Sigma)$ implies

$$\eta := z^\top q^\xi \sim \mathcal{N}(z^\top \mu, z^\top \Sigma z) \quad \forall z \in \mathbb{R}^n,$$

whence

$$\tilde{\eta}_z := \frac{\eta - z^\top \mu}{\sqrt{z^\top \Sigma z}} \sim \mathcal{N}(0,1) \quad \forall z \in \mathbb{R}^n \setminus \{0\}.$$

Therefore, it holds for all $z \in \mathbb{R}^n \setminus \{0\}$ and all $\rho \geq 0$ that

$$\begin{aligned} \mathbb{P}\left(z^\top (q^\xi + Mz) \leq \rho\right) &= \mathbb{P}\left(\tilde{\eta}_z \leq \frac{\rho - z^\top \mu - z^\top M z}{\sqrt{z^\top \Sigma z}}\right) \\ &= \Phi\left(\frac{\rho - z^\top \mu - z^\top M z}{\sqrt{z^\top \Sigma z}}\right). \end{aligned} \tag{19}$$

We infer for all $z \in \mathbb{R}^n \setminus \{0\}$ and all $\rho \geq 0$ the equivalence

$$\mathbb{P}\left(z^\top (q^\xi + Mz) \leq \rho\right) \geq p_2 \iff \frac{\rho - z^\top \mu - z^\top M z}{\sqrt{z^\top \Sigma z}} \geq \Phi^{-1}(p_2) \iff \alpha(z,\rho) \leq 0$$

thanks to Φ being invertible. For $z = 0$, both sides in the relation above always hold true due to $\rho \geq 0$. Hence the equivalence can be extended to all $z \in \mathbb{R}^n$. Summarizing, the original inequality $\varphi_2(z,\rho) \geq p_2$ is equivalent with the inequality $\alpha(z,\rho) \leq 0$. With M being positive semi-definite, the function $z \mapsto z^\top M z$ is convex. Since $\Phi^{-1}(p_2) \geq 0$ due to our assumption $p_2 \geq 0.5$, α is a convex function and the statement of the Proposition is proven. □

Remark 14. We emphasize that the description (6) of the feasible set of Problem (8) fails to be convex. This is due to the fact that probability functions (like φ_1, φ_2) are always bounded in the interval $[0,1]$. Hence, they could be concave (as needed in the original inequalities $\varphi_1 \geq p_1$, $\varphi_2 \geq p_2$) only if they were constant. This, however, is easily seen to be impossible in our setting. It is therefore essential, in particular in the context of optimality conditions, to pass to an equivalent convex description of the feasible set as in Theorem 13.

As an elementary yet useful consequence of Theorem 13 we observe that the optimization in variables (z,ρ) in problem (8) may be reduced to an optimization just in z.

Corollary 15. *Under the assumptions of Theorem 13, the following holds true: $(\bar{z},\bar{\rho})$ is a solution to* (18)*, and thus to* (8)*, if and only if $\bar{z}$ is a solution to the convex optimization problem*

$$\min_z \quad \tilde{\alpha}(z) \quad \textit{s.t.} \quad \log p_1 - \log \tilde{\varphi}_1(z) \leq 0, \; z \geq 0, \tag{20}$$

where $\bar{\rho} = \tilde{\alpha}(\bar{z})$ *and*

$$\tilde{\alpha}(z) := \Phi^{-1}(p_2)\sqrt{z^\top \Sigma z} + z^\top \mu + z^\top M z, \quad \tilde{\varphi}_1(z) := \mathbb{P}(q^\xi + Mz \geq 0).$$

Remark 16. From the last two results, it immediately follows that the objective in (20) is strictly convex if the assumptions in Theorem 13 are slightly tightened by requiring that M is positive definite, not just positive semi-definite. Then, the solution of (20) is unique.

5. Differentiability of the Probability Functions and Gradient Formulae

Turning to the differentiability of the probability functions in (9)–(11), the first result is evident from (15).

Lemma 17. *Suppose that Assumption 1 holds and that* q^ξ *has a Gaussian distribution according to* $q^\xi \sim \mathcal{N}(\mu, \Sigma)$*. Then, the probability function* φ_2 *from* (11) *is continuously differentiable at all* (z, ρ) *with* $z \neq 0$ *and its gradient is*

$$\nabla \varphi_2(z, \rho) = -\phi\left(\frac{\rho - z^\top \mu - z^\top M z}{\sqrt{z^\top \Sigma z}}\right) \cdot \left(\left(z^\top \Sigma z\right)^{-1/2}(\mu + (M + M^\top)z) + \left(z^\top \Sigma z\right)^{-3/2}(\rho - z^\top \mu - z^\top M z)\Sigma z\right),$$

where ϕ *denotes the one-dimensional standard Gaussian density.*

The corresponding statements for φ and φ_1 require some preparation. To this aim, consider the general probability function

$$\hat{\varphi}(x) := \mathbb{P}(A(x)\eta \leq b(x)) \quad (x \in \mathbb{R}^s), \tag{21}$$

for matrix and vector functions $A : \mathbb{R}^s \to \mathbb{R}^{p \times n}$, $b : \mathbb{R}^s \to \mathbb{R}^p$ and for some n-dimensional Gaussian random vector $\eta \sim \mathcal{N}(0, R)$ with a positive definite correlation matrix R. The latter admits a Cholesky decomposition $R = LL^\top$. For a matrix P we denote by P_j its jth row and by $P_{j,i}$ its entry in row j and column i. By χ we denote the one-dimensional density of the Chi-distribution with n degrees of freedom and by ν the law of the uniform distribution on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. We cite the following result, which is Theorem 5.1 by van Ackooij and Henrion (2017).

Theorem 18. *In* (21)*, fix some* $\bar{x}$ *with* $b_j(\bar{x}) > 0$ *for* $j = 1, \dots, p$*. Assume that any two rows of the matrix* $A(\bar{x})$ *are linearly independent. Then,* $\hat{\varphi}$ *is continuously differentiable at* $\bar{x}$ *with*

$$\nabla \hat{\varphi}(\bar{x}) = -\int_{\{v \in \mathbb{S}^{n-1} \mid J^*(v) \neq \emptyset\}} \frac{\chi(\kappa(v))}{A_{j(v)}(\bar{x})Lv} \left(\kappa(v) \sum_{i=1}^{n} \nabla A_{j(v),i}(\bar{x}) L_i v - \nabla b_{j(v)}(\bar{x})\right) \mathrm{d}\nu(v).$$

Here,

$$J^*(v) := \{j \in \{1, \dots, p\} \mid A_j(\bar{x})Lv > 0\}, \quad \kappa(v) := \min_{j \in J^*(v)} \frac{b_j(\bar{x})}{A_j(\bar{x})Lv},$$

and $j(v)$ *denotes an (arbitrary) index realizing the minimum in the definition of* κ*.*

We note that the arbitrary choice of the index $j(v)$ in the theorem above is justified because $\#J^*(v) = 1$ holds ν-almost surely. We are now in a position to state the desired differentiability results for the probability functions φ and φ_1.

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Theorem 19. *Suppose that Assumption 1 holds and q^ξ has a Gaussian distribution according to $q^\xi \sim \mathcal{N}(\mu, \Sigma)$. Let $\Sigma = \tilde{L}\tilde{L}^\top$ be a Cholesky decomposition. Then, given some $\bar{z}$ with $\varphi_1(\bar{z}, \bar{\rho}) > 0.5$, φ_1 is continuously differentiable at $(\bar{z}, \bar{\rho})$ with gradient*

$$\nabla_z \varphi_1(\bar{z}, \bar{\rho}) = -\int_{\{v \in \mathbb{S}^{n-1} | J^*(v) \neq \emptyset\}} \frac{\chi(\kappa(v))}{\tilde{L}_{j(v)} v} M_{j(v)} \, \mathrm{d}\nu(v), \quad \nabla_\rho \varphi_1(\bar{z}, \bar{\rho}) = 0,$$

where

$$J^*(v) := \{j \in \{1, \ldots, n\} \mid \tilde{L}_j v < 0\}, \quad \kappa(v) := \min_{j \in J^*(v)} \frac{-M_j \bar{z} - \mu_j}{\tilde{L}_j v},$$

and $j(v)$ denotes an (arbitrary) index realizing the minimum in the definition of κ.

Similarly, given some $(\bar{z}, \bar{\rho})$ with $\bar{z} \neq 0$ and $\varphi(\bar{z}, \bar{\rho}) > 0.5$, φ is continuously differentiable at $\bar{z}$ with gradient

$$\nabla \varphi(\bar{z}) = -\int_{\{v \in \mathbb{S}^{n-1} | J^*(v) \neq \emptyset\}} \Gamma(v) \, \mathrm{d}\nu(v)$$

where $J^*(v) := J_1^*(v) \cup J_2^*(v)$ with

$$J_1^*(v) := \{j \in \{1, \ldots, n\} \mid \tilde{L}_j v < 0\},$$

$$J_2^*(v) := \begin{cases} \{n+1\}, & \text{if } \bar{z}^\top \tilde{L} v > 0, \\ \emptyset, & \text{else}, \end{cases}$$

$$\Gamma(v) := \begin{cases} \frac{\chi(\kappa(v))}{\tilde{L}_{j(v)} v} \left(M_{j(v)}, 0 \right), & \text{if } j(v) \in J_1^*(v), \\ \frac{\chi(\kappa(v))}{\bar{z}^\top \tilde{L} v} \left(\kappa(v) \tilde{L} v + \mu + (M + M^\top) \bar{z}, -1 \right), & \text{if } j(v) \in J_2^*(v), \end{cases}$$

$$\kappa(v) := \begin{cases} \min_{j \in J_1^*(v)} \frac{-M_j \bar{z} - \mu_j}{\tilde{L}_j v}, & \text{if } J_2^* = \emptyset, \\ \frac{\bar{\rho} - \bar{z}^\top (\mu + M \bar{z})}{\bar{z}^\top \tilde{L} v}, & \text{if } J_1^* = \emptyset, \\ \min \left\{ \min_{j \in J_1^*(v)} \frac{-M_j \bar{z} - \mu_j}{\tilde{L}_j v}, \frac{\bar{\rho} - \bar{z}^\top (\mu + M \bar{z})}{\bar{z}^\top \tilde{L} v} \right\}, & \text{else}, \end{cases}$$

and $j(v)$ denotes an (arbitrary) index realizing the minimum in the definition of κ.

Proof. The vanishing partial derivative of φ_1 with respect to ρ follows from the fact that φ_1 does not depend on ρ. Defining the partial function $\tilde{\varphi}_1(z) := \varphi_1(z, \bar{\rho})$ and observing that $\nabla_z \varphi_1(\bar{z}, \bar{\rho}) = \nabla \tilde{\varphi}_1(\bar{z})$, it is sufficient to show that the latter gradient equals the formula asserted in the statement for the former (partial) gradient. To see this, note that $\tilde{\varphi}_1$ can be represented in the form (21) upon setting

$$s := p := n, \quad x := z, \quad A(x) := -D^{-1}, \quad b(x) := Mx + \mu, \quad \eta := D(q^\xi - \mu),$$

where D is defined as the diagonal matrix with entries $(\Sigma_{i,i})^{-1/2} > 0$. Indeed, $R := D \Sigma D$ is a correlation matrix, i.e., the covariance matrix with diagonal entries equal to one, which has a Cholesky decomposition $R = (D\tilde{L})(D\tilde{L})^\top)$ if the original covariance matrix has a Cholesky decomposition $\Sigma = \tilde{L}\tilde{L}^\top$. Then, $\eta \sim \mathcal{N}(0, R)$ holds as required in Theorem 18 and

$$A(x)\eta \leq b(x) \iff q^\xi + Mx \geq 0,$$

so that $\tilde{\varphi}_1$ coincides with $\hat{\varphi}$ in (21).

We claim that $b_j(\bar{x}) = b_j(\bar{z}) = M_j \bar{z} + \mu_j > 0$ for all $j = 1, \ldots, p$. Indeed, the set $C := \{y \in \mathbb{R}^n | y + M\bar{z} \geq 0\}$ is convex and certainly contains a Slater point. Therefore, $\operatorname{int} C = \{y \in \mathbb{R}^n | y_j + M_j \bar{z} > 0 \ \forall j\}$. Now, if there was an index j' with $b_{j'}(\bar{z}) = M_{j'} \bar{z} + \mu_{j'} \leq 0$, then $\mu \notin \operatorname{int} C$ and so μ could be separated from C by some hyperplane. This entails that C is contained in some closed half space H with $\mu \notin \operatorname{int} H$. The latter implies, by symmetry of the Gaussian distribution, the contradiction

$$\varphi_1(\bar{z}, \bar{\rho}) = \mathbb{P}(q^\xi \in C) \leq \mathbb{P}(q^\xi \in H) \leq 0.5$$

with our assumption $\varphi_1(\bar{z},\bar{\rho}) > 0.5$.

Since, finally, any two rows of the regular matrix $A(\bar{x}) = -D^{-1}$ are linearly independent, all assumptions of Theorem 18 are satisfied. Now, the asserted gradient formula follows from the general formula in Theorem 18 by noting that the Cholesky factor L in that theorem equals $D\tilde{L}$.

The probability function φ can be represented in the form of (21) by setting $s := p := n+1,\ x := (z,\rho),\ \eta := D(q^\xi - \mu)$ and

$$A(x) := A(z,\rho) := \begin{bmatrix} -D^{-1} \\ z^\top D^{-1} \end{bmatrix}, \quad b(x) := b(z,\rho) := \begin{pmatrix} Mz+\mu \\ \rho - z^\top(\mu + Mz) \end{pmatrix},$$

where D is defined as before. We also keep the correlation matrix and the Cholesky decomposition as before. Then,

$$A(x)\eta \le b(x) \iff q^\xi + Mx \ge 0 \text{ and } x^\top(q^\xi + Mx) \le \rho,$$

so that φ coincides with $\hat{\varphi}$ in (21). The claim that $b_j(\bar{x}) = b_j(\bar{z}) > 0$ holds for all $j = 1,\ldots,p$ can be shown exactly the same way as above—this time using the convex set

$$C := \{y \in \mathbb{R}^n | y + M\bar{z} \ge 0,\ \bar{z}^\top(y + \bar{z}^\top M\bar{z}\}.$$

One easily verifies that either C is empty, a singleton, or admits a Slater point. The first two cases would yield the contradiction $\varphi_1(\bar{z}) = \mathbb{P}(q^\xi \in C) = 0$ with our assumption $\varphi(\bar{z}) > 0.5$. With this Slater point at hand one may follow the argumentation above to verify the claim.

Finally, it is easily verified, that under our assumption $\bar{z} \neq \emptyset$, all rows of the matrix $A(\bar{x}) := A(\bar{z},\bar{\rho})$ are pairwise linearly independent. Now, the asserted gradient formula follows from the general formula in Theorem 18 upon recalling the already used relation $L = D\tilde{L}$. □

6. Optimality Conditions

In this section we derive necessary and sufficient optimality conditions for the optimization problem (8) with two separated chance constraints.

Theorem 20. *Suppose that Assumption 1 holds, that M is positive definite, and that q^ξ has a Gaussian distribution according to $q^\xi \sim \mathcal{N}(\mu,\Sigma)$. In Problem* (8)*, let $p_1 \in (0.5, 1)$ and $p_2 \ge 0.5$. Then, a point $(\bar{z},\bar{\rho})$ with $\bar{z} \ge 0$, $\bar{z} \neq 0$, and $\bar{\rho} \ge 0$ is a solution to this problem if and only if*

$$\bar{\rho} = \Phi^{-1}(p_2)\sqrt{\bar{z}^\top\Sigma\bar{z}} + \bar{z}^\top\mu + \bar{z}^\top M\bar{z}$$

and if there exists some $\lambda \ge 0$ such that $\lambda(\mathbb{P}(q^\xi + Mz_{k_1} \ge 0) - p_1) = 0$ and

$$\begin{aligned}\Phi^{-1}(p_2)\left(\frac{1}{\sqrt{z^\top\Sigma z}}\Sigma + M + M^\top\right) z + \mu \\ = -\lambda \int_{\{v\in\mathbb{S}^{n-1} | J^*(v)\neq\emptyset\}} \frac{\chi(\kappa(v))}{\tilde{L}_{j(v)}v} M_{j(v)}\,\mathrm{d}\nu(v),\end{aligned}$$

where the expressions on the right-hand side are explained in the first statement of Theorem 19.

Proof. Under the assumptions made, Proposition 13 and Corollary 15 yield that the given point $(\bar{z},\bar{\rho})$ is a solution to (8) if and only if $\bar{z}$ is a solution to the convex optimization problem (20) and $\bar{\rho} = \tilde{\alpha}(\bar{z})$. We verify that the feasible set of (20) admits a (generalized) Slater point, i.e., a feasible point satisfying the nonlinear constraints strictly: Since our assumptions imply those of Proposition 10, we may set $p := p_1$ and pick up the construction of the point (z_{k_1},ℓ_1) there. This point satisfied $z_{k_1} \ge 0$ and $\ell_1 \ge 0$. Moreover, we observe from (16) that

$$\varphi_1(z_{k_1},\ell_1) = \mathbb{P}(q^\xi + Mz_{k_1} \ge 0) \ge (1+p)/2 > p = p_1.$$

With $\tilde{\varphi}_1$ as introduced in (20), it follows that

$$\log p_1 - \log \tilde{\varphi}_1(z_{k_1}) = \log p_1 - \log \varphi_1(z_{k_1}, \ell_1) < 0$$

so that z_{k_1} is the desired Slater point in (20). This now allows us to formulate the KKT conditions for that problem. Before doing so, we note that the functions $\tilde{\alpha}$ and $\log \tilde{\varphi}_1$ are continuously differentiable at $\bar{z}$. This follows for $\tilde{\alpha}$ from our assumption $\bar{z} \neq 0$. Moreover, as a feasible solution to (20), $\bar{z}$ satisfies

$$\varphi_1(\bar{z}, \bar{\rho}) = \tilde{\varphi}_1(\bar{z}) \geq p_1 > 0.5.$$

Hence, φ_1 is continuously differentiable at $(\bar{z}, \bar{\rho})$ by Theorem 19 (first statement) and so $\tilde{\varphi}_1 = \varphi_1(\cdot, \bar{\rho})$ is continuously differentiable at $\bar{z}$ with $\nabla \tilde{\varphi}_1(\bar{z}) = \nabla_z \varphi_1(\bar{z}, \bar{\rho})$. As already stated in the proof of Proposition 13, we have that $\varphi_1(z, \rho) > 0$ for all (z, ρ), whence $\tilde{\varphi}_1(z) > 0$ for all z. Consequently, $\log \tilde{\varphi}_1$ is continuously differentiable at $\bar{z}$. We therefore finally obtain the following: $\bar{z}$ is a solution to (20) if and only if there exists some $\lambda^* \geq 0$ such that

$$\nabla \tilde{\alpha}(\bar{z}) = \lambda^* \frac{1}{\tilde{\varphi}_1(\bar{z})} \nabla \tilde{\varphi}_1(\bar{z}), \quad \lambda^*(\tilde{\varphi}_1(\bar{z}) - p_1) = 0.$$

Since $\tilde{\varphi}_1(\bar{z}) > 0$ holds, we get that $\bar{z}$ is a solution to (20) if and only if there exists some $\lambda \geq 0$ such that

$$\nabla \tilde{\alpha}(\bar{z}) = \lambda \nabla \tilde{\varphi}_1(\bar{z}) = \lambda \nabla_z \varphi_1(\bar{z}, \bar{\rho}), \quad \lambda(\tilde{\varphi}_1(\bar{z}) - p_1) = 0.$$

Now, the statement of our theorem follows from the definition of $\tilde{\alpha}$ and from the gradient formula in the first statement of Theorem 18. □

7. Case Study: An Cournot Oligopoly Among Energy Producers

We consider n energy producing firms indexed by $i \in \{1, \dots, n\}$. Every firm decides upon a scalar energy production level $x_i \geq 0$ and all firms face a common inverse market demand function

$$p(x) = p(x_1, \dots, x_n) = \alpha - \beta \sum_{j=1}^{n} x_j$$

with a price intercept $\alpha > 0$ and a slope $\beta > 0$. Moreover, $\gamma_i > 0$ is the marginal cost of production of firm i. The optimization problem of firm i is then given by

$$\max_{x_i} \quad p(x)x_i - \gamma_i x_i \qquad \text{s.t.} \quad x_i \geq 0.$$

The second derivative of the firm's objective function is $-2\beta < 0$. Hence, the objective function is strictly concave and the KKT conditions are necessary and sufficient. For player i, global optimality is thus characterized by

$$0 \leq \gamma_i + \beta x_i - \alpha + \beta \sum_{j=1}^{n} x_j \perp x_i \geq 0.$$

Concatenating all n KKT conditions finally leads to the LCP

$$0 \leq Mx + q \perp x \geq 0$$

with

$$M = \begin{bmatrix} 2\beta & \beta & \dots & \beta \\ \beta & 2\beta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \beta \\ \beta & \dots & \beta & 2\beta \end{bmatrix}, \quad q = \begin{pmatrix} \gamma_1 - \alpha \\ \vdots \\ \gamma_n - \alpha \end{pmatrix}.$$

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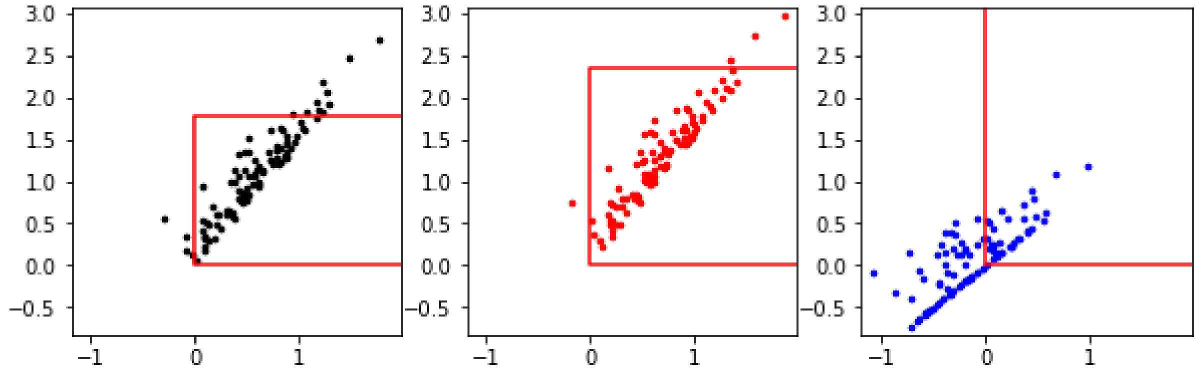

FIGURE 3. Plot of simulated samples q^ξ for the solution of the LCP problem based on separated (left) and joint (middle) chance constraints as well as based on the expected value (right).

By writing $M = \beta(J + I)$ with J being the matrix of all ones and I being the identity matrix, it follows that M is positive definite because J is positive semidefinite and I is positive definite; see Gabriel et al. (2022).

Since our results can mainly deal with uncertainties in the vector q, we can both handle the price intercept α of the inverse market demand function (which is a single scalar parameter) and the marginal costs of production γ_i (which is a scalar parameter per firm) as uncertain.

We consider a numerical example with the data $n = 5$, $\alpha = 3$, $\beta = 1$, and with random costs $\gamma \sim \mathcal{N}(\mu, \Sigma)$, where $\mu = (5, 2, 1, 3, 4)^\top$ and Σ having diagonal elements 0.2 (variances of the γ_i) and off-diagonal elements 0.1 (covariances between the γ_i). Then, the random vector q^ξ defined by $q_i^\xi := \eta_i - \alpha$ for $i = 1, \dots, n$ has the distribution $q^\xi \sim \mathcal{N}((2, -1, -2, 0, 1), \Sigma)$. First, we tackle the optimization problem (8) with separate chance constraints (6) and probability levels $p_1 = p_2 = 0.9$. We solve the problem numerically in its equivalent convex description (20) by using the gradient formula from the first statement of Theorem 18. We obtain the solution vector and associated relaxation parameter

$$\bar{z}^{\text{sep}} = (0, 0.32, 1.16, 0, 0)^\top, \quad \bar{\rho} = 1.77.$$

Second, we consider the optimization problem (7) with joint chance constraints (5) and the probability level $p = 0.9$. We solve the problem by using the gradient formula from the second statement of Theorem 18. We obtain the solution vector and associated relaxation parameter

$$\bar{z}^{\text{joint}} = (0, 0.35, 1.20, 0, 0)^\top, \quad \bar{\rho} = 2.34.$$

We observe that the solutions of the two problems are pretty close to each other, whereas the relaxation parameter in the joint model is significantly higher than in the separate one. This is not surprising, because for the same probability level, the joint model is more restrictive than the separate model. Indeed, it is easy to see that, for probability levels $p = p_1 = p_2$, the feasible set (5) is contained in the feasible set (6).

For the sake of comparison, we also provide the solution of the original (non-relaxed) LCP problem when replacing γ by its expected value:

$$\bar{z}^{\text{mean}} = (0, 0, 1, 0, 0)^\top.$$

Figure 3 illustrates the effect of the different solutions on 100 samples of q^ξ simulated according to the given distribution. The horizontal axis shows the values of $z^\top(Mz + q^\xi)$ and the vertical axis the values of $\min\{(Mz + q^\xi)_i \mid i = 1, \dots, n\}$ for $z = \bar{z}^{\text{sep}}/z = \bar{z}^{\text{joint}}/z = \bar{z}^{\text{mean}}$ (left/middle/right). Feasibility in the LCP sense would amount to positions on the positive x-axis. The relaxation by the parameter ρ

leads to an extension of the feasibility region as in the left and middle picture. Clearly, the expected-value solution (right figure) is not at all robust in that more than half of the point violate the condition $Mz + q^\xi \geq 0$, which, not surprisingly, leads to frequent negative values of the complementarity term $z^\top(Mz + q^\xi)$. In contrast, in the chance-constrained models (left and middle figures) the feasibility of the relation $Mz + q^\xi \geq 0$ is guaranteed with high probability. Hence, the complementarity term stays between zero and the corresponding ρ-value of the relaxation parameter with high probability. The two solutions have a quite similar effect. The separated model has slightly more violations of the relation $Mz + q^\xi \geq 0$ and significantly more violations of the relaxed complementarity relation $z^\top(Mz + q^\xi) \leq \rho$. However, admitting the same relaxation parameter in the separated as in the joint model would make this effect disappear again. This similarity between both solutions already observed in Figure 1 suggests to give some preference to the simpler convex optimization problem with separated chance constraints over the formally more adequate problem with a joint chance constraint.

When increasing the probability levels, say to $p = p_1 = p_2 = 0.99$, we obtain the optimal solutions

$$\bar{z}^{\text{sep}} = (0, 0.44, 1.33, 0, 0)^\top, \quad \bar{\rho} = 3.18; \quad \bar{z}^{\text{joint}} = (0, 0.46, 1.35, 0, 0)^\top, \quad \bar{\rho} = 4.13.$$

Again, the solutions themselves are quite similar, except the value of the relaxation parameter. Compared to the solutions at lower probabilities the production levels have increased. This also means that the consideration of uncertainty seems to mitigate the effect of market power, which usually leads to smaller production values and thus higher scarcity prices. The probabilistic solutions suggest two firms to be active contrary to just one firm in the expected-value case. However, it is interesting to note that the activity of two and inactivity of the remaining three firms seems to be independent of the probability level and hence to be very robust with respect to uncertainties in the production costs.

8. Conclusion

In this paper, LCPs with joint chance constraints are studied for the first time. We provide a detailed analysis of the arising optimization problems (continuity, existence, differentiability, optimality conditions) and demonstrate the applicability of our results for the example of an uncertain Cournot oligopoly among energy producers. Despite the theoretical results of this paper, there are still many open questions for future research. First of all, one could try to generalize our results so that the case of uncertain M is treated as well. Moreover, one may consider other distributions than Gaussian ones, e.g., general elliptically symmetric distributions.

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