Twice degenerate equations in the spaces of vector-valued functions *

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Abstract

New results are suggested which allow to calculate an index at infinity for asymptotically linear and asymptotically homogeneous vector fields in spaces of vector-valued functions. The case is considered where both linear approximation at infinity and "linear + homogeneous" approximation are degenerate. Applications are given to the 2π-periodic problem for a system of two nonlinear first order ODE's and to the two-point BVP for a system of two nonlinear second order ODE's.

1 Introduction

Various problems concerning nonlinear equations (boundary value problems for ordinary and partial differential equations, integral operator equations etc) can be reduced to the calculation of some topological characteristics of completely continuous vector fields in Banach spaces: solvability, multiplicity of solutions, different type bifurcations, justification of approximate methods.

The usual approach to compute these characteristics (degree, rotation, index) goes as following. The vector field is split into the sum of dominating terms and "smaller order" ones. If these dominating terms (often linear) are non-degenerate in some appropriate sense then the required characteristics can be defined by the dominating terms only. If these dominating terms are degenerate, then it is necessary to consider the "next order" homogeneous terms. Again, if the vector field defined as the sum of the dominating and "next order" terms is non-degenerate, then the characteristics can be calculated with the use of these two parts of the initial vector field. And if this sum of the first and the second order terms is also degenerate it is necessary to consider "higher order" terms.

For the index at infinity calculation the first step of this program was done by Leray and Schauder; of course this step is the most productive. The next step was initiated in, about 1970 (see [11, 12]) by E.N. Landesman, A.C. Lazer and D.E. Leach. The last step was probably studied first in some papers by S. Fučik and his co-authors (see [3]) for zero "next order" term. The study of this case was continued in papers in A.M. Krasnosel'skii (e.g. [4]). In the paper [8] the last step was given for non-zero "next order"part, there some citations and history can be found. All these results concern scalar-valued functions.

In the present paper we consider vector fields in the spaces of vector-valued functions. This case (even for non-degenerate homogeneous "next order" part) contains essential difficulties (see [2]) compared with the scalar one.

The paper is organized as following. In the section 2 we reproduce some already known results concerning with the index at infinity calculation of asymptotically linear vector fields in abstract Banach space. Section 3 contains the conditions from [2] of asymptotic homogeneity of a superposition operator acting in spaces of integrable vector-valued functions. The Proposition 5 given there will be used in the proof of Theorem 2 (sections 7 and 8).

The main condition of Proposition 5 from section 3 fails in two natural cases: if the homogeneous part is identically zero and if this part is zero for one normed element from the subspace, where the linear part degenerates. The first case is considered in section 5 with proofs in section 6. For this case the higher order terms can tend to zero at infnity.

The second case is much more cumbersome. For this case higher order terms can not vanish at infinity, the homogeneous part can be discontinuous. This case is studied under rather strong assumptions: the simplicity of the linear part degeneracy and only for special type of principal homogeneous term. The homogeneuous part is considered depending on the signs of some linear functionals only. This type of nonlinearities appears naturally in the study of systems containing a number of scalar-valued Landesman-Lazer type nonlinearities.

Section 8 contains proofs, while some additional remarks are given in section 9. Examples can be found at the end of the paper.

2 Asymptotically homogeneous vector fields in abstract Banach spaces

Consider in a Banach space E some completely continuous operator T.

Definition 1. *Let the vector field $\Phi x = x - Tx$ be non-zero for $\|x\| \geq r_0$. Then the rotation (see [10]) of the field on the boundary of every ball $B(r,0) = \{x \in E;\ \|x\| \leq r\}$ is defined and the value of this rotation is common for all $r > r_0$. This common value is called an* index at infinity *of the field Φx and is denoted as $\operatorname{ind}_\infty \Phi$.*

The main subject of the paper is the calculation of this index at infinity for some special classes of vector fields.

A vector field Φx is called *linear* if the operator T is linear, in this case we shall denote it as A. A linear vector field is always zero at $x = 0$. Except for this singular point a linear vector field either has no other singular points at all (if 1 is a regular point for A) or it degenerates on a non-trivial subspace (if 1 belongs to the spectrum of A).

If 1 is a regular value for a completely continuous linear operator A, then 0 is an isolated (and as a matter of fact the unique) singular point of the vector field $\Phi x = x - Ax$. Its index coincides with the index of the vector field Φx at infinity. The rotation of this vector field on the boundary of a given domain $\mathcal{D}$ either is equal to zero, if $0 \notin \mathcal{D}$, or it coincides with the index of zero, if $0 \in \mathcal{D}$.

Proposition 1. *The equality*

$$\operatorname{ind}_\infty \Phi = (-1)^\beta \tag{1}$$

holds where β denotes the sum of multiplicities of all real eigenvalues of A which are greater than 1.

A proof of this assertion see, for instance, in [10].

Definition 2. *A vector field $\Phi x = x - Tx$ and the operator T are called* asymptotically linear *if the operator T admits the representation $Tx = Ax + Fx$ where A is* *a linear operator and an operator F satisfies the condition*

$$\lim_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|} = 0.$$

The operator A is called the asymptotical derivative *of an asymptotically linear operator T or the* derivative of T at infinity. *A linear vector field $x - Ax$ is called the* principal linear part *of the vector field $x - Tx$. The principal linear part is said to be* non-degenerate *if* 1 *does not belong to the spectrum of the operator A, and is said to be* degenerate *otherwise.*

Asymptotical derivatives of completely continuous operators are always completely continuous [10].

The next theorem of Leray-Schauder follows from theorems on calculating of the rotation of a vector field in terms of its principal part.

Proposition 2 ([10]). *Let a vector field $\Phi x = x - Tx$ be asymptotically linear with the non-degenerate main linear part $x - Ax$. Then the index of the vector field Φ at infinity is defined and*

$$\mathrm{ind}_\infty \Phi = (-1)^\beta,$$

where β denotes the sum of multiplicites of real eigenvalues of A which are greater than 1.

The results below and their proofs can be found in [7] for vector fields in Banach spaces.

Definition 3. *A nonlinear operator Q in the Banach space E is said to be* homogeneous*, or more precisely* homogeneous of degree 0*, if*

$$Q(x) = Q(\lambda x), \qquad \lambda > 0, x \in E.$$

A homogeneous nonlinearity is determined by its values on the unit sphere and at the coordinate origin. If A is a linear operator and Q is a homogeneous one then the operator QA is homogeneous; in fact if F is an arbitrary operator, then FQ is homogeneous.

If a homogeneous operator is not constant, then it must be discontinuous at zero. Moreover such operators can have others discontinuity points and the totality of such points can even be dense in E.[1]

Let a finite dimensional subspace $E_1 \subset E$ be chosen and let P_1 be a fixed projector on this subspace: $PE = E_1$, $P_1^2 = P_1$.

Definition 4 ([7]). *An operator F is said to be* asymptotically homogeneous *in the space E (with respect to the subspace E_1 and the projector P_1) if it can be represented as the sum $F = Q + B$ where the operator Q is homogeneous and the operator B satisfies the following condition of "vanishing at infinity": for each $c > 0$ the equality*

$$\lim_{R\to+\infty} \sup_{e_1\in E_1,\, \|e_1\|=1,\, h\in E,\, \|h\|<c} \|P_1 B(Re_1 + h)\| = 0 \tag{2}$$

[1] This case is natural in applications.

holds.

Consider some completely continuous asymptotically linear vector field $\Phi x = x - Ax - Fx$ with degenerate linear part $x - Ax$. Let 1 belongs to the spectrum $\sigma(A)$ of the operator A, put $E_1 = \mathrm{Ker}\,(I - A)$ and let P_1 be the projector on E_1 which commutes with A. Suppose that there are not any generalized eigenvectors of A, corresponding to the eigenvalue 1: $Ae = e$ for $e \in E_1$.

Proposition 3 ([7]). *Let the operator F be asymptotically homogeneous: $F = Q + B$, where Q is homogeneous and B satisfies condition* (2) *with the finite dimensional subspace E_1 and the projector P_1, to be defined by the linear operator A. Suppose that the finite dimensional vector field P_1Qe on the sphere $U = \{e \in E_1,\ \|e\| = 1\}$ is non-degenerate, i.e.*

$$P_1Qe \neq 0,\ \ e \in U, \tag{3}$$

and that the operator $P_1Qx : E \to E_1$ is continuous at each point of U. Then the index $\mathrm{ind}_\infty \Phi$ is defined and

$$\mathrm{ind}_\infty \Phi = (-1)^\beta \gamma(P_1Q, U),$$

where $\gamma(P_1Q, U)$ denotes the rotation of the finite dimensional vector field P_1Q on the sphere U in the finite dimensional subspace E_1.

In applications the subspace E_1 is often one- or two-dimensional and the rotation $\gamma(P_1Q, U)$ can be calculated in an explicit form.

Now we can formulate exactly the main goal of our paper: to calculate index at infinity when condition (3) is not valid. We do this in two principal cases: $Q \equiv 0$ (section 4) and $P_1Q \not\equiv 0$, but $P_1Qe_0 = 0$ for some $e_0 \in U$ (section 6).

3 Homogeneity of the superposition operator in the space of vector-valued functions

The applicability of Proposition 3 from the previous section is based on concrete examples of asymptotically homogeneous operators. There are various examples of such operators, which are applicable to problems arising in differential equations, boundary value problems, oscillation theory. The most important example is the superposition operator under appropriate conditions, another types of examples are generated by hysteresis operators (see [9] for general mathematical theory of hysteresis and [1] for asymptotic homogeneity of hysteresis operators).

Let Ω be a compact set with a finite continuous measure on it, and let $f(t,x) :\ \Omega \times I\!R \to I\!R$ be a bounded Carathéodorian function.[2] Consider the superposition operator $x(t) \mapsto f(t, x(t))$. This operator maps any measurable function $x(t) :\ \Omega \to I\!R$ into measurable one. If $f(t,x)$ satisfies Landesman-Lazer conditions

$$\lim_{\xi \to +\infty} f(t,\xi) = q^+(t), \qquad \lim_{\xi \to -\infty} f(t,\xi) = q^-(t)$$

[2] Carathéodorian functions are continuous in x for every $t \in \Omega$ and measurable in t for every x.

and functions from E_1 do not equal to zero "very often" in the sense that

$$\text{mes}\,\{t \in \Omega :\ e(t) = 0\} = 0, \qquad e(t) \in E_1,\ e(t) \not\equiv 0,$$

then

- the superposition operator is asymptotically homogeneous in any L^p $(p < \infty)$;
- its homogeneous part has the form $Qx(t) = q(t, x(t))$, where

$$q(t,x) = \begin{cases} q^-(t), & x \le 0, \\ q^+(t), & x > 0; \end{cases} \tag{4}$$

- the operator $Qx(t)$ is continuous on $U \subset L^p$.

Below we represent a result about asymptotic homogeneity of a superposition operator, generated by the function $f(t,x) :\ \Omega \times I\!R^n \to I\!R^n$. The proof of the following results can be found in [2] as well as examples and details.

Let Ω be a closed bounded domain in a finite dimensional space. We will consider operators, vector fields and equations in spaces E of functions $x(t) :\ \Omega \to I\!R^n$. Denote by $\langle \cdot, \cdot \rangle$ the scalar product in the space $I\!R^n$ and by $|\cdot|_n$ the corresponding norm.

Consider an arbitrary finite dimensional subspace $E_1 \subset E$ of continuous on Ω vector-valued functions and denote $U = \{e(t) :\ e(t) \in E_1,\ \|e\| = 1\}$. Suppose that each non-zero function $e(t) \in E_1$ satisfies the condition

$$\text{mes}\,\{t \in \Omega :\ e(t) = 0\} = 0. \tag{5}$$

Let us fix a closed set $\Delta \subset S$ on the unit sphere $S = \{x \in I\!R^n :\ |x|_n = 1\} \subset I\!R^n$. Generally speaking in applications this set is "small": it has the co-dimension 2.

Let $u \in S$. Denote by $\rho(u, \Delta)$ the distance between a point u of the sphere and the set Δ. For each function $e(t) \in E_1$ introduce the notation

$$\chi(\delta, \Delta, e) = \text{mes}\,\{t \in \Omega :\ \rho(\frac{e(t)}{|e(t)|_n}, \Delta) \le \delta\}.$$

The main assumption in the theorem formulated below on asymptotic homogeneity of the superposition operator $x(t) \mapsto f(t, x(t))$ is the following: there exists a set Δ such that

1. the limit

$$\lim_{R \to +\infty} f(t, Ru) = q(t, u) \tag{6}$$

exists for each $u \in S,\ u \notin \Delta$. The limit function $q(t,u)$ satisfies the Carathéodory condition for $u \notin \Delta$: it is continuous in u and measurable in t. The limit in (6) is supposed to be uniform in $t \in \Omega$ and in u belonging to any given closed subset of S which is disjoint with Δ.

2. the equality

$$\chi(0, \Delta, e) = 0. \tag{7}$$

holds for each function $e(t) \in E_1$.

The assumption 1 can be reformulated as follows:

1*. The equality

$$\lim_{R\to+\infty} \sup_{t\in\Omega,\, u\in\Delta_*} |f(t,Ru) - q(t,u)|_n = 0 \tag{8}$$

holds for each $\Delta_* \in S$ such that $\overline{\Delta}_* \bigcap \Delta = \emptyset$.

Equality (5) together with the main assumption guarantee that the operator

$$Qx(t) = \begin{cases} q(t, \dfrac{x(t)}{|x(t)|_n}), & x(t) \neq 0, \\ 0, & x(t) = 0 \end{cases} \tag{9}$$

is continuous as an operator in L^1 (and in others L^p for $p < \infty$) at every point of U (see [9]). The compactness of U guarantees the uniform continuity of this operator on U.

Let us suppose also that the functions $f(t,x)$ and $q(t,u)$ are both uniformly bounded.

Proposition 4. *The operator $x(t) \mapsto f(t,x(t))$ is asymptotically homogeneous in the space $E = L^2 = L^2(\Omega, I\!R^n)$ under the above listed assumptions.*

This theorem was proved in [5] using other terminology for the case $\Delta = \emptyset$. The closure G of the totality of discontinuity points of the function $q(t,u)$ may play the role of the set Δ.

In the end of this section we will use again the space $L^2 = L^2(\Omega, I\!R^n)$ of integrable with the square functions $x(t): \ \Omega \to I\!R^n$ with the usual norm $\|\cdot\|$ generated by the scalar product in $\langle\cdot,\cdot\rangle$ in $I\!R^n$:

$$\|\cdot\| = \sqrt{(\cdot,\cdot)}, \quad (x,y) = \int_\Omega \langle x(t), y(t)\rangle\, dt.$$

Denote by $A: \ L^2 \to L^2$ a linear completely continuous operator. Let us suppose that a bounded function $f(t,x): \ \Omega \times I\!R^n \to I\!R^n$ satisfies the Carathéodory condition. Consider in L^2 the completely continuous vector field

$$\Phi x = x - A(x + f(t,x)). \tag{10}$$

This field is asymptotically linear and its asymptotic derivative is equal to $I - A$.

If $1 \notin \sigma(A)$, then $\operatorname{ind}_\infty \Phi = (-1)^\beta$, where β denotes the sum of multiplicities of all real eigenvalues of the operator A which are greater than 1.

If $1 \in \sigma(A)$, then the asymptotic derivative $I - A$ is degenerate and to compute the index one has to use some properties of the nonlinearity $f(t,x)$.

Denote $E_1 = \operatorname{Ker}(I - A)$ and suppose that $E_1 = \{e(t): \ Ae = e\}$ holds. The last assumption means that the eigenvalue 1 of A does not have generalized eigenvectors. Denote by P_1 a projector on E_1 which commutes with A. This projector P_1 can be constructed in the following way.

Denote by $e_1, \ ,\dots, e_m$ ($m = \dim E_1$) a basis in the finite dimensional space E_1 and denote by $g_1, \ ,\dots, g_m$ a basis in the finite dimensional space $E_1^* = \operatorname{Ker}(I - A^*) \subset L^2$, which satisfies the condition

$$\int_\Omega \langle e_i(t), g_j(t)\rangle \, dt = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. Then the projector P_1 can be defined as

$$P_1 x(\cdot) = \sum_{i=1}^m e_i(\cdot) \int_\Omega \langle g_i(t), x(t)\rangle \, dt.$$

Proposition 5. *Let a bounded nonlinearity $f(t,x)$ satisfy the conditions of Theorem 2 for some set Δ and function $q(t,u)$. Let the vector field $\Psi e = P_1 q(t, e(t)/|e(t)|_n)$ is non-degenerate on U. Then*

$$\operatorname{ind}_\infty \Phi = (-1)^\beta \gamma(\Psi, U).$$

Proposition 5 follows immediately from Propositions 3 and 4.

4 One-side estimates

The main assumption on the function $f(t,x) : \Omega \times I\!\!R^n \to I\!\!R^n$ $(n > 1)$ is one of the two following one-side estimates: either

$$\langle x, f(t,x)\rangle \ge \psi(t, |x|_n), \quad t \in \Omega, \ x \in I\!\!R^n, \ |x|_n \ge u_0, \tag{11}$$

or

$$\langle x, f(t,x)\rangle \le \psi(t, |x|_n), \quad t \in \Omega, \ x \in I\!\!R^n, \ |x|_n \ge u_0 \tag{12}$$

for an appropriate function $\psi(t,u) : \ \Omega \times [u_0, +\infty) \to I\!\!R^+$. Let us remain that $\langle \cdot, \cdot \rangle$ and $|\cdot|_n$ denote scalar product and norm in $I\!\!R^n$.

Suppose that the function $\psi(t,u)$ is Carathéodorian and does not increase with respect to u for each fixed t. Moreover, let this function be strictly positive for $t \in \Omega_0$, where $\Omega_0 \subset \Omega$ and $\operatorname{mes}\Omega_0 > 0$. For such function $\psi(t,u)$ the value

$$\int_\Omega \psi(t, u_0 + u(t)) \, dt$$

is strictly positive for any non-negative scalar function $u(t)$.

Unfortunately, conditions (11) and (12) are too restrictive. In various natural applications these conditions can be weaken, for example they need only be valid in some part of the space $I\!\!R^n$. Consider some closed set $\Delta^* \subset S$, where S again denotes the unit sphere in $I\!\!R^n$. Instead of conditions (11) and (12) we consider the estimates

$$\langle x, f(t,x)\rangle \ge \psi(t, |x|_n), \quad t \in \Omega, \ \frac{x}{|x|_n} \in \Delta^*, \ |x|_n \ge u_0, \tag{13}$$

and

$$\langle x, f(t,x)\rangle \le \psi(t, |x|_n), \quad t \in \Omega,\ \frac{x}{|x|_n} \in \Delta^*,\ |x|_n \ge u_0. \tag{14}$$

We shall use these estimates in the following way. Consider some function $e(t):\Omega \to I\!R^n$ such that $\operatorname{mes}\{t \in \Omega:\ e(t) = 0\} = 0$. Put

$$\Delta_0 = \Delta_0(e) = \{x \in S:\ \exists t \in \Omega:\ e(t) \ne 0,\ x = \frac{e(t)}{|e(t)|_n}\}. \tag{15}$$

Choose a $\delta > 0$ and consider the set

$$\Delta^* = \{x \in S:\ \rho(x, \Delta_0) \le \delta\}.$$

Fix some positive constant c. For sufficiently large $\xi > 0$ for the values of any function $x(t) = \xi e(t) + h(t)$ with arbitrary $h(t)$ such that $|h(t)| \le c$ the inequality $|x(t)|_n \ge u_*$, where $u_* \ge c + u_0$, implies

$$\frac{x(t)}{|x(t)|_n} \in \Delta^*. \tag{16}$$

This means that (if (13) holds)

$$\int_{\{|x(t)|_n \ge u_*\}} \langle x(t), f(t, x(t))\rangle\, dt \ge \int_{\{|x(t)|_n \ge u_*\}} \psi(t, |x(t)|_n)\, dt \tag{17}$$

where $\xi \ge \xi_0$ where ξ_0 depends on $u_0,\ u_*,\ c$ and δ.

Inequalities (11) and (12) can be considered as inequalities (13) and (14) with $\Delta^* = S$.

Functions $e(t)$ will be chosen as normed (in L^2) elements of some finite dimensional subspace. If its dimension is greater than 1, then Δ_0 can coincide with S and it is natural to assume conditions (11) and (12), if this subspace is one-dimensional (or $\dim E_1 \ll n$), then we use weaker conditions (13) and (14) for some Δ^*.

5 Complete degeneracy of homogeneous terms

In this section we consider vector field

$$\Upsilon x = x - A(x + f(t,x) + b(t)) \tag{18}$$

where $x = x(t):\ \Omega \to I\!R^n$. The continuous function $f(t,x)$ suppose to be small for large values of $|x|$: $f(t,x) \to 0$ as $|x|_n \to \infty$. This vector field is asymptotically homogeneous, $q(t,x) \equiv b(t)$. If $P_1 b \ne 0$ then $\operatorname{ind}_\infty \Upsilon = 0$. In this section we consider the case $P_1 b = 0$.

Together with (13) and (14) we suppose that the following assumption is valid:

$$|f(t,x)|_n \le \theta(|x|_n), \qquad t \in \Omega,\ x \in I\!R^n, \tag{19}$$

where the non-increasing function $\theta(u)$ satisfies the relation

$$\lim_{u \to \infty} \theta(u) = 0.$$

Relation (19) is the sufficient condition of the asymptotical homogeneity of $f(t,x)$ with homogeneous part being identically zero. This special case allows to calculate the index at infinity without the use of Proposition 5.

The main assumptions in the following theorems on index have the form of restrictions from below on the function $\psi(t,u)$.

Theorem 1. *Let A be a linear completely continuous normal ($AA^* = A^*A$) operator in the space $L^2 = L^2(\Omega, \mathbb{R}^n)$. Let the operator A act and be continuous as an operator from L^2 to L^∞. Let the bounded function $f(t,x)$ satisfy one of the estimates* (11) *or* (12) *and the relation* (19). *Let the function $\psi(t,u)$ satisfy the following restrictions: for any positive R and u_**

$$\lim_{\delta\to 0} \sup_{e(t)\in E_1\ \|e\|=1} \frac{\chi(\delta, |e|_n)}{\int_\Omega \psi(t, u_* + R\delta^{-1}|e(t)|_n)\, dt} = 0 \tag{20}$$

and

$$\lim_{\delta\to 0} \sup_{e(t)\in E_1\ \|e\|=1} \frac{\int_\Omega \theta(\delta^{-1}|e(t)|_n)\, dt}{\int_\Omega \psi(t, u_* + R\delta^{-1}|e(t)|_n)\, dt} = 0. \tag{21}$$

Finally, let

$$\int_\Omega \langle b(t), e(t)\rangle\, dt = 0, \qquad e(t) \in E_1 = \mathrm{Ker}\,(I - A). \tag{22}$$

Then $\mathrm{ind}_\infty \Upsilon = (-1)^{\sigma+\sigma_0}$, *where σ is the sum of multiplicities of all real eigenvalues of A greater than 1 while $\sigma_0 = 0$ in the case of estimate* (11) *and $\sigma_0 = \dim E_1$ in the case of estimate* (12).

As we said in the end of the previous section if $\dim E_1 = 1$ then it is possible to use conditions (13) and (14) with an appropriate Δ^*. This is natural, e.g. if $n - 1 > \dim E_1$. In the last case Δ_0 has zero measure on the sphere S.

Conditions close to (20) were considered in [4], where one can find examples of how (20) transforms for concrete cases.

Condition (21) connects the projection of the vector $f(t,x) \in \mathbb{R}^n$ on x and its norm. The projection cannot be extremely small.

Let us give here an example. Let E_1 consist from vector-valued functions $e(t)$: $[0,\pi] \to \mathbb{R}^2$ of the type $\{a\sin t, b\sin t\}$. This 2-dimensional subspace E_1 can appear during the study of two-point boundary value problem for two second order ordinary differential equations.

Then for any normed function $e(t) \in E_1$ for some positive constants c^1, c^2 and δ_0 the estimate

$$c^1\delta \le \chi(\delta, |e|_n) \le c^2\delta, \quad 0 \le \delta \le \delta_0$$

holds. Let the function $\psi(t,u)$ be independent from t. Then condition (20) can be rewritten as

$$\int^\infty \psi(u)\, du = \infty,$$

and condition (21) is valid if (for example) for any $R > 0$

$$\lim_{u\to\infty} \frac{\theta(u)}{\psi(Ru)} = 0.$$

Condition (21) follows from the last equality according to l'Hôpital rule. Moreover, this equality and (21) are "almost equivalent".

For instance all these conditions are fulfilled if

$$\langle x, f(t,x)\rangle \geq \frac{c^*}{|x|_n \ln |x|_n} \quad \text{and} \quad |f(t,x)|_n \leq \frac{c_*}{|x|_n^{1-\varepsilon}}, \qquad \varepsilon \in (0,1].$$

6 Proof of Theorem 1

The proof will be carried out for the case where the function $f(t,x)$ satisfies (11). Consider on the sphere $S_\rho = \{\|x\|_n = \rho\} \subset L^2$ of a sufficiently large radius ρ the deformation

$$\Phi(x,\lambda) = x - A(\lambda x + f(t,x) + b(t)) \tag{23}$$

of the field $\Upsilon x = \Phi(x,1)$ to the field $\Phi_0 x = \Phi(x,\lambda_0)$ ($\lambda_0 > 1$). The field $\Phi_0 x$ is asymptotically linear and for values of λ_0 sufficiently close to 1 the asymptotic derivative $I - \lambda_0 A$ is non-degenerate, $\operatorname{ind}_\infty \Phi_0 = (-1)^\sigma$. This means that Theorem 1 follows from an *a priori* estimate of all possible zeros $x(t)$ of the deformation (23) for $\lambda \in [1,\lambda_0]$, $\lambda_0 > 1$.

Let for some λ the function $x(t) = \xi e(t) + h(t)$ (here $\xi e(t) = P_1 x$, $\|e\|_n = 1$, $\xi \geq 0$, $h(t) \perp E_1$) be a zero of the homotopy (23): $\Phi(x,\lambda) = 0$. Then the following two equalities are valid: $P_1\Phi(x,\lambda) = 0$ and $P_2\Phi(x,\lambda) = 0$, where $P_2 = I - P_1$. The second equality implies an *a priori* estimate

$$\|h(t)\|_{L^\infty} \leq r \tag{24}$$

for some r. This estimate follows from the bounded character of the function $f(t,x)$, from strict positivity of the distance between the interval $[1,\lambda_0]$ and the spectrum of the operator AP_2 and from the continuity of A as the operator from L^2 to L^∞. The value r does not depend on either ξ or on λ.

Let us multiply in L^2 the equality $P_1\Phi(x,\lambda)$ by the function $e(t)$. The equality obtained

$$(1-\lambda)\xi - \int_\Omega \langle e(t), f(t,x)\rangle\, dt = 0$$

according to $1 - \lambda \leq 0$ and $\xi \geq 0$ implies

$$\int_\Omega \langle e(t), f(t, \xi e(t) + h(t))\rangle\, dt \leq 0. \tag{25}$$

The last relation together with (24) have to guarantee an *a priori* estimate of possible values of scalar component ξ. Up to here the proof is rather standard.

Multiply (25) on the positive value ξ and add to the inequality obtained the value

$$(h(t), f(t,x(t))) = \int_\Omega \langle h(t), f(t,x(t))\rangle\, dt.$$

We obtain the relation

$$\int_\Omega \langle x(t), f(t, x(t))\rangle\, dt \le \int_\Omega \langle h(t), f(t, \xi e(t) + h(t))\rangle\, dt.$$

Since

$$\int_\Omega \langle x(t), f(t, x(t))\rangle\, dt = \int_{\{|x(t)|_n \le u_0\}} \langle x(t), f(t, x(t))\rangle\, dt + \int_{\{|x(t)|_n > u_0\}} \langle x(t), f(t, x(t))\rangle\, dt$$

$$\ge \int_{\{|x(t)|_n > u_0\}} \psi(t, |x(t)|_n)\, dt - u_0 \sup_{t\in\Omega,\, x\in I\!R^n} |f(t,x)|_n \,\mathrm{mes}\, \{|x(t)|_n \le u_0\}$$

$$\ge \int_{\{|x(t)|_n > u_0\}} \psi(t, u_0 + |x(t)|_n)\, dt - c_1 \mathrm{mes}\, \{\xi |e(t)|_n \le r + u_0\}$$

$$= \int_\Omega \psi(t, u_0 + |x(t)|_n)\, dt - \int_{\{|x(t)|_n \le u_0\}} \psi(t, u_0 + |x(t)|_n)\, dt - c_1 \chi(u_* \xi^{-1}, |e|_n)$$

$$\ge \int_\Omega \psi(t, u_0 + |x(t)|_n)\, dt - c_2 \chi(u_* \xi^{-1}, |e|_n),$$

(we denote $u_* = r + u_0$ and $c_i > 0$ are some constants) and

$$\int_\Omega \langle h(t), f(t, \xi e(t) + h(t))\rangle\, dt \le r \int_\Omega \theta(|x(t)|_n)\, dt$$

we have the estimate

$$\int_\Omega \psi(t, u_0 + |x(t)|_n)\, dt \le r \int_\Omega \theta(|x(t)|_n)\, dt + c_2 \chi(u_* \xi^{-1}, |e|_n). \tag{26}$$

The integral in the left-hand side of (26) can be estimated from below:

$$\int_\Omega \psi(t, u_0 + |x(t)|_n)\, dt \ge \int_\Omega \psi(t, u_* + \xi |e(t)|_n)\, dt. \tag{27}$$

This estimate is close to ones considered in [6]. The integral in the right-hand side of (26) can be estimated from above. Split the set Ω in two nonintersected parts

$$\Omega^+(\xi) = \{t \in \Omega :\ \xi |e(t)|_n > 2r\} \quad \text{and} \quad \Omega^-(\xi) = \{t \in \Omega :\ \xi |e(t)|_n \le 2r\}.$$

¿From the relations

$$\int_\Omega \theta(|\xi e(t) + h(t)|_n)\, dt = \int_{\Omega^+} \cdots + \int_{\Omega^-} \cdots \le \int_{\Omega^+} \theta(\frac{1}{2} \xi |e(t)|_n)\, dt + \int_{\Omega^-} \cdots \le$$

$$\le \int_\Omega \theta(\frac{1}{2} \xi |e(t)|_n)\, dt + c_3 \,\mathrm{mes}\, \Omega^-$$

follows the estimate

$$\int_\Omega \theta(|\xi e(t) + h(t)|_n)\, dt \le \int_\Omega \theta(\frac{1}{2} \xi |e(t)|_n)\, dt + c_3 \chi(2r\xi^{-1}, |e|_n). \tag{28}$$

Estimates (26), (27) and (28) imply

$$\int_\Omega \psi(t, u_* + \xi |e(t)|_n)\, dt \le \int_\Omega \theta(\frac{1}{2} \xi |e(t)|_n)\, dt + c_4 \chi(u^* \xi^{-1}, |e|_n)$$

which contradicts $\xi \to \infty$ due to (20) and (21). $\square$

7 Theorem 2

Again, the main restrictions which allow us to calculate the index at infinity are conditions (13) and (14) on the vector function $f(t,x) : \Omega \times I\!R^n \to I\!R^n$. We again suppose that $\psi(t,u) : \ \Omega \times (u_0, \infty) \to I\!R^+$ is Carathéodorian, does not increase in u for any t and for t from some set Ω_0 of positive measure this function is strictly positive.

Consider the vector field

$$\Upsilon x = x - A(x + f(t,x) + q(t,x)). \tag{29}$$

This field is asymptotically homogeneous and differs from the field (18) with the non-zero term $Aq(t,x)$ (the "input signal" $b(t)$ is "hidden" in the term $q(t,x)$). We suppose that the function

$$g(t,x) = f(t,x) + q(t,x)$$

is continuous with respect to the set of its variables and that the term $q(t,x)$ is homogeneous ($q(t,\lambda x) \equiv q(t,x)$ for $\lambda > 0$). This means the function $f(t,x)$ at zero has a discontinuity compensating the discontinuity of homogeneous term. Below we consider the situation where the functions $f(t,x)$ and $q(t,x)$ have other points of discontinuity. For example, if one of the components of the vector-function $q(t,x)$ is $\operatorname{sign} x_1$ then the set of the points of dicontinuity for this vector-function is at least the whole hyperplane $\{x_1 = 0\}$. The function $f(t,x)$ has to compensate this discontinuity and some possible other ones.

During the investigation of the field (29) we also need to use conditions of smallness of asymptotically zero terms. But if the function $q(t,x)$ has discontinuity not only for $x = 0$ then condition (19) always fails. This means that we need to use some conditions which are different from (19), but comparable with it.

We again suppose that 1 is an eigenvalue of normal completely continuous in L^2 linear operator A. Moreover, we suppose that the eigenvalue 1 is simple. Denote P_1 the orthogonal projector onto the one-dimensional subspace $E_1 = \operatorname{Ker}(I - A) = \{e(t) = ae_0(t),\ a \in I\!R\}$, $\|e_0\| = 1$ and denote $P_2 = I - P_1$ the projector onto the orthogonal complement E_2.

We consider the case where the one-dimensional vector field $P_1 Qe$ is degenerate on the sphere $U \subset E_1$. This sphere U for 1-dimensional space E_1 consists from two points: $e_0(t)$ and $-e_0(t)$. Let

$$\int_\Omega \langle e_0, q(t, e_0(t))\rangle\, dt = 0 \neq \int_\Omega \langle e_0, q(t, -e_0(t))\rangle\, dt \stackrel{\text{def}}{=} \zeta. \tag{30}$$

Here we consider an important partial class of homogeneous nonlinearities: nonlinearities which depend on signs of certain linear functionals.

Let $L_1(x), \dots, L_k(x)$ be a set of linear functionals in $I\!R^n$. Let

$$q(t,x) = \tilde{q}\left(t,\ \operatorname{sign} L_1(x), \dots,\ \operatorname{sign} L_k(x)\right) \tag{31}$$

and let the function $\tilde{q}(t, u_1, \dots, u_k)$ be continuous with respect of the variables $t,\ u_j$.

For the case considered the index of the field (29) is defined by the sign of ζ and by the choice of inequality (13) or (14) as a condition on $f(t,x)$. The following Theorem 2 will be proved by reducing to Proposition 5. Among the other assumptions of this proposition there is one about the existence of the set Δ. For functions (31) this set is defined by the functionals $L_j(x)$ and naturally have the form

$$\Delta = \bigcup \{x \in S : \ L_j(x) = 0, \ j = 1, \dots, k\}.$$

The condition (7) has the form

$$\operatorname{mes}\{t \in \Omega : \ L_j(e_0(t)) = 0\} = 0, \qquad j = 1, \dots, k. \tag{32}$$

Let

$$|f(t,x)|_n \le \theta_0(|x|_n) + \sum_{j=1}^{k} \theta_j\left(|L_j(x)|\right) \tag{33}$$

where the functions $\theta_j(u)$ $(j = 0, 1, \dots, k)$ satisfy

$$\lim_{u \to \infty} \theta_j(u) = 0.$$

Theorem 2. *Let condition* (32) *be valid. Let A act from L^2 to L^∞ and be continuous. Let the bounded function $f(t,x)$ satisfy* (33)*. Let the function $\psi(t,u)$ for any positive R and u^* satisfy for each $j = 1, \dots, k$ the conditions*

$$\lim_{\delta \to 0} \frac{\chi(\delta, |L_j e_0(t)|)}{\displaystyle\int_\Omega \psi(t, u^* + R\delta^{-1}|e_0(t)|)\, dt} = 0, \tag{34}$$

$$\lim_{\delta \to 0} \frac{\displaystyle\int_\Omega \theta_0(\delta^{-1}|e_0|_n)\, dt}{\displaystyle\int_\Omega \psi(t, u^* + R\delta^{-1}|e_0(t)|)\, dt} = 0 \tag{35}$$

and

$$\lim_{\delta \to 0} \frac{\displaystyle\int_\Omega \theta_j(\delta^{-1}|L_j e_0|)\, dt}{\displaystyle\int_\Omega \psi(t, u^* + R\delta^{-1}|e_0(t)|)\, dt} = 0. \tag{36}$$

Let (30) *hold and consider the set* (15)*. Finally, let $\Delta^* \subset S$ be some δ-neighbourhood of this set. Then the following statements are fulfilled:*

(1). *Let either $\zeta > 0$ and* (13) *hold or $\zeta < 0$ and* (14) *hold (with Δ^* chosen as above). Then* $\operatorname{ind}_\infty \Upsilon = 0$.

(2). *Let either $\zeta < 0$ and* (13) *hold or $\zeta > 0$ and* (14) *hold. Then* $\operatorname{ind}_\infty \Upsilon = (-1)^\sigma \operatorname{sign} \zeta$*, where σ is the sum of multiplicities of all real eigenvalues of A which are greater than* 1.

Note, that (32) guarantee the continuity of $\tilde{q}(t,x)$ in L^2 at the points $\pm e_0(t)$.

Theorem 2 is the generalization of Theorems 1 and 2 from [8] for vector fields in the spaces of vector-valued functions.

8 Proof of Theorem 2

Consider the deformation

$$\Phi(x,\lambda) = x - A(x + f(t,x) + q(t,x) + \lambda e_0(t)). \tag{37}$$

Lemma 1. *Put* $\lambda_0 = \dfrac{1}{2}|\zeta|$*. For any zero* $x(t) = \xi e_0(t) + h(t)$ *of the deformation* (37) (*for* $\lambda \in [0,\lambda_0]$ *if* (13) *holds and for* $\lambda \in [-\lambda_0, 0]$ *if* (14) *holds*) *under the assumptions of Theorem* 2 *the following* a priori *estimate is valid:* $|\xi|$, $\|h\|_{L^\infty} \le const.$

Now Theorem 2 follows from Proposition 5 and general topological properties of rotation. The index at infinity of the field $\Phi(x,\lambda_0)$ is equal $(-1)^\sigma \cdot \gamma$ where γ is the rotation of the one-dimensional field $P_1 q(t,e)$ in the points $\pm e_0(t)$. For large ξ this field at the point $-e_0(t)$ is directed as $(\operatorname{sign}\xi)e_0(t)$ and at the point $e_0(t)$ is directed as $(\operatorname{sign}\lambda_0)e_0(t)$.

In case (1) $\operatorname{sign}\lambda_0 = \operatorname{sign}\zeta$ and $\gamma = 0$ in case (2) $\operatorname{sign}\lambda_0 = -\operatorname{sign}\zeta$ and $\gamma = \operatorname{sign}\zeta$.

Let us now prove Lemma 1. The proof will be given for the case (13) and $\lambda \ge 0$. The estimate of the infinite dimensional component $h(t)$ of a zero $x(t)$ of the deformation (37) follows from $h(t) - Ah(t) = AP_2 g(t,x)$ and the bounded behavior of $g(t,x)$: for some r the *a priori* estimate (24) is valid.

Since $P_1\Phi(x,\lambda) = 0$ then

$$\int_\Omega \langle e_0(t), f(t,x) + q(t,x) + \lambda e_0(t)\rangle\, dt = 0$$

and

$$\int_\Omega \langle e_0(t), f(t,x(t))\rangle\, dt + \int_\Omega \langle e_0(t), q(t,x(t))\rangle\, dt + \lambda = 0. \tag{38}$$

Let us prove the *a priori* estimate of ξ. This is done with different way for $\xi > 0$ and for $\xi < 0$.

Let $\xi < 0$. If the esimate of ξ does not exist then we can go to limits as $\xi \to -\infty$ in (38). We have

$$\int_\Omega \langle e_0(t), f(t,x(t))\rangle\, dt \to 0$$

(this relation means asymptotical homogeneity of the field Υ!) and

$$\int_\Omega \langle e_0(t), q(t,x(t))\rangle\, dt \to \int_\Omega \langle e_0(t), q(t,-e_0(t))\rangle\, dt = \zeta,$$

but $|\lambda| \le \lambda_0 < |\zeta|$. The contradiction proves the estimate $\xi > const.$

Let $\xi > 0$. Consider the value

$$s(\xi,h) = \int_\Omega \langle e_0(t), q(t,x(t))\rangle\, dt.$$

If $t \in \{|\xi L_j(e_0(t))| > r\}$, $j = 1,\dots,k$ then $\operatorname{sign} L_j(e_0(t)) = \operatorname{sign} L_j(x(t))$ and for these t the relation $q(t,x) \equiv q(t,e_0)$ holds. Exactly in this step of our proof we use the representation of $q(t,x)$ as a function of signs of linear functionals.

Since $(e_0, q(t, e_0)) = 0$ therefore

$$|s(\xi, h)| = \int_{\bigcup\limits_{j=1,\dots,k} \{t\in\Omega:\, |\xi L_j(e_0(t))|\le r\}} |\langle e_0(t), q(t, x(t)) - q(t, e_0(t))\rangle|\, dt \le$$

$$\le c_1 \sum_{j=1,\dots,k} \operatorname{mes}\{t \in \Omega :\ |\xi L_j(e_0(t))| \le r\} \le c_2 \chi\left(r_1 \xi^{-1}, e_0\right).$$

In the last formulae we have used the inequality $|L_j(e_0)| \le const|e_0|$ and its consequence

$$\{t \in \Omega :\ |\xi L_j(e_0(t))| \le r\} \subset \{t \in \Omega :\ |\xi e_0(t)| \le r_1\}.$$

Now let $\Phi(x, \lambda) = 0$. Then $(e_0, \Phi(x, \lambda)) = 0$, the last equality can be rewritten as

$$\xi \int_\Omega \langle e_0(t), f(t, x) + q(t, x) + \lambda e_0(t)\rangle\, dt = 0.$$

Again (as in the proof of Theorem 1) let us add to both parts of this equality the term $(h(t), f(t, x))$. The equality obtained has the form

$$\int_\Omega \langle x(t), f(t, x(t))\rangle\, dt + s(\xi, h) + \lambda = (h, f(t, x)). \tag{39}$$

Since for sufficiently large values of ξ inclusion (16) is valid, then (17) holds and (39) implies the inequality

$$\int_\Omega \psi(t, u_0 + |x(t)|)\, dt \le c_3 \chi(r_1 \xi^{-1}) + \theta_0(|x|_n) + \sum_{j=1}^{k} \theta_j(|L_j(x)|).$$

Constructions which repeat the proof of Theorem 1 complete the proof of Theorem 2. □

9 Remarks

1. Results close to presented here can be obtained for vector fields with linear operator A which does not possess the property of normality. We use the normality in two steps of the proof: first for the non-existence of generalized eigenvectors and second for the orthogonality of E_1 to other eigenvectors. Both properties can be supposed independently without any normality. In particular, the orthogonality can be obtain by the choice of the scalar product in $\mathbb{R}^n$ and by the choice of the measure on Ω.

2. It will be interesting to obtain some close results for more general classes of functional nonlinearities.

3. It will also be interesting to obtain any result close to Theorem 2 for a non-simple eigenvalue 1.

4. All results can be generalized for vector fields with nonlinearities depending on delays, derivatives, hysteresis, etc.

5. Instead of linear functionals in Theorem 2 it is possible to consider nonlinearities with $q(t,x)$ depending on the signs of nonlinear forms $L_j(x)$ where

$$L_j(\lambda x) = \lambda^{\alpha_j} L_j(x)$$

for some $\alpha_j > 0$. This is useful for example in the study of systems with nonlinearities of the type $\arctan(x_1^2 + x_2^2 - x_3^2)$.

6. Let us underline the principal geometrical difference between the two cases considered in Theorems 1 and 2. In the case of Theorem 1 our vector field is homotopical to some linear one and its index at infinity always equals to ± 1, while in Theorem 2 the index can be equal to zero.

10 Examples

We present two examples in this section. The first is an application of Theorem 1 and the second is an application of Theorem 2. We give here solvability in both examples and for Example 1 we give also a multiplicity result and a statement on asymptotic bifurcation points.

Example 1. Consider the 2π-periodic problem for the system

$$\begin{cases} x_1' + x_2 & = f_1(x,y) + \sin t + \cos 3t, \\ x_2' - x_1 & = f_2(x,y) + \cos t + \sin 2t, \end{cases} \tag{40}$$

where

$$f_1(x_1,x_2) = \frac{x_1 - 1}{(x_1^2 + x_2^2 + 1)^{1998}}, \qquad f_2(x_1,x_2) = \frac{x_2 + \sqrt{|x_1|}}{(x_1^2 + x_2^2 + 1)^{1998}}. \tag{41}$$

The left-hand side of (40) degenerates on the 2-dimensional subspace $E_1 \subset L^2$, which has an orthogonal normed basis

$$\mathbf{e}_1(t) = \frac{1}{\sqrt{2\pi}}\{\sin t,\, -\cos t\}, \qquad \mathbf{e}_2(t) = \frac{1}{\sqrt{2\pi}}\{\cos t,\, \sin t\}.$$

All non-zero functions from E_1 satisfy

$$|a\mathbf{e}_1(t) + b\mathbf{e}_2(t)|_n \equiv \frac{1}{\sqrt{2\pi}}(a^2 + b^2) \qquad\qquad t \in \Omega = [0, 2\pi],$$

and if $\|\mathbf{e}\| = 1$, then the distribution $\chi(\delta, \mathbf{e})$ is identically zero for $\delta < (\sqrt{2\pi})^{-1/2}$.

The functions (41) satisfy

$$x_1 f_1(x_1,x_2) + x_2 f_2(x_1,x_2) = \frac{1}{(x_1^2 + x_2^2)^{1997}} + o\left(\left(x_1^2 + x_2^2\right)^{-1997}\right).$$

So we can take

$$\psi(u) = \frac{1}{2}\, u^{-3994}, \qquad \theta(u) = 3u^{-3995}. \tag{42}$$

Rewrite the system (40) as the operator equation $\mathbf{x} = A(\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t))$, where the normal operator A is the inverse operator to the differential operator $\{x_1, x_2\} \mapsto \{x_1' + x_1 + x_2,\ x_2' - x_1 + x_2\}$ with the 2π-periodical boundary condition and $\mathbf{b}(t) = \{\sin t + \cos 3t,\ \cos t + \sin 2t\}$. The vector field $\Upsilon\mathbf{x} - A(\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t))$ satisfies all the assumptions of Theorem 1: $P_1\mathbf{b} = 0$, the function $\mathbf{f}(\mathbf{x}) = \{f_1(x_1, x_2),\ f_2(x_1, x_2)\}$ satisfies (11), the functions (42) satisfy (20) and (21). According to Theorem 1 the field Υ has non-zero index at infinity, consequently system (40) has at least one 2π-periodic solution.

In the case considered, $\dim E_1 = 2$ and $\sigma = 0$. It means that $\operatorname{ind}_\infty \Upsilon = 1$.

Now consider the system with a parameter

$$\begin{cases} x_1' + x_2 &= f_1(x, y) + \lambda \sin t + \cos 3t, \\ x_2' - x_1 &= f_2(x, y) + \quad \cos t + \sin 2t, \end{cases} \tag{43}$$

If $\lambda = 1$, then systems (40) and (43) coincide, but if $\lambda \neq 1$ then Theorem 1 is unapplicable as $P_1\mathbf{b} \neq 0$. For this case $\operatorname{ind}_\infty \Upsilon = 0$ [6]. The value $\lambda = 1$ of the parameter is an asymptotic bifurcation point [10] for the 2π-periodic problem for system (43). Morover, for $\lambda \neq 1$ and λ close enough to 1 at least two 2π-periodic solutions exist for system (43).

Example 2. Consider the system

$$\begin{cases} x_1'' + \ {}^9/_2 x_1 + \quad x_2 &= b_1(t) + \arctan(x_1 + x_2) \\ \\ x_2'' + \quad x_1 + \ 3x_2 &= b_2(t) + \dfrac{x_2 + \sqrt{|x_1|} + 1}{|x_2| + .1}. \end{cases} \tag{44}$$

We are interested in the solutions of the system satisfying the boundary conditions

$$x_1(0) = x_2(0) = x_1(\pi) = x_2(\pi) = 0. \tag{45}$$

The linear part of system (44) degenerates on the one-dimensional space

$$E_0 = \Big\{\mathbf{e}(t) = \{2a \sin 2t, a \sin 2t\},\ a \in I\!R\Big\}$$

and distributions of the normed functions $\mathbf{e}(t)$ from this subspace satisfy the estimates

$$c^1\delta \le \chi(\delta, |\mathbf{e}|_n) \le c^2\delta \tag{46}$$

for small values of δ.

Denote

$$f_1(x_1, x_2) = \arctan(x_1 + x_2) - \frac{\pi}{2} \operatorname{sign}(x_1 + x_2), \quad f_2(x_1, x_2) = \frac{x_2 + \sqrt{|x_1|} + 1}{|x_2| + .1} - \operatorname{sign} x_2$$

and write $\Delta = \{\mathbf{x} \in S : \ \mathbf{x} = \{x_1, x_2\}, \ x_1/x_2 \in [1.9, 2.1]\}$. Obviously,

$$\mathbf{f}(\xi\mathbf{x}) = \{f_1(\xi x_1, \xi x_2, \ f_2(\xi x_1, \xi x_2)\} \to 0$$

for $\mathbf{x} \in \Delta$ as $\xi \to \infty$. This means that the right-hand side of (44) generates an asymptotically homogeneous superposition operator with the homogeneous part

$$\{x_1(t), x_2(t)\} \mapsto \mathbf{q}(t, \mathbf{x}) \stackrel{\text{def}}{=}$$

$$\{q_1(t, x_1, x_2), \ q_2(t, x_1, x_2)\} = \{b_1(t) + \frac{\pi}{2} \operatorname{sign}(x_1 + x_2), \ b_2(t) + \operatorname{sign} x_2\}.$$

The function $\mathbf{q}(t, \mathbf{x})$ depends on the signs of two linear functionals $L_1(\mathbf{x}) = x_1 + x_2$ and $L_2(\mathbf{x}) = x_2$. Denote $\mathbf{e}(t) = \{e_1(t), e_2(t)\} = ({}^5/{}_{2\pi})^{-1/2}\{2\sin 2t, \sin 2t\}$. If

$$(\mathbf{e}, \mathbf{q}(t, \mathbf{e})) = \int_0^\pi \Big(e_1 q_1(t, e_1, e_2) + e_2 q_2(t, e_1, e_2)\Big) \, dt \neq 0 \tag{47}$$

and

$$(\mathbf{e}, \mathbf{q}(t, -\mathbf{e})) = \int_0^\pi \Big(e_1 q_1(t, -e_1, -e_2) + e_2 q_2(t, -e_1, -e_2)\Big) \, dt \neq 0,$$

then one can apply Proposition 5 to the analysis of system (44): the "linear + homogeneous" terms are non-degenerate. We consider the case where (47) fails:

$$\int_0^\pi \sin 2t \, \Big(2b_1(t) + b_2(t)\Big) \, dt + 2 + 2\pi = 0.$$

For this case

$$(\mathbf{e}, \mathbf{q}(t, -\mathbf{e})) = \zeta = ({}^5/{}_{2\pi})^{-1/2}(-4 - 4\pi).$$

Put

$$\psi(u) = \frac{1}{10}, \qquad \theta(u) = 2u^{-.5}.$$

It is possible to check that

$$x_1 f_1(x_1, x_2) + x_2 f_2(x_1, x_2) \geq \psi(\sqrt{x_1^2 + x_2^2}), \quad \mathbf{x} = \{x_1, x_2\} \in \Delta^* = \Delta, \ |\mathbf{x}|_n \geq u_0$$

and

$$|f_1(x_1, x_2)|, \ |f_2(x_1, x_2)| \leq \theta(\sqrt{x_1^2 + x_2^2}) + \theta(|x_1 + x_2|) + \theta(|x_2|),$$

and that the functions ψ and θ satisfy all the assumptions of Theorem 2 due to (46).

Again, let us rewrite our system as the operator equation $\mathbf{x} = A(x + \mathbf{q}(t, \mathbf{x}) + \mathbf{f}(x))$, where the linear self-adjont operator A is inverse to the corresponding differential one. All the assumptions of Theorem 2 are fulfilled, (13) holds, $\zeta < 0$, the index at infinity of the vector field $\mathbf{x} - A(x + \mathbf{q}(t, \mathbf{x}) + \mathbf{f}(x))$ is equal to ± 1, the system (44) has at least one solution satisfying the boundary conditions (45).

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