Optimal velocity control of a Brinkman–Cahn–Hilliard system with curvature effects

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Abstract

We address an optimal control problem governed by a system coupling a Brinkman-type momentum equation for the velocity field with a sixth-order Cahn–Hilliard equation for the phase variable, incorporating curvature effects in the free energy. The control acts as a distributed velocity control, allowing for the manipulation of the flow field and, consequently, the phase separation dynamics. We establish the existence of optimal controls, prove the Fréchet differentiability of the control-to-state operator, and derive first-order necessary optimality conditions in terms of a variational inequality involving the adjoint state variables. We also discuss the aspect of sparsity. Beyond its analytical novelty, this work provides a rigorous control framework for Brinkman–Cahn–Hilliard systems incorporating a curvature regularization, offering a foundation for applications in microfluidic design and controlled pattern formation.

1 Introduction

Diffuse-interface models have greatly advanced the study of multiphase flows by providing an energetic framework for interfacial phenomena and curvature effects. In particular, higher-order Cahn–Hilliard systems effectively describe phase separation and membrane dynamics in soft matter and biological contexts. In the recent contribution [10], a Brinkman–Cahn–Hilliard model with curvature effects was introduced, coupling a sixth-order phase field equation with a Brinkman-type momentum balance to capture viscous dissipation and curvature-induced stresses in two-phase incompressible flows.

The present paper aims to take a further step in this line of research by investigating the corresponding optimal control problem. The possibility of influencing the order parameter dynamics through a distributed control via the velocity field provides a natural bridge between theoretical analysis and applications, ranging from microfluidic design to the controlled patterning of soft materials. Accordingly, we consider here an optimal control problem governed by the aforementioned sixth-order Brinkman–Cahn–Hilliard system, which models phase separation processes in incompressible viscous mixtures and incorporates curvature effects within the free energy functional. The corresponding state system is given by

$$-\operatorname{div}\mathbb{T}(\varphi,\boldsymbol{v},p) + \lambda(\varphi)\boldsymbol{v} = \mu\nabla\varphi + \boldsymbol{u} \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } Q, \tag{1.1}$$

$$\partial_t\varphi + \boldsymbol{v}\cdot\nabla\varphi - \operatorname{div}(m(\varphi)\nabla\mu) = S(\varphi) \qquad\qquad \text{in } Q, \tag{1.2}$$

$$-\varepsilon\Delta w + \tfrac{1}{\varepsilon}f'(\varphi)w + \nu w = \mu \qquad\qquad \text{in } Q, \tag{1.3}$$

$$-\varepsilon\Delta\varphi + \tfrac{1}{\varepsilon}f(\varphi) = w \qquad\qquad \text{in } Q. \tag{1.4}$$

Here, Q stands for $\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^3$ is the spatial bounded domain with smooth boundary Γ and $T>0$ indicates a fixed final time. The system is then complemented with the boundary and initial

conditions

$$\mathbb{T}(\varphi, \boldsymbol{v}, p)\boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}w = \partial_{\boldsymbol{n}}\varphi = 0 \qquad\qquad \text{on } \Sigma, \qquad (1.5)$$

$$\varphi(0) = \varphi_0 \qquad\qquad \text{in } \Omega, \qquad (1.6)$$

where $\boldsymbol{n}$ and $\partial_{\boldsymbol{n}}$ denote the outer unit normal vector to Γ and the associated normal derivative, respectively, $\Sigma := \Gamma \times (0, T)$, and φ_0 is a given function acting as initial datum. We observe that, in view of (1.4) and (1.5), the unknown φ also (at least formally) satisfies the boundary condition $\partial_{\boldsymbol{n}}\Delta\varphi = 0$ on Σ.

The first equation (1.1) is a Brinkman-type momentum balance, where the viscosity coefficient $\lambda(\varphi)$ in (1.1) is given in terms of a positive function λ, and the Brinkman stress tensor $\mathbb{T}(\varphi, \boldsymbol{v}, p)$ is defined by

$$\mathbb{T}(\varphi, \boldsymbol{v}, p) = \eta(\varphi)D\boldsymbol{v} - p\mathbb{I}$$

with phase-dependent positive viscosity $\eta(\varphi)$; moreover, p is the pressure, $\mathbb{I} \in \mathbb{R}^{3\times3}$ the identity matrix, and $D\boldsymbol{v}$ denotes the symmetric part of the gradient of the velocity field $\boldsymbol{v}$, that is,

$$D\boldsymbol{v} := \frac{1}{2}\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^{\top}\right). \qquad (1.7)$$

This formulation interpolates between the Stokes ($\lambda \equiv 0$) and Darcy ($\eta \equiv 0$) regimes, making it well suited for modeling flows in porous or heterogeneous media. The variable φ serves as the order parameter, representing the local phase concentration, and is normalized so that $\varphi = \pm1$ correspond to the pure phases, while $\{-1 < \varphi < 1\}$ describes the diffuse interfacial layer of thickness ε. Its evolution is governed by a sixth-order Cahn–Hilliard-type equation with a phase-dependent source term $S(\varphi)$ and a positive mobility function $m(\varphi)$ in equation (1.2), whereas $\boldsymbol{u}$ on the right-hand side of (1.1) denotes the distributed control variable. Furthermore, in the above equations the quantities μ and w represent the first variations of the total free energy $\mathcal{E}(\varphi)$ and of the Ginzburg–Landau energy, respectively, that is, $\mu = \frac{\delta\mathcal{E}}{\delta\varphi}$ and $w = \frac{\delta\mathcal{G}}{\delta\varphi}$, with

$$\mathcal{E}(\varphi) := \mathcal{F}(\varphi) + \nu\mathcal{G}(\varphi) = \frac{1}{2}\int_{\Omega}\left(-\varepsilon\Delta\varphi + \tfrac{1}{\varepsilon}f(\varphi)\right)^2 + \nu\int_{\Omega}\left(\tfrac{\varepsilon}{2}|\nabla\varphi|^2 + \tfrac{1}{\varepsilon}F(\varphi)\right). \qquad (1.8)$$

In (1.8), F denotes a smooth double-well potential, $f = F'$, and ν is a real parameter, which is not necessarily positive. A typical example for F is given by the classical quartic potential

$$F_{\text{reg}}(s) := \frac{1}{4}\left(s^2 - 1\right)^2, \quad s \in \mathbb{R}. \qquad (1.9)$$

The well-posedness of problem (1.1)–(1.6) has been established and discussed in [10] within a suitable analytic framework that also includes the analysis of the Darcy limit as the viscosity $\eta(\varphi)$ tends to zero. The purpose of the present contribution is to build on those results and to provide a rigorous analysis of a corresponding optimal control problem. For the optimal control application, we consider the tracking-type cost functional

$$\mathcal{J}(\varphi, \boldsymbol{u}) := \frac{b_1}{2}\int_Q|\varphi - \varphi_Q|^2 + \frac{b_2}{2}\int_{\Omega}|\varphi(T) - \varphi_{\Omega}|^2 + \frac{b_3}{2}\int_Q|\boldsymbol{u}|^2 + G(\boldsymbol{u}),$$

which is to be minimized over the set of admissible controls

$$\mathcal{U}_{ad} := \left\{\boldsymbol{u} = (u_1, u_2, u_3) \in (L^{\infty}(Q))^3 : \underline{u}_i \le u_i \le \overline{u}_i \text{ for } i = 1, 2, 3\right\},$$

subject to the state system (1.1)–(1.6). In the cost functional, the constants b_i are nonnegative, and G represents an additional regularization or a sparsity-enhancing term, where a prototypical choice is given by

$$G(\boldsymbol{u}) := \kappa \int_Q \big(|u_1| + |u_2| + |u_3|\big) \tag{1.10}$$

with some $\kappa > 0$. In the set $\mathcal{U}_{ad}$ above, the bounds $\underline{u}_i$, $\overline{u}_i$ are given measurable functions on Q satisfying $\underline{u}_i \leq \overline{u}_i$ a.e. in Q, ensuring that $\mathcal{U}_{ad}$ is nonempty. At this point, we remark that the notion of sparsity, i.e., the possibility that any locally optimal control may vanish in subregions of positive measure of the space-time cylinder Q, is very important in practical applications, in particular, in the practical numerical solution of control problems. In connection with partial differential equations, sparsity was first investigated in [46]. For an overview of the existing literature in this field, we refer the reader to the references given in the recent papers [44] and [15].

A physical interpretation of the model is discussed in [10] (see the references therein). Here, we briefly recall its relation to diffuse-interface formulations of curvature-driven flows and to applications in optimal control. A key motivation stems from modeling bilayer membranes and soft-matter systems, where curvature plays a central energetic role. The classical Helfrich model [5,28] describes the elastic bending energy of a smooth membrane Γ_0 by

$$\mathcal{E}_{\text{elastic}} = \frac{k}{2} \int_{\Gamma_0} (H - H_0)^2 \, \mathrm{d}S,$$

where H is the mean curvature, H_0 the spontaneous curvature, and k the bending rigidity. An alternative diffuse-interface representation replaces the sharp-interface formulation by introducing an order parameter φ distinguishing interior ($\varphi = 1$) and exterior ($\varphi = -1$) regions, leading to a modified Willmore functional of the form

$$\mathcal{E}_\varepsilon(\varphi) = \frac{k}{2\varepsilon} \int_\Omega \Big(-\varepsilon \Delta \varphi + \frac{1}{\varepsilon}(\varphi^2 - 1)\varphi \Big)^2, \tag{1.11}$$

where $\varepsilon > 0$, as above, represents the interfacial thickness. It is known that $\mathcal{E}_\varepsilon(\varphi)$ converges to the sharp-interface energy as $\varepsilon \to 0$ (see [18,19]). Since sharp-interface asymptotics are not considered here, we fix $\varepsilon = 1$ from now on for simplicity.

The energy functional defined in (1.8), which characterizes our system, can be regarded as a higher-order extension of the classical Ginzburg–Landau free energy $\mathcal{G}(\varphi)$, see [10] for further details. For $\nu = 0$, $\mathcal{E}(\varphi)$ reduces to the Willmore-type energy (1.11), recovering the Canham–Helfrich bending description. When $\nu > 0$, it acts as a curvature-penalized regularization of $\mathcal{G}(\varphi)$, while for $\nu < 0$ it yields the functionalized Cahn–Hilliard energy [25], relevant for amphiphilic mixtures and membrane models. These regimes highlight the flexibility of the framework in describing interfacial phenomena ranging from membrane elasticity to pattern formation.

For the aforementioned models, a substantial body of analytical and numerical literature has investigated diffuse-interface formulations. Rigorous mathematical analyses can be found in [3, 5–7, 13, 14, 25, 28–30, 40, 49, 50], while numerical studies have been carried out in [1, 4, 18–20, 32, 47].

Sixth-order Cahn–Hilliard-type equations have been extensively explored, for instance, in oil-water-surfactant dynamics [35, 36, 39] and in phase-field crystal models [8, 26, 27, 33, 34, 48]. Related optimal control and optimization problems have been studied in [9, 11, 15, 24, 37, 38, 41, 43–46], whereas the coupling with the standard Cahn–Hilliard equation and Brinkman flows has been analyzed in [2, 10, 12, 16, 17, 21–23, 31].

Furthermore, these contributions include studies on velocity control for hydrodynamically coupled systems involving either classical or nonlocal Cahn–Hilliard equations [11, 24, 37, 38, 45], as well as sixth-order Cahn–Hilliard systems without fluid coupling [9]. Overall, this literature highlights the richness of both analytical and numerical approaches in high-order diffuse-interface models and their optimal control applications, encompassing vesicle-fluid interactions, phase-field crystal dynamics, and complex multiphase flows.

The nonlinear coupling between the flow and the sixth-order Cahn–Hilliard subsystem, together with the curvature term in the energy (1.8), makes the analysis of the problem particularly challenging. The paper is organized as follows. In Section 2, we introduce the notation, assumptions, and main theoretical results. Section 3 is devoted to proving the existence of at least one optimal control for the considered minimization problem. In Section 4, we study the differentiability properties of the control-to-state operator and establish its Fréchet differentiability in a suitable functional framework. As a preliminary step, we analyze the corresponding linearized system. In Section 5, we derive the first-order necessary conditions for optimality, which are then reformulated in terms of an adjoint system that we introduce and solve. The final Section 6 then brings the derivation of sparsity results for the prototypical sparsity function G given in (1.10). Overall, these results constitute a first step toward a rigorous optimal control theory for Brinkman–Cahn–Hilliard systems with curvature effects and lay the groundwork for further analytical and numerical studies.

2   Notation, assumptions and main results

In this section, we fix our notation, list the assumptions on the state system, give it a precise form, and recall some results for it that are already known. As for the set Ω, its boundary Γ, the normal derivative $\partial_{\boldsymbol{n}}$, the space-time cylinder Q and its lateral surface Σ, we keep the notation used in the Introduction. More precisely, Ω is an open, bounded and connected set in $\mathbb{R}^3$ having a smooth boundary. The symbol $|\Omega|$ denotes its Lebesgue measure. For any Banach space X, the symbols $\|\cdot\|_X$ and X^* denote the corresponding norm and its dual space. However, some exceptions to this notation are listed below. We introduce the spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{z \in H^2(\Omega) : \partial_{\boldsymbol{n}} z = 0 \text{ on } \Gamma\}, \tag{2.1}$$

$$\boldsymbol{H} := H \times H \times H, \quad \boldsymbol{V} := V \times V \times V \quad \text{and} \quad \boldsymbol{V}_0 := \{\boldsymbol{\zeta} \in \boldsymbol{V} : \operatorname{div} \boldsymbol{\zeta} = 0\}. \tag{2.2}$$

Similarly, we use the boldface characters, like $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^1(\Omega)$, to denote powers of the Lebesgue and Sobolev spaces. To simplify the notation, the norms in the special cases H and $\boldsymbol{H}$ are simply indicated by $\|\cdot\|$. The symbol $\|\cdot\|_\infty$ might denote the norm in each of the spaces $L^\infty(\Omega)$, $L^\infty(Q)$ and $L^\infty(0, T)$, if no confusion can arise. Furthermore, we use the same symbol for the norm in some space and the norm in any power thereof.

Since W is dense in V, and V is dense in H, we can make standard identifications and adopt the usual framework of Hilbert triplets. Namely, we have that

$$\langle y, z \rangle_V = \int_\Omega yz \quad \text{and} \quad \langle y, z \rangle_W = \langle y, z \rangle_V$$
$$\text{for every } y \in H \text{ and } z \in V \text{ and every } y \in V^* \text{ and } z \in W, \text{ respectively}$$

so that

$$W \hookrightarrow V \hookrightarrow H \hookrightarrow V^* \hookrightarrow W^*. \tag{2.3}$$

The symbols $\langle\,\cdot\,,\,\cdot\,\rangle_V$ and $\langle\,\cdot\,,\,\cdot\,\rangle_W$ denote the duality pairings between V^* and V and between W^* and W, respectively. However, thanks to the above identifications, there is no confusion if we avoid using subscripts and adopt the simpler symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ for the above pairings. Besides the space $\boldsymbol{V}_0$ already introduced, we also define

$$\boldsymbol{H}_0 := \{\boldsymbol{\zeta} \in \boldsymbol{H} : \operatorname{div}\boldsymbol{\zeta} = 0\}, \tag{2.4}$$

where the divergence is understood in the sense of distributions. Notice that the embedding

$$\boldsymbol{V}_0 \hookrightarrow \boldsymbol{H}_0 \tag{2.5}$$

is dense (see, e.g., [30, Cor. 2.3]) and that all of the embeddings in (2.3) and (2.5) are compact. Next, we recall the symbol $D\boldsymbol{v}$ introduced in (1.7) for the symmetrized gradient of the velocity $\boldsymbol{v}$, whose use will be extended to any vector field $\boldsymbol{\zeta} \in \boldsymbol{V}$. Finally, we adopt the standard notation

$$A : B := \sum_{i,j=1}^{3} a_{ij}b_{ij} \quad \text{and} \quad |A|^2 := A : A, \quad \text{for } A = (a_{ij}),\ B = (b_{ij}) \in \mathbb{R}^{3\times 3}, \tag{2.6}$$

for the scalar product and the norm of matrices.

We are ready to list our assumptions on the structure of the state system. For the involved functions and parameters, we postulate:

$\lambda : \mathbb{R} \to \mathbb{R}$ is of class C^1 and satisfies $\quad \lambda_* \leq \lambda(s) \leq \lambda^*$

    for every $s \in \mathbb{R}$ and some positive constants λ_* and λ^*. $\tag{2.7}$

$\eta(\cdot) \equiv \eta_0,\ m(\cdot) \equiv m$ with $\eta_0, m > 0$, and ν is real constant. $\tag{2.8}$

$S(s) := -\sigma s + h(s) \quad \text{for } s \in \mathbb{R}, \quad \text{with} \quad \sigma \in \mathbb{R} \quad \text{and a function}$

    $h : \mathbb{R} \to \mathbb{R}$ of class C^2, which is bounded and Lipschitz continuous. $\tag{2.9}$

$F : \mathbb{R} \to \mathbb{R}$ is of class C^4, and F and $f := F'$ satisfy $\tag{2.10}$

$$\lim_{|s|\to+\infty} \frac{f(s)}{s} = +\infty, \quad \text{as well as} \tag{2.11}$$

$f'(s) \geq -C_1, \quad |F(s)| \leq C_2\left(|sf(s)| + 1\right), \quad \text{and} \quad |sf'(s)| \leq C_3(|f(s)| + 1),$

    for every $s \in \mathbb{R}$ and some positive constants C_1, C_2 and C_3. $\tag{2.12}$

Notice that our assumptions on the potential are satisfied if $F = F_{reg}$, the classical regular potential defined in (1.9). For the data, we assume that

$$\boldsymbol{u} \in L^2(0, T; \boldsymbol{H}), \tag{2.13}$$

$$\varphi_0 \in W. \tag{2.14}$$

As stated in the Introduction, a well-posedenss result was proved in [10] for an equivalent problem in which the pressure p no longer appears. Namely, in the quoted paper it was shown that, instead of looking for the pair $(\boldsymbol{v}, p)$ solving (1.1), we can equivalently look for its first component $\boldsymbol{v}$ as a divergence-free solution to the variational equation

$$\int_\Omega \left(\eta_0 D\boldsymbol{v} : \nabla\boldsymbol{\zeta} + \lambda(\varphi)\boldsymbol{v} \cdot \boldsymbol{\zeta}\right) = \int_\Omega (\mu\nabla\varphi + \boldsymbol{u}) \cdot \boldsymbol{\zeta}$$

    for every $\boldsymbol{\zeta} \in \boldsymbol{V}_0$ and a.e. in $(0, T)$. $\tag{2.15}$

The argument was based on the identity

$$D\boldsymbol{\zeta} : \nabla\boldsymbol{\zeta} = |D\boldsymbol{\zeta}|^2 \quad \text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}, \tag{2.16}$$

and the Korn inequality

$$\|\boldsymbol{\zeta}\|_V^2 \leq \mathcal{C}_K \int_\Omega \left(|D\boldsymbol{\zeta}|^2 + |\boldsymbol{\zeta}|^2 \right) \quad \text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}, \tag{2.17}$$

which holds true for some constant $\mathcal{C}_K > 0$ depending only on Ω. The combination of these facts yields the coerciveness inequality

$$\int_\Omega \left(\eta_0 D\boldsymbol{\zeta} : \nabla\boldsymbol{\zeta} + \lambda(\phi)|\boldsymbol{\zeta}|^2 \right) \geq \alpha \, \|\boldsymbol{\zeta}\|_V^2 \quad \text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}_0, \tag{2.18}$$

$$\text{where} \quad \alpha := \frac{\min\{\eta_0, \lambda_*\}}{\mathcal{C}_K}, \tag{2.19}$$

which allows to apply the Lax–Milgram theorem. Next, following the lines of [10], we present another possible version of the problem. Namely, one can eliminate w by inserting (1.4) in (1.3) (where now $\varepsilon = 1$ in both equations) to obtain

$$-\Delta\big(-\Delta\varphi + f(\varphi)\big) + \big(f'(\varphi) + \nu\big)\big(-\Delta\varphi + f(\varphi)\big) = \mu \quad \text{in } Q. \tag{2.20}$$

Here is the precise formulation of the problem under consideration. We look for a quadruplet $(\boldsymbol{v}, \varphi, \mu, w)$ with the regularity

$$\boldsymbol{v} \in L^2(0, T; \boldsymbol{V}_0), \tag{2.21}$$

$$\varphi \in H^1(0, T; V^*) \cap L^\infty(0, T; W) \cap L^2(0, T; H^5(\Omega)), \tag{2.22}$$

$$\mu \in L^2(0, T; V), \tag{2.23}$$

$$w \in L^\infty(0, T; H) \cap L^2(0, T; H^3(\Omega) \cap W), \tag{2.24}$$

that solves the variational equations

$$\int_\Omega \big(\eta_0 D\boldsymbol{v} : \nabla\boldsymbol{\zeta} + \lambda(\varphi)\boldsymbol{v} \cdot \boldsymbol{\zeta} \big) = \int_\Omega (\mu\nabla\varphi + \boldsymbol{u}) \cdot \boldsymbol{\zeta}$$

$$\text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}_0 \text{ and a.e. in } (0, T), \tag{2.25}$$

$$\langle \partial_t\varphi, z \rangle + \int_\Omega \boldsymbol{v} \cdot \nabla\varphi \, z + \int_\Omega m\nabla\mu \cdot \nabla z = \int_\Omega S(\varphi)z$$

$$\text{for every } z \in V \text{ and a.e. in } (0, T), \tag{2.26}$$

$$\int_\Omega \nabla w \cdot \nabla z + \int_\Omega \big(f'(\varphi) + \nu \big) wz = \int_\Omega \mu z$$

$$\text{for every } z \in V \text{ and a.e. in } (0, T), \tag{2.27}$$

$$\int_\Omega \nabla\varphi \cdot \nabla z + \int_\Omega f(\varphi)z = \int_\Omega wz$$

$$\text{for every } z \in V \text{ and a.e. in } (0, T), \tag{2.28}$$

as well as the initial condition

$$\varphi(0) = \varphi_0. \tag{2.29}$$

Observe that (2.27) and (2.28) can be replaced by their strong forms

$$-\Delta w + f'(\varphi)w + \nu w = \mu \quad \text{a.e. in } Q, \tag{2.30}$$

$$-\Delta \varphi + f(\varphi) = w \qquad \text{a.e. in } Q, \tag{2.31}$$

thanks to the regularity of w and φ required in (2.24) and (2.22) (which also encode the homogeneous Neumann boundary condition for these components).

Taking (2.31) into account, we can rewrite (2.27) as

$$\int_\Omega \nabla(-\Delta\varphi + f(\varphi)) \cdot \nabla z + \int_\Omega \left(f'(\varphi) + \nu\right)\left(-\Delta\varphi + f(\varphi)\right)z = \int_\Omega \mu\, z$$
$$\text{for every } z \in V \text{ and a.e. in } (0,T), \tag{2.32}$$

and keep (2.28) or (2.31) as the definition of w. Note that (2.32) provides a weak formulation of (2.20).

Here are the already known well-posedness and continuous dependence results (see [10, Thm. 2.1 and Thm. 2.3]).

Theorem 2.1 *Assume* (2.7)–(2.12) *on the structure, and suppose that the data satisfy* (2.13)–(2.14). *Then there exists a unique quadruplet* $(\boldsymbol{v}, \varphi, \mu, w)$ *with the regularity* (2.21)–(2.24) *that solves Problem* (2.25)–(2.29). *Moreover, this solution satisfies the estimate*

$$\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{V})} + \|\varphi\|_{H^1(0,T;V^*) \cap L^\infty(0,T;W) \cap L^2(0,T;H^5(\Omega))}$$
$$+ \|\mu\|_{L^2(0,T;V)} + \|w\|_{L^\infty(0,T;H) \cap L^2(0,T;H^3(\Omega))} \leq K_1\,, \tag{2.33}$$

with a constant K_1 *that depends only on the structure of the system,* Ω, T *and an upper bound for the norms of the data related to* (2.13)–(2.14).

Theorem 2.2 *Assume* (2.7)–(2.12) *for the structure and* (2.14) *for the initial datum. Then the following holds true: if* $\boldsymbol{u}_i \in L^2(0,T;\boldsymbol{H})$, $i = 1,2$, *are given and* $(\boldsymbol{v}_i, \varphi_i, \mu_i, w_i)$ *are the corresponding solutions, then the estimate*

$$\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{V})} + \|\varphi\|_{C^0([0,T];V) \cap L^2(0,T;H^4(\Omega))} + \|\mu\|_{L^2(0,T;H)}$$
$$+ \|w\|_{L^2(0,T;W)} \leq K_2 \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{H})} \tag{2.34}$$

holds true for the differences $(\boldsymbol{v}, \varphi, \mu, w) = (\boldsymbol{v}_1, \varphi_1, \mu_1, w_1) - (\boldsymbol{v}_2, \varphi_2, \mu_2, w_2)$ *and* $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, *with a constant* K_2 *that depends only on the structure of the system,* Ω, T, *the initial datum* φ_0, *and an upper bound for the norms of* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$ *in* $L^2(0,T;\boldsymbol{H})$.

Once well-posedness of the state system is established, one can deal with the control problem. We present the cost functional mentioned in the Introduction in a precise form. We set

$$\mathcal{J}(\varphi, \boldsymbol{u}) := J(\varphi, \boldsymbol{u}) + G(\boldsymbol{u})\,, \quad \text{where} \tag{2.35}$$

$$J(\varphi, \boldsymbol{u}) := \frac{b_1}{2} \int_Q |\varphi - \varphi_Q|^2 + \frac{b_2}{2} \int_\Omega |\varphi(T) - \varphi_\Omega|^2 + \frac{b_3}{2} \int_Q |\boldsymbol{u}|^2\,, \tag{2.36}$$

$$\text{and} \quad G : \boldsymbol{L}^2(Q) \to [0, +\infty] \quad \text{is convex, proper and lower semicontinuous.} \tag{2.37}$$

In (2.36), the coefficients and the target functions satisfy

$$b_1\,,\ b_2 \in [0, +\infty) \quad \text{and} \quad b_3 \in (0, +\infty), \tag{2.38}$$

$$\varphi_Q \in L^2(Q) \quad \text{and} \quad \varphi_\Omega \in H\,. \tag{2.39}$$

The set of admissible controls is defined by

$$\mathcal{U}_{ad} := \{\boldsymbol{u} = (u_1, u_2, u_3) \in \boldsymbol{L}^\infty(Q) : \ \underline{u}_i \le u_i \le \overline{u}_i \ \text{a.e. in } Q\,, \text{for } i = 1, 2, 3\}, \tag{2.40}$$

where $\underline{u}_i\,,\ \overline{u}_i \in L^\infty(Q)$ satisfy

$$\underline{u}_i \le \overline{u}_i \quad \text{a.e. in } Q\,, \text{for } i = 1, 2, 3\,. \tag{2.41}$$

Then, the control problem under investigation reads:

$$\text{Minimize} \quad \mathcal{J}(\varphi, \boldsymbol{u}) \quad \text{over } \mathcal{U}_{ad} \quad \text{under the constraint:}$$
$$\varphi \text{ is the second component of the solution}$$
$$\text{to the problem (2.25)–(2.29) corresponding to } \boldsymbol{u}\,. \tag{2.42}$$

We are ready to present the existence theorem for an optimal control.

Theorem 2.3 *Assume* (2.7)–(2.12) *for the structure of the state system and* (2.14) *for the initial data. Moreover, assume* (2.35)–(2.39) *for the cost functional and* (2.40)–(2.41) *for the set of the admissible controls. Then there exists at least one optimal control to the control problem* (2.42)*, that is, there is some $\boldsymbol{u}^* \in \mathcal{U}_{ad}$ such that $\mathcal{J}(\varphi^*, \boldsymbol{u}^*) \le \mathcal{J}(\varphi, \boldsymbol{u})$ for every $\boldsymbol{u} \in \mathcal{U}_{ad}$, where φ^* and φ are the second components of the solutions $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*)$ and $(\boldsymbol{v}, \varphi, \mu, w)$ to the state systems* (2.25)–(2.29) *corresponding to $\boldsymbol{u}^*$ and $\boldsymbol{u}$, respectively.*

The next steps consists in finding a significant necessary condition for a given admissible control to be optimal. To this end, we follow a standard procedure. Namely, we introduce the so-called *control-to-state operator*, termed $\mathcal{S}$, that maps some neighborhood $\mathcal{U}_R$ of $\mathcal{U}_{ad}$ in $L^2(0, T; \boldsymbol{H})$ to a suitable Banach space and associates to every $\boldsymbol{u} \in \mathcal{U}_R$ the corresponding state, i.e., the solution $(\boldsymbol{v}, \varphi, \mu, w)$ to problem (2.25)–(2.29). Then, we introduce the composite map $\tilde{J}$, also referred to as the *reduced cost functional*, by setting, for $\boldsymbol{u} \in \mathcal{U}_R$,

$$\tilde{J}(\boldsymbol{u}) := J(\varphi, \boldsymbol{u})\,, \quad \text{where } \varphi \text{ is the second component of } (\boldsymbol{v}, \varphi, \mu, w) := \mathcal{S}(\boldsymbol{u})\,, \tag{2.43}$$

so that the functional to be minimized on $\mathcal{U}_{ad}$ is just $\tilde{J} + G$. Then, a standard argument from Convex Analysis ensures that, if $\boldsymbol{u}^*$ is an optimal control, then there exists some $\boldsymbol{\Lambda}^*$ in the subdifferential $\partial G(\boldsymbol{u}^*) \subset \boldsymbol{L}^2(Q)$ such that

$$D\tilde{J}(\boldsymbol{u}^*)[\boldsymbol{u} - \boldsymbol{u}^*] + \int_Q \boldsymbol{\Lambda}^* \cdot (\boldsymbol{u} - \boldsymbol{u}^*) \ge 0 \quad \text{for every } \boldsymbol{u} \in \mathcal{U}_{ad}\,, \tag{2.44}$$

where $D\tilde{J}(\boldsymbol{u}^*)$ is the Fréchet derivative of $\tilde{J}$ at $\boldsymbol{u}^*$, provided that it exists. In order to justify this procedure, we prove that the control-to-state operator $\mathcal{S}$ is Fréchet differentiable, so that $D\tilde{J}(\boldsymbol{u}^*)$ can be computed via the chain rule. However, evaluating the Fréchet derivative of $\mathcal{S}$ involves the so-called *linearized system*, which depends on an arbitrary increment. To address this difficulty, we introduce a proper *adjoint system* that allows us to give (2.44) a more suitable form. Namely, we prove that the above inequality can be expressed as

$$\int_Q \left(b_3 \boldsymbol{u}^* + \boldsymbol{\omega}\right) \cdot (\boldsymbol{u} - \boldsymbol{u}^*) + \int_Q \boldsymbol{\Lambda}^* \cdot (\boldsymbol{u} - \boldsymbol{u}^*) \ge 0 \quad \text{for every } \boldsymbol{u} \in \mathcal{U}_{ad}\,, \tag{2.45}$$

where $\boldsymbol{\omega}$ is the first component of the solution $(\boldsymbol{\omega}, p, q, r)$ to the adjoint problem introduced in Section 5. In Section 6, we exploit the variational inequality (2.45) to derive a sparsity result for the optimal controls if the sparsity functional G is given in the special form (1.10) (see Theorem 6.1 below).

The remainder of the paper is organized as follows. The existence of an optimal control is proved in Section 3. The following Section 4 is devoted to the analysis of the control-to-state operator, namely, to its Fréchet differentiability. The necessary condition (2.45) is then proved in Section 5, where a natural adjoint problem is introduced and solved, and the sparsity is addressed in the final Section 6.

Throughout the paper, we assume that $\varepsilon = m = \eta_0 = 1$ without loss of generality and, besides the Hölder and Cauchy–Schwarz inequalities and the Korn inequality (2.17) already mentioned, we widely use Young's inequality

$$ ab \leq \delta a^2 + \frac{1}{4\delta}\, b^2 \quad \text{for every } a, b \in \mathbb{R} \text{ and } \delta > 0. \tag{2.46}$$

We also account for the Sobolev inequalities, as well as for some inequalities associated to the elliptic regularity theory and to the compact embeddings between Sobolev spaces (via Ehrling's lemma). In particular, we have

$$ \|z\|_q \leq \mathcal{C}_S\, \|z\|_V \quad \text{for every } z \in V \text{ and } q \in [1, 6], \tag{2.47}$$

$$ \|z\|_\infty \leq \mathcal{C}_S\, \|z\|_W \quad \text{for every } z \in W, \tag{2.48}$$

$$ \|z\|_W \leq \mathcal{C}_E\left(\|\Delta z\| + \|z\|\right) \quad \text{for every } z \in W, \tag{2.49}$$

$$ \|z\|_{H^3(\Omega)} \leq \mathcal{C}_E\left(\|\nabla \Delta z\| + \|z\|\right) \quad \text{for every } z \in H^3(\Omega) \cap W, \tag{2.50}$$

$$ \|z\|_{H^4(\Omega)} \leq \mathcal{C}_E\left(\|\Delta^2 z\| + \|z\|\right) \quad \text{for every } z \in H^4(\Omega) \text{ with } z, \Delta z \in W, \tag{2.51}$$

$$ \|z\|_V \leq \delta\, \|\Delta z\| + \mathcal{C}_\delta\, \|z\| \quad \text{for every } z \in W \text{ and every } \delta > 0, \tag{2.52}$$

$$ \|z\|_{H^2(\Omega)} \leq \delta\, \|\nabla \Delta z\| + \mathcal{C}_\delta\, \|z\| \quad \text{for every } z \in H^3(\Omega) \cap W \text{ and every } \delta > 0, \tag{2.53}$$

$$ \|z\|_{H^3(\Omega)} \leq \delta\, \|\Delta^2 z\| + \mathcal{C}_\delta\, \|z\|_V $$
$$ \text{for every } z \in H^4(\Omega) \text{ with } z, \Delta z \in W \text{ and every } \delta > 0\,. \tag{2.54}$$

The constants on the right-hand sides of (2.47)–(2.51) depend only on Ω, while $\mathcal{C}_\delta$ in (2.52)–(2.54) depends on both Ω and δ. Similar inequalities hold, of course, for vector-and matrix-valued functions. In connection with (2.49) and (2.51), we recall that for z to belong to W it suffices that $z \in V$, $\Delta z \in H$, and the homogeneous Neumann boundary condition is satisfied in the usual weak sense; moreover, z belongs to $H^4(\Omega)$ whenever both z and Δz belong to W. Concerning the equivalence of norms actually stated by (2.50), we observe that the inequality

$$ \|z\|_{H^3(\Omega)} \leq \mathcal{C}_E\big(\|\Delta z\|_{H^1(\Omega)} + \|z\|\big) \quad \text{for all } z \in H^3(\Omega) \cap W $$

follows by elliptic regularity. Furthermore, since $z \in W$ satisfies $\int_\Omega \Delta z = \int_\Gamma \partial_{\boldsymbol{n}} z = 0$, the Poincaré–Wirtinger inequality implies that $\|\Delta z\|_{H^1(\Omega)} \leq C\, \|\nabla \Delta z\|$ for some constant $C > 0$. Combining the above estimates yields (2.50).

We conclude this section by introducing a convention for the notation of the constants that enter the estimates we are going to perform. The lowercase symbol c denotes a generic constant that depends only on Ω, T, the structure of the system, and an upper bound for the norms of the data. Notice that

the value of c may change from line to line and even within the same line. Furthermore, whenever a positive constant depends on a specific parameter such as δ, we indicate this dependence by using a subscript and writing c_δ instead of a general c. On the contrary, specific constants we want to refer to are denoted by different symbols, like in (2.33) and (2.47), where different characters are used.

3   The existence of an optimal control

We now proceed with the proof of Theorem 2.3. We denote by Λ the infimum of the functional to be minimized. Then, Λ is a nonnegative real number and there exists a sequence $\{\boldsymbol{u}_n\}$ in $\mathcal{U}_{ad}$ such that

$$\Lambda \leq \mathcal{J}(\varphi_n, \boldsymbol{u}_n) \leq \Lambda + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \tag{3.1}$$

where φ_n is the second component of the solution $(\boldsymbol{v}_n, \varphi_n, \mu_n, w_n)$ to the state system corresponding to $\boldsymbol{u}_n$. Since $\mathcal{U}_{ad}$ is bounded and closed in $\boldsymbol{L}^\infty(Q)$, we can without loss of generality assume that

$$\boldsymbol{u}_n \to \boldsymbol{u}^* \quad \text{weakly star in } \boldsymbol{L}^\infty(Q)$$

for some $\boldsymbol{u}^* \in \mathcal{U}_{ad}$. On the other hand, $(\boldsymbol{v}_n, \varphi_n, \mu_n, w_n)$ satisfies the stability estimate (2.33) with a constant K_1 that does not depend on n. Therefore, we may also assume that, as $n \to \infty$,

$$\boldsymbol{v}_n \to \boldsymbol{v}^* \quad \text{weakly in } L^2(0, T; \boldsymbol{V}), \tag{3.2}$$

$$\varphi_n \to \varphi^* \quad \text{weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; W) \cap L^2(0, T; H^5(\Omega)), \tag{3.3}$$

$$\mu_n \to \mu^* \quad \text{weakly in } L^2(0, T; V), \tag{3.4}$$

$$w_n \to w^* \quad \text{weakly star in } L^\infty(0, T; H) \cap L^2(0, T; H^3(\Omega) \cap W). \tag{3.5}$$

In particular, we have that $\varphi^*(0) = \varphi_0$. Moreover, by applying, e.g., [42, Sect. 8, Cor. 4], the regularity of f and well-known compact embeddings, we deduce the strong convergence properties, as $n \to \infty$,

$$\varphi_n \to \varphi^*, \quad f(\varphi_n) \to f(\varphi^*), \quad \text{and} \quad f'(\varphi_n) \to f'(\varphi^*), \quad \text{uniformly in } \overline{Q},$$

$$\varphi_n \to \varphi^* \quad \text{strongly in } C^0([0, T]; W^{1,4}(\Omega)),$$

whence also

$$\mu_n \nabla \varphi_n \to \mu^* \nabla \varphi^* \quad \text{weakly in } L^2(0, T; \boldsymbol{H}).$$

Therefore, it is straightforward to conclude that $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*)$ verifies the time-integrated versions of the variational equations (2.25)–(2.28) associated with $\boldsymbol{u}^*$, for time-dependent test functions. This means that $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*)$ is the solution to the state system corresponding to $\boldsymbol{u}^*$. Moreover, the above strong convergence implies that $\varphi_n(T)$ converges to $\varphi^*(T)$ strongly in H. Recalling (3.1), and using the lower semicontinuity of $\mathcal{J}$, we then obtain

$$\mathcal{J}(\varphi^*, \boldsymbol{u}^*) \leq \liminf_{n\to\infty} \mathcal{J}(\varphi_n, \boldsymbol{u}_n) \leq \liminf_{n\to\infty} \left(\Lambda + \frac{1}{n}\right) = \lim_{n\to\infty} \left(\Lambda + \frac{1}{n}\right) = \Lambda,$$

which readily yields that $\mathcal{J}(\varphi^*, \boldsymbol{u}^*) = \Lambda$, and the proof is complete.

4 The control-to-state mapping

The *control-to-state operator* was introduced in Section 2 in a preliminary form. We now provide its precise definition. To this end, we assume that

$$\mathcal{U}_R \text{ is an open ball in } L^2(0,T;\boldsymbol{H}) \text{ containing } \mathcal{U}_{ad}, \tag{4.1}$$

and we introduce the solution space

$$\mathcal{X} := L^2(0,T;\boldsymbol{V}_0) \times \big(H^1(0,T;W^*) \cap C^0([0,T];V) \cap L^2(0,T;H^4(\Omega) \cap W)\big) \times$$
$$\times L^2(0,T;H) \times L^2(0,T;W) \,. \tag{4.2}$$

We then define

$$\mathcal{S} : \mathcal{U}_R \to \mathcal{X} \,, \text{ with } \mathcal{U}_R \ni \boldsymbol{u} \mapsto (\boldsymbol{v}, \varphi, \mu, w) =: \mathcal{S}(\boldsymbol{u})$$
$$= \text{ the solution to the state system (2.25)–(2.29) corresponding to } \boldsymbol{u}. \tag{4.3}$$

Our proof of the Fréchet differentiability of $\mathcal{S}$ relies on an improvement of the continuous dependence inequality (2.34). We have the following result.

Lemma 4.1 *Under the same assumptions, and with the notation used in Theorem 2.2, we have that*

$$\|\varphi\|_{L^\infty(0,T;W) \cap L^2(0,T;H^5(\Omega))} \le K_3 \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{H})} \,, \tag{4.4}$$

where the constant K_3 has the same dependencies as the constant K_2 in (2.34).

Proof. We proceed formally, for brevity, since a rigorous proof would be rather lengthy. A rigorous approach could follow the procedure used in [10] to prove Theorem 2.1, which is based on the approximation of Problem (2.25)–(2.29) by a Faedo–Galerkin scheme constructed by means of the eigenfunctions of related eigenvalue problems. In particular, as far as the component φ is concerned, the eigenvalue problem for the Laplace operator with homogeneous Neumann boundary conditions has been chosen. This provides a high regularity to the discrete solutions φ_n in the n-dimensional approximating spaces. In particular, $\Delta^m \varphi_n$ belongs to W for every nonnegative integer $m \in \mathbb{N}$. Since the solution to Problem (2.25)–(2.29) is unique, the solutions appearing in the statement of the lemma coincide with those obtained through this discretization procedure and the subsequent passage to the limit as $n \to \infty$. Consequently, formal estimates performed directly on the solution to Problem (2.25)–(2.29) can be justified via the discrete scheme. Accordingly, we confine ourselves to providing a formal proof of (4.4) by directly testing certain equations with functions related to the solution, whose discrete counterparts would be admissible in the Faedo–Galerkin scheme.

We take the difference of (2.26) written for both solutions, and we do the same for (2.32). We have, a.e. in $(0,T)$, that

$$\int_\Omega \partial_t \varphi \, z + \int_\Omega \big(\boldsymbol{v} \cdot \nabla\varphi_1 + \boldsymbol{v}_2 \cdot \nabla\varphi\big)z + \int_\Omega \nabla\mu \cdot \nabla z = \int_\Omega \big(S(\varphi_1) - S(\varphi_2)\big)z \,, \tag{4.5}$$

$$\int_\Omega \nabla\big(-\Delta\varphi + f(\varphi_1) - f(\varphi_2)\big) \cdot \nabla z + \int_\Omega \big[\big(f'(\varphi_1) - f'(\varphi_2)\big)(-\Delta\varphi_1) + f'(\varphi_2)(-\Delta\varphi)\big]z$$
$$+ \int_\Omega \big(f(\varphi_1)f'(\varphi_1) - f(\varphi_2)f'(\varphi_2)\big)z + \nu \int_\Omega \big(-\Delta\varphi + f(\varphi_1) - f(\varphi_2)\big)z = \int_\Omega \mu z \,, \tag{4.6}$$

where $z \in V$ is arbitrary in (4.5) and (4.6). No duality is needed for the time derivative, as it reduces to an integral in the corresponding discrete framework. Now, we test (4.5) by $\Delta^2\varphi$ and (4.6) by $-\Delta^3\varphi$, take the sum of the resulting equalities, and rearrange a little. On the left-hand side, we retain the terms

$$\frac{1}{2}\frac{d}{dt}\|\Delta\varphi\|^2 + \int_\Omega |\nabla\Delta^2\varphi|^2 \,. \tag{4.7}$$

The sum of the terms involving μ on the right-hand side is given by

$$-\int_\Omega \nabla\mu \cdot \nabla\Delta^2\varphi + \int_\Omega \mu(-\Delta^3\varphi) = 0\,,$$

using integration by parts and the homogeneous Neumann boundary conditions. Now, we estimate, a.e. in $(0, T)$, the remaining terms on the right-hand side. More precisely, we prepare the estimates that produce terms that can be controlled by the left-hand side (4.7) (either directly or via the Gronwall lemma after time integration) and terms whose integrals over $(0, T)$ can be estimated by the right-hand side of (2.34). We also make a direct use of the L^∞ in time estimates for the solutions ensured by (2.33). In particular, we can assume that all of the nonlinearities are Lipschitz continuous and bounded. We repeatedly account for the Young and Hölder inequalities and for some of the inequalities (2.47)–(2.54). Finally, δ is a positive parameter whose value will be chosen later on. We then have

$$-\int_\Omega (\boldsymbol{v} \cdot \nabla\varphi_1)\Delta^2\varphi \leq c\,\|\boldsymbol{v}\|_4\,\|\nabla\varphi_1\|_4\,\|\Delta^2\varphi\| \leq c\,\|\boldsymbol{v}\|_V^2 + c\,\|\varphi\|_{H^4(\Omega)}^2\,,$$

where we have also used the continuity of the embedding

$$L^\infty(0, T; W) \hookrightarrow L^\infty(0, T; W^{1,4}(\Omega))$$

and the fact that φ_1 is estimated in $L^\infty(0, T; W)$. Next, we have

$$-\int_\Omega (\boldsymbol{v}_2 \cdot \nabla\varphi)\Delta^2\varphi \leq \|\boldsymbol{v}_2\|_4\,\|\nabla\varphi\|\,\|\Delta^2\varphi\|_4 \leq \delta\,\|\Delta^2\varphi\|_V^2 + c_\delta\,\|\boldsymbol{v}_2\|_V^2\,\|\varphi\|_V^2\,.$$

The integral involving S is straightforward to handle and therefore omitted. The next term, however, requires a careful treatment. Noting that in the discrete case $f(\varphi_i)$, $\Delta\varphi_i$, $\Delta^2\varphi_i$ and $\Delta f(\varphi_i)$ satisfy the homogeneous Neumann boundary condition (for the last one, see below for the computation of its gradient), the first two integrations by parts are justified, yielding

$$-\int_\Omega \nabla\big(f(\varphi_1) - f(\varphi_2)\big) \cdot \nabla\Delta^3\varphi = \int_\Omega \Delta\big(f(\varphi_1) - f(\varphi_2)\big)\Delta^3\varphi$$
$$= -\int_\Omega \nabla\Delta\big(f(\varphi_1) - f(\varphi_2)\big) \cdot \nabla\Delta^2\varphi$$
$$\leq \delta\,\|\nabla\Delta^2\varphi\|^2 + c_\delta\,\|\nabla\Delta\big(f(\varphi_1) - f(\varphi_2)\big)\|^2\,.$$

The treatment of the last term needs a preliminary computation. Namely, we have that

$$\Delta\big(f(\varphi_1) - f(\varphi_2)\big)$$
$$= f''(\varphi_1)|\nabla\varphi_1|^2 - f''(\varphi_2)|\nabla\varphi_2|^2 + f'(\varphi_1)\Delta\varphi_1 - f'(\varphi_2)\Delta\varphi_2\,,$$

whence, denoting by $D^2\varphi_i$ the Hessian matrix of φ_i,

$$\nabla\Delta\big(f(\varphi_1) - f(\varphi_2)\big)$$
$$= f^{(3)}(\varphi_1)|\nabla\varphi_1|^2\nabla\varphi_1 - f^{(3)}(\varphi_2)|\nabla\varphi_2|^2\nabla\varphi_2 + 2f''(\varphi_1)D^2\varphi_1\nabla\varphi_1 - 2f''(\varphi_2)D^2\varphi_2\nabla\varphi_2$$
$$+ f''(\varphi_1)\Delta\varphi_1\nabla\varphi_1 - f''(\varphi_2)\Delta\varphi_2\nabla\varphi_2 + f'(\varphi_1)\nabla\Delta\varphi_1 - f'(\varphi_2)\nabla\Delta\varphi_2\,.$$

Hence, we need to estimate these four differences. However, we consider just the first two, since the others can be dealt with in a similar way.

At this point, we use the fact that there exists some constant $\hat{c} > 0$ (actually, a little calculus shows that the minimal such constant is $\hat{c} = 3/2$) such that

$$\big||x|^2 x - |y|^2 y\big| \leq \hat{c}\,|x-y|(|x|^2 + |y|^2), \quad \text{for all } x, y \in \mathbb{R}^3\,.$$

We therefore can estimate as follows:

$$\begin{aligned}
&\|f^{(3)}(\varphi_1)|\nabla\varphi_1|^2\nabla\varphi_1 - f^{(3)}(\varphi_2)|\nabla\varphi_2|^2\,\nabla\varphi_2\|^2 \\
&= \|(f^{(3)}(\varphi_1) - f^{(3)}(\varphi_2))|\nabla\varphi_1|^2\nabla\varphi_1 + f^{(3)}(\varphi_2)(|\nabla\varphi_1|^2\nabla\varphi_1 - |\nabla\varphi_2|^2\nabla\varphi_2)\|^2 \\
&\leq c\,\||\varphi|\,|\nabla\varphi_1|^3\|^2 + c\,\||\nabla\varphi|\,(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2)\|^2 \\
&\leq c\,\|\varphi\|_6^2\,\|\nabla\varphi_1\|_\infty^2\,\|\nabla\varphi_1\|_6^4 + c\,\|\nabla\varphi\|_6^2\big(\|\nabla\varphi_1\|_6^4 + \|\nabla\varphi_2\|_6^4\big) \\
&\leq c\,\|\varphi\|_V^2\,\|\varphi_1\|_{H^3(\Omega)}^2\,\|\varphi_1\|_W^4 + c\,\|\varphi\|_W^2\big(\|\varphi_1\|_W^4 + \|\varphi_2\|_W^4\big) \\
&\leq c\,\|\varphi\|_V^2\,\|\varphi_1\|_{H^3(\Omega)}^2 + c\,\|\varphi\|_W^2\,.
\end{aligned}$$

As for the second difference, we have that

$$\begin{aligned}
&\|2f''(\varphi_1)D^2\varphi_1\nabla\varphi_1 - 2f''(\varphi_2)D^2\varphi_2\nabla\varphi_2\|^2 \\
&\leq c\,\|(f''(\varphi_1) - f''(\varphi_2))D^2\varphi_1\nabla\varphi_1\|^2 + c\,\|f''(\varphi_2)(D^2\varphi_1 - D^2\varphi_2)\nabla\varphi_1\|^2 \\
&\quad + c\,\|f''(\varphi_2)D^2\varphi_2(\nabla\varphi_1 - \nabla\varphi_2)\|^2 \\
&\leq c\,\|\varphi\|_6^2\,\|D^2\varphi_1\|_6^2\,\|\nabla\varphi_1\|_6^2 + c\,\|D^2\varphi\|_4^2\,\|\nabla\varphi_1\|_4^2 + c\,\|D^2\varphi_2\|_4^2\,\|\nabla\varphi\|_4^2 \\
&\leq c\,\|\varphi\|_V^2\,\|\varphi_1\|_{H^3(\Omega)}^2\,\|\varphi_1\|_W^2 + c\,\|\varphi\|_{H^3(\Omega)}^2\,\|\varphi_1\|_W^2 + c\,\|\varphi_2\|_{H^3(\Omega)}^2\,\|\varphi\|_W^2 \\
&\leq c\,\|\varphi\|_V^2\,\|\varphi_1\|_{H^3(\Omega)}^2 + c\,\|\varphi\|_{H^3(\Omega)}^2 + c\,\|\varphi_2\|_{H^3(\Omega)}^2\,\|\varphi\|_W^2\,.
\end{aligned}$$

Let us come to the next two terms which originate from (4.6). Concerning the first, we have

$$\begin{aligned}
&-\int_\Omega \big(f'(\varphi_1) - f'(\varphi_2)\big)(-\Delta\varphi_1)(-\Delta^3\varphi) \\
&= \int_\Omega \nabla\big[(-\Delta\varphi_1)\big(f'(\varphi_1) - f'(\varphi_2)\big)\big]\cdot\nabla\Delta^2\varphi \\
&= \int_\Omega (-\Delta\varphi_1)\big(f''(\varphi_1) - f''(\varphi_2)\big)\nabla\varphi_1\cdot\nabla\Delta^2\varphi + \int_\Omega (-\Delta\varphi_1)f''(\varphi_2)\nabla\varphi\cdot\nabla\Delta^2\varphi \\
&\quad + \int_\Omega \big(f'(\varphi_1) - f'(\varphi_2)\big)\nabla(-\Delta\varphi_1)\cdot\nabla\Delta^2\varphi \\
&\leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\,\|\Delta\varphi_1\|_6^2\,\|\varphi\|_6^2\,\|\nabla\varphi_1\|_6^2 + c_\delta\,\|\Delta\varphi_1\|_4^2\,\|\nabla\varphi\|_4 + c_\delta\,\|\varphi\|_4^2\,\|\nabla\Delta\varphi_1\|_4^2 \\
&\leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 \\
&\quad + c_\delta\,\|\varphi_1\|_{H^3(\Omega)}^2\,\|\varphi\|_V^2 + c_\delta\,\|\varphi_1\|_{H^3(\Omega)}^2\big(\|\varphi\|^2 + \|\Delta\varphi\|^2\big) + c_\delta\,\|\varphi\|_V^2\,\|\varphi_1\|_{H^4(\Omega)}^2\,,
\end{aligned}$$

where we also have repeatedly used that the mapping $t \mapsto \|\varphi_1(t)\|_W$ belongs to $L^\infty(0, T)$. Next,

we deduce that

$$-\int_\Omega f'(\varphi_2)(-\Delta\varphi)(-\Delta^3\varphi) = \int_\Omega \nabla\big(f'(\varphi_2)(-\Delta\varphi)\big)\cdot\nabla\Delta^2\varphi$$

$$\leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\,\big\|\nabla\big(f'(\varphi_2)\Delta\varphi\big)\big\|^2$$

$$= \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\,\big\|\Delta\varphi f''(\varphi_2)\nabla\varphi_2 + f'(\varphi_2)\nabla\Delta\varphi\big\|^2$$

$$\leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\,\|\Delta\varphi\|^2\|\nabla\varphi_2\|_\infty^2 + c_\delta\,\|\nabla\Delta\varphi\|^2$$

$$\leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\,\|\Delta\varphi\|^2\,\|\varphi_2\|_{H^3(\Omega)}^2 + c_\delta\,\|\varphi\|_{H^3(\Omega)}^2\,.$$

As for the next term, we replace ff' by a generic smooth function g. The inequality we obtain, when applied to $g = -\nu f$, yields an estimate for the last term to be considered. We have

$$-\int_\Omega \big(g(\varphi_1) - g(\varphi_2)\big)(-\Delta^3\varphi) = \int_\Omega \nabla\big(g(\varphi_1) - g(\varphi_2)\big)\cdot\nabla\Delta^2\varphi$$

$$= \int_\Omega \big((g'(\varphi_1) - g'(\varphi_2))\nabla\varphi_1 + g'(\varphi_2)\nabla\varphi\big)\cdot\nabla\Delta^2\varphi$$

$$\leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\,\big(\big\||\varphi|\,|\nabla\varphi_1|\big\|^2 + \|\nabla\varphi\|^2\big)$$

$$\leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\,\big(\|\varphi\|_4^2\,\|\nabla\varphi_1\|_4^2 + \|\nabla\varphi\|^2\big)$$

$$\leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\,\big(\|\varphi_1\|_W^2 + 1\big)\|\varphi\|_V^2$$

$$\leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\|\varphi\|_V^2\,.$$

Finally, it is clear that

$$-\nu\int_\Omega (-\Delta\varphi)(-\Delta^3\varphi) = \nu\int_\Omega \nabla\Delta\varphi\cdot\nabla\Delta^2\varphi \leq \delta\int_\Omega |\nabla\Delta^2\varphi|^2 + c_\delta\,\|\varphi\|_{H^3(\Omega)}^2\,.$$

At this point, we recall the left-hand side (4.7) of the identity obtained as said at the beginning, and collect all the above inequalities to estimate the corresponding right-hand side. Then, we choose δ small enough, rearrange, and integrate over $(0, t)$ for an arbitrary $t \in (0, T]$. Taking also (2.34) into account, we finally obtain that

$$\|\Delta\varphi(t)\|^2 + \int_{Q_t} |\nabla\Delta^2\varphi|^2 \leq c\,\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{H})}^2 + \int_0^t \psi(s)\,\|\Delta\varphi(s)\|^2\,ds\,,$$

where ψ is naturally defined by the above estimates. We observe that the most delicate terms arising in ψ involve the expressions $\|\varphi_1\|_{H^4(\Omega)}^2$ and $\|\varphi_2\|_{H^3(\Omega)}^2$. Therefore, in view of (2.33), the mapping $t \mapsto \psi(t)$ is bounded in $L^1(0, T)$. Integrating in time and applying Gronwall's lemma, we can therefore deduce that

$$\|\Delta\varphi\|_{L^\infty(0,T;H)} + \|\nabla\Delta^2\varphi\|_{L^2(0,T;H)} \leq c\,\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{H})}\,,$$

whence (4.4) follows from (2.34), (2.49), (2.50) and elliptic regularity. $\qquad\square$

Remark 4.2 Having proved Lemma 4.4, and adopting the same notation as in Theorem 2.2, we observe that, by first taking the difference of equations (2.31) and then of (2.30), and arguing as in the above proof in order to control the derivatives of the nonlinear terms, we are led to the further estimate

$$\|w\|_{L^\infty(0,T;H)\cap L^2(0,T;H^3(\Omega))} + \|\mu\|_{L^2(0,T;V)} \le K_4 \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{H})},$$

which holds for the differences $w = w_1 - w_2$, $\mu = \mu_1 - \mu_2$, and $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$. The constant K_4 has the same dependencies as K_3 in (4.4).

From now on, we generally assume:

> *All of the above assumptions on the structure on the original system,*
> *the initial datum, the cost functional, and the set of admissible controls,*
> *are satisfied.*   (4.8)

For the reader's convenience, we recall that these assumptions are given in (2.7)–(2.12), (2.14), and (2.35)–(2.41).

We now turn our interest to the study of the Fréchet differentiability of the solution operator $\mathcal{S}$. As a preliminary step, we observe that such a derivative is characterized through the associated *linearized system*, which we therefore introduce and analyze next. For its solution, we can use the same notation as that adopted for the solution to problem (2.25)–(2.29), since no confusion can arise. Indeed, the solution $\mathcal{S}(\boldsymbol{u}^*)$ to the original problem corresponding to a fixed $\boldsymbol{u}^* \in \mathcal{U}_R$ is denoted by $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*)$. Given an element $\boldsymbol{h} \in L^2(0,T;\boldsymbol{H})$, the associated linearized problem consists in looking for a quadruplet $(\boldsymbol{v}, \varphi, \mu, w)$ with the regularity

$$\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}_0), \tag{4.9}$$

$$\varphi \in H^1(0,T;W^*) \cap C^0([0,T];V) \cap L^2(0,T;H^4(\Omega) \cap W), \tag{4.10}$$

$$\mu \in L^2(0,T;H), \tag{4.11}$$

$$w \in L^2(0,T;W), \tag{4.12}$$

that solves the variational equations

$$\int_\Omega D\boldsymbol{v} : \nabla\boldsymbol{\zeta} + \int_\Omega \lambda(\varphi^*)\boldsymbol{v} \cdot \boldsymbol{\zeta} + \int_\Omega \lambda'(\varphi^*)\varphi \boldsymbol{v}^* \cdot \boldsymbol{\zeta}$$

$$= \int_\Omega \left(\mu \nabla\varphi^* + \mu^* \nabla\varphi + \boldsymbol{h}\right) \cdot \boldsymbol{\zeta} \qquad \text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}_0 \text{ and a.e. in } (0,T), \tag{4.13}$$

$$\langle \partial_t \varphi, z \rangle + \int_\Omega \boldsymbol{v} \cdot \nabla\varphi^* \, z + \int_\Omega \boldsymbol{v}^* \cdot \nabla\varphi \, z - \int_\Omega \mu \, \Delta z$$

$$= \int_\Omega S'(\varphi^*)\varphi z \qquad \text{for every } z \in W \text{ and a.e. in } (0,T), \tag{4.14}$$

$$\int_\Omega \nabla w \cdot \nabla z + \int_\Omega f''(\varphi^*)\varphi w^* z + \int_\Omega (f'(\varphi^*) + \nu)wz$$

$$= \int_\Omega \mu z \qquad \text{for every } z \in V \text{ and a.e. in } (0,T), \tag{4.15}$$

$$\int_\Omega \nabla\varphi \cdot \nabla z + \int_\Omega f'(\varphi^*)\varphi z = \int_\Omega wz \qquad \text{for every } z \in V \text{ and a.e. in } (0,T) \tag{4.16}$$

and satisfies the initial condition

$$\varphi(0) = 0 \,. \tag{4.17}$$

Also in this case, one can eliminate the fourth component w of the solution $(\boldsymbol{v}, \varphi, \mu, w)$ whenever this is convenient. Indeed, (4.16) and the regularities (4.10) and (4.12) yield

$$-\Delta\varphi + f'(\varphi^*)\varphi = w \quad \text{a.e. in } Q \,. \tag{4.18}$$

Hence, if $(\boldsymbol{v}, \varphi, \mu, w)$ is a solution to (4.13)–(4.16), then the triplet $(\boldsymbol{v}, \varphi, \mu)$ solves (4.13), (4.14) and the variational equation

$$\int_\Omega \nabla\big(-\Delta\varphi + f'(\varphi^*)\varphi\big) \cdot \nabla z + \int_\Omega f''(\varphi^*)\varphi \big(-\Delta\varphi^* + f(\varphi^*)\big) z$$

$$+ \int_\Omega \big(f'(\varphi^*) + \nu\big)\big(-\Delta\varphi + f'(\varphi^*)\varphi\big) z$$

$$= \int_\Omega \mu z \quad \text{for every } z \in V \text{ and a.e. in } (0, T) \,. \tag{4.19}$$

Conversely, if the pair (φ, μ) solves (4.19) and (4.16) is taken as a definition of w, then (4.15) is satisfied as well. So, we can consider either of these equivalent problems. We also notice that the equations (4.15), (4.16) and (4.19) can be written in strong form as boundary value problems, thanks to the regularity of the solution required in (4.9)–(4.12). For instance, regarding (4.19), we have

$$-\Delta\big(-\Delta\varphi + f'(\varphi^*)\varphi\big) + f''(\varphi^*)\varphi \big(-\Delta\varphi^* + f(\varphi^*)\big)$$
$$+ \big(f'(\varphi^*) + \nu\big)\big(-\Delta\varphi + f'(\varphi^*)\varphi\big) = \mu \quad \text{a.e. in } Q \,, \tag{4.20}$$

with the boundary conditions

$$\partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\Delta\varphi = 0 \quad \text{on } \Sigma \,. \tag{4.21}$$

Recalling the definition of $\mathcal{X}$ given in (4.2), we have the following result.

Theorem 4.3 *Let $\boldsymbol{u}^* \in \mathcal{U}_R$ be given and $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*) := \mathcal{S}(\boldsymbol{u}^*)$ be the corresponding state. Then, for every $\boldsymbol{h} \in L^2(0, T; \boldsymbol{H})$, the linearized problem (4.13)–(4.17) has a unique solution $(\boldsymbol{v}, \varphi, \mu, w)$ with the regularity (4.9)–(4.12). Moroever, the inequality*

$$\|(\boldsymbol{v}, \varphi, \mu, w)\|_{\mathcal{X}} \le K \|\boldsymbol{h}\|_{L^2(0,T;\boldsymbol{H})} \tag{4.22}$$

holds true with a constant K that does not depend on $\boldsymbol{h}$, that is,

$$\text{the linear mapping } \boldsymbol{h} \mapsto (\boldsymbol{v}, \varphi, \mu, w) \text{ belongs to } \mathcal{L}(L^2(0, T; \boldsymbol{H}), \mathcal{X}). \tag{4.23}$$

Proof. Also in this case, we proceed formally. However, we make some observations at the end of the proof. We recall that $\mathcal{S}(\boldsymbol{u}^*)$ satisfies the stability estimate (2.33), which, in particular, implies that $f(\varphi^*)$, $f'(\varphi^*)$ and $f''(\varphi^*)$ are bounded since f is smooth and φ^* is bounded. Moreover, λ, λ' and S' are bounded by assumption. We test (4.13) by $\boldsymbol{v}$ and use the identity (2.16). At the same time, we test (4.14) by $M(\varphi - \Delta\varphi)$ and (4.19) by $-M\Delta\varphi + M\Delta^2\varphi - N\mu$, where M and N are positive parameters whose values will be chosen later. By rearranging a little, we obtain the identities

$$\int_\Omega |D\boldsymbol{v}|^2 + \int_\Omega \lambda(\varphi^*)|\boldsymbol{v}|^2$$

$$= -\int_\Omega \lambda'(\varphi^*)\varphi\boldsymbol{v}^* \cdot \boldsymbol{v} + \int_\Omega \big(\mu\nabla\varphi^* + \mu^*\nabla\varphi + \boldsymbol{h}\big) \cdot \boldsymbol{v} \,, \tag{4.24}$$

$$\frac{M}{2}\frac{d}{dt}\|\varphi\|_V^2 - M\int_\Omega \mu\Delta(\varphi - \Delta\varphi)$$

$$= -M\int_\Omega \boldsymbol{v}\cdot\nabla\varphi^*(\varphi - \Delta\varphi) - M\int_\Omega \boldsymbol{v}^*\cdot\nabla\varphi\,(\varphi - \Delta\varphi)$$

$$+ M\int_\Omega S'(\varphi^*)\varphi(\varphi - \Delta\varphi)\,, \tag{4.25}$$

$$M\int_\Omega |\nabla\Delta\varphi|^2 + M\int_\Omega |\Delta^2\varphi|^2 + N\int_\Omega |\mu|^2 - M\int_\Omega \mu(-\Delta\varphi + \Delta^2\varphi)$$

$$= -N\int_\Omega \Delta^2\varphi\,\mu - \int_\Omega (-\Delta)\big(f'(\varphi^*)\varphi\big)\big(-M\Delta\varphi + M\Delta^2\varphi - N\mu\big)$$

$$- \int_\Omega f''(\varphi^*)\varphi\big(-\Delta\varphi^* + f(\varphi^*)\big)\big(-M\Delta\varphi + M\Delta^2\varphi - N\mu\big)$$

$$- \int_\Omega \big(f'(\varphi^*) + \nu\big)\big(-\Delta\varphi + f'(\varphi^*)\varphi\big)\big(-M\Delta\varphi + M\Delta^2\varphi - N\mu\big)\,. \tag{4.26}$$

Now, we sum up and notice that two terms involving μ cancel each other. Then, by applying the coerciveness inequality (2.18), we obtain an inequality whose left-hand side is given by the expression

$$\alpha\|\boldsymbol{v}\|_V^2 + \frac{M}{2}\frac{d}{dt}\|\varphi\|_V^2 + M\int_\Omega |\nabla\Delta\varphi|^2 + M\int_\Omega |\Delta^2\varphi|^2 + N\int_\Omega |\mu|^2\,. \tag{4.27}$$

We now estimate the terms of the resulting right-hand side individually. More precisely, we derive the estimates so that every term can be controlled by the left-hand side of (4.27), either directly or, after integrating in time, via Gronwall's lemma, since the coefficients are bounded in $L^1(0,T)$. We make a wide use of the Young and Hölder inequalities, as well as of some of the inequalities (2.47)–(2.54), even with the precise values of the constants that appear there. At first, we have

$$-\int_\Omega \lambda'(\varphi^*)\varphi\boldsymbol{v}^*\cdot\boldsymbol{v} \leq c\|\varphi\|\|\boldsymbol{v}^*\|_4\|\boldsymbol{v}\|_4 \leq \frac{\alpha}{8}\|\boldsymbol{v}\|_V^2 + c\|\boldsymbol{v}^*\|_V^2\|\varphi\|^2\,.$$

Next, we deduce that

$$\int_\Omega \mu\nabla\varphi^*\cdot\boldsymbol{v} \leq \|\mu\|\,\|\nabla\varphi^*\|_4\,\|\boldsymbol{v}\|_4 \leq \mathcal{C}_S\|\mu\|\,\|\nabla\varphi^*\|_4\,\|\boldsymbol{v}\|_V$$

$$\leq \frac{\alpha}{8}\|\boldsymbol{v}\|_V^2 + \frac{2\mathcal{C}_S^2}{\alpha}\|\nabla\varphi^*\|_4^2\|\mu\|^2 \leq \frac{\alpha}{8}\|\boldsymbol{v}\|_V^2 + \hat{C}\|\mu\|^2\,,$$

where we have set

$$\hat{C} := \frac{2\mathcal{C}_S^2}{\alpha}\|\nabla\varphi^*\|_{L^\infty(0,T;L^4(\Omega))}^2\,, \tag{4.28}$$

recalling that $\varphi^* \in L^\infty(0,T;W)$ and that $W \hookrightarrow W^{1,4}(\Omega)$. The next two terms can be estimated as follows:

$$\int_\Omega \mu^*\nabla\varphi\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{h}\cdot\boldsymbol{v} \leq \|\mu^*\|_4\,\|\nabla\varphi\|\,\|\boldsymbol{v}\|_4 + \|\boldsymbol{h}\|\,\|\boldsymbol{v}\|$$

$$\leq \frac{\alpha}{8}\|\boldsymbol{v}\|_V^2 + c\|\mu^*\|_V^2\,\|\varphi\|_V^2 + c\|\boldsymbol{h}\|^2\,.$$

Let us come to the terms that appear on the right-hand side of (4.25). We have

$$
-M \int_\Omega \boldsymbol{v} \cdot \nabla \varphi^* (\varphi - \Delta \varphi) \le M \, \|\boldsymbol{v}\|_4 \, \|\nabla \varphi^*\|_4 \, \|\varphi - \Delta \varphi\|
$$

$$
\le \frac{\alpha}{8} \, \|\boldsymbol{v}\|_V^2 + c \, M^2 \, \|\nabla \varphi^*\|_4^2 \, \|\varphi\|_W^2 \le \frac{\alpha}{8} \, \|\boldsymbol{v}\|_V^2 + c \, M^2 \, \|\varphi\|_W^2
$$

$$
\le \frac{\alpha}{8} \, \|\boldsymbol{v}\|_V^2 + \frac{M}{16} \, \|\Delta^2 \varphi\|^2 + c_M \, \|\varphi\|_V^2 \,,
$$

and, similarly,

$$
-M \int_\Omega \boldsymbol{v}^* \cdot \nabla \varphi \, (\varphi - \Delta \varphi) \le M \, \|\boldsymbol{v}^*\|_4 \, \|\nabla \varphi\| \, \|\varphi - \Delta \varphi\|_4
$$

$$
\le c \, M \, \|\varphi\|_{H^3(\Omega)}^2 + c \, M \, \|\boldsymbol{v}^*\|_V^2 \, \|\varphi\|_V^2 \le \frac{M}{16} \, \|\Delta^2 \varphi\|^2 + c_M \, \|\varphi\|_V^2 + c \, M \, \|\boldsymbol{v}^*\|_V^2 \, \|\varphi\|_V^2 \,.
$$

Finally, we easily see that

$$
M \int_\Omega S'(\varphi^*) \varphi (\varphi - \Delta \varphi) \le c \, M \, \|\varphi\| \, \|\varphi - \Delta \varphi\| \le \frac{M}{16} \, \|\Delta^2 \varphi\|^2 + c_M \, \|\varphi\|_V^2 \,.
$$

The first term on the right-hand side term of (4.26) is handled similarly, using Young's inequality to obtain

$$
-N \int_\Omega \Delta^2 \varphi \, \mu \le \frac{N}{8} \, \|\mu\|^2 + 2N \, \|\Delta^2 \varphi\|^2 \,.
$$

The last three terms, however, require a more careful analysis. In order to minimize the calculation, we introduce a generic function $z \in L^2(0, T; H)$ and a positive parameter κ that are specified later. We have

$$
-\int_\Omega (-\Delta)\big(f'(\varphi^*)\varphi\big) z = \int_\Omega \operatorname{div}\big(f''(\varphi^*)\nabla \varphi^* \varphi + f'(\varphi^*)\nabla \varphi\big) z
$$

$$
= \int_\Omega \big(f^{(3)}(\varphi^*)|\nabla \varphi^*|^2 \varphi + f''(\varphi^*)\Delta \varphi^* \, \varphi + 2f''(\varphi^*)\nabla \varphi^* \cdot \nabla \varphi + f'(\varphi^*)\Delta \varphi\big) z
$$

$$
\le c\big(\|\nabla \varphi^*\|_6^2 \, \|\varphi\|_6 + \|\Delta \varphi^*\|_4 \, \|\varphi\|_4 + \|\nabla \varphi^*\|_4 \, \|\nabla \varphi\|_4 + \|\Delta \varphi\|\big) \|z\|
$$

$$
\le \frac{\kappa}{32} \, \|z\|^2 + c_\kappa \big(\|\varphi^*\|_W^2 \, \|\varphi\|_V^2 + \|\varphi^*\|_{H^3(\Omega)}^2 \, \|\varphi\|_V^2 + \|\varphi\|_W^2\big)
$$

$$
\le \frac{\kappa}{32} \, \|z\|^2 + c_\kappa \big(\|\varphi\|_W^2 + \|\varphi^*\|_{H^3(\Omega)}^2 \, \|\varphi\|_V^2\big)
$$

$$
\le \frac{\kappa}{32} \, \|z\|^2 + \frac{M}{8} \, \|\Delta^2 \varphi\|^2 + c_{\kappa,M} \, \|\varphi\|_V^2 + c_\kappa \, \|\varphi^*\|_{H^3(\Omega)}^2 \, \|\varphi\|_V^2
$$

$$
\le \frac{\kappa}{32} \, \|z\|^2 + \frac{M}{8} \, \|\Delta^2 \varphi\|^2 + c_{\kappa,M}\big(\|\varphi^*\|_{H^3(\Omega)}^2 + 1\big)\|\varphi\|_V^2 \,.
$$

Next, we find that

$$
-\int_\Omega f''(\varphi^*)\varphi \big(-\Delta \varphi^* + f(\varphi^*)\big) z \le c \, \|\varphi\|_4 \big(\|\Delta \varphi^*\|_4 + 1\big) \|z\|
$$

$$
\le c \, \|\varphi\|_V \big(\|\varphi^*\|_{H^3(\Omega)} + 1\big) \|z\| \le \frac{\kappa}{32} \, \|z\|^2 + c_\kappa \big(\|\varphi^*\|_{H^3(\Omega)}^2 + 1\big) \|\varphi\|_V^2 \,.
$$

Finally, it turns out that

$$-\int_\Omega \big(f'(\varphi^*)+\nu\big)\big(-\Delta\varphi+f'(\varphi^*)\varphi\big)z \le c\,\|\varphi\|_W\,\|z\|$$

$$\le \frac{\kappa}{32}\,\|z\|^2 + c_\kappa\,\|\varphi\|_W^2 \le \frac{\kappa}{32}\,\|z\|^2 + \frac{M}{8}\,\|\Delta^2\varphi\|^2 + c_{\kappa,M}\,\|\varphi\|_V^2\,.$$

At this point, we make a suitable choice of z and κ in each of the three last estimates. Namely, we choose first $z=M(-\Delta\varphi+\Delta^2\varphi)$ and $\kappa=1/M$, and then $z=-N\mu$ and $\kappa=1/N$, and add the two resulting inequalities. Moreover, we observe that

$$\|-\Delta\varphi+\Delta^2\varphi\|^2 \le 2\,\|\Delta\varphi\|^2 + 2\,\|\Delta^2\varphi\|^2 \le 3\,\|\Delta^2\varphi\|^2 + c\,\|\varphi\|_V^2\,.$$

We then obtain the following three corresponding estimates:

$$-\int_\Omega (-\Delta)\big(f'(\varphi^*)\varphi\big)\big(-M\Delta\varphi+M\Delta^2\varphi-N\mu\big)$$

$$\le \frac{M}{32}\,\|-\Delta\varphi+\Delta^2\varphi\|^2 + \frac{N}{32}\,\|\mu\|^2 + \frac{M}{4}\,\|\Delta^2\varphi\|^2 + c_{M,N}\big(\|\varphi^*\|_{H^3(\Omega)}^2+1\big)\|\varphi\|_V^2$$

$$\le \frac{11M}{32}\,\|\Delta^2\varphi\|^2 + \frac{N}{32}\,\|\mu\|^2 + c_{M,N}\big(\|\varphi^*\|_{H^3(\Omega)}^2+1\big)\|\varphi\|_V^2\,,$$

$$-\int_\Omega f''(\varphi^*)\varphi\,\big(-\Delta\varphi^*+f(\varphi^*)\big)\big(-M\Delta\varphi+M\Delta^2\varphi-N\mu\big)$$

$$\le \frac{3M}{32}\,\|\Delta^2\varphi\|^2 + \frac{N}{32}\,\|\mu\|^2 + c_{M,N}\big(\|\varphi^*\|_{H^3(\Omega)}^2+1\big)\|\varphi\|_V^2\,,$$

$$-\int_\Omega \big(f'(\varphi^*)+\nu\big)\big(-\Delta\varphi+f'(\varphi^*)\varphi\big)\big(-M\Delta\varphi+M\Delta^2\varphi-N\mu\big)$$

$$\le \frac{7M}{32}\,\|\Delta^2\varphi\|^2 + \frac{N}{32}\,\|\mu\|^2 + c_{M,N}\,\|\varphi\|_V^2\,.$$

These are the last estimates that were needed regarding the right-hand side of the inequality obtained by adding (4.24)–(4.26) to each other. Then, we deduce an estimate for the whole right-hand side by just adding the above inequalities to each other. Since the related left-hand side is given by (4.27), we arrive at the estimate

$$\frac{\alpha}{2}\,\|\boldsymbol{v}\|_V^2 + \frac{M}{2}\frac{d}{dt}\,\|\varphi\|_V^2 + M\int_\Omega |\nabla\Delta\varphi|^2$$

$$+ \Big(\frac{8M}{32}-2N\Big)\int_\Omega |\Delta^2\varphi|^2 + \Big(\frac{25N}{32}-\hat{C}\Big)\int_\Omega |\mu|^2$$

$$\le c_{M,N}\big(\|\boldsymbol{v}^*\|_V^2 + \|\mu^*\|_V^2 + \|\varphi^*\|_{H^3(\Omega)}^2+1\big)\|\varphi\|_V^2 + c\,\|\boldsymbol{h}\|^2\,.$$

Now we recall that $\hat{C}$ is fixed and defined in (4.28). We therefore can first choose N, and subsequently M, in such way that all of the coefficients on the left-hand side become positive. We then integrate in time and apply the Gronwall lemma, observing again that the terms containing norms of $(\boldsymbol{v}^*,\varphi^*,\mu^*,w^*)$ on the right-hand side are time-integrable due to (2.33). This yields that

$$\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{H})} + \|\varphi\|_{L^\infty(0,T;V)} + \|\nabla\Delta\varphi\|_{L^2(0,T;H)} + \|\Delta^2\varphi\|_{L^2(0,T;H)}$$

$$+ \|\mu\|_{L^2(0,T;H)} \le c\,\|\boldsymbol{h}\|_{L^2(0,T;\boldsymbol{H})}\,,$$

and we immediately deduce that

$$\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{H})} + \|\varphi\|_{L^\infty(0,T;V)\cap L^2(0,T;H^4(\Omega)\cap W)} + \|\mu\|_{L^2(0,T;H)} \le c\,\|\boldsymbol{h}\|_{L^2(0,T;\boldsymbol{H})}\,.$$

This estimate is rigorous when performed on an arbitrary solution $(\boldsymbol{v}, \varphi, \mu, w)$, and it implies uniqueness. Indeed, by linearity, one can assume $\boldsymbol{h} = \boldsymbol{0}$ and deduce that the three first components of the solution vanish. Then, w vanishes as well, due to (4.16).

Concerning the existence of a solution and the estimate (4.22), the formal procedure leading to the above bound can be rigorously justified by performing the same computations on a suitable Faedo–Galerkin approximation (in close analogy with the proof of [10, Thm. 2.1]). Passage to the limit then yields the result for the actual solution. The same is true for the estimate

$$\|\partial_t\varphi\|_{L^2(0,T;W^*)} \le c\left(\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{V})} + \|\varphi\|_{L^\infty(0,T;V)} + \|\mu\|_{L^2(0,T;H)}\right),$$

which can be derived by (the discrete version of) (4.14). Then, (4.22) with $\mathcal{X}$ given by (4.2) follows by combining the above inequalities and taking (4.16) as a definition of w. □

At this point, we recall that $\mathcal{U}_R$ is endowed with the topology of $L^2(0,T;\boldsymbol{H})$. Here is the main result of this section.

Theorem 4.4 *The control-to-state operator $\mathcal{S} : \mathcal{U}_R \to \mathcal{X}$ is Fréchet differentiable in $\mathcal{U}_R$. More precisely, for every $\boldsymbol{u}^* \in \mathcal{U}_R$, the Frechet derivative $D\mathcal{S}(\boldsymbol{u}^*) \in \mathcal{L}(L^2(0,T;\boldsymbol{H}), \mathcal{X})$ acts as follows: if $\boldsymbol{h} \in L^2(0,T;\boldsymbol{H})$, then the value $D\mathcal{S}(\boldsymbol{u}^*)[\boldsymbol{h}]$ is the solution $(\boldsymbol{v}, \varphi, \mu, w)$ to the linearized problem (4.13)–(4.17), where $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*) := \mathcal{S}(\boldsymbol{u}^*)$.*

Proof. In order to keep the paper at a reasonable length, we do not provide a complete proof. Instead, we briefly outline the argument and highlight the most delicate steps. Let $\boldsymbol{u}^* \in \mathcal{U}_R$ be fixed. In the following, we consider increments $\boldsymbol{h} \in L^2(0,T;\boldsymbol{H})$ having a sufficiently small norm such that $\boldsymbol{u}^* + \boldsymbol{h} \in \mathcal{U}_R$, and we denote by c positive constants that do not depend on the special choice of such increments. Then the corresponding states $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*) := \mathcal{S}(\boldsymbol{u}^*)$ and $(\boldsymbol{v}^h, \varphi^h, \mu^h, w^h) := \mathcal{S}(\boldsymbol{u}^* + \boldsymbol{h})$, as well as any convex combination thereof, satisfy the stability estimate (2.33). Finally, we consider the solution $(\boldsymbol{v}, \varphi, \mu, w)$ to the linearized system (4.13)–(4.17) corresponding to $\boldsymbol{u}^*$ and $\boldsymbol{h}$ and set, for convenience,

$$(\boldsymbol{\xi}, \psi, \eta, \omega) := (\boldsymbol{v}^h, \varphi^h, \mu^h, w^h) - (\boldsymbol{v}^*, \varphi^*, \mu^*, w^*) - (\boldsymbol{v}, \varphi, \mu, w)\,. \tag{4.29}$$

At this point, we note that $(\boldsymbol{v}, \varphi, \mu, w)$, and therefore also $(\boldsymbol{\xi}, \psi, \eta, \omega)$, depends on $\boldsymbol{h}$. In order not to overload the exposition with indices, we have chosen not to indicate this dependence in our notation throughout this proof. Also for other quantities like Λ_i or R_i, which will be introduced below, we will suppress the dependence on $\boldsymbol{h}$ in the notation.

Returning to the proof, we recall (4.23), which implies that the assertion will be completely proved once we can show that

$$\|(\boldsymbol{\xi}, \psi, \eta, \omega)\|_{\mathcal{X}} \le c\,\|\boldsymbol{h}\|^2_{L^2(0,T;H)} \tag{4.30}$$

with a constant c that does not depend on $\boldsymbol{h}$. So, we sketch the proof of (4.30). To this end, we consider the problem solved by $(\boldsymbol{\xi}, \psi, \eta, \omega)$, which is obtained by subtracting both the systems solved by $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*)$ and $(\boldsymbol{v}, \varphi, \mu, w)$ from the one solved by $(\boldsymbol{v}^h, \varphi^h, \mu^h, w^h)$. We have

$$(\boldsymbol{\xi}, \psi, \eta, \omega) \in \mathcal{X}\,, \tag{4.31}$$

and, with the notation given below, the variational equalities

$$\int_\Omega D\boldsymbol{\xi} : \nabla\boldsymbol{\zeta} + \int_\Omega \lambda(\varphi^*)\boldsymbol{\xi}\cdot\boldsymbol{\zeta} = \int_\Omega \boldsymbol{\Lambda}_1\cdot\boldsymbol{\zeta} + \int_\Omega \boldsymbol{\Lambda}_2\cdot\boldsymbol{\zeta}$$

for every $\boldsymbol{\zeta}\in\boldsymbol{V}_0$ and a.e. in $(0,T)$, (4.32)

$$\langle\partial_t\psi, z\rangle - \int_\Omega \eta\,\Delta z = \int_\Omega \Lambda_3 z + \int_\Omega \Lambda_4 z$$

for every $z\in W$ and a.e. in $(0,T)$, (4.33)

$$\int_\Omega \nabla\omega\cdot\nabla z + \nu\int_\Omega \omega z = \int_\Omega \eta z + \int_\Omega \Lambda_5 z$$

for every $z\in V$ and a.e. in $(0,T)$, (4.34)

$$\int_\Omega \nabla\psi\cdot\nabla z = \int_\Omega \omega z + \int_\Omega \Lambda_6 z$$

for every $z\in V$ and a.e. in $(0,T)$, (4.35)

are satisfied, as well as the initial condition

$$\psi(0) = 0\,. \tag{4.36}$$

In the above equations, we have introduced the abbreviating notation

$$\boldsymbol{\Lambda}_1 := -\big(\lambda(\varphi^{\boldsymbol{h}}) - \lambda(\varphi^*)\big)(\boldsymbol{v}^{\boldsymbol{h}} - \boldsymbol{v}^*)$$
$$\qquad - \big(\lambda(\varphi^{\boldsymbol{h}}) - \lambda(\varphi^*) - \lambda'(\varphi^*)\varphi\big)\boldsymbol{v}^*\,, \tag{4.37}$$

$$\boldsymbol{\Lambda}_2 := \mu^*\nabla\psi + (\mu^{\boldsymbol{h}} - \mu^*)\nabla(\varphi^{\boldsymbol{h}} - \varphi^*) + \eta\nabla\varphi^*, \tag{4.38}$$

$$\Lambda_3 := -\boldsymbol{v}^*\cdot\nabla\psi - (\boldsymbol{v}^{\boldsymbol{h}} - \boldsymbol{v}^*)\cdot\nabla(\varphi^{\boldsymbol{h}} - \varphi^*) - \boldsymbol{\xi}\cdot\nabla\varphi^*, \tag{4.39}$$

$$\Lambda_4 := S(\varphi^{\boldsymbol{h}}) - S(\varphi^*) - S'(\varphi^*)\varphi\,, \tag{4.40}$$

$$\Lambda_5 := -\big(f'(\varphi^{\boldsymbol{h}}) - f'(\varphi^*) - f''(\varphi^*)\varphi\big)w^*$$
$$\qquad - \big(f'(\varphi^{\boldsymbol{h}}) - f'(\varphi^*)\big)(w^{\boldsymbol{h}} - w^*) - f'(\varphi^*)\omega\,, \tag{4.41}$$

$$\Lambda_6 := -\big(f(\varphi^{\boldsymbol{h}}) - f(\varphi^*) - f'(\varphi^*)\varphi\big)\,. \tag{4.42}$$

Clearly, due to the regularity of the solution, some of the above equations can be written in their strong form. This is the case for (4.35), so that ω can be explicitly written in terms of ψ and Λ_6. Since it turns out that $\Delta\Lambda_6$ belongs to $L^2(0,T;H)$ (see the computation below), we obtain after a substitution and an integration by parts in (4.34) that

$$\int_\Omega (-\Delta)(-\Delta\psi - \Lambda_6)z + \nu\int_\Omega (-\Delta\psi - \Lambda_6)z = \int_\Omega \eta z + \int_\Omega \Lambda_5\,z$$

for every $z\in H$ and a.e. in $(0,T)$. (4.43)

In performing the estimates that are needed to prove (4.30), it is convenient to rewrite certain differences in suitable forms. By applying standard Taylor expansions with integral remainder, we easily find that

$$\lambda(\varphi^{\boldsymbol{h}}) - \lambda(\varphi^*) = \lambda'(\varphi^*)(\varphi^{\boldsymbol{h}} - \varphi^*) + R_1\,,$$
$$S(\varphi^{\boldsymbol{h}}) - S(\varphi^*) = S'(\varphi^*)(\varphi^{\boldsymbol{h}} - \varphi^*) + R_2\,,$$
$$f'(\varphi^{\boldsymbol{h}}) - f'(\varphi^*) = f''(\varphi^*)(\varphi^{\boldsymbol{h}} - \varphi^*) + R_3\,,$$
$$f(\varphi^{\boldsymbol{h}}) - f(\varphi^*) = f'(\varphi^*)(\varphi^{\boldsymbol{h}} - \varphi^*) + R_4\,,$$

where the remainders satisfy, a.e. in Q,

$$|R_1| + |R_2| + |R_3| + |R_4| \leq c\,|\varphi^{\boldsymbol{h}} - \varphi^*|^2\,. \tag{4.44}$$

On the other hand, it results that

$$\boldsymbol{\Lambda}_1 = -\big(\lambda(\varphi^{\boldsymbol{h}}) - \lambda(\varphi^*)\big)(\boldsymbol{v}^{\boldsymbol{h}} - \boldsymbol{v}^*) - \lambda'(\varphi^*)\psi\boldsymbol{v}^* - R_1\,\boldsymbol{v}^*, \tag{4.45}$$

$$\Lambda_4 = S'(\varphi^*)\psi + R_2\,, \tag{4.46}$$

$$\begin{aligned}\Lambda_5 = &-\big(f''(\varphi^*)\psi + R_3\big)w^* - f'(\varphi^*)(-\Delta\psi - \Lambda_6)\\&-\big(f'(\varphi^{\boldsymbol{h}}) - f'(\varphi^*)\big)\big(-\Delta(\varphi^{\boldsymbol{h}} - \varphi^*) + f(\varphi^{\boldsymbol{h}}) - f(\varphi^*)\big),\end{aligned} \tag{4.47}$$

$$\Lambda_6 = -f'(\varphi^*)\psi - R_4\,. \tag{4.48}$$

At this point, we start estimating. As in the proof of Theorem 4.3, it is convenient to introduce two positive parameters M and N. We test (4.32) by $\boldsymbol{\xi}$ and (4.33) by $M(\psi - \Delta\psi)$. At the same time, we test (4.43) by $-N\eta$, $-M\Delta\psi$, and $M\Delta^2\psi$. By also recalling the identity (2.16), we obtain

$$\int_\Omega |D\boldsymbol{\xi}|^2 + \int_\Omega \lambda(\varphi^*)|\boldsymbol{\xi}|^2 = \int_\Omega \boldsymbol{\Lambda}_1 \cdot \boldsymbol{\xi} + \int_\Omega \boldsymbol{\Lambda}_2 \cdot \boldsymbol{\xi}\,, \tag{4.49}$$

$$\begin{aligned}&\frac{M}{2}\frac{d}{dt}\|\psi\|_V^2 - M\int_\Omega \eta\,\Delta(\psi - \Delta\psi)\\&\qquad = M\int_\Omega \Lambda_3(\psi - \Delta\psi) + M\int_\Omega \Lambda_4(\psi - \Delta\psi),\end{aligned} \tag{4.50}$$

$$\begin{aligned}&-N\int_\Omega (-\Delta)(-\Delta\psi - \Lambda_6)\eta - N\nu\int_\Omega (-\Delta\psi - \Lambda_6)\eta\\&\qquad = -N\int_\Omega |\eta|^2 - N\int_\Omega \Lambda_5\,\eta\,,\end{aligned} \tag{4.51}$$

$$\begin{aligned}&-M\int_\Omega (-\Delta)(-\Delta\psi - \Lambda_6)\Delta\psi - M\nu\int_\Omega (-\Delta\psi - \Lambda_6)\Delta\psi\\&\qquad = -M\int_\Omega \eta\Delta\psi - M\int_\Omega \Lambda_5\,\Delta\psi\,,\end{aligned} \tag{4.52}$$

$$\begin{aligned}&M\int_\Omega (-\Delta)(-\Delta\psi - \Lambda_6)\Delta^2\psi + M\nu\int_\Omega (-\Delta\psi - \Lambda_6)\Delta^2\psi\\&\qquad = M\int_\Omega \eta\Delta^2\psi + M\int_\Omega \Lambda_5\,\Delta^2\psi\,.\end{aligned} \tag{4.53}$$

Next, we add these four identities, noticing that four of the terms involving both η and ψ cancel out. Then, rearranging terms and invoking the coerciveness inequality (2.18), we obtain an inequality whose left-hand side is given by the expression

$$\alpha\,\|\boldsymbol{\xi}\|_V^2 \;+\; \frac{M}{2}\frac{d}{dt}\|\psi\|_V^2 + N\int_\Omega |\eta|^2 + M\int_\Omega |\nabla\Delta\psi|^2 + M\int_\Omega |\Delta^2\psi|^2\,. \tag{4.54}$$

We now have to estimate the terms on the resulting right-hand side by using the definitions (4.38)–(4.39) of $\boldsymbol{\Lambda}_2$ and Λ_3 and the new expressions (4.45)–(4.48) for the other Λ_i, as well as the estimate (4.44) of the remainders. However, the techniques that are needed are quite similar to those employed in the proof of Theorem 4.3, i.e., a wide use of the Hölder, Sobolev and Young inequalities and of the

compactness inequality (2.54). For this reason, we provide the details only for the most delicate terms, namely, those involving the Laplacian of Λ_6 (see (4.51)–(4.53)), which, by (4.48), is the sum of two contributions. We have

$$\Delta f'(\varphi^*) = f^{(3)}(\varphi^*)|\nabla\varphi^*|^2 + f'(\varphi^*)\Delta\varphi^*\,,$$

whence

$$\Delta(f'(\varphi^*)\psi) = f^{(3)}(\varphi^*)|\nabla\varphi^*|^2\psi + f'(\varphi^*)\Delta\varphi^*\,\psi + 2f''(\varphi^*)\nabla\varphi^*\cdot\nabla\psi + f'(\varphi^*)\Delta\psi\,.$$

To handle the remaining contribution, we use the explicit representation of R_4, which is given by

$$R_4 = \Phi\,|\varphi^{\boldsymbol{h}}-\varphi^*|^2\,,\quad\text{with the notation}\quad\Phi:=\int_0^1(1-s)f''(\varphi^*+s(\varphi^{\boldsymbol{h}}-\varphi^*))\,ds\,.$$

It results that

$$\nabla\Phi = \int_0^1(1-s)f^{(3)}(\varphi^*+s(\varphi^{\boldsymbol{h}}-\varphi^*))\,[\nabla\varphi^*+s\nabla(\varphi^{\boldsymbol{h}}-\varphi^*)]\,ds\,,\quad\text{whence}$$

$$\Delta\Phi = \int_0^1(1-s)f^{(4)}(\varphi^*+s(\varphi^{\boldsymbol{h}}-\varphi^*))\,|\nabla\varphi^*+s\nabla(\varphi^{\boldsymbol{h}}-\varphi^*)|^2\,ds$$
$$+\int_0^1(1-s)f^{(3)}(\varphi^*+s(\varphi^{\boldsymbol{h}}-\varphi^*))[\Delta\varphi^*+s\Delta(\varphi^{\boldsymbol{h}}-\varphi^*)]\,ds\,.$$

Therefore, we derive that

$$\Delta R_4 = \Delta\Phi\,|\varphi^{\boldsymbol{h}}-\varphi^*|^2 + 2\,\nabla\Phi\cdot\nabla(|\varphi^{\boldsymbol{h}}-\varphi^*|^2) + \Phi\,\Delta(|\varphi^{\boldsymbol{h}}-\varphi^*|^2)$$
$$= \int_0^1(1-s)f^{(4)}(\varphi^*+s(\varphi^{\boldsymbol{h}}-\varphi^*))\,|\nabla\varphi^*+s\nabla(\varphi^{\boldsymbol{h}}-\varphi^*)|^2\,ds\,|\varphi^{\boldsymbol{h}}-\varphi^*|^2$$
$$+\int_0^1(1-s)f^{(3)}(\varphi^*+s(\varphi^{\boldsymbol{h}}-\varphi^*))\,[\Delta\varphi^*+s\Delta(\varphi^{\boldsymbol{h}}-\varphi^*)]\,ds\,|\varphi^{\boldsymbol{h}}-\varphi^*|^2$$
$$+2\int_0^1(1-s)f^{(3)}(\varphi^*+s(\varphi^{\boldsymbol{h}}-\varphi^*))[\nabla\varphi^*+s\nabla(\varphi^{\boldsymbol{h}}-\varphi^*)]\,ds\cdot[2(\varphi^{\boldsymbol{h}}-\varphi^*)\nabla(\varphi^{\boldsymbol{h}}-\varphi^*)]$$
$$+\int_0^1(1-s)f''(\varphi^*+s(\varphi^{\boldsymbol{h}}-\varphi^*))\,ds\,[\,2\,|\nabla(\varphi^{\boldsymbol{h}}-\varphi^*)|^2 + 2\,(\varphi^{\boldsymbol{h}}-\varphi^*)\Delta(\varphi^{\boldsymbol{h}}-\varphi^*)\,]\,.$$

We are now in a position to estimate the terms in (4.51)–(4.53) involving $\Delta\Lambda_6$. Using the previous rearrangement, their sum can be written as

$$\int_\Omega(\Delta\Lambda_6)\big(N\eta + M\Delta\psi - M\Delta^2\psi\big)$$
$$\leq\Big(\frac{N}{2}\,\|\eta\|^2 + \frac{M}{2}\,\|\Delta^2\psi\|^2 + c_M\,\|\psi\|_V^2\Big) + c_{M,N}\,\|\Delta\Lambda_6\|^2\,,$$

and it remains to estimate the last norm. We have

$$\|\Delta\Lambda_6\| \leq \|\Delta(f'(\varphi^*)\psi)\| + \|\Delta R_4\|\,.$$

At first, we see that

$$\|\Delta(f'(\varphi^*)\psi)\| \leq c\,\|\nabla\varphi^*\|_6^2\,\|\psi\|_6 + c\,\|\Delta\varphi^*\|_4\,\|\psi\|_4 + c\,\|\nabla\varphi^*\|_4\,\|\nabla\psi\|_4 + c\,\|\Delta\psi\|\,,$$

and the last two terms can be dealt with by invoking the compactness inequality (2.54). On the other hand, we have a.e. in Q that

$$
\begin{aligned}
|\Delta R_4| \le &\, c\,(|\nabla\varphi^{\boldsymbol{h}}|^2 + |\nabla\varphi^*|^2)|\varphi^{\boldsymbol{h}} - \varphi^*|^2 + c\,(|\Delta\varphi^{\boldsymbol{h}}| + |\Delta\varphi^*|)|\varphi^{\boldsymbol{h}} - \varphi^*|^2 \\
&+ c\,(|\nabla\varphi^{\boldsymbol{h}}| + |\nabla\varphi^*|)|\varphi^{\boldsymbol{h}} - \varphi^*|\,|\nabla(\varphi^{\boldsymbol{h}} - \varphi^*)| \\
&+ c\,|\nabla(\varphi^{\boldsymbol{h}} - \varphi^*)|^2 + c\,|\varphi^{\boldsymbol{h}} - \varphi^*|\,|\Delta(\varphi^{\boldsymbol{h}} - \varphi^*)|\,,
\end{aligned}
$$

whence

$$
\begin{aligned}
\|\Delta R_4\| \le &\, c\,(\|\nabla\varphi^{\boldsymbol{h}}\|_\infty \|\nabla\varphi^{\boldsymbol{h}}\|_6 + \|\nabla\varphi^*\|_\infty \|\nabla\varphi^*\|_6)\|\varphi^{\boldsymbol{h}} - \varphi^*\|_6^2 \\
&+ c(\|\Delta\varphi^{\boldsymbol{h}}\|_6 + \|\Delta\varphi^*\|_6)\|\|\varphi^{\boldsymbol{h}} - \varphi^*\|\|_6^2 \\
&+ c\,(\|\nabla\varphi^{\boldsymbol{h}}\|_6 + \|\nabla\varphi^*\|_6)\|\varphi^{\boldsymbol{h}} - \varphi^*\|_6 \|\nabla(\varphi^{\boldsymbol{h}} - \varphi^*)\|_\infty \\
&+ c\,\|\nabla(\varphi^{\boldsymbol{h}} - \varphi^*)\|_\infty^2 + c\,\|\varphi^{\boldsymbol{h}} - \varphi^*\|_4 \|\Delta(\varphi^{\boldsymbol{h}} - \varphi^*)\|_4\,. \quad (4.55)
\end{aligned}
$$

This ends the preliminary estimate of the most delicate terms, and we apply it in a while. As said before, the other terms are much easier to estimate. Here, we briefly comment on another point. As in the proof of Theorem 4.3, we still have to choose the values of the parameters M and N. As an example, we consider the last term appearing in (4.49). We have

$$
\begin{aligned}
\int_\Omega \boldsymbol{\Lambda}_2 \cdot \boldsymbol{\xi} \le &\, \int_\Omega |\mu^*|\,|\nabla\psi|\,|\boldsymbol{\xi}| + \int_\Omega |\mu^{\boldsymbol{h}} - \mu^*|\,|\nabla(\varphi^{\boldsymbol{h}} - \varphi^*)|\,|\boldsymbol{\xi}| + \int_\Omega |\eta|\,|\nabla\varphi^*|\,|\boldsymbol{\xi}| \\
\le &\, \|\mu^*\|_4 \|\nabla\psi\| \|\boldsymbol{\xi}\|_4 + \|\mu^{\boldsymbol{h}} - \mu^*\| \|\nabla(\varphi^{\boldsymbol{h}} - \varphi^*)\|_4 \|\boldsymbol{\xi}\|_4 + \|\eta\| \|\nabla\varphi^*\|_4 \|\boldsymbol{\xi}\|_4 \\
\le &\, \frac{\alpha}{16} \|\boldsymbol{\xi}\|_V^2 + c\,\|\mu^*\|_V^2 \|\psi\|_V^2 + c\,\|\mu^{\boldsymbol{h}} - \mu^*\|^2 \|\varphi^{\boldsymbol{h}} - \varphi^*\|_W^2 + C^* \|\varphi^*\|_W^2 \|\eta\|^2\,,
\end{aligned}
$$

where the special symbol C^* is used in place of the generic c. By a comparison with the left-hand side (4.54), it is now clear that we need N to be larger than $C^* \|\varphi^*\|_{L^\infty(0,T;W)}^2$ (and even much larger, since similar situations appear also in other terms), where we note that the above norm of φ^* is controlled by the stability inequality (2.33).

At the end of the estimates, one recalls all the inequalities that regard the right-hand side corresponding to the left-hand side (4.54) and rearranges. Then, one integrates over $(0,t)$ with an arbitrary $t \in (0,T]$. The integral of the left-hand side (4.54) is essentially given by

$$
\int_0^t \|\boldsymbol{\xi}(s)\|_V^2\,ds + \|\psi(t)\|_V^2 + \int_{Q_t} \left(|\eta|^2 + |\nabla\Delta\psi|^2 + |\Delta^2\psi|^2\right). \quad (4.56)
$$

Among the integrals that should appear on the corresponding right-hand side, we consider only the one involving ΔR_4. We have, from (4.55),

$$
\begin{aligned}
\int_{Q_t} |\Delta R_4|^2 \le &\, c \int_0^t \left(\|\nabla\varphi^{\boldsymbol{h}}(s)\|_\infty^2 \|\nabla\varphi^{\boldsymbol{h}}(s)\|_6^2 + \|\nabla\varphi^*(s)\|_\infty^2 \|\nabla\varphi^*(s)\|_6^2\right)\|(\varphi^{\boldsymbol{h}} - \varphi^*)(s)\|_6^4\,ds \\
&+ c \int_0^t (\|\Delta\varphi^{\boldsymbol{h}}(s)\|_6^2 + \|\Delta\varphi^*(s)\|_6^2)\|(\varphi^{\boldsymbol{h}} - \varphi^*)(s)\|_6^4\,ds \\
&+ c \int_0^t (\|\nabla\varphi^{\boldsymbol{h}}(s)\|_6^2 + \|\nabla\varphi^*(s)\|_6^2)\|(\varphi^{\boldsymbol{h}} - \varphi^*)(s)\|_6^2 \|\nabla(\varphi^{\boldsymbol{h}} - \varphi^*)(s)\|_\infty^2\,ds \\
&+ c \int_0^t \|\nabla(\varphi^{\boldsymbol{h}} - \varphi^*)(s)\|_\infty^4\,ds + c \int_0^t \|(\varphi^{\boldsymbol{h}} - \varphi^*)(s)\|_4^2 \|\Delta(\varphi^{\boldsymbol{h}} - \varphi^*)(s)\|_4^2\,ds
\end{aligned}
$$

$$
\leq c \left(\|\varphi^{\boldsymbol{h}}\|^2_{L^2(0,T;H^3(\Omega))} \|\varphi^{\boldsymbol{h}}\|^2_{L^\infty(0,T;W)} + \|\varphi^*\|^2_{L^2(0,T;H^3(\Omega))} \|\varphi^*\|^2_{L^\infty(0,T;W)} \right) \|\varphi^{\boldsymbol{h}} - \varphi^*\|^4_{L^\infty(0,T;V)}
$$

$$
+ c \left(\|\varphi^{\boldsymbol{h}}\|^2_{L^2(0,T;H^3(\Omega))} + \|\varphi^*\|^2_{L^2(0,T;H^3(\Omega))} \right) \|\varphi^{\boldsymbol{h}} - \varphi^*\|^4_{L^\infty(0,T;V)}
$$

$$
+ c \left(\|\varphi^{\boldsymbol{h}}\|^2_{L^\infty(0,T;W)} + \|\varphi^*\|^2_{L^\infty(0,T;W)} \right) \|\varphi^{\boldsymbol{h}} - \varphi^*\|^2_{L^\infty(0,T;V)} \|\varphi^{\boldsymbol{h}} - \varphi^*\|^2_{L^2(0,T;H^3(\Omega))}
$$

$$
+ c \|\nabla(\varphi^{\boldsymbol{h}} - \varphi^*)\|^4_{L^4(0,T;L^\infty(\Omega))} + c \|\varphi^{\boldsymbol{h}} - \varphi^*\|^2_{L^\infty(0,T;V)} \|\varphi^{\boldsymbol{h}} - \varphi^*\|^2_{L^2(0,T;H^3(\Omega))} \,,
$$

and the continuous embedding $L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)) \hookrightarrow L^4(0,T;L^\infty(\Omega))$ can be used to treat the first term on the last line. Therefore, all the above norms of the difference $\varphi^{\boldsymbol{h}} - \varphi^*$ can be estimated in terms of the norm of $\boldsymbol{h}$ in $L^2(0,T;H)$, thanks to (4.4). On the other hand, the occurring norms of $\varphi^{\boldsymbol{h}}$ and φ^* are controlled by the stability estimate (2.33). Hence, we conclude that

$$
\int_{Q_t} |\Delta R_4|^2 \leq c \,\|\boldsymbol{h}\|^4_{L^2(0,T;H)} \,.
$$

This ends the treatment of the most complicated term that enters the right-hand side corresponding to the left-hand side (4.56). The others are simpler to handle, where some of them lead to similar inequalities while the other ones can be dealt with using Gronwall's lemma, since their coefficients are bounded in $L^1(0,T)$ thanks to (2.33). We eventually can conclude that

$$
\|\boldsymbol{\xi}\|_{L^2(0,T;V)} + \|\psi\|_{L^\infty(0,T;V)} + \|\eta\|_{L^2(0,T;H)}
$$
$$
+ \|\nabla\Delta\psi\|_{L^2(0,T;H)} + \|\Delta^2\psi\|_{L^2(0,T;H)} \leq c \,\|\boldsymbol{h}\|^2_{L^2(0,T;H)} \,.
$$

By applying elliptic regularity inequalities and a comparison in (4.33) and (4.34) (to recover estimates of $\partial_t\psi$ and ω), and recalling the definition (4.2) of $\mathcal{X}$, one obtains (4.30). The assertion is proved. $\square$

5 Necessary conditions for optimality

In this section, we give a necessary condition for an admissible control $\boldsymbol{u}^*$ to be optimal. The first result is a simple application of Convex Analysis, using the convexity of $\mathcal{U}_{ad}$ and the assumptions (2.35)–(2.39) on the cost functional $\mathcal{J}$. Since both the control-to-state operator $\mathcal{S}$ and the first part J of the cost functional $\mathcal{J}$ are Fréchet differentiable, the same is true for the composite map $\tilde{J}$ defined in (2.43), and we can compute its Fréchet derivative by the chain rule. Therefore, we have the following result, which gives a precise form to the variational inequality (2.44).

Corollary 5.1 *Let $\boldsymbol{u}^* \in \mathcal{U}_{ad}$ be an optimal control for the control problem (2.42) with associated state $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*) := \mathcal{S}(\boldsymbol{u}^*)$. Then, there exists some $\boldsymbol{\Lambda}^*$ in the subdifferential $\partial G(\boldsymbol{u}^*)$ such that*

$$
b_1 \int_Q (\varphi^* - \varphi_Q)\varphi + b_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\varphi(T) + b_3 \int_Q \boldsymbol{u}^* \cdot (\boldsymbol{u} - \boldsymbol{u}^*)
$$
$$
+ \int_Q \boldsymbol{\Lambda}^* \cdot (\boldsymbol{u} - \boldsymbol{u}^*) \geq 0 \qquad \text{for every } \boldsymbol{u} \in \mathcal{U}_{ad} \,, \tag{5.1}
$$

where φ is the second component of the solution $(\boldsymbol{v}, \varphi, \mu, w)$ to the linearized problem (4.13)–(4.17) corresponding to $\boldsymbol{h} := \boldsymbol{u} - \boldsymbol{u}^*$.

This result is not yet satisfactory. Indeed, it involves the solutions to infinitely many linearized problems, since $\boldsymbol{u}$ is arbitrary in $\mathcal{U}_{ad}$. To overcome this difficulty, we follow the standard procedure and introduce the proper adjoint problem associated with $\boldsymbol{u}^*$ and $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*)$. It consists in finding a quadruplet $(\boldsymbol{\omega}, p, q, r)$ with the regularity

$$\boldsymbol{\omega} \in L^\infty(0,T;\boldsymbol{V}_0) \cap L^2(0,T;\boldsymbol{H}^2(\Omega)), \tag{5.2}$$
$$p \in H^1(0,T;(H^3(\Omega)\cap W)^*) \cap C^0([0,T];H) \cap L^2(0,T;H^3(\Omega)\cap W), \tag{5.3}$$
$$q \in L^2(0,T;V), \tag{5.4}$$
$$r \in L^2(0,T;V^*), \tag{5.5}$$

that solves the variational equations

$$\int_\Omega D\boldsymbol{\omega} : \nabla\boldsymbol{\zeta} + \int_\Omega \lambda(\varphi^*)\boldsymbol{\omega}\cdot\boldsymbol{\zeta} + \int_\Omega p\,\nabla\varphi^*\cdot\boldsymbol{\zeta} = 0$$
$$\text{for every } \boldsymbol{\zeta}\in\boldsymbol{V}_0 \text{ and a.e. in } (0,T), \tag{5.6}$$

$$-\langle\partial_t p, z\rangle + \int_\Omega \lambda'(\varphi^*)\boldsymbol{v}^*\cdot\boldsymbol{\omega}\,z - \int_\Omega S'(\varphi^*)pz + \int_\Omega f''(\varphi^*)w^*qz$$
$$+ \langle r, -\Delta z + f'(\varphi^*)z\rangle + \int_\Omega p\boldsymbol{v}^*\cdot\nabla z - \int_\Omega \mu^*\boldsymbol{\omega}\cdot\nabla z$$
$$= \int_\Omega g_Q\,z \quad \text{for every } z\in H^3(\Omega)\cap W \text{ and a.e. in } (0,T), \tag{5.7}$$

where the duality pairing in (5.7) is understood between $(H^3(\Omega)\cap W)^*$ and $H^3(\Omega)\cap W$,

$$\int_\Omega (-\nabla\varphi^*)\cdot\boldsymbol{\omega}\,z + \int_\Omega (-\Delta p)\,z = \int_\Omega qz$$
$$\text{for every } z\in H \text{ and a.e. in } (0,T), \tag{5.8}$$

$$\int_\Omega \nabla q\cdot\nabla z + \int_\Omega (f'(\varphi^*)+\nu)qz = \langle r,z\rangle$$
$$\text{for every } z\in V \text{ and a.e. in } (0,T), \tag{5.9}$$

and satisfies the final condition
$$p(T) = g_\Omega, \tag{5.10}$$

where we have set, for convenience,

$$g_Q := b_1(\varphi^* - \varphi_Q) \quad \text{and} \quad g_\Omega := b_2\big(\varphi^*(T) - \varphi_\Omega\big). \tag{5.11}$$

It is worth noting that, in view of (2.33) and (2.39), it holds that

$$g_Q \in L^2(0,T;H) \quad \text{and} \quad g_\Omega \in H. \tag{5.12}$$

We have written all the equations in their variational forms. However, due to the regularity assumed on the functions that occur, it is clear that (5.8) can be written as a partial differential equation on Q. Namely, we have that
$$-\nabla\varphi^*\cdot\boldsymbol{\omega} - \Delta p = q \quad \text{a.e. in } Q,$$

so that (5.7) may be equivalently replaced by

$$- \langle \partial_t p, z \rangle + \int_\Omega \lambda'(\varphi^*) \boldsymbol{v}^* \cdot \boldsymbol{\omega}\, z - \int_\Omega S'(\varphi^*) pz + \int_\Omega f''(\varphi^*) w^* (-\nabla\varphi^* \cdot \boldsymbol{\omega} - \Delta p) z$$

$$+ \langle r, -\Delta z + f'(\varphi^*) z \rangle + \int_\Omega p \boldsymbol{v}^* \cdot \nabla z - \int_\Omega \mu^* \boldsymbol{\omega} \cdot \nabla z = \int_\Omega g_Q\, z$$

$$\text{for every } z \in H^3(\Omega) \cap W \text{ and a.e. in } (0, T), \tag{5.13}$$

and (5.9) becomes

$$\int_\Omega \nabla(-\nabla\varphi^* \cdot \boldsymbol{\omega} - \Delta p) \cdot \nabla\zeta + \int_\Omega (f'(\varphi^*) + \nu)(-\nabla\varphi^* \cdot \boldsymbol{\omega} - \Delta p)\zeta = \langle r, \zeta \rangle$$

$$\text{for every } \zeta \in V \text{ and a.e. in } (0, T). \tag{5.14}$$

Taking now an arbitrary $z \in H^3(\Omega) \cap W$ and letting $\zeta = -\Delta z + f'(\varphi^*) z$ in (5.14), the resulting identity can be used to replace the fifth term on the left-hand side of (5.13). In this way, we obtain

$$- \langle \partial_t p, z \rangle + \int_\Omega \lambda'(\varphi^*) \boldsymbol{v}^* \cdot \boldsymbol{\omega}\, z - \int_\Omega S'(\varphi^*) pz + \int_\Omega f''(\varphi^*) w^* (-\nabla\varphi^* \cdot \boldsymbol{\omega} - \Delta p) z$$

$$+ \int_\Omega \nabla(-\nabla\varphi^* \cdot \boldsymbol{\omega}) \cdot \nabla(-\Delta z + f'(\varphi^*) z) + \int_\Omega \nabla(-\Delta p) \cdot \nabla(-\Delta z + f'(\varphi^*) z)$$

$$- \int_\Omega (f'(\varphi^*) + \nu)(\nabla\varphi^* \cdot \boldsymbol{\omega})(-\Delta z + f'(\varphi^*) z) - \int_\Omega (f'(\varphi^*) + \nu)\Delta p(-\Delta z + f'(\varphi^*) z)$$

$$+ \int_\Omega p \boldsymbol{v}^* \cdot \nabla z - \int_\Omega \mu^* \boldsymbol{\omega} \cdot \nabla z = \int_\Omega g_Q\, z \quad \text{for every } z \in V \text{ and a.e. in } (0, T). \tag{5.15}$$

The first result concerns the well-posedness of the adjoint problem.

Theorem 5.2 *Let $\boldsymbol{u}^* \in \mathcal{U}_R$ be given, and let $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*) := \mathcal{S}(\boldsymbol{u}^*)$ be the corresponding state. Then, with the notation* (5.11)*, the adjoint problem* (5.6)–(5.10) *has a unique solution satisfying* (5.2)–(5.5).

Proof. For the sake of brevity, we are forced to perform just formal estimates also in this case. However, we will make some observations at the end of the proof. We test (5.6) by $\boldsymbol{\omega}$. From the coerciveness inequality (2.18) we then infer that

$$\alpha \|\boldsymbol{\omega}\|_V^2 \leq - \int_\Omega p \nabla\varphi^* \cdot \boldsymbol{\omega}\,. \tag{5.16}$$

At the same time, we test (5.15) by p to obtain that

$$- \frac{1}{2} \frac{d}{dt} \|p\|^2 + \int_\Omega |\nabla\Delta p|^2$$

$$= - \int_\Omega \lambda'(\varphi^*) \boldsymbol{v}^* \cdot \boldsymbol{\omega}\, p + \int_\Omega S'(\varphi^*)|p|^2 - \int_\Omega f''(\varphi^*) w^* (-\nabla\varphi^* \cdot \boldsymbol{\omega} - \Delta p) p$$

$$- \int_\Omega \nabla(-\nabla\varphi^* \cdot \boldsymbol{\omega}) \cdot \nabla(-\Delta p + f'(\varphi^*) p) - \int_\Omega \nabla(-\Delta p) \cdot \nabla(f'(\varphi^*) p)$$

$$+ \int_\Omega (f'(\varphi^*) + \nu)(\nabla\varphi^* \cdot \boldsymbol{\omega})(-\Delta p + f'(\varphi^*) p)$$

$$+ \int_\Omega (f'(\varphi^*) + \nu) \Delta p (-\Delta p + f'(\varphi^*)p)$$

$$- \int_\Omega p \boldsymbol{v}^* \cdot \nabla p + \int_\Omega \mu^* \boldsymbol{\omega} \cdot \nabla p + \int_\Omega g_Q \, p \,. \tag{5.17}$$

Each term on the right-hand side of (5.17) will be estimated separately. In the whole proof, we account for the stability estimate (2.33) satisfied by the solution $\mathcal{S}(\boldsymbol{u}^*)$. In particular, φ^* is estimated in $L^\infty(0, T; W)$. However, prior to this, we make some preliminary observations. First, we deduce from (5.16) that

$$\alpha \, \|\boldsymbol{\omega}\|_V^2 \le \|p\| \, \|\nabla \varphi^*\|_4 \, \|\boldsymbol{\omega}\|_4 \le c \, \|p\| \, \|\boldsymbol{\omega}\|_V \,,$$

whence we immediately conclude that

$$\|\boldsymbol{\omega}\|_V \le c \, \|p\| \,. \tag{5.18}$$

Next, for a given $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$, we consider the problem of finding $\boldsymbol{v} \in \boldsymbol{V}_0$ satisfying the variational equation

$$\int_\Omega (D\boldsymbol{v} : \nabla \boldsymbol{\zeta} + \boldsymbol{v} \cdot \boldsymbol{\zeta}) = \int_\Omega \boldsymbol{g} \cdot \boldsymbol{\zeta} \quad \text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}_0 \,.$$

This problem has a unique solution $\boldsymbol{v}$ by the Lax–Milgram theorem. Moreover, it turns out that $\boldsymbol{v}$ belongs to $\boldsymbol{H}^2(\Omega)$ and satisfies the estimate $\|\boldsymbol{v}\|_{\boldsymbol{H}^2(\Omega)} \le C \, \|\boldsymbol{g}\|$ with a constant C that depends only on Ω. This result can be derived, e.g., from [17, Lemma 2.49], and we apply it to the equation (5.6) written in the form

$$\int_\Omega (D\boldsymbol{\omega} : \nabla \boldsymbol{\zeta} + \boldsymbol{\omega} \cdot \boldsymbol{\zeta}) = \int_\Omega \bigl(-\lambda(\varphi^*)\boldsymbol{\omega} - p \, \nabla \varphi^* + \boldsymbol{\omega}\bigr) \cdot \boldsymbol{\zeta} \quad \text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}_0 \,,$$

for a fixed time. Owing to (5.18), we easily see that

$$\|-\lambda(\varphi^*)\boldsymbol{\omega} - p \, \nabla \varphi^* + \boldsymbol{\omega}\| \le c \, \|\boldsymbol{\omega}\| + \|p\|_4 \, \|\nabla \varphi^*\|_4 \le c \, \|p\|_V \,,$$

so that

$$\|\boldsymbol{\omega}\|_{\boldsymbol{H}^2(\Omega)} \le c \, \|p\|_V \quad \text{a.e. in } (0, T) \,. \tag{5.19}$$

We can now proceed in the estimates of the terms on the right-hand side of (5.17). In the sequel, δ is a positive parameter. We owe to the above estimates of $\boldsymbol{\omega}$ and widely account for some of the inequalities (2.47)–(2.54). To begin with, we have that

$$- \int_\Omega \lambda'(\varphi^*)\boldsymbol{v}^* \cdot \boldsymbol{\omega} \, p \le c \, \|\boldsymbol{v}^*\|_4 \, \|\boldsymbol{\omega}\|_4 \|p\| \le c \, \|\boldsymbol{v}^*\|_V \, \|p\|^2$$

as well as

$$\int_\Omega S'(\varphi^*) |p|^2 \le c \, \|p\|^2.$$

We continue with the next term, finding that

$$- \int_\Omega f''(\varphi^*) w^* (-\nabla \varphi^* \cdot \boldsymbol{\omega} - \Delta p) p$$

$$\le c \, \|w^*\|_\infty \, \|\nabla \varphi^*\|_4 \, \|\boldsymbol{\omega}\|_4 \|p\| + c \, \|w^*\|_\infty \, \|\Delta p\| \, \|p\|$$

$$\le c \|w^*\|_{H^2(\Omega)} \|p\|^2 + \|p\|_{H^2(\Omega)}^2 + c \, \|w^*\|_{H^2(\Omega)}^2 \, \|p\|^2$$

$$\le \delta \, \|\nabla \Delta p\|^2 + c_\delta \bigl(1 + \|w^*\|_{H^2(\Omega)}^2\bigr) \|p\|^2 \,.$$

The treatment of the next term needs the little computation

$$\nabla(\nabla\varphi^* \cdot \boldsymbol{\omega}) = (D^2\varphi^*)\boldsymbol{\omega} + (\nabla\boldsymbol{\omega})\nabla\varphi^*,$$

where D^2 denotes the Hessian operator. Hence, we have that

$$-\int_\Omega \nabla(-\nabla\varphi^* \cdot \boldsymbol{\omega}) \cdot \nabla(-\Delta p + f'(\varphi^*)p)$$
$$= \int_\Omega (D^2\varphi^*)\boldsymbol{\omega} \cdot \nabla(-\Delta p) + \int_\Omega (\nabla\boldsymbol{\omega})\nabla\varphi^* \cdot \nabla(-\Delta p)$$
$$+ \int_\Omega ((D^2\varphi^*)\boldsymbol{\omega} + (\nabla\boldsymbol{\omega})\nabla\varphi^*) \cdot (f''(\varphi^*)\nabla\varphi^*p + f'(\varphi^*)\nabla p),$$

and we estimate these integrals individually. At first, invoking (5.18) and (5.19), we infer that

$$\int_\Omega (D^2\varphi^*)\boldsymbol{\omega} \cdot \nabla(-\Delta p) + \int_\Omega (\nabla\boldsymbol{\omega})\nabla\varphi^* \cdot \nabla(-\Delta p)$$
$$\le c\,\|\varphi^*\|_{W^{2,4}(\Omega)}\,\|\boldsymbol{\omega}\|_4\,\|\nabla\Delta p\| + c\,\|\nabla\boldsymbol{\omega}\|_4\,\|\nabla\varphi^*\|_4\|\nabla\Delta p\|$$
$$\le \delta\,\|\nabla\Delta p\|^2 + c_\delta\,\|\varphi^*\|_{H^3(\Omega)}^2\,\|p\|^2 + c_\delta\,\|\boldsymbol{\omega}\|_{\boldsymbol{H}^2(\Omega)}^2$$
$$\le \delta\,\|\nabla\Delta p\|^2 + c_\delta\,\|\varphi^*\|_{H^3(\Omega)}^2\,\|p\|^2 + c_\delta\,\|p\|_V^2$$
$$\le 2\delta\,\|\nabla\Delta p\|^2 + c_\delta\left(1 + \|\varphi^*\|_{H^3(\Omega)}^2\right)\|p\|^2.$$

Similarly, we deduce that

$$\int_\Omega ((D^2\varphi^*)\boldsymbol{\omega} + (\nabla\boldsymbol{\omega})\nabla\varphi^*) \cdot (f''(\varphi^*)\nabla\varphi^*p + f'(\varphi^*)\nabla p)$$
$$\le c\left(\|\varphi^*\|_{W^{2,4}(\Omega)}\,\|\boldsymbol{\omega}\|_4 + \|\nabla\boldsymbol{\omega}\|_4\,\|\nabla\varphi^*\|_4\right)\left(\|\nabla\varphi^*\|_4\|p\|_4 + \|\nabla p\|\right)$$
$$\le c\left(\|\varphi^*\|_{H^3(\Omega)}\,\|p\| + \|p\|_V\right)\|p\|_V \le c\,\|\varphi^*\|_{H^3(\Omega)}^2\,\|p\|^2 + c\,\|p\|_V^2$$
$$\le \delta\,\|\nabla\Delta p\|^2 + c_\delta\left(1 + \|\varphi^*\|_{H^3(\Omega)}^2\right)\|p\|^2.$$

Now, about the next term in (5.17), we point out that

$$-\int_\Omega \nabla(-\Delta p) \cdot \nabla(f'(\varphi^*)p) = \int_\Omega \nabla(\Delta p) \cdot (f''(\varphi^*)\nabla\varphi^*p + f'(\varphi^*)\nabla p)$$
$$\le c\,\|\nabla\Delta p\|\left(\|\nabla\varphi^*\|_4\|p\|_4 + \|\nabla p\|\right)$$
$$\le \delta\,\|\nabla\Delta p\|^2 + c_\delta\|p\|_V^2 \le 2\delta\,\|\nabla\Delta p\|^2 + c_\delta\|p\|^2.$$

We continue with the two terms containing the factor $f'(\varphi^*)+\nu$. Using the same arguments as before, we find

$$\int_\Omega (f'(\varphi^*) + \nu)(\nabla\varphi^* \cdot \boldsymbol{\omega})(-\Delta p + f'(\varphi^*)p) + \int_\Omega (f'(\varphi^*) + \nu)\Delta p(-\Delta p + f'(\varphi^*)p)$$
$$\le c\|\nabla\varphi^*\|_4\,\|\boldsymbol{\omega}\|_4\left(\|\Delta p\| + \|p\|\right) + c\,\|\Delta p\|\left(\|\Delta p\| + \|p\|\right)$$
$$\le c\,\|p\|_{H^2(\Omega)}^2 \le \delta\,\|\nabla\Delta p\|^2 + c_\delta\|p\|^2.$$

Next, thanks to (5.18), we obtain that

$$
\begin{aligned}
- \int_\Omega p \boldsymbol{v}^* \cdot \nabla p &+ \int_\Omega \mu^* \boldsymbol{\omega} \cdot \nabla p \\
&\le c(\|p\| \, \|\boldsymbol{v}^*\|_4 + \|\mu^*\| \, \|\boldsymbol{\omega}\|_4) \|\nabla p\|_4 \\
&\le c(\|p\| \, \|\boldsymbol{v}^*\|_V + \|\mu^*\| \, \|p\|) \|p\|_{H^2(\Omega)} \\
&\le \|p\|_{H^2(\Omega)}^2 + c\big(\|\boldsymbol{v}^*\|_V^2 + \|\mu^*\|^2\big) \|p\|^2 \\
&\le \delta \, \|\nabla \Delta p\|^2 + c_\delta \big(1 + \|\boldsymbol{v}^*\|_V^2 + \|\mu^*\|^2\big) \|p\|^2 \, .
\end{aligned}
$$

Finally, we have

$$
\int_\Omega g_Q \, p \le \frac{1}{2} \|g_Q\|^2 + \frac{1}{2} \|p\|^2 \, .
$$

This ends the preliminary estimates of the terms on the right-hand side of (5.17). Thus, we rearrange, choose $\delta > 0$ small enough, and integrate with respect to time over (t, T), with an arbitrary $t \in [0, T)$. Hence, we obtain

$$
\begin{aligned}
\frac{1}{2} \|p(t)\|^2 + \frac{1}{2} \int_{Q_t} |\nabla \Delta p|^2 &\le \frac{1}{2} \|g_\Omega\|^2 + \frac{1}{2} \int_{Q_t} |g_Q|^2 \\
&+ c \int_t^T \Big(1 + \|\boldsymbol{v}^*(s)\|_V^2 + \|w^*(s)\|_{H^2(\Omega)}^2 + \|\varphi^*(s)\|_{H^3(\Omega)}^2 + \|\mu^*(s)\|^2 \Big) \|p(s)\|^2 ds \, ,
\end{aligned}
$$

where now $Q_t := \Omega \times (t, T)$. Since the function

$$
s \mapsto 1 + \|\boldsymbol{v}^*(s)\|_V^2 + \|w^*(s)\|_{H^2(\Omega)}^2 + \|\varphi^*(s)\|_{H^3(\Omega)}^2 + \|\mu^*(s)\|^2
$$

is bounded in $L^1(0, T)$ by virtue of (2.33), the (backward) Gronwall lemma allows us to infer that

$$
\|p\|_{L^\infty(0,T;H)} + \|\nabla \Delta p\|_{L^2(0,T;H)} \le c\big(\|g_Q\|_{L^2(0,T;H)} + \|g_\Omega\|\big) \, .
$$

This estimate is improved using the estimate (2.50) from elliptic regularity theory and, consequently, the inequalities (5.18) and (5.19), in order to conclude that

$$
\begin{aligned}
\|p\|_{L^\infty(0,T;H) \cap L^2(0,T;H^3(\Omega) \cap W)} &+ \|\boldsymbol{\omega}\|_{L^\infty(0,T;\boldsymbol{V}_0) \cap L^2(0,T;\boldsymbol{H}^2(\Omega))} \\
&\le c\big(\|g_Q\|_{L^2(0,T;H)} + \|g_\Omega\|\big) \, .
\end{aligned} \tag{5.20}
$$

Then, additional estimates for $\partial_t p$, q and r are easily obtained via comparison in (5.7), (5.8) and (5.9), respectively. We thus deduce that

$$
\begin{aligned}
\|p\|_{H^1(0,T;(H^3(\Omega) \cap W)^*)} &+ \|q\|_{L^2(0,T;V)} + \|r\|_{L^2(0,T;V^*)} \\
&\le c\big(\|g_Q\|_{L^2(0,T;H)} + \|g_\Omega\|\big) \, .
\end{aligned} \tag{5.21}
$$

We conclude with some observation for the reader's convenience. First, the procedure that led us to the estimates (5.20) and (5.21) is rigorous if performed on a solution satisfying (5.2)–(5.5). In the particular case when $g_Q = 0$ and $g_\Omega = 0$, this estimate implies that the solution vanishes. This proves the uniqueness of the solution by linearity. As for existence, the formal estimates we have derived can be performed on an approximating problem based on a Faedo–Galerkin scheme by assuming that g_Q and g_Ω are given by (5.11). Then, by standard compactness arguments, one can construct a solution possessing the asserted regularity. □

Our next result yields the following first-order necessary optimality condition.

Theorem 5.3 *Assume that $\boldsymbol{u}^* \in \mathcal{U}_{ad}$ is an optimal control for the control problem (2.42), and let $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*) := \mathcal{S}(\boldsymbol{u}^*)$ be the corresponding state. Then there exists some $\boldsymbol{\Lambda}^*$ in the subdifferential $\partial G(\boldsymbol{u}^*)$ such that*

$$\int_Q \big(b_3 \boldsymbol{u}^* + \boldsymbol{\omega}\big) \cdot (\boldsymbol{u} - \boldsymbol{u}^*) + \int_Q \boldsymbol{\Lambda}^* \cdot (\boldsymbol{u} - \boldsymbol{u}^*) \geq 0 \qquad \text{for every } \boldsymbol{u} \in \mathcal{U}_{ad}, \qquad (5.22)$$

where $\boldsymbol{\omega}$ is the first component of the solution $(\boldsymbol{\omega}, p, q, r)$ to the adjoint problem (5.6)–(5.10).

Proof. We fix $\boldsymbol{u} \in \mathcal{U}_{ad}$ and consider the linearized problem (4.13)–(4.17) corresponding to $\boldsymbol{h} := \boldsymbol{u} - \boldsymbol{u}^*$. Let us emphasize that (4.16) is equivalent to (4.18), whose terms are all in V, a.e. in $(0,T)$. We test (4.13) by $\boldsymbol{\omega}$, (4.14) by p, (4.15) by q, and take the duality pairing between r and (4.18), respectively. Then, we integrate over $(0,T)$ with respect to time and take the sum of the resulting equalities. It then follows the identity

$$\int_Q D\boldsymbol{v} : \nabla\boldsymbol{\omega} + \int_Q \lambda(\varphi^*)\boldsymbol{v} \cdot \boldsymbol{\omega} + \int_Q \lambda'(\varphi^*)\varphi\boldsymbol{v}^* \cdot \boldsymbol{\omega} - \int_Q \big(\mu\nabla\varphi^* + \mu^*\nabla\varphi + \boldsymbol{u} - \boldsymbol{u}^*\big) \cdot \boldsymbol{\omega}$$

$$+ \int_0^T \langle \partial_t\varphi(t), p(t) \rangle \, dt + \int_Q \boldsymbol{v} \cdot \nabla\varphi^* \, p + \int_Q \boldsymbol{v}^* \cdot \nabla\varphi \, p + \int_Q \mu(-\Delta p) - \int_Q S'(\varphi^*)\varphi p$$

$$+ \int_Q \nabla w \cdot \nabla q + \int_Q f''(\varphi^*)\varphi w^* q + \int_Q f'(\varphi^*) w q + \nu \int_Q w q - \int_Q \mu q$$

$$+ \int_0^T \langle r(t), (-\Delta\varphi + f'(\varphi^*)\varphi - w)(t) \rangle \, dt = 0. \qquad (5.23)$$

At the same time, we consider the adjoint problem (5.6)–(5.10) and test its equations by $\boldsymbol{v}$, φ, μ and w, integrate over $(0,T)$ and sum up. We obtain

$$\int_Q D\boldsymbol{\omega} : \nabla\boldsymbol{v} + \int_Q \lambda(\varphi^*)\boldsymbol{\omega} \cdot \boldsymbol{v} + \int_Q p \nabla\varphi^* \cdot \boldsymbol{v}$$

$$- \int_0^T \langle \partial_t p(t), \varphi(t) \rangle \, dt + \int_Q \lambda'(\varphi^*)\boldsymbol{v}^* \cdot \boldsymbol{\omega} \, \varphi - \int_Q S'(\varphi^*) p\varphi + \int_Q f''(\varphi^*) w^* q\varphi$$

$$+ \int_0^T \langle r(t), (-\Delta\varphi + f'(\varphi^*)\varphi)(t) \rangle \, dt + \int_\Omega p\boldsymbol{v}^* \cdot \nabla\varphi - \int_Q \mu^*\boldsymbol{\omega} \cdot \nabla\varphi - \int_Q g_Q \varphi$$

$$+ \int_Q (-\nabla\varphi^*) \cdot \boldsymbol{\omega} \, \mu + \int_Q (-\Delta p) \, \mu - \int_Q q\mu$$

$$+ \int_Q \nabla q \cdot \nabla w + \int_Q f'(\varphi^*) q w + \nu \int_Q q w - \int_0^T \langle r(t), w(t) \rangle \, dt = 0. \qquad (5.24)$$

At this point, we take the difference between (5.23) and (5.24). Several terms cancel each other, and it remains the following identity:

$$- \int_Q (\boldsymbol{u} - \boldsymbol{u}^*) \cdot \boldsymbol{\omega} + \int_0^T \langle \partial_t\varphi(t), p(t) \rangle \, dt + \int_0^T \langle \partial_t p(t), \varphi(t) \rangle \, dt + \int_Q g_Q\varphi = 0.$$

Since both φ and p belong to $H^1(0,T;(H^3(\Omega)\cap W)^*) \cap L^2(0,T;H^3(\Omega)\cap W)$, the sum of the two integrals in the middle of the above line is the time integral of the time derivative of $\int_\Omega \varphi \, p$. Thus, on

account of the initial condition (4.17), the final condition (5.10) and the definitions (5.11), we deduce that

$$-\int_Q (\boldsymbol{u} - \boldsymbol{u}^*) \cdot \boldsymbol{\omega} + \int_\Omega b_2\big(\varphi^*(T) - \varphi_\Omega\big)\,\varphi(T) + \int_Q b_1(\varphi^* - \varphi_Q)\,\varphi = 0\,.$$

By combining this with the inequality (5.1), we obtain (5.22). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.4 The variational inequality (5.22) is equivalent to a system of three variational inequalities that have to be satisfied simultaneously. Indeed, if $\boldsymbol{\Lambda}^* = (\Lambda_1^*, \Lambda_2^*, \Lambda_3^*)$ is given, then it is easily seen that (5.22) holds true if and only if the components of $\boldsymbol{u}^* = (u_1^*, u_2^*, u_3^*)$ satisfy for $i = 1, 2, 3$ the variational inequalities

$$\int_Q (b_3 u_i^* + \omega_i + \Lambda_i^*)(u - u_i^*) \geq 0 \quad \text{for every } u \in U_{ad}^i\,, \qquad\qquad (5.25)$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and

$$U_{ad}^i := \{u \in L^\infty(Q) : \underline{u}_i \leq u \leq \overline{u}_i \text{ a.e. in } Q\}, \quad i = 1, 2, 3. \qquad\qquad (5.26)$$

A standard argument then shows that the following pointwise projection conditions are valid:

$$u_i^* = \max\big\{\underline{u}_i,\, \min\big\{\overline{u}_i, -b_3^{-1}(\omega_i + \Lambda_i^*)\big\}\big\} \quad \text{a.e. in } Q, \text{ for } i = 1, 2, 3. \qquad (5.27)$$

Remark 5.5 It is worth noting that the argumentation used to show the validity of Corollary 5.1, Theorem 5.3, and Remark 5.4 does not only work for (globally) optimal controls, but also for controls that are only locally optimal in the sense of $\boldsymbol{L}^\infty(Q)$. The results shown above therefore hold true correspondingly in the locally optimal case. We recall in this connection that a control $\boldsymbol{u}^* \in \mathcal{U}_{ad}$ is termed locally optimal in the sense of $\boldsymbol{L}^p(Q)$ for some $p \in [1, +\infty]$ if there is some $\gamma > 0$ such that

$$\mathcal{J}(\varphi^*, \boldsymbol{u}^*) \leq \mathcal{J}(\varphi, \boldsymbol{u}) \quad \text{for all } \boldsymbol{u} \in \mathcal{U}_{ad} \text{ with } \|\boldsymbol{u} - \boldsymbol{u}^*\|_{\boldsymbol{L}^p(Q)} \leq \gamma\,, \qquad (5.28)$$

where φ and φ^* denote the second components of $\mathcal{S}(\boldsymbol{u})$ and $\mathcal{S}(\boldsymbol{u}^*)$, respectively. Observe that every control, which is locally optimal in the sense of $\boldsymbol{L}^p(Q)$ for some $p \in [1, +\infty)$, is also locally optimal in the sense of $\boldsymbol{L}^\infty(Q)$.

6 Remarks on sparsity

In this section, we discuss the aspect of sparsity in the optimal control problem under investigation. We again assume that the general assumption (4.8) is fulfilled and that $\boldsymbol{u}^* \in \mathcal{U}_{ad}$ is an optimal control with associated state $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*) = \mathcal{S}(\boldsymbol{u}^*)$ and adjoint state $(\boldsymbol{\omega}, p, q, r)$. We remark at this place that all of the following results would remain valid correspondingly if $\boldsymbol{u}^*$ were only locally optimal in the sense of $\boldsymbol{L}^\infty(Q)$.

The convex functional G in the objective functional is responsible for the occurrence of sparsity, i.e., the possibility that optimal controls may vanish in some subregion of Q having a positive measure. The special sparsity properties are determined by the form of the subdifferential ∂G via the variational inequality (5.22). There are several concepts of sparsity, which are each induced by a specific functional G. We confine ourselves to *full sparsity*, which is induced by the $L^1(Q)$ norm

$$j(u) := \|u\|_{L^1(Q)} = \int_Q |u|\,, \quad \text{for } u \in L^2(Q)\,, \qquad\qquad (6.1)$$

which is nonnegative, convex and continuous (and thus sequentially lower semicontinuous) on the space $L^2(Q)$. It is well known (see, e.g., [29]) that its subdifferential $\partial j(u)$ is for every $u \in L^2(Q)$ given by the set of all $\lambda \in L^2(Q)$ that satisfy, for almost every $(x,t) \in Q$,

$$\lambda(x,t) \in \begin{cases} \{+1\}, & \text{if } u(x,t) > 0, \\ [-1,+1], & \text{if } u(x,t) = 0, \\ \{-1\}, & \text{if } u(x,t) < 0. \end{cases} \tag{6.2}$$

In the following, we consider a sparsity term which is slightly more general than that given in (1.10), namely

$$G(\boldsymbol{u}) = \int_Q \big(\kappa_1 |u_1| + \kappa_2 |u_2| + \kappa_3 |u_3|\big) = \sum_{i=1}^3 \kappa_i j(I_i(\boldsymbol{u}))$$
$$\text{for } \boldsymbol{u} = (u_1, u_2, u_3) \in \boldsymbol{L}^2(Q), \tag{6.3}$$

with given sparsity parameters $\kappa_i > 0$, $i = 1, 2, 3$, and the linear and continuous projection operators $I_i : \boldsymbol{L}^2(Q) \to L^2(Q)$, $\boldsymbol{u} = (u_1, u_2, u_3) \mapsto u_i$, $i = 1, 2, 3$. Using the well-known rules for subdifferentials (see, e.g., [29, Sect. 4.2.2, Thms. 1 and 2]), and denoting by I_i^* the dual operator of I_i, $i = 1, 2, 3$, we then conclude that the subdifferential of G is given by

$$\partial G(\boldsymbol{u}) = \sum_{i=1}^3 \kappa_i \, I_i^* \, \partial j(I_i(\boldsymbol{u}))$$
$$= \{(\kappa_1 \lambda_1, \kappa_2 \lambda_2, \kappa_3 \lambda_3) \in \boldsymbol{L}^2(Q) : \lambda_i \in \partial j(u_i), \; i = 1, 2, 3\}. \tag{6.4}$$

Therefore, the components of the multiplier $\boldsymbol{\Lambda}^* = (\Lambda_1^*, \Lambda_2^*, \Lambda_3^*) \in \partial G(\boldsymbol{u}^*)$ occurring in (5.22) are, for $i = 1, 2, 3$, of the form $\Lambda_i^* = \kappa_i \lambda_i^*$ with some $\lambda_i^* \in \partial j(u_i^*)$. From (5.25) and (5.27) in Remark 5.4 we then infer that for $i = 1, 2, 3$ the variational inequality

$$\int_Q \big(b_3 u_i^* + \omega_i + \kappa_i \lambda_i^*\big)(u - u_i^*) \geq 0 \quad \text{for every } u \in U_{ad}^i \tag{6.5}$$

and the projection condition

$$u_i^* = \max\big\{\underline{u}_i, \min\big\{\overline{u}_i, -b_3^{-1}(\omega_i + \kappa_i \lambda_i^*)\big\}\big\} \quad \text{a.e. in } Q, \tag{6.6}$$

are satisfied, with suitable $\lambda_i^* \in \partial j(u_i^*)$. We then arrive at the following sparsity result, which is in line with analogous results established in [15, 43, 44].

Theorem 6.1 *Suppose that the general assumptions for our optimal control problem are satisfied, and assume that the thresholds satisfy the following condition:*

$$\underline{u}_i \text{ and } \overline{u}_i \text{ are constants such that } \underline{u}_i < 0 < \overline{u}_i \text{ for } i = 1, 2, 3. \tag{6.7}$$

Moreover, let $\boldsymbol{u}^* \in \mathcal{U}_{ad}$ be an optimal control with associated state $(\boldsymbol{v}^*, \varphi^*, \mu^*, w^*) = \mathcal{S}(\boldsymbol{u}^*)$ and adjoint state $(\boldsymbol{\omega}, p, q, r)$. Then, for every $i \in \{1, 2, 3\}$, it holds that for a.e. $(x,t) \in Q$ the following equivalence is valid:

$$u_i^*(x,t) = 0 \quad \Longleftrightarrow \quad |\omega_i(x,t)| \leq \kappa_i. \tag{6.8}$$

Proof. Let $i \in \{1,2,3\}$ be fixed. For almost every $(x,t) \in Q$ it holds: if $u_i^*(x,t) = 0$, then, by virtue of (6.6), $0 = -b_3^{-1}(\omega_i(x,t) + \kappa_i \lambda_i^*(x,t))$, whence we immediately obtain that $|\omega_i(x,t)| = \kappa_i|\lambda_i^*(x,t)|$. Since $\lambda_i^* \in \partial j(u_i^*)$, it then follows from (6.2) that $|\omega_i(x,t)| \leq \kappa_i$.

Conversely, we have for almost every $(x,t) \in Q$ the following implication: if $|\omega_i(x,t)| \leq \kappa_i$ and $u_i^*(x,t) > 0$, then, by (6.2), it follows that $\lambda_i^*(x,t) = +1$ and thus, again by (6.6), $0 < -b_3^{-1}(\omega_i(x,t) + \kappa_i)$. But then

$$\omega_i(x,t) + \kappa_i < 0 \text{ and therefore } |\omega_i(x,t)| = -\omega_i(x,t) > \kappa_i,$$

leading to a contradiction. Similar reasoning yields a contradiction also if $u_i^*(x,t) < 0$ is assumed. In conclusion, we must have for almost every $(x,t) \in Q$: if $|\omega_i(x,t)| \leq \kappa_i$ then $u_i^*(x,t) = 0$. This concludes the proof of the assertion. $\square$

Remark 6.2 Observe that the adjoint variable $\boldsymbol{\omega}$ appearing in the sparsity condition (6.8) depends on the special optimal control $\boldsymbol{u}^*$. It is therefore natural to rise the question whether there exists a global sparsity parameter $\kappa^* > 0$ such that all optimal controls vanish a.e. in Q whenever the parameters κ_i exceed the value κ^*. Apparently, this requires to establish a global $\boldsymbol{L}^\infty(Q)$-bound for the adjoint variable $\boldsymbol{\omega}$. However, recalling the global estimate (2.33) established in Theorem 2.1, we can conclude from a closer inspection of the estimates performed in the proof of Theorem 5.2 that there exists a global constant $\widehat{C} > 0$ such that

$$\|\boldsymbol{\omega}\|_{L^\infty(0,T;\boldsymbol{V}_0)} \leq \widehat{C},$$

whenever $\boldsymbol{\omega}$ is the first component of the solution to the adjoint system associated with an arbitrary optimal control $\boldsymbol{u}^*$. Hence, the existence of a suitable global sparsity parameter κ^* can be guaranteed at least in the spatially one-dimensional case.

Acknowledgments

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