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Abstract

We study a Cahn–Hilliard two-phase model describing the flow of two viscoelastoplastic fluids, which arises in geodynamics. A phase-field variable indicates the proportional distribution of the two fluids in the mixture. The motion of the incompressible mixture is described in terms of the volume-averaged velocity. Besides a volume-averaged Stokes-like viscous contribution, the Cauchy stress tensor in the momentum balance contains an additional volume-averaged internal stress tensor to model the elastoplastic behavior. This internal stress has its own evolution law featuring the nonlinear Zaremba-Jaumann time-derivative and the subdifferential of a non-smooth plastic potential. The well-posedness of this system is studied in two cases: Based on a regularization by stress-diffusion we obtain the existence of Leray-Hopf-type weak solutions. In order to deduce existence results also in the absence of the regularization, we introduce the concept of dissipative solutions, which is based on an estimate for the relative energy. We discuss general properties of dissipative solutions and show their existence for the viscoelastoplastic two-phase model in the setting of stress-diffusion. By a limit passage in the relative energy inequality for vanishing stress-diffusion, we conclude an existence result for the non-regularized model.

1 Introduction

The movement of tectonic plates driven by convective processes within the Earth's mantle is a generally recognized theory which explains many geological phenomena, e.g., earthquakes, volcanoes, formation of mountains, ocean trenches, mid-ocean ridges, and island arcs, see [41, 37, 25, 27] for details. These tectonic plates form the Earth's lithosphere, consisting of the Earth's crust and the uppermost part of its mantle. On geological time scales of millions of years, the moving and mechanically deforming plates in the mantle are treated as non-Newtonian viscoelastoplastic fluids.

In this paper, we consider a system of equations which describes a two-phase flow of an incompressible mixture of two viscoelastoplastic fluids arising from geodynamics. In a time interval $(0,T)$, where $T \in (0,\infty]$, and a bounded C^2-domain $\Omega \subseteq \mathbb{R}^3$, the system is given as:

$$\partial_t(\rho v) + \mathrm{div}(v \otimes (\rho v + J)) - \mathrm{div}(\mathbb{T}) = f - \varepsilon \mathrm{div}(\nabla\varphi \otimes \nabla\varphi) \quad \text{in } \Omega \times (0,T), \tag{1.1a}$$

$$\mathrm{div}(v) = 0 \quad \text{in } \Omega \times (0,T), \tag{1.1b}$$

$$\mathbb{T} = \eta(\varphi)S + 2\nu(\varphi)(\nabla v)_{\mathrm{sym}} - p\mathbb{I} \quad \text{in } \Omega \times (0,T), \tag{1.1c}$$

$$\overset{\triangledown}{S} + \partial P(\varphi; S) - \gamma\Delta S \ni \eta(\varphi)(\nabla v)_{\mathrm{sym}} \quad \text{in } \Omega \times (0,T), \tag{1.1d}$$

$$\partial_t\varphi + v \cdot \nabla\varphi = \Delta\mu \quad \text{in } \Omega \times (0,T), \tag{1.1e}$$

$$\mu = \frac{1}{\varepsilon}W'(\varphi) - \varepsilon\Delta\varphi, \quad \text{in } \Omega \times (0,T). \tag{1.1f}$$

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The system is complemented by the following boundary and initial conditions

$$v|_{\partial\Omega} = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.1g}$$

$$\gamma \vec{n} \cdot \nabla S|_{\partial\Omega} = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.1h}$$

$$\vec{n} \cdot \nabla\varphi|_{\partial\Omega} = \vec{n} \cdot \nabla\mu|_{\partial\Omega} = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.1i}$$

$$(v, S, \varphi)|_{t=0} = (v_0, S_0, \varphi_0) \text{ in } \Omega, \tag{1.1j}$$

where $\vec{n}$ denotes the outward unit normal vector of Ω.

Equation (1.1a) describes the momentum balance for fluids, phrased in terms of the volume-averaged Eulerian velocity field $v : [0,T] \times \Omega \to \mathbb{R}^3$, the mass density $\rho : [0,T] \times \Omega \to \mathbb{R}$, the volume-averaged mass flux $J : [0,T] \times \Omega \to \mathbb{R}^3$ given as $J := -\frac{\rho_2 - \rho_1}{2} \nabla\mu$ which occurs due to the unmatched mass density, the Cauchy stress tensor $\mathbb{T} : [0,T] \times \Omega \to \mathbb{R}^{3\times3}$, an external loading $f : [0,T] \times \Omega \to \mathbb{R}^3$, and the Korteweg stress representing the capillarity stress, which is modeled as $\varepsilon \nabla\varphi \otimes \nabla\varphi$ with a parameter $\varepsilon > 0$. Equation (1.1b) states the incompressibility condition. Equation (1.1c) provides the Cauchy stress tensor for viscoelastoplastic fluids, consisting of two parts: a radial part, given by the pressure $p : [0,T] \times \Omega \to \mathbb{R}$, and a deviatoric part, which is in the form of a symmetric matrix with zero trace including the viscous part given as $2\nu(\varphi)(\nabla v)_{\text{sym}}$ and an extra contribution by the internal S given as $\eta(\varphi)S$ where coefficients ν and η are functions of φ arising from the viscosity and the elasticity of the fluids. The strain rate $(\nabla v)_{\text{sym}} = \frac{1}{2}(\nabla v + \nabla v^\top)$, which is the symmetric part of the velocity gradient, describes the relative motion between the particles.

Equation (1.1d) characterizes the evolution law of the volume-averaged internal stress $S : [0,T] \times \Omega \to \mathbb{R}^{3\times3}_{\mathrm{sym,Tr}}$, which takes the form of a Maxwell-type stress-strain relation. Moreover, the rate of the internal stress is controlled by the Zaremba-Jaumann rate $\overset{\triangledown}{S}$ defined as

$$\overset{\triangledown}{S} := \partial_t S + v \cdot \nabla S + S(\nabla v)_{\text{skw}} - (\nabla v)_{\text{skw}} S, \tag{1.2}$$

which is a notion of time derivative widely used in geophysical models, c.f., [37, 25, 27]. Here, the spin tensor $(\nabla v)_{\text{skw}} = \frac{1}{2}(\nabla v - \nabla v^\top)$ denotes the skew-symmetric part of the velocity gradient. To model plastic effects, an additional term $\partial P(\varphi; S)$ is incorporated into equation (1.1d). An example of the plastic potential, which is used in geodynamics for the plastic deformation in lithospheric plates, is given as

$$P(\varphi; S) := G(\varphi) P_1(S) + (1 - G(\varphi)) P_2(S). \tag{1.3}$$

Here,

$$P_i(S) := \begin{cases} \frac{a_i}{2}|S|^2 + b_i|S| & \text{if } |S| \le \sigma_{\mathrm{yield,i}}, \\ \infty & \text{if } |S| > \sigma_{\mathrm{yield,i}}, \end{cases} \tag{1.4}$$

where $a_i > 0$, $b_i > 0$ are constants of each pure phase i and $\sigma_{\mathrm{yield,i}}$ is the yield stress of each pure phase i, $i = 1, 2$, which determines the onset of plastic flow behavior see [37, 25]. Moreover, $\partial P(\varphi; S)$ denotes the subdifferential of the convex potential $P(\varphi; \cdot)$ in S. It is defined by

$$\partial P(\varphi; S) := \left\{ \xi \in \mathbb{R}^{3\times3}_{\mathrm{sym,Tr}} : \langle \xi, \tilde{S} - S \rangle_{\mathbb{R}^{3\times3}} + P(\varphi; S) \le P(\varphi; \tilde{S}) \right\}. \tag{1.5}$$

In addition, the plastic potential is given as

$$\mathcal{P}(\varphi; S) := \int_\Omega P(\varphi; S) \, \mathrm{d}x \, . \tag{1.6}$$

The Cahn-Hilliard type equation (1.1e)- (1.1f) describes the evolution law of the phase-field variable $\varphi : [0,T] \times \Omega \to \mathbb{R}$ which indicates the presence of each of the two phases. Hence, in the physical sense, we expect, for $(t,x) \in [0,T] \times \Omega$

$$\varphi(t,x) \begin{cases} = -1, & \text{pure fluid 1,} \\ \in (-1,1), & \text{mixture of fluid 1 and fluid 2,} \\ = 1, & \text{pure fluid 2.} \end{cases} \tag{1.7}$$

Moreover, in equation (1.1e)-(1.1f), $\mu : [0,T] \times \Omega \to \mathbb{R}$ is the associated chemical potential, $W : \mathbb{R} \to [0,\infty]$ is a singular potential, and $\varepsilon > 0$ is a small parameter related to the thickness of the interface.

In particular, the mass density is modeled as

$$\rho(\varphi) := \frac{1-\varphi}{2}\rho_1 + \frac{1+\varphi}{2}\rho_2, \tag{1.8}$$

where $\rho_i > 0$ is the constant mass density of each pure fluid i, $i = 1,2$. Hence, equation (1.1e) provides the additional continuity equation

$$\partial_t \rho + \mathrm{div}(\rho v + J) = 0. \tag{1.9}$$

To have an understanding of this system, let us first define the total energy of the system at time $t \in [0,T]$, i.e., the sum of the kinetic energy depending on ρ and v, the elastic energy depending on S and the phase-field energy depending on φ:

$$E(t) := \mathcal{E}(v(t), S(t), \varphi(t)) := \int_\Omega \frac{\rho(\varphi)}{2}|v(x,t)|^2 + \frac{1}{2}|S(x,t)|^2 + \frac{1}{2}|\nabla\varphi|^2 + W(\varphi)\,\mathrm{d}x\,. \tag{1.10}$$

Assume for now that we have a smooth solution (v, S, φ, μ) of system (1.1). Then we multiply (1.1a) by v, (1.1d) by S, (1.1e) by μ and (1.1f) by $\partial_t\varphi$, integrate over space and time, and perform an integration by parts, so that we obtain the following energy-dissipation balance:

$$E(t) + \int_0^t \int_\Omega 2\nu(\varphi)|(\nabla v)_{\mathrm{sym}}|^2 + \gamma|\nabla S|^2 + \xi : S + |\nabla\mu|^2 \,\mathrm{d}x\,\mathrm{d}\tau = E(0) + \int_0^t \langle f, v\rangle_{H^1}\,\mathrm{d}\tau \tag{1.11}$$

for all $t \in [0,T]$ and where $\xi \in \partial P(\varphi; S)$. From this energy-dissipation balance, we can see that the total energy of the system is dissipated by four parts: a Newtonian viscosity, a quadratic stress diffusion, dissipation due the plastic deformation stemming from the non-smooth potential P and a quadratic term due to phase separation involving the chemical potential μ.

The mathematical challenges associated with analysis of this system stem from the following: First of all, the momentum balance (1.1a) comes with all the difficulties arising from the three-dimensional Navier–Stokes equations. Based on the large body of analytical results on Navier–Stokes equations, cf. e.g. [17, 35, 24, 43, 22, 15, 39, 13, 38], we cannot expect a better class of solutions for the velocity field than Leray–Hopf solutions, which were introduced in [33, 28].

Moreover, the stress evolution equation (1.1d) includes a nonlinearity in the Zaremba-Jaumann derivative (1.2) of S and a multi-valued derivative due to the non-smoothness of the dissipation potential P. [36] studies the global existence of solutions based on the Zaremba-Jaumann derivative together with a smooth dissipation potential such that $\partial P(S) = aS$. In addition, the stress diffusion $-\gamma\Delta S$ with $\gamma > 0$ is introduced as a regularization for analytical reasons as in [19]. However, in order to get closer to models used in geoscientific applications, e.g. [37], we aim to avoid it in our analysis.

Furthermore, this model contains more than one phase, which also gives us the difficulties arising from the coupled Cahn–Hilliard–Navier–Stokes system. We refer to [20, 4] for the study of Cahn–Hilliard equations and to [1, 16, 3, 2] for the study of the Cahn–Hilliard–Navier–Stokes system. In contrast to the above-mentioned works, we additionally face the difficulties of the multi-valued stress evolution (1.1d). A single-phase system of the momentum and stress equations has already been studied in [19] and [18].

Our goal is to study the existence results for system (1.1) not only in the case $\gamma > 0$ but, more importantly, also in the case $\gamma = 0$. This will be achieved with the help of an alternative concept of solution: the dissipative solution. The idea of dissipative solutions is that the solutions do not satisfy the equation in the distributional sense anymore, but, instead, one controls the difference between the solutions and smooth test functions satisfying the equations in terms of the relative energy and the relative dissipation. The concept of dissipative solutions was introduced by P.L. Lions [35, Sec. 4.4] in the context of the incompressible Euler equations and his motivation stemmed from the consideration of the singular limit in the Boltzmann equation and identification of its limits [34]. Since then, dissipative solutions have been applied in different contexts, for instance to singular limits in the Boltzman equations [40], incompressible viscous electro-magneto-hydrodynamics [6], equations of viscoelastic diffusion in polymers [44], the Ericksen–Leslie equations [30, 31], or finite-element approximation of nematic electrolytes [7]. Moreover, it is also applied to isothermal damped Hamiltonian systems in [32] and to a viscoelastoplastic single-phase models in [18], where this notion of solutions is called energy-variational solution. The term dissipative solutions is also used for other solution concepts. On the one hand, it is used in [23] in the context of the Navier-Stokes-Fourier system, where it basically denotes a weak solution. On the other hand the term dissipative solution is also used for different measure-valued solution frameworks, see for instance [14, 10]. These concepts are different from the dissipative solutions in [35] and we rather refer with this term to the original definition by Lions.

2 General notations, preliminaries and assumptions

In this section, we fix the notation that will be used throughout this work and recall some useful results that will be applied for our analysis.

2.1 General notations

By default, we use Einstein's summation convention for vectors and tensors. Let $a = (a_j)_{j=1}^3, b = (b_j)_{j=1}^3 \in \mathbb{R}^3$ be two vectors, then their inner product is written as $a \cdot b := a_j b_j$, and their tensor product is written as $a \otimes b = (a_j b_k)_{j,k=1}^3$. Similarly, let $A = (A_{jk})_{j,k=1}^3, B = (B_{jk})_{j,k=1}^3 \in \mathbb{R}^{3\times 3}$ be two second-order tensors, the tensor inner product is written as $A : B = A_{jk}B_{jk}$. Besides, for two third-order tensors $C = (C_{jkl})_{j,k,l=1}^3, D = (D_{jkl})_{j,k,l=1}^3 \in \mathbb{R}^{3\times 3\times 3}$, we denote the inner product by $C \,\vdots\, D = C_{jkl}D_{jkl}$. The tensor product between a second-order tensor $A \in \mathbb{R}^{3\times 3}$ and a vector $a \in \mathbb{R}^3$ gives a third-order tensor and it is defined as $(A \otimes a)_{jkl} = (A_{jk}a_l)_{j,k,l=1}^3$. We write the transpose and trace of a matrix $A \in \mathbb{R}^{3\times 3}$ in the usual way that $A^\top$ and TrA. Moreover, we set the space of the symmetric and trace-free second order tensors as

$$\mathbb{R}^{3\times 3}_{\mathrm{sym,Tr}} := \left\{ A \in \mathbb{R}^{3\times 3} : A = A^\top, \mathrm{Tr}A = 0 \right\}. \tag{2.1}$$

The point $(x,t) \in \Omega \times (0,T)$ is defined by the spatial variable $x \in \Omega$ and the time variable $t \in (0,T)$. Thus, we write the partial time derivative and partial spatial derivative of a (sufficiently regular) function

$u : \Omega \times (0, t) \to \mathbb{R}$ as $\partial_t u$ and $\partial_{x_i} u$, where $i = 1, 2, 3$. Moreover, ∇u and Δu denote the gradient and Laplace of u with respect to spatial variable. The symmetrized and skew-symmetrized part of ∇v of a vector field $v : \Omega \to \mathbb{R}^3$ is given by

$$(\nabla v)_{\text{sym}} := \frac{1}{2}\left(\nabla v + \nabla v^\top\right) \text{ and } (\nabla v)_{\text{skw}} := \frac{1}{2}\left(\nabla v - \nabla v^\top\right). \tag{2.2}$$

We also write $\text{div}(v) = \partial_{x_i} v_i$ as the divergence of vector field v. Similarly, for a second-order tensor S, the divergence is defined by $\text{div}(S) = \partial_{x_k} S_{jk}$.

Function spaces. Let X be a Banach space with norm $\|\cdot\|_X$ and dual space X'. The same notation is used also for X^3 and $X^{3\times3}$. When the dimension is clear, we simply write X instead of X^3 or $X^{3\times3}$. We use $\langle x', x\rangle_X$ to denote the duality pairing of $x' \in X'$ and $x \in X$.

The space $C^\infty(\Omega)$ denotes the class of smooth functions in Ω and the space $C_0^\infty(\Omega)$ denotes the class of smooth functions with compact support in Ω. The Lebesgue spaces and Sobolev spaces are denoted as $L^p(\Omega)$ and $W^{k,p}(\Omega)$ for $p \in [1, \infty]$ and $k \in \mathbb{N}$, in particular, for $p = 2$, we write $W^{k,2}(\Omega) = H^k(\Omega)$. Moreover, we write $H_0^1(\Omega)$ as the space of functions in $H^1(\Omega)$ whose boundary value is zero in the trace sense and $H^{-1}(\Omega) := (H_0^1(\Omega))'$ is the dual space.

Now let $(0, T) \subseteq \mathbb{R}$ be an interval, the space $C^0(0, T; X)$ consists of the class of continuous functions in time with values in the Banach space X. For $p \in [1, \infty]$, the corresponding Lebesgue-Bochner spaces are denoted by $L^p(0, T; X)$. Moreover, we write $W^{1,p}(I; X) := \{u \in L^p(0, T; X) : \partial_t u \in L^p(0, T; X)\}$ and $H^1(0, T; X) = W^{1,2}(0, T; X)$. The local Lebesgue-Bochner spaces $L^p_{\text{loc}}(0, T; \Omega)$ and $H^1_{\text{loc}}(0, T; \Omega)$ consist of the class of functions in $L^p(J; X)$ and $H^1(J; X)$ for every compact subinterval $J \subseteq (0, T)$ respectively. Besides, we will simply write $u(t) := u(\cdot, t)$ for u defined on $\Omega \times I$.

For any $u \in L^1(\Omega)$

$$u_\Omega := \frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x \tag{2.3}$$

is the mean value of u in Ω.

Solenoidal vector fields and symmetric deviatoric fields. We introduce function spaces for solenoidal (divergence-free) vector fields and symmetric, deviatoric (trace-free) fields. The corresponding classes of smooth functions on Ω are given by

$$C^\infty_{0,\text{div}}(\Omega) := \left\{\varphi \in C_0^\infty(\Omega)^3 : \text{div}(\varphi) = 0 \text{ in } \Omega\right\}, \tag{2.4a}$$

$$C^\infty_{\text{sym},\text{Tr}}(\bar{\Omega}) := \left\{\psi \in C^\infty(\bar{\Omega})^{3\times3} : \psi = \psi^\top, \text{Tr}(\psi) = 0 \text{ in } \Omega\right\}. \tag{2.4b}$$

We further write the time-dependent solenoidal vector fields and symmetric, deviatoric fields as

$$C^\infty_{0,\text{div}}(\Omega \times I) := \left\{\Phi \in C_0^\infty(\Omega \times I)^3 : \text{div}(\Phi) = 0 \text{ in } \Omega \times I\right\}, \tag{2.4c}$$

$$C^\infty_{0,\text{sym},\text{Tr}}(\Omega \times I) := \left\{\Psi \in C_0^\infty(\Omega \times I)^{3\times3} : \Psi = \Psi^\top, \text{Tr}(\Psi) = 0 \text{ in } \Omega \times I\right\}, \tag{2.4d}$$

where $I \subseteq [0, \infty)$ is an interval. The corresponding Lebesgue space of space-integrable functions on Ω are defined by

$$L^2_{\text{div}}(\Omega) := \left\{v \in L^2(\Omega)^3 : \text{div}(v) = 0 \text{ in } \Omega\right\}, \tag{2.4e}$$

$$L^2_{\mathrm{sym,Tr}}(\Omega) := \left\{ S \in L^2(\Omega)^{3\times 3} : S = S^\top, \mathrm{Tr}(S) = 0 \text{ in } \Omega \right\}. \tag{2.4f}$$

The Sobolev spaces obtained as the closure of $C^\infty_{0,\mathrm{div}}(\Omega)$ and $C^\infty_{0,\mathrm{sym,Tr}}(\Omega)$ with respect to the $H^1(\Omega)$-norm are denoted by

$$H^1_{0,\mathrm{div}}(\Omega) := \left\{ v \in H^1_0(\Omega)^3 : \mathrm{div}(v) = 0 \text{ in } \Omega \right\}, \tag{2.4g}$$

$$H^1_{\mathrm{sym,Tr}}(\Omega) := \left\{ S \in H^1(\Omega)^{3\times 3} : S = S^\top, \mathrm{Tr}(S) = 0 \text{ in } \Omega \right\}. \tag{2.4h}$$

Notice that all the boundary conditions are identified in the trace sense.

2.2 General assumptions and further notations

In the following, we collect and discuss the mathematical assumptions on the domain, the given data, and the material parameters.

Assumption 2.1 (on the domain)**.** We assume that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with C^2-boundary $\partial\Omega$ and write $\vec{n}$ as the outward unit normal vector.

Also in view of (1.10), we make the following hypothesis for the non-smooth dissipation potential $\mathcal{P}$ in (1.1d):

Assumption 2.2 (on the plastic potential)**.** For the plastic potential

$$\begin{aligned} \mathcal{P} : L^2(\Omega) \times L^2_{\mathrm{sym,Tr}}(\Omega) &\to [0,\infty] \\ (\varphi; S) &\mapsto \int_\Omega P(x, \varphi(x), S(x))\,\mathrm{d}x\,, \end{aligned} \tag{2.5}$$

we make the following assumptions: The density $P : \Omega \times \mathbb{R} \times \mathbb{R}^{3\times 3}_{\mathrm{sym,Tr}} \to [0,\infty]$ is proper and measurable with $P(x,\varphi,0) = 0$ for all $x \in \Omega$ and $\varphi \in \mathbb{R}$. Moreover,

- for all $x \in \Omega$, the mapping $(y,z) \mapsto P(x,y,z)$ is lower semicontinuous,
- for all $(x,y) \in \Omega \times \mathbb{R}$, the mapping $z \mapsto P(x,y,z)$ is convex,
- for all $(x,z) \in \Omega \times \mathbb{R}^{3\times 3}_{\mathrm{sym,Tr}}$, the mapping $y \mapsto P(x,y,z)$ is continuous.

Besides, the convex partial subdifferential of $\mathcal{P}(\varphi;\cdot)$ at point S is given by

$$\begin{aligned} \partial\mathcal{P}(\varphi;S) := \Big\{ \xi \in (L^2_{\mathrm{sym,Tr}}(\Omega))' : \langle \xi, \tilde{S} - S \rangle_{L^2(\Omega)} + \mathcal{P}(\varphi;S) \\ \le \mathcal{P}(\varphi;\tilde{S}) \text{ for all } \tilde{S} \in L^2_{\mathrm{sym,Tr}}(\Omega) \Big\}. \end{aligned} \tag{2.6}$$

Notice that, by definition, $\partial\mathcal{P}(\varphi;S) = \emptyset$ if $\mathcal{P}(\varphi;S) = \infty$.

Furthermore, we make the following hypotheses for the initial data and the external loading:

Assumption 2.3 (on the given data)**.** Assume that $v_0 \in L^2_{\mathrm{div}}(\Omega)$, $S_0 \in L^2_{\mathrm{sym,Tr}}(\Omega)$ and $f \in L^2_{\mathrm{loc}}([0,T); H^{-1}(\Omega)^3)$. Moreover, assume that $\varphi_0 \in H^1(\Omega)$ with $|\varphi_0| \le 1$ almost everywhere in Ω and

$$\frac{1}{|\Omega|}\int_\Omega \varphi_0\,\mathrm{d}x \in (-1,1).$$

In order to guarantee the existence of weak solutions, we make the following assumptions on the coefficients and the singular potential. The idea of these assumptions is inspired directly by [2, Assumption 3.1] and [4, Assumption 1.1].

Assumption 2.4 (on the material parameters)**.** The dependence of the material parameters on the composition of the mixture, i.e., on φ is assumed to be as follows:
(1) The dependence of the mass density ρ on the phase-field variable φ is given by

$$\rho(\varphi) = \frac{\rho_1 + \rho_2}{2} + \frac{\rho_2 - \rho_1}{2}\varphi \tag{2.7}$$

where the constant $\rho_i > 0$ is the mass density of fluid i, $i = 1, 2$.
(2) For the viscosity parameter ν and the elastic modulus η, we assume that $\nu \in C^0(\mathbb{R})$, $\eta \in C^1(\mathbb{R})$ and

$$0 < \nu_1 \leq \nu(\varphi) \leq \nu_2 \quad 0 \leq \eta_1 \leq \eta(\varphi) \leq \eta_2, \text{ and } |\eta'(\varphi)| \leq C \text{ for all } \varphi \in \mathbb{R} \tag{2.8}$$

for some positive constants $C, \eta_1, \eta_2, \nu_1, \nu_2$. Herein, the constants η_i, ν_i can be viewed as the constants associated with the pure fluid i, $i = 1, 2$.

Assumption 2.5 (on the singular potential)**.** For the singular potential W, we assume that $W \in C([-1, 1]) \cap C^3(-1, 1)$ such that

$$\lim_{\varphi \to -1} W'(\varphi) = -\infty,\ \lim_{\varphi \to 1} W'(\varphi) = \infty,\ W''(\varphi) \geq -\kappa \text{ for some } \kappa \geq 0. \tag{2.9}$$

Moreover, we extend $W(\varphi)$ to $+\infty$ for $\varphi \in \mathbb{R}\backslash[-1, 1]$. Without loss of generality, we also assume that $W(\varphi) \geq 0$ for all $\varphi \in [-1, 1]$.

Remark 2.6. (1) As an example for such a singular potential, one can consider

$$W(\varphi) = \frac{1}{2}\left((1+\varphi)\ln(1+\varphi) + (1-\varphi)\ln(1-\varphi)\right) - \frac{\lambda}{2}\varphi^2,\ \varphi \in [-1, 1],$$

for fixed $\lambda \geq 0$.
(2) Since we will show that a solution φ takes values in $(-1, 1)$, we actually only need the functions ν, η to be defined on this interval and then extend them outside of this interval in a sufficiently smooth way by constants. So, by the assumptions of continuity and continuous differentiability, respectively, the bounds (2.8) are natural.

Energy functionals. We define the kinetic energy, the elastic energy and the phase-field energy as follows:

$$\mathcal{E}_{\mathrm{kin}} : L^\infty(\Omega) \times L^2_{\mathrm{div}}(\Omega) \to [0, \infty],\ (\varphi, v) \mapsto \int_\Omega \rho(\varphi)\frac{|v|^2}{2}\,\mathrm{d}x\,, \tag{2.10a}$$

$$\mathcal{E}_{\mathrm{el}} : L^2_{\mathrm{sym,Tr}}(\Omega) \to [0, \infty],\ S \mapsto \int_\Omega \frac{|S|^2}{2}\,\mathrm{d}x\,, \tag{2.10b}$$

$$\mathcal{E}_{\mathrm{pf}} : H^1(\Omega) \to [0, \infty],\ \varphi \mapsto \int_\Omega \varepsilon\frac{|\nabla\varphi|^2}{2} + \varepsilon^{-1}W(\varphi)\,\mathrm{d}x\,. \tag{2.10c}$$

Then, the total energy of this system is given by

$$\mathcal{E}_{\mathrm{tot}}(v, S, \varphi) := \mathcal{E}_{\mathrm{kin}}(\varphi, v) + \mathcal{E}_{\mathrm{el}}(S) + \mathcal{E}_{\mathrm{pf}}(\varphi). \tag{2.10d}$$

Dissipation functionals: We define the viscous Stokes-type dissipation, the quadratic stress diffusion, and the quadratic diffusion of the chemical potential according to the Cahn–Hilliard equation as follows:

$$\mathcal{D}_{\mathrm{s}} : L^1(\Omega) \times H^1_{0,\mathrm{div}}(\Omega) \to [0,\infty],\ (\varphi, v) \mapsto \int_\Omega 2\nu(\varphi)|(\nabla v)_{\mathsf{sym}}|^2 \, \mathrm{d}x\,, \tag{2.11a}$$

$$\mathcal{D}_{\mathrm{ch}} : H^1(\Omega) \to [0,\infty],\ \mu \mapsto \int_\Omega |\nabla \mu|^2 \, \mathrm{d}x\,, \tag{2.11b}$$

$$\mathcal{D}_{\mathrm{sd},\gamma} : H^1_{\mathrm{sym,Tr}}(\Omega) \to [0,\infty],\ S \mapsto \int_\Omega \gamma |\nabla S|^2 \, \mathrm{d}x\,. \tag{2.11c}$$

Then, the sum of Stokes-type dissipation and Cahn–Hilliard dissipation and the total dissipation potential are given by

$$\mathcal{D}_{\mathrm{chs}}(v,\varphi,\mu) := \mathcal{D}_s(\varphi, v) + \mathcal{D}_{\mathrm{ch}}(\mu), \tag{2.11d}$$

$$\mathcal{D}_{\mathrm{tot}}(v,S,\varphi,\mu) := \mathcal{D}_{\mathrm{chs}}(v,\varphi,\mu) + \mathcal{D}_{\mathrm{sd},\gamma}(S) + \mathcal{P}(\varphi; S). \tag{2.11e}$$

Note that (2.11e) also involves the plastic potential $\mathcal{P}$, which is specified in more details in (2.5).

2.3 Reformulation of system (1.1)

Now, we reformulate the Korteweg stress that appears on the right-hand side of (1.1a) with the aid of (1.1f). Formally, for a smooth solution φ, by substituting (1.1f) in (1.1a), we find

$$\begin{aligned} -\varepsilon \mathrm{div}(\nabla\varphi \otimes \nabla\varphi) &= -\varepsilon\Delta\varphi \cdot \nabla\varphi - \varepsilon\nabla\left(\frac{|\nabla\varphi|^2}{2}\right) \\ &= \mu\nabla\varphi - \varepsilon^{-1}W'(\varphi)\nabla\varphi - \varepsilon\nabla\left(\frac{|\nabla\varphi|^2}{2}\right) \\ &= \mu\nabla\varphi - \varepsilon^{-1}\nabla(W(\varphi)) - \varepsilon\nabla\left(\frac{|\nabla\varphi|^2}{2}\right). \end{aligned} \tag{2.12}$$

This allows us to define a new pressure term

$$g = p + \varepsilon^{-1}W(\varphi) + \varepsilon\frac{1}{2}|\nabla\varphi|^2 \tag{2.13}$$

to rewrite (1.1a) as

$$\partial_t(\rho v) + \mathrm{div}((v \otimes (\rho v + J))) - \mathrm{div}((\eta(\varphi)S + 2\nu(\varphi)(\nabla v)_{\mathsf{sym}})) + \nabla g = \mu\nabla\varphi. \tag{2.14}$$

Hence, when testing (2.14) with divergence-free vectors, the additional pressure term also vanish. Notice that the singular potential is not convex. But thanks to the assumption that W'' is bounded from below by $-\kappa$ with $\kappa > 0$ cf. (2.9), we define

$$W_\kappa(\varphi) := W(\varphi) + \frac{\kappa}{2}\varphi^2. \tag{2.15}$$

Thanks to the bound $W''(\varphi) \geq -\kappa$, we find that $W''_\kappa(\varphi) = W''(\varphi) + \kappa \geq 0$, for all $\varphi \in [-1,1]$ which ensures that W_κ is convex. In particular, it holds that $W'_\kappa(\varphi) = W'(\varphi) + \kappa\varphi$. Using W_κ in (1.1f), this is equivalent to

$$\mu + \frac{\kappa}{\varepsilon}\varphi = \varepsilon^{-1}W'_\kappa(\varphi) - \varepsilon\Delta\varphi. \tag{2.16}$$

Now, we can consider the energy $\mathcal{E}_{\mathrm{pf},\kappa} : L^2(\Omega) \to \mathbb{R}$ with its domain given by

$$\mathrm{dom}(\mathcal{E}_{\mathrm{pf},\kappa}) := \{\varphi \in H^1(\Omega) : -1 \le \varphi \le 1 \text{ a.e.}\}, \tag{2.17a}$$

$$\mathcal{E}_{\mathrm{pf},\kappa}(\varphi) := \begin{cases} \frac{1}{2}\int_\Omega \varepsilon|\nabla\varphi|^2 \,\mathrm{d}x + \int_\Omega \varepsilon^{-1} W_\kappa(\varphi)\,\mathrm{d}x & \text{for } u \in \mathrm{dom}(\mathcal{E}_{\mathrm{pf},\kappa}), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.17b}$$

According to [4, Theorem 4.3], the domain of the subdifferential is given by

$$\mathrm{dom}(\partial\mathcal{E}_{\mathrm{pf},\kappa}) = \left\{\varphi \in H^2(\Omega) : W_\kappa'(\varphi) \in L^2(\Omega), W_\kappa''(\varphi)|\nabla\varphi|^2 \in L^1(\Omega), \vec{n}\cdot\nabla\varphi|_{\partial\Omega} = 0\right\} \tag{2.18}$$

as well as

$$\partial\mathcal{E}_{\mathrm{pf},\kappa}(\varphi) = \left\{-\varepsilon\Delta\varphi + \varepsilon^{-1}W_\kappa'(\varphi)\right\} \text{ for all } \varphi \in \mathrm{dom}(\partial\mathcal{E}_{\mathrm{pf},\kappa}). \tag{2.19}$$

In this case, since $\partial\mathcal{E}_{\mathrm{pf},\kappa}$ is single-valued, it coincides with the Gâteaux derivative $\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi) = -\varepsilon\Delta\varphi + \varepsilon^{-1}W_\kappa'(\varphi)$. Additionally, we have the following estimate

$$\|\varphi\|_{H^2}^2 + \|W_\kappa'(\varphi)\|_{L^2}^2 + \int_\Omega W_\kappa''(\varphi)|\nabla\varphi|^2\,\mathrm{d}x \le C\left(\|\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi)\|_{L^2}^2 + \|\varphi\|_{L^2}^2 + 1\right) \tag{2.20}$$

for all $\varphi \in \mathrm{dom}(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa})$ as well as

$$\mu + \frac{\kappa}{\varepsilon}\varphi = -\varepsilon\Delta\varphi + \varepsilon^{-1}W_\kappa'(\varphi) = \mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi) \tag{2.21}$$

for all $\varphi \in \mathrm{dom}(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa})$. Thus, we have the following relation between the original phase-field energy $\mathcal{E}_{\mathrm{pf}}$ from (2.10) and the convexified phase-field energy $\mathcal{E}_{\mathrm{pf},\kappa}$ from (2.17b)

$$\mathcal{E}_{\mathrm{pf}}(\varphi) = \mathcal{E}_{\mathrm{pf},\kappa}(\varphi) - \frac{\kappa}{2\varepsilon}\|\varphi\|_{L^2}^2. \tag{2.22}$$

Therefore, after this reformulation, system (1.1) can be written in the following formally equivalent form:

$$\begin{aligned}
\partial_t(\rho v) + \mathrm{div}(v \otimes (\rho v + J)) - \mathrm{div}(\mathbb{T}) &= f + \mu\nabla\varphi && \text{in } \Omega\times(0,T), && (2.23a)\\
\mathrm{div}(v) &= 0 && \text{in } \Omega\times(0,T), && (2.23b)\\
\mathbb{T} &= \eta(\varphi) + 2\nu(\varphi)(\nabla v)_{\mathrm{sym}} - g\mathbb{I} && \text{in } \Omega\times(0,T), && (2.23c)\\
\overset{\triangledown}{S} + \partial P(\varphi;S) - \gamma\Delta S &\ni \eta(\varphi)(\nabla v)_{\mathrm{sym}} && \text{in } \Omega\times(0,T), && (2.23d)\\
\partial_t\varphi + v\cdot\nabla\varphi &= \Delta\mu && \text{in } \Omega\times(0,T), && (2.23e)\\
\mu + \kappa\varepsilon^{-1}\varphi &= -\varepsilon\Delta\varphi + \varepsilon^{-1}W_\kappa'(\varphi) && \text{in } \Omega\times(0,T), && (2.23f)\\
v|_{\partial\Omega} &= 0 && \text{on } \partial\Omega\times(0,T), && (2.23g)\\
\gamma\vec{n}\cdot\nabla S|_{\partial\Omega} &= 0 && \text{on } \partial\Omega\times(0,T), && (2.23h)\\
\vec{n}\cdot\nabla\varphi|_{\partial\Omega} = \vec{n}\cdot\nabla\mu|_{\partial\Omega} &= 0 && \text{on } \partial\Omega\times(0,T), && (2.23i)\\
(v,S,\varphi)|_{t=0} &= (v_0,S_0,\varphi_0) && \text{in } \Omega. && (2.23j)
\end{aligned}$$

2.4 Weak solutions of system (1.1) with respect to (2.23) for $\gamma > 0$

Observe that the partial subdifferential $\partial\mathcal{P}(\varphi;S)$ may be multi-valued for non-smooth potentials $\mathcal{P}$. By using the definition of the partial subdifferential, we can introduce the variational inequality

$$\langle\xi, \tilde{S} - S\rangle_{L^2_{\mathrm{sym,Tr}}(\Omega)} \le \mathcal{P}(\varphi;\tilde{S}) - \mathcal{P}(\varphi;S), \tag{2.24}$$

that is equivalent to all $\xi \in \partial\mathcal{P}(\varphi; S)$.

Accordingly, a weak solution of system (2.23) is defined by a weak formulation and an energy estimate of equation (2.23a)-(2.23c), an evolutionary variational inequality of equation (2.23d), and a weak formulation and an energy estimate of equation (2.23e)-(2.23f).

Definition 2.7 (Weak solutions for system (2.23))**.** Let $\gamma > 0$. Let the assumptions 2.1-2.4 hold true. A quadruplet (v, S, φ, μ) is called a weak solution of the two-phase system (2.23) if the following properties are satisfied:
1. The quadruplet (v, S, φ, μ) has the following regularity:

$$\begin{aligned}
&v \in L^\infty_{\mathrm{loc}}([0,T); L^2_{\mathrm{div}}(\Omega)) \cap L^2_{\mathrm{loc}}([0,T); H^1_{0,\mathrm{div}}(\Omega)),\\
&S \in L^\infty_{\mathrm{loc}}([0,T); L^2_{\mathrm{sym,Tr}}(\Omega)) \cap L^2_{\mathrm{loc}}([0,T); H^1_{\mathrm{sym,Tr}}(\Omega)),\\
&\varphi \in L^\infty_{\mathrm{loc}}([0,T); H^1(\Omega)) \cap L^2_{\mathrm{loc}}([0,T); H^2(\Omega)), W'(\varphi) \in L^2_{\mathrm{loc}}([0,T); L^2(\Omega)),\\
&\mu \in L^2_{\mathrm{loc}}([0,T); H^1(\Omega))
\end{aligned}$$

2. The quadruplet (v, S, φ, μ) satisfies the following weak formulations and energy estimates:
2.1. Weak formulation of the momentum balance:

$$\begin{aligned}
&\int_0^T \int_\Omega -\rho v \cdot \partial_t \Phi - (\rho v \otimes v) : \nabla\Phi + 2\nu(\varphi)(\nabla v)_{\mathsf{sym}} : (\nabla\Phi)_{\mathsf{sym}} + \eta(\varphi) S : (\nabla\Phi)_{\mathsf{sym}} \,\mathrm{d}x\,\mathrm{d}t\\
-&\int_0^T \int_\Omega (v \otimes J) : \nabla\Phi + \mu\nabla\varphi \cdot \Phi \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \rho_0 v_0 \cdot \Phi(\cdot, 0)\,\mathrm{d}x + \int_0^T \langle f, \Phi\rangle_{H^1}\,\mathrm{d}t
\end{aligned} \tag{2.25a}$$

for all $\Phi \in C^\infty_{0,\mathrm{div}}(\Omega \times [0,T))$, and the partial energy inequality

$$\begin{aligned}
&\mathcal{E}_{\mathrm{kin}}(\varphi(t), v(t)) + \int_0^t \int_\Omega 2\nu(\varphi)|(\nabla v)_{\mathsf{sym}}|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \eta(\varphi) S : \nabla v \,\mathrm{d}x\,\mathrm{d}s\\
\le& \mathcal{E}_{\mathrm{kin}}(\varphi_0, v_0) + \int_0^t \langle f, v\rangle_{H^1}\,\mathrm{d}s + \int_0^t \int_\Omega \mu(\nabla\varphi \cdot v)\,\mathrm{d}x\,\mathrm{d}s\,,
\end{aligned} \tag{2.25b}$$

for almost all $t \in (0,T)$.
2.2. The evolutionary variational inequality for the stress

$$\begin{aligned}
&\frac{1}{2}\|S(t) - \tilde{S}(t)\|^2_{L^2} + \int_0^t \int_\Omega \partial_t \tilde{S} : (S - \tilde{S}) - v \cdot \nabla S : \tilde{S} - (S(\nabla v)_{\mathsf{skw}} - (\nabla v)_{\mathsf{skw}} S) : \tilde{S}\,\mathrm{d}x\,\mathrm{d}s\\
+&\int_0^t \mathcal{P}(\varphi; S) - \mathcal{P}(\varphi; \tilde{S})\,\mathrm{d}s + \int_0^t \int_\Omega \gamma\nabla S \,\vdots\, \nabla(S - \tilde{S}) - \eta(\varphi)(\nabla v)_{\mathsf{sym}} : (S - \tilde{S})\,\mathrm{d}x\,\mathrm{d}s\\
\le&\frac{1}{2}\|S_0 - \tilde{S}(0)\|^2_{L^2}
\end{aligned} \tag{2.25c}$$

for all $\tilde{S} \in C^\infty_{0,\mathrm{sym,Tr}}(\Omega \times [0,T)) \cap \mathrm{dom}(\mathcal{P}(\varphi; \cdot))$, and a.e. $t \in (0,T)$.
2.3. Weak formulation of the phase-field evolution law:

$$\int_0^T \int_\Omega -\varphi \cdot \partial_t \zeta + (v \cdot \nabla\varphi)\zeta \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \varphi_0 \cdot \zeta(\cdot, 0)\,\mathrm{d}x - \int_0^T \int_\Omega \nabla\mu : \nabla\zeta \,\mathrm{d}x\,\mathrm{d}t \tag{2.25d}$$

for all $\zeta \in C^\infty_0([0,T); C^1(\bar{\Omega}))$ as well as

$$\mu = \varepsilon^{-1} W'(\varphi) - \varepsilon\Delta\varphi \tag{2.25e}$$

almost everywhere in $\Omega \times (0,T)$, and the partial energy inequality

$$\mathcal{E}_{\rm pf}(\varphi(t)) + \int_0^t \int_\Omega |\nabla \mu|^2 \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega \mu(\nabla\varphi \cdot v) \, \mathrm{d}x \, \mathrm{d}s \le \mathcal{E}_{\rm pf}(\varphi_0), \tag{2.25f}$$

for almost all $t \in (0,T)$.

Remark 2.8. Notice that the evolutionary variational inequality (2.25c), is defined by testing (2.23d) with $S - \tilde{S}$, integrating over $\Omega \times (0,t)$, using (2.24) together with an integration by parts in space and time.

Further notice that, by choosing $\tilde{S} \equiv 0$ in (2.25c) and summing with (2.25b) and (2.25f), a weak solution (v, S, φ, μ) satisfies the following total energy-dissipation estimate:

$$\mathcal{E}_{\rm tot}(v(t), S(t), \varphi(t)) + \int_0^t \mathcal{D}_{\rm tot}(v, S, \varphi, \mu) \, \mathrm{d}\tau \le \mathcal{E}_{\rm tot}(v_0, S_0, \varphi_0) + \int_0^t \langle f, v\rangle_{H^1} \, \mathrm{d}\tau \tag{2.26}$$

for almost all $t \in [0,T)$.

3 Dissipative solutions

In this section, we introduce the notion of dissipative solutions for $\gamma \ge 0$ and investigate their relation to the weak solutions from Definition 2.7 in the case $\gamma > 0$.

3.1 General concept of dissipative solutions

The notion of dissipative solutions is based on a relative energy inequality. In order to better explain the concept, we follow the general approach proposed in [5]. For this, we consider two reflexive Banach spaces $\mathbb{V}$ and $\mathbb{Y}$ with dual spaces $\mathbb{V}'$ and $\mathbb{Y}'$ such that $\mathbb{Y} \subseteq \mathbb{V} \subseteq \mathbb{V}' \subseteq \mathbb{Y}'$ and a general evolutionary PDE on time interval $(0,T)$ of the form

$$\partial_t \boldsymbol{U}(t) + A(\boldsymbol{U}(t)) = \boldsymbol{0} \quad \text{in } \mathbb{Y}' \text{ with} \quad \boldsymbol{U}(0) = \boldsymbol{U}_0 \quad \text{in } \mathbb{V}. \tag{3.1a}$$

Here, $A : \mathbb{V} \to \mathbb{Y}'$ denotes a differential operator and $\boldsymbol{U}_0 \in \mathbb{V}$ the initial datum. Let $\mathcal{E} : \mathbb{V} \to [0,\infty]$ and $\Psi : \mathbb{Y} \to [0,\infty]$ be energy functional which is assumed to be twice Gâteaux differentiable and dissipation functional associated with (3.1a). Furthermore, we introduce a space of test functions $\mathfrak{T}$ such that $\mathrm{D}\mathcal{E}(\tilde{\boldsymbol{U}}(t)), \mathcal{A}(\tilde{\boldsymbol{U}}(t)) \in \mathbb{Y}$ for a.e. $t \in (0,T)$ and all $\tilde{\boldsymbol{U}} \in \mathfrak{T}$. For any initial value $\boldsymbol{U}_0 \in \mathbb{V}$, a sufficiently regular solution $\boldsymbol{U} \in \mathbb{V}$ with the property $\mathrm{D}\mathcal{E}(\boldsymbol{U}(t)) \in \mathbb{Y}$ for all $t \in [0,T]$ of (3.1a) formally fulfills the energy-dissipation mechanism

$$\mathcal{E}(\boldsymbol{U})\Big|_0^t + \int_0^t \Psi(\boldsymbol{U}) \, \mathrm{d}\tau \le 0 \tag{3.1b}$$

for all $t \in [0,T]$. This is obtained by testing (3.1a) by $\mathrm{D}\mathcal{E}(\boldsymbol{U}(t)) \in \mathbb{Y}$, and it also implies that

$$\Psi(\boldsymbol{U}) := \langle A(\boldsymbol{U}), \mathrm{D}\mathcal{E}(\boldsymbol{U})\rangle_{\mathbb{Y}}. \tag{3.1c}$$

Moreover, we introduce a so-called regularity weight

$$\mathcal{K} : \mathfrak{T} \to [0,\infty] \text{ with } \mathcal{K}(0) = 0, \tag{3.1d}$$

which is to be chosen such that both sides of the relative energy inequality remain finite.

In addition, we define the system operator $\mathcal{A}$ directly following (3.1a) as

$$\mathcal{A} : \mathfrak{T} \to \mathbb{Y}', \, \mathcal{A}(\tilde{\boldsymbol{U}}) := \partial_t \tilde{\boldsymbol{U}} + A(\tilde{\boldsymbol{U}}). \tag{3.1e}$$

We refer to [5] for the full list of assumptions. A major assumption in [5] is that the energy $\mathcal{E}$ is convex, which gives the positivity of the first-order Taylor expansion of the energy functional. Following the assertions in [5, Prop. 3.6], we define the relative energy $\mathcal{R}$ and the relative dissipation $\mathcal{W}^{(\mathcal{K})}$ as the first-order Taylor expansion of the energy $\mathcal{E}$ and the term $\Psi(\boldsymbol{U}) - \langle A(\boldsymbol{U}), \mathrm{D}\mathcal{E}(\tilde{\boldsymbol{U}})\rangle_{\mathbb{Y}} + \mathcal{K}(\tilde{\boldsymbol{U}})\mathcal{E}(\boldsymbol{U})$ given as

$$\mathcal{R}(\boldsymbol{U}|\tilde{\boldsymbol{U}}) := \mathcal{E}(\boldsymbol{U}) - \mathcal{E}(\tilde{\boldsymbol{U}}) - \langle \mathrm{D}\mathcal{E}(\tilde{\boldsymbol{U}}), \boldsymbol{U} - \tilde{\boldsymbol{U}}\rangle_{\mathbb{V}} \tag{3.1f}$$

as well as

$$\begin{aligned} \mathcal{W}^{(\mathcal{K})}(\boldsymbol{U}|\tilde{\boldsymbol{U}}) := \Psi(\boldsymbol{U}) - \langle A(\boldsymbol{U}), \mathrm{D}\mathcal{E}(\tilde{\boldsymbol{U}})\rangle_{\mathbb{Y}} - \langle A(\tilde{\boldsymbol{U}}), \mathrm{D}^2\mathcal{E}(\tilde{\boldsymbol{U}})(\boldsymbol{U} - \tilde{\boldsymbol{U}})\rangle_{\mathbb{Y}} \\ + \mathcal{K}(\tilde{\boldsymbol{U}})\mathcal{R}(\boldsymbol{U}|\tilde{\boldsymbol{U}}) \, , \end{aligned} \tag{3.1g}$$

where we use (3.1c). Therefore, we can formally compute

$$\begin{aligned} \mathcal{R}(\boldsymbol{U}|\tilde{\boldsymbol{U}})\Big|_0^t + \int_0^t \Psi(\boldsymbol{U}) \,\mathrm{d}\tau &\leq - \int_0^t \langle \partial_t \boldsymbol{U}, \mathrm{D}\mathcal{E}(\tilde{\boldsymbol{U}})\rangle_{\mathbb{Y}} + \langle \boldsymbol{U} - \tilde{\boldsymbol{U}}, \mathrm{D}^2\mathcal{E}(\tilde{\boldsymbol{U}})\partial_t\tilde{\boldsymbol{U}}\rangle_{\mathbb{Y}} \,\mathrm{d}\tau \\ &= \int_0^t \langle A(\boldsymbol{U}), \mathrm{D}\mathcal{E}(\tilde{\boldsymbol{U}})\rangle_{\mathbb{Y}} + \langle \boldsymbol{U} - \tilde{\boldsymbol{U}}, \mathrm{D}^2\mathcal{E}(\tilde{\boldsymbol{U}})A(\tilde{\boldsymbol{U}})\rangle_{\mathbb{Y}} \,\mathrm{d}\tau \\ &\quad - \int_0^t \langle \mathrm{D}^2\mathcal{E}(\tilde{\boldsymbol{U}})(\boldsymbol{U} - \tilde{\boldsymbol{U}}), \mathcal{A}(\tilde{\boldsymbol{U}})\rangle_{\mathbb{Y}} \,\mathrm{d}\tau \end{aligned}$$

By rearranging terms on both sides and using (3.1e) and (3.1g), we infer

$$\mathcal{R}(\boldsymbol{U}|\tilde{\boldsymbol{U}})\Big|_0^t + \int_0^t \mathcal{W}^{(\mathcal{K})}(\boldsymbol{U}|\tilde{\boldsymbol{U}}) + \langle \mathrm{D}^2\mathcal{E}(\tilde{\boldsymbol{U}})(\boldsymbol{U} - \tilde{\boldsymbol{U}}), \mathcal{A}(\tilde{\boldsymbol{U}})\rangle_{\mathbb{Y}} \,\mathrm{d}\tau \leq \int_0^t \mathcal{K}(\tilde{\boldsymbol{U}})\mathcal{R}(\boldsymbol{U}|\tilde{\boldsymbol{U}}) \,\mathrm{d}\tau \, . \tag{3.1h}$$

Applying Gronwall's inequality provides the relative energy inequality in this general setting as

$$\begin{aligned} \mathcal{R}(\boldsymbol{U}(t)|\tilde{\boldsymbol{U}}(t)) + \int_0^t \exp\Big(\int_s^t \mathcal{K}(\tilde{\boldsymbol{U}}) \,\mathrm{d}\tau\Big) \Big[\mathcal{W}^{(\mathcal{K})}(\boldsymbol{U}|\tilde{\boldsymbol{U}}) \\ + \langle \mathrm{D}^2\mathcal{E}(\tilde{\boldsymbol{U}})(\boldsymbol{U} - \tilde{\boldsymbol{U}}), \mathcal{A}(\tilde{\boldsymbol{U}})\rangle_{\mathbb{Y}}\Big] \,\mathrm{d}s \leq \mathcal{R}(\boldsymbol{U}_0|\tilde{\boldsymbol{U}}(0))\exp\Big(\int_0^t \mathcal{K}(\tilde{\boldsymbol{U}}) \,\mathrm{d}s\Big) \end{aligned} \tag{3.1i}$$

for a.e. $t \in (0, T)$ and smooth test functions $\tilde{\boldsymbol{U}} \in \mathfrak{T}$.

Definition 3.1 (Dissipative solutions for general system (3.1a))**.** A function $\boldsymbol{U} : (0, T) \to \mathbb{V}$ is called a dissipative solution for system (3.1a), if $\boldsymbol{U}$ satisfies (3.1i) for all sufficiently regular test functions $\tilde{\boldsymbol{U}} \in \mathfrak{T}$ and for a.e. $t \in (0, T)$.

3.2 Dissipative solutions for system (2.23)

A key ingredient for the notion of dissipative solutions in [5] is the convexity of the system's energy $\mathcal{E}$. This means for our system (2.23) that the total energy should be convex. But note that the kinetic energy $\mathcal{E}_{\mathrm{kin}}$, from (2.10) as a function of (φ, v) is not convex. In order to overcome the non-convexity in the kinetic energy $\mathcal{E}_{\rm kin}$, we use transformation of variables. Expressing the energy in terms of the mass density ρ and the momentum p with $p = \rho v$ instead of mass density ρ and velocity v makes the kinetic energy convex, i.e.,

$$\tilde{\mathcal{E}}_{\rm kin}(\rho, p) := \int_\Omega \tilde{\mathcal{E}}_{\rm kin}(\rho, p)\mathrm{d}x \text{ with } \tilde{\mathcal{E}}_{\rm kin}(\rho, p) := \begin{cases} \frac{p^2}{2\rho} & \text{if } \rho > 0, \\ 0 & \text{if } \rho = 0 \text{ and } p = 0, \\ \infty & \text{otherwise.} \end{cases} \tag{3.2}$$

Accordingly, we also denote the the total energy that depends on the momentum as a variable by $\tilde{\mathcal{E}}_{\rm tot,\kappa}$. Using the general definitions of the relative energy (3.1f), we find the following expression for the relative kinetic energy

$$\begin{aligned} \tilde{\mathcal{R}}_{\rm kin}(\rho, p|\tilde\rho, \tilde p) &= \frac{1}{2}\int_\Omega \frac{p^2}{\rho} - \frac{\tilde p^2}{\tilde \rho} - \frac{2\tilde p}{\tilde\rho}(p - \tilde p) + \frac{\tilde p^2}{\tilde\rho^2}(\rho - \tilde\rho)\,\mathrm{d}x = \frac{1}{2}\int_\Omega \rho\left|\frac{p}{\rho} - \frac{\tilde p}{\tilde \rho}\right|^2 \mathrm{d}x \\ &= \int_\Omega \rho\frac{|v - \tilde v|^2}{2}\,\mathrm{d}x =: \mathcal{R}_{\rm kin}(\varphi; v|\tilde v) \end{aligned} \tag{3.3}$$

with $p = \rho v$ and $\tilde p = \tilde\rho\tilde v$. One can see that

$$\mathcal{R}_{\rm kin}(\varphi; v|\tilde v) = \mathcal{E}_{\rm kin}(\varphi, v - \tilde v).$$

Moreover, since the elastic energy is convex, in particular quadratic, we define the relative elastic energy as:

$$\mathcal{R}_{\rm el}(S|\tilde S) := \int_\Omega \frac{|S - \tilde S|^2}{2}\,\mathrm{d}x = \mathcal{E}_{\rm el}(S - \tilde S). \tag{3.4}$$

In addition, by (2.9), the singular potential W is κ-convex, i.e., $W_\kappa(\varphi) = W(\varphi) + \frac{\kappa}{2}|\varphi|^2$ is convex. The gradient part of the phase-field energy is convex, in particular quadratic. According to (2.19), we define the relative phase-field energy as

$$\begin{aligned} \mathcal{R}_{\rm pf,\kappa}(\varphi|\tilde\varphi) &:= \mathcal{E}_{\rm pf,\kappa}(\varphi) - \mathcal{E}_{\rm pf,\kappa}(\tilde\varphi) - \mathrm{D}\mathcal{E}_{\rm pf,\kappa}(\tilde\varphi)(\varphi - \tilde\varphi) \\ &= \varepsilon\int_\Omega \frac{|\nabla\varphi - \nabla\tilde\varphi|^2}{2}\,\mathrm{d}x + \varepsilon^{-1}\int_\Omega W_\kappa(\varphi) - W_\kappa(\tilde\varphi) - W_\kappa'(\tilde\varphi)(\varphi - \tilde\varphi)\,\mathrm{d}x \end{aligned} \tag{3.5}$$

for all $\tilde\varphi \in \mathrm{dom}(\mathrm{D}\mathcal{E}_{\rm pf,\kappa})$.

Altogether, we define the relative total energy as:

$$\mathcal{R}\left(v, S, \varphi\middle|\tilde v, \tilde S, \tilde\varphi\right) := \mathcal{R}_{\rm kin}(\varphi, v|\tilde v) + \mathcal{R}_{\rm el}(S|\tilde S) + \mathcal{R}_{\rm pf,\kappa}(\varphi|\tilde\varphi). \tag{3.6}$$

We introduce the space of admissible test functions as

$$\begin{aligned} \mathfrak{T} := \{(v, S, \varphi) : v \in C^\infty_{0,\rm div}(\Omega \times [0,T)), S \in C^\infty_{0,\rm sym,Tr}(\Omega \times [0,T)) \cap \mathrm{dom}(\mathcal{P}(\varphi;\cdot)), \\ \varphi \in C_0^\infty(\Omega \times [0,T)), |\varphi| < 1\}. \end{aligned} \tag{3.7}$$

Notice that, in our setting, the spaces

$$\begin{aligned} \mathbb{V} &:= L^2_{\rm div}(\Omega) \times L^2_{\rm sym,Tr}(\Omega) \times L^2(\Omega), \\ \mathbb{Y} &:= H^1_{0,\rm div}(\Omega) \times H^1_{\rm sym,Tr}(\Omega) \times H^1(\Omega). \end{aligned} \tag{3.8}$$

Now, the derivation of the relative energy inequality for system (2.23) mostly follows the general approach of (3.1i). However, in the definition (3.1e) of the system operator $\mathcal{A}_\gamma$ and (3.1g) of the relative dissipation $\mathcal{W}_\gamma^{(\mathcal{K})}$, we have to slightly deviate from the general form for the following reasons: The first difference arises from the non-smoothness of the plastic potential $\mathcal{P}$, which compels us to treat the multi-valued subdifferential $\partial\mathcal{P}(\varphi; S)$ separately, instead of directly including it in the system operator $\mathcal{A}_\gamma$ and in the relative dissipation $\mathcal{W}_\gamma^{(\mathcal{K})}$. More precisely, for this term we shall keep the variational inequality and add it to the relative energy inequality, see (3.16b) below. Another difference stems from the dependence of ρ on φ, which requires to use the momentum p as a variable in the kinetic energy for convexity reasons, cf. (3.2), and for compensation, additional terms are created in the relative dissipation, see (3.10).

In the following, we introduce $\mathcal{A}_\gamma$ and $\mathcal{W}_\gamma^{(\mathcal{K})}$ for system (2.23) and discuss the difference to (3.1g) in more detail in Remark 3.2.

System operator $\mathcal{A}_\gamma$. Due to the non-smoothness of the dissipation potential $\mathcal{P}$, the multi-valued subdifferential $\partial\mathcal{P}(\varphi; S)$ will not be included in the system operator $\mathcal{A}_\gamma$. For the remaining terms we follow (3.1e) and define the system operator

$$\mathcal{A}_\gamma = \left(\mathcal{A}^{(1)}, \mathcal{A}_\gamma^{(2)}, \mathcal{A}^{(3)}\right)^\top : \mathfrak{T} \to (H^1_{0,\mathrm{div}}(\Omega))' \times (H^1_{\mathrm{sym,Tr}}(\Omega))' \times (L^2(\Omega))' \tag{3.9a}$$

from (2.23) as follows:

$$\begin{aligned}\langle \mathcal{A}^{(1)}(\tilde{v}, \tilde{S}, \tilde{\varphi}), \Phi\rangle_{H^1_{0,\mathrm{div}}} :=& \int_\Omega \partial_t(\tilde{\rho}\tilde{v}) \cdot \Phi - \tilde{\rho}\tilde{v} \otimes \tilde{v} : \nabla\Phi - \tilde{v} \otimes \tilde{J} : \nabla\Phi + \eta(\tilde{\varphi})\tilde{S} : \nabla\Phi \,\mathrm{d}x \\ &+ \int_\Omega 2\nu(\tilde{\varphi})(\nabla\tilde{v})_{\mathsf{sym}} : \nabla\Phi - \tilde{\mu}\nabla\tilde{\varphi} \cdot \Phi \,\mathrm{d}x - \langle f, \Phi\rangle_{H^1}\end{aligned} \tag{3.9b}$$

for all $\Phi \in H^1_{0,\mathrm{div}}(\Omega)$;

$$\begin{aligned}\langle \mathcal{A}_\gamma^{(2)}(\tilde{v}, \tilde{S}, \tilde{\varphi}), \Psi\rangle_{H^1_{\mathrm{sym,Tr}}} :=& \int_\Omega \partial_t\tilde{S} : \Psi + \tilde{v} \cdot \nabla\tilde{S} : \Psi + \left(\tilde{S}(\nabla\tilde{v})_{\mathsf{skw}} - (\nabla\tilde{v})_{\mathsf{skw}}\tilde{S}\right) : \Psi \,\mathrm{d}x \\ &+ \int_\Omega \gamma\nabla\tilde{S} \,\vdots\, \nabla\Psi - \eta(\tilde{\varphi})(\nabla\tilde{v})_{\mathsf{sym}} : \Psi \,\mathrm{d}x\end{aligned} \tag{3.9c}$$

for all $\Psi \in H^1_{\mathrm{sym,Tr}}(\Omega)$;

$$\langle \mathcal{A}^{(3)}(\tilde{v}, \tilde{\varphi}), \zeta\rangle_{H^1} := \int_\Omega \partial_t\tilde{\varphi}\zeta + \tilde{v} \cdot \nabla\tilde{\varphi}\zeta - \Delta\tilde{\mu}\zeta \,\mathrm{d}x\,, \tag{3.9d}$$

for all $\zeta \in H^1(\Omega)$, and where

$$\tilde{\mu} := -\varepsilon\Delta\tilde{\varphi} + \varepsilon^{-1}W'(\tilde{\varphi}). \tag{3.9e}$$

Relative dissipation. Due to the non-smoothness of the plastic potential, we also do not include the multi-valued subdifferential $\partial\mathcal{P}(\varphi; S)$ into the relative dissipation. For the remaining terms and for a given regularity weight $\mathcal{K}$ as in (3.1d), we now introduce the relative dissipation following (3.1g). However, as already mentioned, in order to ensure the convexity of the total energy, hence the positive definiteness of its second derivative, we have to use $\tilde{\mathcal{E}}_{\mathrm{tot},\kappa}$, cf. (3.2) and below. To compensate this change of variables, two additional terms are created in the definition of the relative dissipation:

$$
\begin{aligned}
\mathcal{W}_\gamma^{(\mathcal{K})}\left(v,S,\varphi|\tilde v,\tilde S,\tilde\varphi\right) :=&\mathcal{D}_{\mathrm{chs}}(v,\varphi,\mu)+\mathcal{D}_{\mathrm{sd},\gamma}(S)-\langle A_\gamma(v,S,\varphi,\mu),\mathrm{D}\tilde{\mathcal{E}}_{\mathrm{tot},\kappa}(\tilde p,\tilde S,\tilde\varphi)\rangle_{\mathbb{Y}}\\
&-\left\langle \mathcal{A}_\gamma(\tilde v,\tilde S,\tilde\varphi),\mathrm{D}^2\tilde{\mathcal{E}}_{\mathrm{tot},\kappa}(\tilde p,\tilde S,\tilde\varphi)\begin{pmatrix}p-\tilde p\\ S-\tilde S\\ \varphi-\tilde\varphi\end{pmatrix}\right\rangle_{\mathbb{Y}}\\
&+\mathcal{K}(\tilde v,\tilde S,\tilde\varphi)\mathcal{R}(v,S,\varphi|\tilde v,\tilde S,\tilde\varphi)\\
&+\left\langle \mathcal{A}_\gamma(\tilde v,\tilde S,\tilde\varphi),\begin{pmatrix}\frac{\rho-\tilde\rho}{\tilde\rho}(v-\tilde v)\\ 0\\ -\frac{\rho_2-\rho_1}{2}\tilde v\frac{\rho-\tilde\rho}{\tilde\rho}(v-\tilde v)\end{pmatrix}\right\rangle_{\mathbb{Y}}\\
&+\int_\Omega(\rho-\tilde\rho)(v-\tilde v)\cdot\partial_t\tilde v\,\mathrm{d}x\ ,
\end{aligned}
\tag{3.10}
$$

where $\mathcal{A}_\gamma$ is defined as in (3.9) and, using (2.10), we set

$$
\left\langle A_\gamma(v,S,\varphi),\mathrm{D}\tilde{\mathcal{E}}_{\mathrm{tot},\kappa}(\tilde p,\tilde S,\tilde\varphi)\right\rangle_{\mathbb{Y}} := \left\langle \begin{pmatrix}A^{(1)}(v,S,\varphi)\\ A_\gamma^{(2)}(v,S,\varphi)\\ A^{(3)}(v,\varphi)\end{pmatrix},\begin{pmatrix}\tilde v\\ \tilde S\\ \tilde\mu+\frac{\kappa}{\varepsilon}\tilde\varphi-\frac{\tilde v^2}{2}\cdot\frac{\rho_2-\rho_1}{2}\end{pmatrix}\right\rangle_{\mathbb{Y}}.
$$

In particular,

$$
\begin{aligned}
\langle A^{(1)}(v,S,\varphi),\tilde v\rangle_{H^1_{0,\mathrm{div}}} :=&\int_\Omega -\rho v\otimes v:\nabla\tilde v-v\otimes J:\nabla\tilde v+\eta(\varphi)S:\nabla\tilde v\,\mathrm{d}x\\
&+\int_\Omega 2\nu(\varphi)(\nabla v)_{\mathsf{sym}}:\nabla\tilde v-\mu\nabla\varphi\cdot\tilde v\,\mathrm{d}x-\langle f,\tilde v\rangle_{H^1}\ ,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
\langle A_\gamma^{(2)}(v,S,\varphi),\tilde S\rangle_{H^1_{\mathrm{sym,Tr}}} :=&\int_\Omega -S\otimes v\,\vdots\,\nabla\tilde S+(S(\nabla v)_{\mathsf{skw}}-(\nabla v)_{\mathsf{skw}}S):\tilde S\,\mathrm{d}x\\
&+\int_\Omega\gamma\nabla S\,\vdots\,\nabla\tilde S-\eta(\varphi)(\nabla v)_{\mathsf{sym}}:\tilde S\,\mathrm{d}x\\
=&\int_\Omega -S\otimes v\,\vdots\,\nabla\tilde S+2S(\nabla v)_{\mathsf{skw}}:\tilde S+\gamma\nabla S\,\vdots\,\nabla\tilde S-\eta(\varphi)(\nabla v)_{\mathsf{sym}}:\tilde S\,\mathrm{d}x\,,
\end{aligned}
\tag{3.12}
$$

where we have used the fact that $\tilde S$ is symmetric, hence

$$
(S(\nabla v)_{\mathsf{skw}}-(\nabla v)_{\mathsf{skw}}S):\tilde S=2S(\nabla v)_{\mathsf{skw}}:\tilde S,
$$

as well as

$$
\begin{aligned}
\langle A^{(3)}(v,\varphi),\tilde\mu+\frac{\kappa}{\varepsilon}\tilde\varphi-\frac{\rho_2-\rho_1}{2}\frac{\tilde v^2}{2}\rangle_{H^1} :=&\int_\Omega v\cdot\nabla\varphi(\tilde\mu+\frac{\kappa}{\varepsilon}\tilde\varphi-\frac{\rho_2-\rho_1}{2}\frac{\tilde v^2}{2})\,\mathrm{d}x\\
&+\int_\Omega\nabla\mu:\nabla(\tilde\mu+\frac{\kappa}{\varepsilon}\tilde\varphi-\frac{\rho_2-\rho_1}{2}\frac{\tilde v^2}{2})\,\mathrm{d}x\ .
\end{aligned}
\tag{3.13}
$$

By collecting all the terms from (3.10)–(3.13), using the definition of the dissipation potentials from (2.11), by adding and subtracting additional terms in order to create differences of the solution and the test function, and by applying integration by parts, we arrive at

$$
\begin{aligned}
\mathcal{W}_\gamma^{(\mathcal{K})}\left(v,S,\varphi|\tilde v,\tilde S,\tilde\varphi\right) :=& \int_\Omega \gamma\left|\nabla S-\nabla\tilde S\right|^2 \mathrm{d}x + \int_\Omega \frac12|\nabla\mu-\nabla\tilde\mu|^2\,\mathrm{d}x \\
&+\int_\Omega 2\nu(\varphi)|(\nabla v)_{\text{sym}}-(\nabla\tilde v)_{\text{sym}}|^2 + 2(\nu(\varphi)-\nu(\tilde\varphi))(\nabla\tilde v)_{\text{sym}} : (\nabla v-\nabla\tilde v)\,\mathrm{d}x \\
&+\int_\Omega \frac12|\nabla\mu|^2-\frac12|\nabla\tilde\mu|^2+\Delta\tilde\mu\left(-\Delta(\varphi-\tilde\varphi)+W''(\tilde\varphi)(\varphi-\tilde\varphi)\right)\mathrm{d}x \\
&+\int_\Omega (v-\tilde v)\otimes(\rho v-\tilde\rho\tilde v+J-\tilde J):\nabla\tilde v+(\rho-\tilde\rho)(v-\tilde v)\cdot\partial_t\tilde v\,\mathrm{d}x \\
&-\int_\Omega (\eta(\varphi)-\eta(\tilde\varphi))(S-\tilde S):\nabla\tilde v+(\eta(\varphi)-\eta(\tilde\varphi))\tilde S:(\nabla v-\nabla\tilde v)\,\mathrm{d}x \\
&-\int_\Omega (S-\tilde S)\otimes(v-\tilde v) \vdots \nabla\tilde S+2(S-\tilde S)(\nabla v-\nabla\tilde v)_{\text{skw}}:\tilde S\,\mathrm{d}x \\
&-\int_\Omega \tilde\mu(\nabla\varphi-\nabla\tilde\varphi)\cdot(v-\tilde v)-\varepsilon(\nabla\varphi-\nabla\tilde\varphi)\otimes(\nabla\varphi-\nabla\tilde\varphi):\nabla\tilde v\,\mathrm{d}x \\
&+\int_\Omega \frac{\kappa}{\varepsilon}(\nabla\mu-\nabla\tilde\mu)\cdot(\nabla\varphi-\nabla\tilde\varphi)+\frac{\kappa}{\varepsilon}(v-\tilde v)\cdot\nabla\tilde\varphi(\varphi-\tilde\varphi)\,\mathrm{d}x \\
&+\mathcal{K}(\tilde v,\tilde S,\tilde\varphi)\mathcal{R}(v,S,\varphi|\tilde v,\tilde S,\tilde\varphi) \\
=&\mathcal{D}_{\mathrm{sd},\gamma}(S-\tilde S)+\mathcal{W}_0^{(\mathcal{K})}(v,S,\varphi|\tilde v,\tilde S,\tilde\varphi).
\end{aligned} \tag{3.14}
$$

Remark 3.2. In order to see the connection of (3.10) and (3.1g), in particular, how the two additional terms arise in (3.10), we calculate the second derivative of the adapted energy $\mathcal{E}_{\text{tot},\kappa}(p,S,\varphi) := \mathcal{E}_{\text{kin}}(\varphi,p)+\mathcal{E}_{\text{el}}(S)+\mathcal{E}_{\text{pf},\kappa}(\varphi)$ such that we observe that

$$
\begin{aligned}
\mathrm{D}^2\mathcal{E}_{\text{tot},\kappa}(\tilde p,\tilde S,\tilde\varphi)\begin{pmatrix}p-\tilde p\\ S-\tilde S\\ \varphi-\tilde\varphi\end{pmatrix} &= \begin{pmatrix}\frac{1}{\tilde\rho}I & 0 & -\frac{\rho_2-\rho_1}{2\tilde\rho^2}\tilde p\\ 0 & I & 0\\ -\frac{\rho_2-\rho_1}{2\tilde\rho^2}\tilde p & 0 & -\Delta+W''(\tilde\varphi)+\frac{\kappa}{\varepsilon}+\frac{|\tilde p|^2}{\tilde\rho^3}\rho'(\tilde\varphi)^2\end{pmatrix}\begin{pmatrix}p-\tilde p\\ S-\tilde S\\ \varphi-\tilde\varphi\end{pmatrix} \\
&= \begin{pmatrix}\frac{\rho}{\tilde\rho}(v-\tilde v)\\ S-\tilde S\\ (-\Delta+W''(\tilde\varphi)+\frac{\kappa}{\varepsilon})(\varphi-\tilde\varphi)-\frac{\rho_2-\rho_1}{2}\frac{\rho}{\tilde\rho}\tilde v\cdot(v-\tilde v)\end{pmatrix}.
\end{aligned}
$$

By following the abstract approach of (3.1i) and by adding and subtracting the term

$$
\left\langle \mathcal{A}_\gamma(\tilde v,\tilde S,\tilde\varphi), \begin{pmatrix}\frac{\rho-\tilde\rho}{\tilde\rho}(v-\tilde v)\\ 0\\ -\frac{\rho_2-\rho_1}{2}\tilde v\frac{\rho-\tilde\rho}{\tilde\rho}(v-\tilde v)\end{pmatrix}\right\rangle_{\mathbb{Y}},
$$

we arrive at the formulation

$$
\left\langle \mathcal{A}_\gamma(\tilde v,\tilde S,\tilde\varphi), \begin{pmatrix}v-\tilde v\\ S-\tilde S\\ -\Delta(\varphi-\tilde\varphi)+W''(\tilde\varphi)(\varphi-\tilde\varphi)+\frac{\kappa}{\varepsilon}(\varphi-\tilde\varphi)-\frac{\rho_2-\rho_1}{2}(v-\tilde v)\cdot\tilde v\end{pmatrix}\right\rangle_{\mathbb{Y}}. \tag{3.15}
$$

Notice that this substitution gives an extra term

$$
\int_\Omega \partial_t(\tilde\rho\tilde v)\left(-\frac{\rho}{\tilde\rho}(v-\tilde v)+(v-\tilde v)\right)+\partial_t\tilde\varphi\left(\frac{\rho_2-\rho_1}{2}\frac{\rho}{\tilde\rho}\tilde v(v-\tilde v)-\frac{\rho_2-\rho_1}{2}\tilde v(v-\tilde v)\right)\mathrm{d}x
$$

$$= \int_\Omega -(\rho - \tilde{\rho})(v - \tilde{v})\partial_t \tilde{v}\,\mathrm{d}x\,,$$

which also contributes to $\mathcal{W}_\gamma^{(\mathcal{K})}$ in (3.14) and thus, to the definition of dissipative solutions.

Following here the original definition of dissipative solutions according to Lions [35, Sec. 4.4], the solution fulfills the relative energy inequality for all smooth enough test functions. There are other solvability concepts that are coined dissipative in the literature. For instance certain measure-valued solutions (e.g., [10]) or essentially weak solutions [23].

Now, we can introduce the dissipative solution for system (2.23) according to Definition 3.1.

Definition 3.3 (Dissipative solution for system (2.23))**.** Let the assumptions 2.1-2.4 hold true. Let $\mathcal{K}$ be a regularity weight satisfying (3.1d). A quadruplet (v, S, φ, μ) is called a dissipative solution of type $\mathcal{K}$ of the two-phase system (2.23) if the following properties are satisfied:
1. The quadruplet (v, S, φ, μ) has the following regularity:

$$\begin{aligned} &v \in L^\infty_{\mathrm{loc}}([0,T), L^2_{\mathrm{div}}(\Omega)) \cap L^2_{\mathrm{loc}}([0,T), H^1_{0,\mathrm{div}}(\Omega)),\\ &S \in L^\infty_{\mathrm{loc}}([0,T), L^2_{\mathrm{sym,Tr}}(\Omega)),\\ &\varphi \in L^\infty_{\mathrm{loc}}([0,T); H^1(\Omega)) \cap L^2_{\mathrm{loc}}([0,T); H^2(\Omega)), W'(\varphi) \in L^2_{\mathrm{loc}}([0,T); L^2(\Omega)),\\ &\mu \in L^2_{\mathrm{loc}}([0,T); H^1(\Omega)). \end{aligned} \tag{3.16a}$$

2. With the relative energy $\mathcal{R}$ from (3.6), the system operator $\mathcal{A}_\gamma$ from (3.9), and the relative dissipation $\mathcal{W}_\gamma^{(\mathcal{K})}$ from (3.14), the quadruplet (v, S, φ, μ) satisfies the following relative energy-dissipation estimate:

$$\begin{aligned} &\mathcal{R}(v(t), S(t), \varphi(t)|\tilde{v}(t), \tilde{S}(t), \tilde{\varphi}(t))\\ &+ \int_0^t \Bigg(\Bigg\langle \mathcal{A}_\gamma(\tilde{v}, \tilde{S}, \tilde{\varphi}), \begin{pmatrix} v - \tilde{v}\\ S - \tilde{S}\\ -\Delta(\varphi - \tilde{\varphi}) + W''(\tilde{\varphi})(\varphi - \tilde{\varphi}) + \frac{\kappa}{\varepsilon}(\varphi - \tilde{\varphi}) - \frac{\rho_2 - \rho_1}{2}(v - \tilde{v})\cdot\tilde{v} \end{pmatrix} \Bigg\rangle_{\mathbb{Y}}\\ &\quad + \mathcal{P}(\varphi; S) - \mathcal{P}(\varphi; \tilde{S}) + \mathcal{W}_\gamma^{(\mathcal{K})}(v, S, \varphi|\tilde{v}, \tilde{S}, \tilde{\varphi}) \Bigg) \exp\left(\int_s^t \mathcal{K}(\tilde{v}, \tilde{S}, \tilde{\varphi})\,\mathrm{d}\tau \right) \mathrm{d}s\\ &\le \mathcal{R}(v_0, S_0, \varphi_0|\tilde{v}(0), \tilde{S}(0), \tilde{\varphi}(0)) \exp\left(\int_0^t \mathcal{K}(\tilde{v}, \tilde{S}, \tilde{\varphi})\,\mathrm{d}s \right) \end{aligned} \tag{3.16b}$$

for all $(\tilde{v}, \tilde{S}, \tilde{\varphi}) \in \mathfrak{T}$ and for a.e. $t \in (0, T)$.

Remark 3.4. One can still obtain an inequality for the relative energy even without the regularity weight term, but this term turns out to be crucial when carrying out the limit passage $\gamma \to 0$ for the relative dissipation. Recall that, when we pass to the limit $\gamma \to 0$, we will lose the control on the term ∇S which is essential for the limit passage of the nonlinear terms $S(\nabla v)_{\mathsf{skw}} - (\nabla v)_{\mathsf{skw}} S$. However, with the help of the chosen regularity weights, $\mathcal{W}_\gamma^{(\mathcal{K})}$ can be convex and even continuous and therefore weakly lower semicontinuous. This will allow us to perform the limit passage.

For notational simplicity, we assume without loss of generality that $\varepsilon = 1$.

3.3 Properties of the dissipative solutions

We first provide a lemma that will be useful for passing to the limit in energy inequalities.

Lemma 3.5 ([18, Lemma 2.2])**.** Let $g_0 \in \mathbb{R}$. Let $f \in L^1(0,T)$ and $g \in L^\infty(0,T)$. Then the following two inequalities are equivalent:

$$-\int_0^T \phi'(t)g(t)\,\mathrm{d}t + \int_0^T \phi(t)f(t)\,\mathrm{d}t \le g_0, \text{ for all } \phi \in \tilde{C}([0,T]), \tag{3.17}$$

where

$$\tilde{C}([0,T]) := \left\{\phi \in C^1([0,T]) : \phi \ge 0 \text{ on } [0,T], \phi' \le 0, \phi(0) = 1, \phi(T) = 0\right\}, \tag{3.18}$$

and

$$g(t) + \int_0^t f(s)\,\mathrm{d}s \le g_0 \text{ for a.e. } t \in (0,T). \tag{3.19}$$

Proof. See [18, Lemma 2.2]. Notice that this proof remains valid for g negative. □

We now investigate how dissipative solutions from Definition 3.3 and weak solutions from Definition 2.7 are connected. The first result shows that a weak solution is also a dissipative solution for arbitrary regularity weight $\mathcal{K}$, given that $\gamma > 0$.

Proposition 3.6. For $\gamma > 0$, let (v, S, φ, μ) be a weak solution in the sense of Definition 2.7 and let $\mathcal{K}$ satisfy (3.1d). Then (v, S, φ, μ) satisfies the relative energy estimate (3.16b) for a.e. $t \in (0,T)$, any given regularity weight $\mathcal{K}$ and all $(\tilde{v}, \tilde{S}, \tilde{\varphi}) \in \mathfrak{T}$. Hence, (v, S, φ, μ) is also a dissipative solution.

Proof. Let $(\tilde{v}, \tilde{S}, \tilde{\varphi}) \in \mathfrak{T}$ and for a.e. $t \in (0,T)$, let $\phi \in \tilde{C}([0,t])$ cf. (3.18). Since (v, S, φ, μ) is a weak solution in the sense of Definition 2.7, it satisfies the weak formulation (2.25a). Besides, since $\tilde{v} \in C^\infty_{0,\mathrm{div}}(\Omega \times [0,T))$ is admissible test function for (2.25a). Therefore, we choose the test functions in the weak formulation to be $-\phi\tilde{v}$, in order to obtain that

$$\begin{aligned}
&-\int_0^t \phi' \int_\Omega -\rho v \cdot \tilde{v}\,\mathrm{d}x\,\mathrm{d}s \\
&+\int_0^t \phi \int_\Omega \rho v \cdot \partial_t \tilde{v} + (\rho v \otimes v) : \nabla\tilde{v} - 2\nu(\varphi)(\nabla v)_{\mathsf{sym}} : \nabla\tilde{v}\,\mathrm{d}x\,\mathrm{d}s \\
&-\int_0^t \phi \int_\Omega \eta(\varphi)S : \nabla\tilde{v} - (v \otimes J) : \nabla\tilde{v} - \mu\nabla\varphi \cdot \tilde{v}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \phi\langle f, \tilde{v}\rangle_{H^1}\,\mathrm{d}s = \int_\Omega -\rho_0 v_0 \cdot \tilde{v}(0)\,\mathrm{d}x\,.
\end{aligned} \tag{3.20}$$

Furthermore, with the help of Lemma 3.5, the partial energy inequalities (2.25b) and (2.25f) can be equivalently written as

$$\begin{aligned}
&-\int_0^t \phi' \int_\Omega \frac{\rho(t)}{2}|v(t)|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \phi \int_\Omega 2\nu(\varphi)|(\nabla v)_{\mathsf{sym}}|^2 + \eta(\varphi)S : \nabla v - \mu(\nabla\varphi \cdot v)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_\Omega \frac{\rho_0}{2}|v_0|^2\,\mathrm{d}x + \int_0^t \phi\langle f, v\rangle_{H^1}\,\mathrm{d}s\,,
\end{aligned} \tag{3.21}$$

as well as

$$\begin{aligned}
&-\int_0^t \phi' \int_\Omega \frac{|\nabla\varphi(t)|^2}{2} + W(\varphi(t))\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \phi \int_\Omega |\nabla\mu|^2 + \mu(\nabla\varphi \cdot v)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_\Omega \frac{|\nabla\varphi_0|^2}{2} + W(\varphi_0)\,\mathrm{d}x\,,
\end{aligned} \tag{3.22}$$

Besides, by Lemma 3.5, the evolutionary variational inequality for the stress (2.25c) can be also transformed into the following

$$
\begin{aligned}
&-\int_0^t \phi' \int_\Omega \frac{1}{2}\Big|S(t)-\tilde{S}(t)\Big|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad+\int_0^t \phi \int_\Omega \partial_t\tilde{S} : (S-\tilde{S}) - v\cdot\nabla S : \tilde{S} - (S(\nabla v)_{\mathsf{skw}} - (\nabla v)_{\mathsf{skw}} S) : \tilde{S}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad+\int_0^t \phi(\mathcal{P}(\varphi;S)-\mathcal{P}(\varphi;\tilde{S}))\,\mathrm{d}s + \int_0^t \phi \int_\Omega \gamma\nabla S \vdots \nabla(S-\tilde{S}) - \eta(\varphi)(\nabla v)_{\mathsf{sym}} : (S-\tilde{S})\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \int_\Omega \frac{1}{2}\Big|S_0-\tilde{S}(0)\Big|^2 \,\mathrm{d}x
\end{aligned} \tag{3.23}
$$

Moreover, by integration-by-parts in time and space, we calculate the terms in the relative phase-field energy as follows:

$$
\begin{aligned}
&-\int_0^t \phi' \int_\Omega -\nabla\varphi : \nabla\tilde{\varphi} - W'(\tilde{\varphi})(\varphi-\tilde{\varphi}) - 2W(\tilde{\varphi})\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_\Omega -\nabla\varphi_0 : \nabla\tilde{\varphi}(0) - W'(\tilde{\varphi}(0))(\varphi_0-\tilde{\varphi}(0)) - 2W(\tilde{\varphi}(0))\,\mathrm{d}x \\
&\quad-\int_0^t \phi \int_\Omega \nabla\partial_t\tilde{\varphi} : \nabla\varphi - \partial_t\varphi\Delta\tilde{\varphi} + W'(\tilde{\varphi})(\partial_t\varphi-\partial_t\tilde{\varphi}) + W''(\tilde{\varphi})\partial_t\tilde{\varphi}(\varphi-\tilde{\varphi}) + 2W'(\tilde{\varphi})\partial_t\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}s\,,
\end{aligned} \tag{3.24}
$$

where we used that $\phi(0)=1$ and $\phi(t)=0$ for $\phi\in\tilde{C}([0,t])$ and the Neumann boundary condition (2.23i). Besides, by testing (2.25d) with $\phi\tilde{\mu}$ and an integration by parts in time, we obtain

$$
\int_0^t \phi \int_\Omega \partial_t\varphi\tilde{\mu} + v\cdot\nabla\varphi\tilde{\mu} + \nabla\mu : \nabla\tilde{\mu}\,\mathrm{d}x\,\mathrm{d}s = 0. \tag{3.25}
$$

By testing (2.25e) with $-\phi\partial_t\tilde{\varphi}$ and integrating in time and space, we derive

$$
-\int_0^t \phi \int_\Omega \mu\partial_t\tilde{\varphi} - W'(\varphi)\partial_t\tilde{\varphi} + \Delta\varphi\partial_t\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}s = 0. \tag{3.26}
$$

Summing (3.25) and (3.26) implies

$$
\begin{aligned}
&-\int_0^t \phi \int_\Omega \nabla\mu : \nabla\tilde{\mu}\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t \phi \int_\Omega \partial_t\varphi\tilde{\mu} + v\cdot\nabla\varphi\tilde{\mu} - \mu\partial_t\tilde{\varphi} + W'(\varphi)\partial_t\tilde{\varphi} + \nabla\varphi\cdot\nabla\partial_t\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned} \tag{3.27}
$$

In addition, we can calculate the contribution of non-convexity of the singular potential, i.e.

$$
\begin{aligned}
&-\kappa\int_0^t \phi' \int_\Omega \frac{|\varphi-\tilde{\varphi}|^2}{2}\,\mathrm{d}x\,\mathrm{d}s \\
&=\kappa\int_\Omega \frac{|\varphi_0-\tilde{\varphi}(0)|^2}{2}\,\mathrm{d}x + \kappa\int_0^t \phi \int_\Omega (\varphi-\tilde{\varphi})(\partial_t\varphi-\partial_t\tilde{\varphi})\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned} \tag{3.28}
$$

Notice that, with the help of (2.25d), the second term on the right-hand side of (3.28) can be written as

$$\begin{aligned}&\int_\Omega -\kappa(\varphi-\tilde\varphi)(\partial_t\varphi-\partial_t\tilde\varphi)\,\mathrm{d}x\\=&\int_\Omega \kappa(\nabla\mu-\nabla\tilde\mu)\cdot(\nabla\varphi-\nabla\tilde\varphi)\,\mathrm{d}x\\&+\int_\Omega \kappa\left(v\cdot(\nabla\varphi-\nabla\tilde\varphi)+(v-\tilde v)\cdot\nabla\tilde\varphi\right)(\varphi-\tilde\varphi)\,\mathrm{d}x\\&+\langle\mathcal{A}^{(3)}(\tilde v,\tilde\varphi),\kappa(\varphi-\tilde\varphi)\rangle\end{aligned}$$

Also, since v is a divergence-free function, thanks to integration by parts, we know

$$\int_\Omega v\cdot(\nabla\varphi-\nabla\tilde\varphi)(\varphi-\tilde\varphi)\,\mathrm{d}x=0.$$

Inserting these two equalities into (3.28) yields

$$\begin{aligned}&-\kappa\int_0^t\phi'\int_\Omega\frac{|\varphi-\tilde\varphi|^2}{2}\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\phi\int_\Omega\kappa(\nabla\mu-\nabla\tilde\mu)\cdot(\nabla\varphi-\nabla\tilde\varphi)+\kappa(v-\tilde v)\cdot\nabla\tilde\varphi(\varphi-\tilde\varphi)\,\mathrm{d}x\,\mathrm{d}s\\&+\int_0^t\phi\langle\mathcal{A}^{(3)}(\tilde v,\tilde\varphi),\kappa(\varphi-\tilde\varphi)\rangle\,\mathrm{d}s=\kappa\int_\Omega\frac{|\varphi_0-\tilde\varphi(0)|^2}{2}\,\mathrm{d}x\,.\end{aligned}\tag{3.29}$$

Now, observe that $v-\tilde v$ is an admissible test function for (3.9b), $S-\tilde S$ is admissible for (3.9c) and $-\Delta(\varphi-\tilde\varphi)+W''(\tilde\varphi)(\varphi-\tilde\varphi)$ is admissible for (3.9d). Moreover, notice that $-\frac{\rho_2-\rho_1}{2}\tilde v\cdot(v-\tilde v)$ is also admissible for (3.9d). Hence, choosing them as the test functions for each operator and summing up (3.20)-(3.24), (3.27), and (3.29) yields that

$$\begin{aligned}&-\int_0^t\phi'\mathcal{R}(v(t),S(t),\varphi(t)|\tilde v(t),\tilde S(t),\tilde\varphi(t))\,\mathrm{d}s\\&+\int_0^t\phi\left\langle\mathcal{A}_\gamma(\tilde v,\tilde S,\tilde\varphi),\begin{pmatrix}v-\tilde v\\S-\tilde S\\-\Delta(\varphi-\tilde\varphi)+W''(\tilde\varphi)(\varphi-\tilde\varphi)+\kappa(\varphi-\tilde\varphi)-\frac{\rho_2-\rho_1}{2}\tilde v\cdot(v-\tilde v)\end{pmatrix}\right\rangle_{\mathbb{Y}}\mathrm{d}s\\&+\int_0^t\phi\left(\mathcal{P}(\varphi;S)-\mathcal{P}(\varphi;\tilde S)\right)\mathrm{d}s+\int_0^t\phi\widetilde{\mathcal{W}}(v,S,\varphi|\tilde v,\tilde S,\tilde\varphi)\,\mathrm{d}s\\\le&\mathcal{R}(v_0,S_0,\varphi_0|\tilde v(0),\tilde S(0),\tilde\varphi(0)),\end{aligned}\tag{3.30}$$

where $\widetilde{\mathcal{W}}$ is given by

$$\widetilde{\mathcal{W}}(v,S,\varphi|\tilde v,\tilde S,\tilde\varphi)=\mathcal{W}_\gamma^{(\mathcal{K})}\left(v,S,\varphi|\tilde v,\tilde S,\tilde\varphi\right)-\mathcal{K}(\tilde v,\tilde S,\tilde\varphi)\mathcal{R}(v,S,\varphi|\tilde v,\tilde S,\tilde\varphi).\tag{3.31}$$

Finally, by choosing $\phi(t)=\psi(t)\exp\left(-\int_0^t\mathcal{K}(\tilde v,\tilde S,\tilde\varphi)\,\mathrm{d}\tau\right)$ for $\psi\in\tilde C([0,t])$, we arrive at the relative energy estimate (3.16b). □

The next two results state that the velocity component and the phase-field variable of a dissipative solution of type $\mathcal{K}$ are also a weak solution of the momentum balance (1.1a) and a weak solution of the evolution law (1.1e), respectively, under certain assumptions on $\mathcal{K}$.

Proposition 3.7. Suppose that the regularity weight $\mathcal{K}$ satisfies $\mathcal{K}(0,0,\tilde\varphi)=0$ for all $\tilde\varphi\in C_0^\infty(\Omega\times[0,T))$ such that $\tilde\varphi\in(-1,1)$. Moreover, assume that $\partial_t\varphi\in L^2(0,T;H^{-1}(\Omega))$. Then a dissipative solution of type $\mathcal{K}$ is a weak solution of the phase-field evolutionary law, i.e. (2.25d) is satisfied for all ζ such that $\zeta\in L^2([0,T);H^1(\Omega))$, $\partial_t\zeta\in L^2([0,T);L^2(\Omega))$ and $\zeta(T)=0$.

Proof. From the assumption on $\mathcal{K}$, we have that $\exp\Big(\int_0^t \mathcal{K}(\tilde{v},0,0)\,\mathrm{d}\tau\Big) = e^0 = 1$ for all $\tilde{v} \in C^\infty_{0,\mathrm{div}}(\Omega\times[0,T))$. Setting $\tilde{v}\equiv 0$ and $\tilde{S}\equiv 0$ in (3.16b) and with the fact that $\mathcal{P}(\varphi;0)\equiv 0$, we obtain

$$\begin{aligned}
&\mathcal{R}(v(t),S(t),\varphi(t)|0,0,\tilde{\varphi}(t)) + \int_0^t \mathcal{P}(\varphi;S) + \mathcal{W}^{(\mathcal{K})}_\gamma(v,S,\varphi|0,0,\tilde{\varphi})\,\mathrm{d}s \\
&+\int_0^t \left\langle \mathcal{A}_\gamma(0,0,\tilde{\varphi}), \begin{pmatrix} v \\ S \\ -\Delta(\varphi-\tilde{\varphi}) + W''(\tilde{\varphi})(\varphi-\tilde{\varphi}) + \kappa(\varphi-\tilde{\varphi})\end{pmatrix}\right\rangle_{\mathbb{Y}} \mathrm{d}s \\
\le&\mathcal{R}(v_0,S_0,\varphi_0|0,0,\tilde{\varphi}(0))
\end{aligned} \tag{3.32}$$

for all $\tilde{\varphi}\in C^\infty_0(\Omega\times[0,T))$ with $\tilde{\varphi}\in(-1,1)$ and a.e. $t\in(0,T)$. In view of (3.9b)-(3.9d), we have

$$\langle \mathcal{A}^{(1)}(0,0,\tilde{\varphi}), v\rangle_{H^1_{0,\mathrm{div}}} = -\langle f, v\rangle_{H^1} - \int_\Omega \tilde{\mu}\nabla\tilde{\varphi}\cdot v\,\mathrm{d}x\,, \tag{3.33a}$$

and

$$\langle \mathcal{A}^{(2)}_\gamma(0,0,\tilde{\varphi}), S\rangle_{H^1_{\mathrm{sym,Tr}}} = 0, \tag{3.33b}$$

and

$$\begin{aligned}
&\langle \mathcal{A}^{(3)}(0,\tilde{\varphi}), -\Delta(\varphi-\tilde{\varphi}) + W''(\tilde{\varphi})(\varphi-\tilde{\varphi}) + \kappa(\varphi-\tilde{\varphi})\rangle_{L^2} \\
=&\int_\Omega \partial_t\tilde{\varphi}\,(-\Delta(\varphi-\tilde{\varphi}) + W''(\tilde{\varphi})(\varphi-\tilde{\varphi}) + \kappa(\varphi-\tilde{\varphi}))\,\mathrm{d}x \\
&-\int_\Omega \Delta\tilde{\mu}\,(-\Delta(\varphi-\tilde{\varphi}) + W''(\tilde{\varphi})(\varphi-\tilde{\varphi}) + \kappa(\varphi-\tilde{\varphi}))\,\mathrm{d}x\,,
\end{aligned} \tag{3.33c}$$

Moreover, by (3.14), it is

$$\begin{aligned}
\mathcal{W}^{(\mathcal{K})}_\gamma(v,S,\varphi|0,0,\tilde{\varphi}) =& \int_\Omega 2\nu(\varphi)|(\nabla v)_{\mathsf{sym}}|^2 + \gamma|\nabla S|^2 + |\nabla\mu|^2\,\mathrm{d}x \\
&-\int_\Omega \nabla\mu\cdot\nabla\tilde{\mu} + \tilde{\mu}(\nabla\varphi-\nabla\tilde{\varphi})\cdot v\,\mathrm{d}x \\
&+\int_\Omega \Delta\tilde{\mu}(-\Delta(\varphi-\tilde{\varphi}) + W''(\tilde{\varphi})(\varphi-\tilde{\varphi}))\,\mathrm{d}x \\
&+\int_\Omega \kappa(\nabla\mu-\nabla\tilde{\mu})\cdot(\nabla\varphi-\nabla\tilde{\varphi}) + \kappa v\cdot\nabla\tilde{\varphi}(\varphi-\tilde{\varphi})\,\mathrm{d}x\,.
\end{aligned} \tag{3.34}$$

Inserting (3.33a)-(3.33c) and (3.34) into (3.32) and applying Lemma 3.5 results in the estimate

$$\begin{aligned}
&-\int_0^T \phi' \int_\Omega \rho\frac{|v|^2}{2} + \frac{|S|^2}{2} + \frac{|\nabla\varphi-\nabla\tilde{\varphi}|^2}{2} + W(\varphi) - W(\tilde{\varphi}) - W'(\tilde{\varphi})(\varphi-\tilde{\varphi}) + \frac{\kappa}{2}|\varphi-\tilde{\varphi}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_0^T \phi\int_\Omega 2\nu(\varphi)|(\nabla v)_{\mathsf{sym}}|^2 + \gamma|\nabla S|^2 + |\nabla\mu|^2\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \phi\langle f,v\rangle_{H^1}\,\mathrm{d}t + \int_0^T \phi\mathcal{P}(\varphi;S)\,\mathrm{d}t \\
&+\int_0^T \phi\int_\Omega \partial_t\tilde{\varphi}\,(-\Delta(\varphi-\tilde{\varphi}) + W''(\tilde{\varphi})(\varphi-\tilde{\varphi}) + \kappa(\varphi-\tilde{\varphi}))\,\mathrm{d}x\,\mathrm{d}t \\
&-\int_0^T \phi\int_\Omega \nabla\mu\cdot\nabla\tilde{\mu} + \tilde{\mu}\nabla\varphi\cdot v - \kappa\nabla\mu\cdot(\nabla\varphi-\nabla\tilde{\varphi}) - \kappa v\cdot\nabla\varphi(\varphi-\tilde{\varphi})\,\mathrm{d}x\,\mathrm{d}t \\
\le&\int_\Omega \frac{|\nabla\varphi_0-\nabla\tilde{\varphi}(0)|^2}{2} + W(\varphi_0) - W(\tilde{\varphi}(0)) - W'(\tilde{\varphi}(0))(\varphi_0-\tilde{\varphi}(0)) + \frac{\kappa}{2}|\varphi_0-\tilde{\varphi}(0)|^2\,\mathrm{d}x \\
&+\int_\Omega \rho_0\frac{|v_0|^2}{2} + \frac{|S_0|^2}{2}\,\mathrm{d}x\,,
\end{aligned} \tag{3.35}$$

for all $\phi \in \tilde{C}([0,T])$. Notice that, by integration by parts in time, rearranging the terms in the above inequality and exploiting cancellations in terms that depend on $\tilde{\mu} + \kappa\tilde{\varphi}$, we then get

$$\begin{aligned}
&-\int_0^T \phi' \int_\Omega \rho\frac{|v|^2}{2} + \frac{|S|^2}{2}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0\frac{|v_0|^2}{2} + \frac{|S_0|^2}{2}\,\mathrm{d}x \\
&+\int_0^T \phi \int_\Omega 2\nu(\varphi)|(\nabla v)_{\text{sym}}|^2 + \gamma|\nabla S|^2\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \phi\langle f, v\rangle_{H^1}\,\mathrm{d}t + \int_0^T \phi\mathcal{P}(\varphi;S)\,\mathrm{d}t \\
&+\int_0^T \phi\langle\partial_t\varphi, \mu+\kappa\varphi\rangle_{H^1}\,\mathrm{d}t + \int_0^T \phi\int_\Omega \kappa v\cdot\nabla\varphi\varphi + \nabla\mu(\nabla\mu+\kappa\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \\
&-\int_0^T \phi\langle\partial_t\varphi, \tilde{\mu}+\kappa\tilde{\varphi}\rangle_{H^1}\,\mathrm{d}t - \int_0^T \phi\int_\Omega v\cdot\nabla\varphi(\tilde{\mu}+\kappa\tilde{\varphi}) + \nabla\mu\cdot(\nabla\tilde{\mu}+\kappa\nabla\tilde{\varphi})\,\mathrm{d}x\,\mathrm{d}t \le 0.
\end{aligned} \tag{3.36}$$

First, notice that in (3.36), the regularity of $\tilde{\varphi}$ can be reduced such that $\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\tilde{\varphi}) = \tilde{\mu} + \kappa\tilde{\varphi} \in L^2([0,T];H^1(\Omega))$. Hence, choose $\tilde{\varphi}_\alpha$ such that $\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\tilde{\varphi}_\alpha) = \alpha\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\tilde{\varphi}) \in L^2([0,T];H^1(\Omega))$ with $\alpha > 0$ (where the existence of such $\tilde{\varphi}_\alpha$ can be guaranteed by the surjectivity of the subdifferential of a proper, convex, lower semicontinuous and coercive functional, see [9, Chapter 2.2] for details) and multiply both sides of (3.36) by $\frac{1}{\alpha}$. This gives

$$\begin{aligned}
&-\frac{1}{\alpha}\int_0^T \phi' \int_\Omega \rho\frac{|v|^2}{2} + \frac{|S|^2}{2}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0\frac{|v_0|^2}{2} + \frac{|S_0|^2}{2}\,\mathrm{d}x \\
&+\frac{1}{\alpha}\int_0^T \phi \int_\Omega 2\nu(\varphi)|(\nabla v)_{\text{sym}}|^2 + \gamma|\nabla S|^2\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \phi\langle f, v\rangle_{H^1}\,\mathrm{d}t + \int_0^T \phi\mathcal{P}(\varphi;S)\,\mathrm{d}t \\
&+\frac{1}{\alpha}\int_0^T \phi\langle\partial_t\varphi, \mu+\kappa\varphi\rangle\,\mathrm{d}t + \int_0^T \phi\int_\Omega \kappa v\cdot\nabla\varphi\varphi + \nabla\mu(\nabla\mu+\kappa\nabla\varphi)\,\mathrm{d}x \\
&-\int_0^T \phi\langle\partial_t\varphi, \tilde{\mu}+\kappa\tilde{\varphi}\rangle_{H^1}\,\mathrm{d}t - \int_0^T \phi\int_\Omega v\cdot\nabla\varphi(\tilde{\mu}+\kappa\tilde{\varphi}) + \nabla\mu\cdot(\nabla\tilde{\mu}+\kappa\nabla\tilde{\varphi})\,\mathrm{d}x\,\mathrm{d}t \le 0.
\end{aligned} \tag{3.37}$$

Letting $\alpha \to \infty$ gives

$$-\int_0^T \phi\langle\partial_t\varphi, \tilde{\mu}+\kappa\tilde{\varphi}\rangle_{H^1}\,\mathrm{d}t - \int_0^T \phi\int_\Omega v\cdot\nabla\varphi(\tilde{\mu}+\kappa\varphi) + \nabla\mu\cdot(\nabla\tilde{\mu}+\kappa\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \le 0 \tag{3.38}$$

for all $\phi \in \tilde{C}([0,T])$. By Lemma 3.5, we conclude

$$\int_0^T \langle\partial_t\varphi, \tilde{\mu}+\kappa\tilde{\varphi}\rangle_{H^1}\,\mathrm{d}t + \int_0^T\int_\Omega v\cdot\nabla\varphi(\tilde{\mu}+\kappa\varphi) + \nabla\mu\cdot(\nabla\tilde{\mu}+\kappa\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \ge 0. \tag{3.39}$$

This inequality is linear with respect to $\tilde{\mu}+\kappa\tilde{\varphi}$. Thus, choosing $\tilde{\varphi}_-$ such that $\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\tilde{\varphi}_-) = -\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\tilde{\varphi})$ and integrating by parts in time yields the desired equality. □

Proposition 3.8. Suppose that the regularity weight $\mathcal{K}$ satisfies $\mathcal{K}(\tilde{v},0,0) = 0$ for all $\tilde{v} \in C^\infty_{0,\mathrm{div}}(\Omega\times[0,T))$. Moreover, assume that $\partial_t\varphi \in L^2([0,T];H^{-1}(\Omega))$ and that the weak formulation (2.25d) holds true for all ζ such that $\zeta \in L^2([0,T];H^1(\Omega))$, $\partial_t\zeta \in L^2([0,T];L^2(\Omega))$ and $\zeta(T) = 0$. Then a dissipative solution of type $\mathcal{K}$ is a weak solution of the momentum balance, i.e. (2.25a) is satisfied for all $\Phi \in C^\infty_{0,\mathrm{div}}(\Omega\times[0,T))$.

Proof. This proposition can be shown by following the idea of the proof of Proposition 3.7. See also [18, Proposition 4.3]. □

Next, we establish that dissipative solutions are transitive with respect to the regularity weights, given that the regularity weights satisfy a certain monotonicity.

Proposition 3.9. Let (v, S, φ, μ) be a dissipative solution of type $\mathcal{K}$. Let $\mathcal{K}$, $\mathcal{L}$ be regularity weights with the property $\mathcal{K}(\tilde{v}, \tilde{S}, \tilde{\varphi}) \leq \mathcal{L}(\tilde{v}, \tilde{S}, \tilde{\varphi})$ for all $(\tilde{v}, \tilde{S}, \tilde{\varphi}) \in \mathfrak{T}$ and a.e. $t \in (0, T)$. Then (v, S, φ, μ) is also a dissipative solution of type $\mathcal{L}$.

Proof. This proposition can be proved by applying Lemma 3.5 to (3.16b) with the test function

$$\Phi(t) := \phi(t) \exp\bigg(- \int_0^t \mathcal{L}(\tilde{v}, \tilde{S}, \tilde{\varphi}) - \mathcal{K}(\tilde{v}, \tilde{S}, \tilde{\varphi}) \,\mathrm{d}\tau \bigg)$$

for $\phi \in \tilde{C}([0, T])$ and using Lemma 3.5 again to cancel ϕ. See also [18, Proposition 4.4]. Notice that Φ is only a valid test function for $\mathcal{K}(\tilde{v}, \tilde{S}, \tilde{\varphi}) \leq \mathcal{L}(\tilde{v}, \tilde{S}, \tilde{\varphi})$. □

4 Global existence result for the regularized two phase system $\gamma > 0$

4.1 Implicit time discretization

In this section, we will use an implicit time discretization to show the existence of weak solutions.

To start with, we first define another dissipation potential $\widetilde{\mathcal{P}}$ as

$$\begin{aligned} &\widetilde{\mathcal{P}} : L^2(\Omega) \times H^1_{\mathrm{sym,Tr}}(\Omega) \to [0, +\infty] \\ &\qquad \widetilde{\mathcal{P}}(\varphi; S) := \begin{cases} \mathcal{P}(\varphi; S) & (\varphi, S) \in L^2(\Omega) \times H^1_{\mathrm{sym,Tr}}(\Omega) \cap \mathrm{dom}(\mathcal{P}), \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{4.1}$$

$\widetilde{\mathcal{P}}$ can be viewed as the restriction of $\mathcal{P}$ in $L^2(\Omega) \times H^1_{\mathrm{sym,Tr}}(\Omega)$. Notice that $\widetilde{\mathcal{P}}$ is proper with $\widetilde{\mathcal{P}}(\varphi; 0) = 0$ for all $\varphi \in L^2(\Omega)$. Moreover, for all $\varphi \in L^2(\Omega)$, the mapping $S \mapsto \widetilde{\mathcal{P}}(\varphi; S)$ is convex and lower semicontinuous in $H^1_{\mathrm{sym,Tr}}(\Omega)$. We write dom$(\partial\widetilde{\mathcal{P}}(\varphi; \cdot))$ to represent the domain of the convex partial subdifferential.

To begin with the time discretization, let $h = \frac{T}{N}$ for $N \in \mathbb{N}$ and let $t_0 = 0$, $t_k = kh$, $t_N = T$ and assume that the initial data v_0, S_0, φ_0 satisfy Assumption 2.3. For all $k = 0, 1, \ldots, N-1$, let $v_k \in L^2_{\mathrm{div}}(\Omega)$, $S_k \in L^2_{\mathrm{sym,Tr}}(\Omega)$, $\varphi_k \in H^1(\Omega)$ with $W'(\varphi_k) \in L^2(\Omega)$ and $\rho_k = \frac{1}{2}(\rho_1 + \rho_2) + \frac{1}{2}(\rho_2 - \rho_1)\varphi_k$. Moreover, let $f_{k+1} = h^{-1} \int_{t_k}^{t_{k+1}} f \,\mathrm{d}\tau$. We determine $(v_{k+1}, S_{k+1}, \varphi_{k+1}, \mu_{k+1})$ based on the given data with

$$J_{k+1} = -\frac{\rho_2 - \rho_1}{2} \nabla \mu_{k+1},$$

i.e., we aim to find some $(v_{k+1}, S_{k+1}, \varphi_{k+1}, \mu_{k+1})$ such that

$$v_{k+1} \in H^1_{0,\mathrm{div}}(\Omega), S_{k+1} \in H^1_{\mathrm{sym,Tr}}(\Omega) \cap \mathrm{dom}(\partial\widetilde{\mathcal{P}}(\varphi_k; \cdot)), \varphi_{k+1} \in \mathrm{dom}(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}), \mu_{k+1} \in H^2_{\vec{n}}(\Omega), \tag{4.2a}$$

where

$$H^2_{\vec{n}}(\Omega) := \{u \in H^2(\Omega) : \vec{n} \cdot \nabla u|_{\partial\Omega} = 0\}, \tag{4.2b}$$

satisfying:
1. The weak formulation of discrete momentum balance:

$$\begin{aligned}&\langle\frac{\rho_{k+1}v_{k+1}-\rho_k v_k}{h}+\mathsf{div}(\rho_k v_{k+1}\otimes v_{k+1}),\Phi\rangle_{L^2}\\&+\langle 2\nu(\varphi_k)(\nabla v_{k+1})_{\mathsf{sym}},(\nabla\Phi)_{\mathsf{sym}}\rangle_{L^2}-\langle\mathsf{div}(\eta(\varphi_k)S_{k+1}),\Phi\rangle_{L^2}\\&+\langle(\mathsf{div}(J_{k+1})-\frac{\rho_{k+1}-\rho_k}{h}-v_{k+1}\cdot\nabla\rho_k)\frac{v_{k+1}}{2},\Phi\rangle_{L^2}\\&+\langle J_{k+1}\cdot\nabla v_{k+1}-\mu_{k+1}\nabla\varphi_k,\Phi\rangle_{L^2}=\langle f_{k+1},\Phi\rangle_{H^1}\end{aligned}\tag{4.2c}$$

for all $\Phi\in C^\infty_{0,\mathrm{div}}(\Omega)$.
2. The weak formulation of the discrete evolution law for the stress:

$$\begin{aligned}&\langle\frac{S_{k+1}-S_k}{h}+(v_{k+1}\cdot\nabla S_{k+1}),\Psi\rangle_{L^2}\\&+\langle(S_{k+1}(\nabla v_{k+1})_{\mathsf{skw}}-(\nabla v_{k+1})_{\mathsf{skw}}S_{k+1}),\Psi\rangle_{L^2}\\&+\langle\xi^k_{k+1},\Psi\rangle_{H^1_{\mathrm{sym,Tr}}}+\langle\gamma\nabla S_{k+1},\nabla\Psi\rangle_{L^2}=\langle\eta(\varphi_k)(\nabla v_{k+1})_{\mathsf{sym}},\Psi\rangle_{L^2}\end{aligned}\tag{4.2d}$$

for all $\Psi\in C^\infty_{\mathrm{sym,Tr}}(\bar\Omega)$, and here $\xi^k_{k+1}\in\partial\widetilde{\mathcal{P}}(\varphi_k;S_{k+1})\subseteq(H^1_{\mathrm{sym,Tr}}(\Omega))'$.
3. The discrete evolution law for the phase-field variable:

$$\frac{\varphi_{k+1}-\varphi_k}{h}+v_{k+1}\cdot\nabla\varphi_k=\Delta\mu_{k+1},\tag{4.2e}$$

as well as

$$\mu_{k+1}+\kappa\frac{\varphi_{k+1}+\varphi_k}{2}=-\Delta\varphi_{k+1}+W'_\kappa(\varphi_{k+1}),\tag{4.2f}$$

almost everywhere in Ω.

Remark 4.1. (1) Integrating (4.2e) in space and performing an integration by parts gives

$$\int_\Omega\varphi_{k+1}\,\mathrm{d}x=\int_\Omega\varphi_k\,\mathrm{d}x+h\int_\Omega-v_{k+1}\cdot\nabla\varphi_k+\Delta\mu_{k+1}\,\mathrm{d}x=\int_\Omega\varphi_k\,\mathrm{d}x\,,\tag{4.3}$$

where we made use of the Neumann boundary condition (2.23i) and the fact that v_{k+1} is divergence-free. Moreover, this equality implies that $\int_\Omega\varphi_k\,\mathrm{d}x=\int_\Omega\varphi_0\,\mathrm{d}x$ is a constant, i.e., the total volume conserved.
(2) By multiplying (4.2e) with $-\frac{1}{2}(\rho_2-\rho_1)$, we obtain that

$$-\frac{\rho_{k+1}-\rho_k}{h}-v_{k+1}\nabla\rho_k=\mathsf{div}(J_{k+1}).\tag{4.4}$$

Notice that $\mathsf{div}(v_{k+1}\otimes J_{k+1})=(\mathsf{div}(J_{k+1}))v_{k+1}+J_{k+1}\cdot\nabla v_{k+1}$. Inserting (4.4) into (4.2c) results in

$$\begin{aligned}&\langle\frac{\rho_{k+1}v_{k+1}-\rho_k v_k}{h}+\mathsf{div}(\rho_k v_{k+1}\otimes v_{k+1}),\Phi\rangle_{L^2}\\&+\langle 2\nu(\varphi_k)(\nabla v_{k+1})_{\mathsf{sym}},(\nabla\Phi)_{\mathsf{sym}}\rangle_{L^2}-\langle\mathsf{div}(\eta(\varphi_k)S_{k+1}),\Phi\rangle_{L^2}\\&+\langle\mathsf{div}(v_{k+1}\otimes J_{k+1}),\Phi\rangle_{L^2}-\langle f_{k+1},\Phi\rangle_{H^1}=\langle\mu_{k+1}\nabla\varphi_k,\Phi\rangle_{L^2},\end{aligned}\tag{4.5}$$

which is the direct weak formulation of the momentum balance (2.23a) with time discretization.

Before we prove the existence of solutions for the time discrete problem (4.2), let us first deduce an estimate for the terms in the Cahn–Hilliard part. This inequality uses a similar method as in the proof of [2, Lemma 4.2].

Lemma 4.2. Assume that $\varphi_{k+1} \in \mathrm{dom}(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa})$ and $\mu_{k+1} \in H^1(\Omega)$ are solutions to (4.2f) for given $\varphi_k \in H^2(\Omega)$ satisfying $|\varphi_k| \le 1$ in Ω and

$$\frac{1}{|\Omega|}\int_\Omega \varphi_{k+1}\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega \varphi_k\,\mathrm{d}x \in (-1,1).$$

Then there exists a positive constant C depending on $\int_\Omega \varphi_k\,\mathrm{d}x$, such that

$$\|W_\kappa'(\varphi_{k+1})\|_{L^2} + \left|\int_\Omega \mu_{k+1}\,\mathrm{d}x\right| \le C\left(\|\nabla\mu_{k+1}\|_{L^2} + \|\nabla\varphi_{k+1}\|_{L^2}^2 + \|\nabla\varphi_k\|_{L^2}^2 + 1\right) \tag{4.6}$$

$$\|\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi_{k+1})\|_{L^2} \le C\left(\|\mu_{k+1}\|_{L^2} + 1\right) \tag{4.7}$$

Proof. Recall from (2.3) that u_Ω denotes the mean value of a function u in Ω. Hence, for φ_{k+1}, we write $\varphi_{k+1,\Omega}$ for the mean value. First, we test (4.2f) with $\zeta = (\varphi_{k+1} - \varphi_{k+1,\Omega})$ to obtain that

$$\begin{aligned}&\int_\Omega \mu_{k+1}(\varphi_{k+1} - \varphi_{k+1,\Omega})\,\mathrm{d}x + \int_\Omega \kappa\frac{\varphi_{k+1}+\varphi_k}{2}(\varphi_{k+1} - \varphi_{k+1,\Omega})\,\mathrm{d}x \\ &= -\int_\Omega \Delta\varphi_{k+1}\cdot(\varphi_{k+1} - \varphi_{k+1,\Omega})\,\mathrm{d}x + \int_\Omega W_\kappa'(\varphi_{k+1})(\varphi_{k+1} - \varphi_{k+1,\Omega})\,\mathrm{d}x\,.\end{aligned} \tag{4.8}$$

Writing $\mu_0 = \mu_{k+1} - \mu_{k+1,\Omega}$, we can see that

$$\int_\Omega \mu_{k+1}(\varphi_{k+1} - \varphi_{k+1,\Omega})\,\mathrm{d}x = \int_\Omega \mu_0\varphi_{k+1}\,\mathrm{d}x\,. \tag{4.9}$$

In view of the homogeneous Neumann boundary condition, notice that

$$\left|-\int_\Omega \Delta\varphi_{k+1}(\varphi_{k+1} - \varphi_{k+1,\Omega})\,\mathrm{d}x\right| = \left|\int_\Omega \nabla\varphi_{k+1}\cdot\nabla\varphi_{k+1}\,\mathrm{d}x\right| = \|\nabla\varphi_{k+1}\|_{L^2}^2. \tag{4.10}$$

Besides, by the assumption $W_\kappa' \in C^1(-1,1)$ and $\lim\limits_{s\to\pm1} W_\kappa'(s) = \pm\infty$ in (2.9), we obtain that

$$W_\kappa'(\varphi_{k+1})(\varphi_{k+1} - \varphi_{k+1,\Omega}) \ge C|W_\kappa'(\varphi_{k+1})| - \tilde{C} \tag{4.11}$$

for all $\varphi_{k+1} \in [-1,1]$, see [2, Lemma 4.2] for details. Inserting (4.9)-(4.11) into (4.8) yields

$$\begin{aligned}\int_\Omega |W_\kappa'(\varphi_{k+1})|\,\mathrm{d}x \le& C(\|\mu_0\|_{L^2}\|\varphi_{k+1}\|_{L^2} + \int_\Omega \kappa\frac{\varphi_{k+1}+\varphi_k}{2}(\varphi_{k+1} - \varphi_{k+1,\Omega})\,\mathrm{d}x + \|\nabla\varphi_{k+1}\|_{L^2}^2 + 1) \\ \le& C\left(\|\nabla\mu\|_{L^2} + \|\nabla\varphi_{k+1}\|_{L^2}^2 + \|\nabla\varphi_k\|_{L^2}^2 + 1\right),\end{aligned} \tag{4.12}$$

where we used the fact that $|\varphi_{k+1}|, |\varphi_k| \le 1$ and Poinaré's inequality. Next, by directly integrating (4.2f), we can see that

$$\int_\Omega \mu_{k+1}\,\mathrm{d}x + \int_\Omega \kappa\frac{\varphi_{k+1}+\varphi_k}{2}\,\mathrm{d}x = \int_\Omega -\Delta\varphi_{k+1}\,\mathrm{d}x + \int_\Omega W_\kappa'(\varphi_{k+1})\,\mathrm{d}x\,.$$

This implies

$$\begin{aligned}\left|\int_\Omega \mu_{k+1}\,\mathrm{d}x\right| &\le \int_\Omega |W_\kappa'(\varphi_{k+1})|\,\mathrm{d}x + \int_\Omega \kappa\left|\frac{\varphi_{k+1}+\varphi_k}{2}\right|\mathrm{d}x \\ &\le C\left(\|\nabla\mu_{k+1}\|_{L^2} + \|\nabla\varphi_{k+1}\|_{L^2}^2 + \|\nabla\varphi_k\|_{L^2}^2 + 1\right),\end{aligned} \tag{4.13}$$

where we used integration by parts, Poincaré's inequality, and the fact that

$$\frac{1}{|\Omega|}\int_\Omega \varphi_{k+1}\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega \varphi_k\,\mathrm{d}x\,.$$

Finally, since $\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi_{k+1}) = -\Delta\varphi_{k+1} + W_\kappa'(\varphi_{k+1}) = \mu_{k+1} + \frac{\kappa}{2}(\varphi_{k+1}+\varphi_k)$, we obtain

$$\|\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi_{k+1})\|_{L^2} \le \|\mu_{k+1}\|_{L^2} + \frac{\kappa}{2}(\|\varphi_{k+1}\|_{L^2} + \|\varphi_k\|_{L^2}) \le C\left(\|\mu_{k+1}\|_{L^2}+1\right). \tag{4.14}$$

Besides, using (2.20), we deduce that

$$\|W_\kappa'(\varphi_{k+1})\|_{L^2}^2 \le C\left(\|\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi_{k+1})\|_{L^2}^2 + \|\varphi_{k+1}\|_{L^2}^2 + 1\right) \le C\left(\|\mu_{k+1}\|_{L^2}+1\right)^2,$$

which implies

$$\|W_\kappa'(\varphi_{k+1})\|_{L^2} \le C\left(\|\mu_{k+1}\|_{L^2}+1\right) \le C\left(\|\nabla\mu_{k+1}\|_{L^2} + \left|\int_\Omega \mu_{k+1}\,\mathrm{d}x\right| + 1\right). \tag{4.15}$$

Therefore, combining (4.15) with (4.13) yields the desired inequality, that is

$$\|W_\kappa'(\varphi_{k+1})\|_{L^2} + \left|\int_\Omega \mu_{k+1}\,\mathrm{d}x\right| \le C\left(\|\nabla\mu_{k+1}\|_{L^2} + \|\nabla\varphi_{k+1}\|_{L^2}^2 + \|\nabla\varphi_k\|_{L^2}^2 + 1\right).$$

□

Now, we show the existence of solutions to the time discrete problem (4.2). We adapt the proof of [2, Lemma 4.3] to our case. Notice that in [2, Lemma 4.3], the extra stress tensor S is not present, while, below the stress tensor S will be the main difficulty because of the set-valued subdifferential.

Lemma 4.3 (Existence of solutions to the time discrete problem)**.** For $k \in \{0,1,\ldots,N-1\}$, let $v_k \in L^2_{\mathrm{div}}(\Omega)$, $S_k \in L^2_{\mathrm{sym,Tr}}(\Omega)$, $\varphi_k \in H^2(\Omega)$ with $|\varphi_k| \le 1$ and $\rho_k = \frac{\rho_2-\rho_1}{2}\varphi_k + \frac{\rho_2+\rho_1}{2}$ be given, let $\widetilde{\mathcal{P}}$ be as in (4.1), and set

$$X := H^1_{0,\mathrm{div}}(\Omega) \times H^1_{\mathrm{sym,Tr}}(\Omega) \cap \mathrm{dom}(\partial\widetilde{\mathcal{P}}(\varphi_k;\cdot)) \times \mathrm{dom}(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}) \times H^2_{\vec{n}}(\Omega). \tag{4.16}$$

Then there exists a quadruplet $(v_{k+1}, S_{k+1}, \varphi_{k+1}, \mu_{k+1}) \in X$ solving (4.2c)-(4.2f). Moreover, this solution satisfies the energy dissipation estimate

$$\begin{aligned}&\mathcal{E}_{\mathrm{tot}}(v_{k+1}, S_{k+1}, \varphi_{k+1}) + h\mathcal{D}_{\mathrm{chs}}(v_{k+1}, \varphi_k, \mu_{k+1}) + h\mathcal{D}_{\mathrm{sd},\gamma}(S_{k+1}) + h\langle \xi^k_{k+1}, S_{k+1}\rangle_{H^1_{\mathrm{sym,Tr}}}\\ \le&\mathcal{E}_{\mathrm{tot}}(v_k, S_k, \varphi_k) + h\langle f_{k+1}, v_{k+1}\rangle_{H^1}.\end{aligned} \tag{4.17}$$

Proof. **Step 1: A priori estimate** (4.17)**.** Let $(v_{k+1}, S_{k+1}, \varphi_{k+1}, \mu_{k+1}) \in H^1_{0,\mathrm{div}}(\Omega) \times H^1_{\mathrm{sym,Tr}}(\Omega) \cap \mathrm{dom}(\partial\widetilde{\mathcal{P}}(\varphi_k;\cdot)) \times \mathrm{dom}(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}) \times H^2_{\vec{n}}(\Omega)$ be a solution to the time discrete problem (4.2). Observe that $v_{k+1} \in H^1_{0,\mathrm{div}}(\Omega)$ is a suitable test function for (4.2c). With this in mind, we calculate that

$$\int_\Omega \left((\mathrm{div}(J_{k+1}))\frac{v_{k+1}}{2} + J_{k+1}\cdot\nabla v_{k+1}\right) v_{k+1}\,\mathrm{d}x = \int_\Omega \mathrm{div}(J_{k+1}\frac{|v_{k+1}|^2}{2})\,\mathrm{d}x = 0, \tag{4.18}$$

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as well as

$$
\begin{aligned}
&\int_\Omega \Big(\mathsf{div}(\rho_k v_{k+1} \otimes v_{k+1}) - (\nabla \rho_k \cdot v_{k+1})\frac{v_{k+1}}{2}\Big)\, v_{k+1} \,\mathrm{d}x \\
=&\int_\Omega \Big(\mathsf{div}(\rho_k v_{k+1} \otimes v_{k+1}) - \mathsf{div}(\rho_k v_{k+1})\frac{v_{k+1}}{2}\Big)\, v_{k+1} \,\mathrm{d}x \\
=&\int_\Omega \Big(\mathsf{div}(\rho_k v_{k+1})|v_{k+1}|^2 + \rho_k v_{k+1} \cdot \nabla(\frac{|v_{k+1}|^2}{2}) - \mathsf{div}(\rho_k v_{k+1})\frac{|v_{k+1}|^2}{2}\Big) \,\mathrm{d}x \\
=&\int_\Omega \mathsf{div}(\rho_k v_{k+1}\frac{|v_{k+1}|^2}{2}) \,\mathrm{d}x = 0,
\end{aligned}
\tag{4.19}
$$

where we used integration-by-parts and the homogeneous boundary condition (2.23h). Moreover, notice that

$$
\begin{aligned}
&\frac{1}{h}(\rho_{k+1}v_{k+1} - \rho_k v_k)v_{k+1} \\
=&\frac{1}{h}(\rho_{k+1} - \rho_k)|v_{k+1}|^2 + \frac{1}{h}\rho_k(v_{k+1} - v_k)v_{k+1} \\
=&\frac{1}{h}(\rho_{k+1} - \rho_k)|v_{k+1}|^2 + \frac{1}{h}\rho_k\frac{|v_{k+1}|^2}{2} - \frac{1}{h}\rho_k\frac{|v_k|^2}{2} + \frac{1}{h}\rho_k\frac{|v_{k+1} - v_k|^2}{2} \\
=&\frac{1}{h}\frac{\rho_{k+1}|v_{k+1}|^2}{2} - \frac{1}{h}\frac{\rho_k|v_k|^2}{2} + \frac{1}{h}\frac{(\rho_{k+1} - \rho_k)|v_{k+1}|^2}{2} + \frac{1}{h}\frac{\rho_k|v_{k+1} - v_k|^2}{2}.
\end{aligned}
\tag{4.20}
$$

Thanks to (4.18), (4.19) and (4.20), by testing (4.2c) with $\Phi = v_{k+1}$, we obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{h}\frac{\rho_{k+1}|v_{k+1}|^2}{2}\,\mathrm{d}x + \int_\Omega \frac{1}{h}\frac{\rho_k|v_{k+1} - v_k|^2}{2}\,\mathrm{d}x \\
&+ \int_\Omega 2\nu(\varphi_k)|(\nabla v_{k+1})_{\mathsf{sym}}|^2\,\mathrm{d}x + \int_\Omega \eta(\varphi_k)S_{k+1} : (\nabla v_{k+1})_{\mathsf{sym}}\,\mathrm{d}x \\
=&\int_\Omega \mu_{k+1}(\nabla\varphi_k \cdot v_{k+1})\,\mathrm{d}x + \int_\Omega \frac{1}{h}\frac{\rho_k|v_k|^2}{2}\,\mathrm{d}x + \langle f_{k+1}, v_{k+1}\rangle_{H^1}.
\end{aligned}
\tag{4.21}
$$

Next, observe that $S_{k+1} \in H^1_{\mathrm{sym,Tr}}(\Omega)$ is a suitable test function for (4.2d). For the first term in (4.2d), we calculate that

$$
\frac{1}{h}(S_{k+1} - S_k) : S_{k+1} = \frac{1}{h}\frac{|S_{k+1}|^2}{2} - \frac{1}{h}\frac{|S_k|^2}{2} + \frac{1}{h}\frac{|S_{k+1} - S_k|^2}{2}.
\tag{4.22}
$$

With the help of (4.22) and due to the fact that many terms cancel out, since v is divergence-free and S is symmetric, we also obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{h}\frac{|S_{k+1}|^2}{2}\,\mathrm{d}x + \int_\Omega \frac{1}{h}\frac{|S_{k+1} - S_k|}{2}\,\mathrm{d}x + \langle \xi^k_{k+1} : S_{k+1}\rangle_{H^1_{\mathrm{sym,Tr}}} + \int_\Omega \gamma|\nabla S_{k+1}|^2\,\mathrm{d}x \\
=&\int_\Omega \eta(\varphi_k)(\nabla v_{k+1})_{\mathsf{sym}} : S_{k+1}\,\mathrm{d}x + \int_\Omega \frac{1}{h}\frac{|S_k|^2}{2}\,\mathrm{d}x\,.
\end{aligned}
\tag{4.23}
$$

Further, observe that $\mu_{k+1} \in H^2_{\vec{n}}(\Omega)$ and $\frac{1}{h}(\varphi_{k+1} - \varphi_k) \in H^2(\Omega)$ are suitable test functions for (4.2e) and (4.2f), respectively. With this test, we obtain

$$
\int_\Omega \frac{1}{h}(\varphi_{k+1} - \varphi_k)\mu_{k+1}\,\mathrm{d}x + \int_\Omega (v_{k+1} \cdot \nabla\varphi_k)\mu_{k+1}\,\mathrm{d}x = -\int_\Omega |\nabla\mu_{k+1}|^2\,\mathrm{d}x\,,
\tag{4.24}
$$

as well as

$$\begin{aligned}
&\int_\Omega \frac{1}{h}\nabla\varphi_{k+1}\cdot(\nabla\varphi_{k+1}-\nabla\varphi_k)\,\mathrm{d}x+\int_\Omega W_0'(\varphi_{k+1})\frac{\varphi_{k+1}-\varphi_k}{h}\,\mathrm{d}x\\
&=\int_\Omega \mu_{k+1}\frac{\varphi_{k+1}-\varphi_k}{h}\,\mathrm{d}x+\int_\Omega \kappa\frac{\varphi_{k+1}^2-\varphi_k^2}{2h}\,\mathrm{d}x\,.
\end{aligned}\tag{4.25}$$

Now summing up (4.21)-(4.25), we derive

$$\begin{aligned}
&\int_\Omega \frac{1}{h}\frac{\rho_k|v_k|^2}{2}\,\mathrm{d}x+\int_\Omega \frac{1}{h}\frac{|S_k|^2}{2}\,\mathrm{d}x+\langle f_{k+1},v_{k+1}\rangle_{H^1}\\
&=\int_\Omega \frac{1}{h}\frac{\rho_{k+1}|v_{k+1}|^2}{2}\,\mathrm{d}x+\int_\Omega \frac{1}{h}\frac{\rho_k|v_{k+1}-v_k|^2}{2}\,\mathrm{d}x+\int_\Omega \frac{1}{h}\frac{|S_{k+1}|^2}{2}\,\mathrm{d}x+\int_\Omega \frac{1}{h}\frac{|S_{k+1}-S_k|^2}{2}\,\mathrm{d}x\\
&\quad+\int_\Omega 2\nu(\varphi_k)|(\nabla v_{k+1})_{\mathrm{sym}}|^2\,\mathrm{d}x+\int_\Omega \gamma|\nabla S_{k+1}|^2\,\mathrm{d}x+\langle \xi_{k+1}^k,S_{k+1}\rangle_{H^1_{\mathrm{sym,Tr}}}+\int_\Omega|\nabla\mu_{k+1}|^2\,\mathrm{d}x\\
&\quad+\int_\Omega W_\kappa'(\varphi_{k+1})\frac{\varphi_{k+1}-\varphi_k}{h}\,\mathrm{d}x-\int_\Omega \kappa\frac{\varphi_{k+1}^2-\varphi_k^2}{2h}\,\mathrm{d}x+\frac{1}{h}\int_\Omega \nabla\varphi_{k+1}\cdot(\nabla\varphi_{k+1}-\nabla\varphi_k)\,\mathrm{d}x\\
&\geq\int_\Omega \frac{1}{h}\frac{\rho_{k+1}|v_{k+1}|^2}{2}\,\mathrm{d}x+\int_\Omega \frac{1}{h}\frac{\rho_k|v_{k+1}-v_k|^2}{2}\,\mathrm{d}x+\int_\Omega \frac{1}{h}\frac{|S_{k+1}|^2}{2}\,\mathrm{d}x+\int_\Omega \frac{1}{h}\frac{|S_{k+1}-S_k|^2}{2}\,\mathrm{d}x\\
&\quad+\int_\Omega 2\nu(\varphi_k)|(\nabla v)_{\mathrm{sym}}|^2\,\mathrm{d}x+\int_\Omega \gamma|\nabla S_{k+1}|^2\,\mathrm{d}x+\langle \xi_{k+1}^k:S_{k+1}\rangle_{H^1_{\mathrm{sym,Tr}}}+\int_\Omega|\nabla\mu_{k+1}|^2\,\mathrm{d}x\\
&\quad+\frac{1}{h}\int_\Omega W_\kappa(\varphi_{k+1})-\kappa\frac{\varphi_{k+1}^2}{2}\,\mathrm{d}x-\frac{1}{h}\int_\Omega W_\kappa(\varphi_k)-\kappa\frac{\varphi_k^2}{2}\,\mathrm{d}x\\
&\quad+\frac{1}{h}\int_\Omega\frac{|\nabla\varphi_{k+1}-\nabla\varphi_k|^2}{2}\,\mathrm{d}x+\frac{1}{h}\int_\Omega\frac{|\nabla\varphi_{k+1}|^2}{2}\,\mathrm{d}x-\frac{1}{h}\int_\Omega\frac{|\nabla\varphi_k|^2}{2}\,\mathrm{d}x\,,
\end{aligned}$$

where we have used the convexity of W_κ, i.e.,

$$W_\kappa'(\varphi_{k+1})(\varphi_{k+1}-\varphi_k)\geq W_\kappa(\varphi_{k+1})-W_\kappa(\varphi_k),$$

and

$$\nabla\varphi_{k+1}\cdot(\nabla\varphi_{k+1}-\nabla\varphi_k)=\frac{|\nabla\varphi_{k+1}|^2}{2}-\frac{|\nabla\varphi_k|^2}{2}+\frac{|\nabla\varphi_{k+1}-\nabla\varphi_k|^2}{2}.$$

Multiplying both sides by h and rearranging terms to the left-hand side results in (4.17).

Step 2: Existence result via Schaefer's fixed-point theorem [21, Chapter 9.2.2, Theorem 2]. Suppose that

$$\mathscr{K}_k:\tilde{Y}\to\tilde{Y}$$

is a continuous and compact mapping. Assume further that the set

$$\Big\{u\in\tilde{Y}:u=\lambda\mathscr{K}_k(u)\text{ for some }0\leq\lambda\leq 1\Big\}$$

is bounded. Then $\mathscr{K}_k$ has a fixed point. In order to apply Schaefer's fixed point theorem, we will determine $\tilde{Y}$ and the operator $\mathscr{K}_k$ based on the discrete weak formulation (4.2). For this, using X from (4.16) and the space

$$Y:=\big(H^1_{0,\mathrm{div}}(\Omega)\big)'\times\big(H^1_{\mathrm{sym,Tr}}(\Omega)\big)'\times L^2(\Omega)\times L^2(\Omega),\tag{4.26}$$

we define operators $\mathscr{L}_k, \mathscr{F}_k : X \to Y$ as follows:

$$\mathscr{L}_k : X \to Y,\; w := (v, S, \varphi, \mu) \mapsto \mathscr{L}_k(w) := \begin{pmatrix} L_k^v(v) \\ L_k^s(S) \\ -\Delta\mu + \int_\Omega \mu \,\mathrm{d}x \\ \varphi + \mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi) \end{pmatrix} \tag{4.27}$$

where $L_k^v(v)$ and $L_k^s(S)$ are defined in the weak sense, i.e.,

$$\langle L_k^v(v), \Phi\rangle = \langle 2\nu(\varphi_k)(\nabla v)_{\mathsf{sym}}, (\nabla\Phi)_{\mathsf{sym}}\rangle_{L^2} - \langle f_{k+1}, \Phi\rangle_{H^1} \tag{4.28}$$

$$\langle L_k^s(S), \Psi\rangle = \langle \gamma S, \Psi\rangle_{H^1} + \langle \xi^k, \Psi\rangle_{H^1_{\mathrm{sym,Tr}}} \text{ with } \xi^k \in \partial\widetilde{\mathcal{P}}(\varphi_k; S) \tag{4.29}$$

for all test functions $\Phi \in H^1_{0,\mathrm{div}}(\Omega)$ and $\Psi \in H^1_{\mathrm{sym,Tr}}(\Omega)$, while the third and forth entries in(4.27) are identified pointwise. We further introduce $\mathscr{F}_k$ as follows

$$\mathscr{F}_k : X \to Y,\; w := (v, S, \varphi, \mu) \mapsto \mathscr{F}_k(w), \tag{4.30}$$

where

$$\mathscr{F}_k(w) := \begin{pmatrix} -\frac{\rho v - \rho_k v_k}{h} - \mathrm{div}(\rho_k v \otimes v) + \mu\nabla\varphi_k - \big(\mathrm{div}(J) - \frac{\rho-\rho_k}{h} - v\cdot\nabla\rho_k\big)\frac{v}{2} - J\cdot\nabla v - \mathrm{div}(\eta(\varphi_k)S) \\ -\frac{S-S_k}{h} - v\cdot\nabla S - S(\nabla v)_{\mathsf{skw}} + (\nabla v)_{\mathsf{skw}} S + \eta(\varphi_k)(\nabla v)_{\mathsf{sym}} + \gamma S \\ -\frac{\varphi-\varphi_k}{h} - v\cdot\nabla\varphi_k + \int_\Omega \mu\,\mathrm{d}x \\ \varphi + \mu + \kappa\frac{\varphi+\varphi_k}{2} \end{pmatrix}. \tag{4.31}$$

From these two definitions, we can see that $w = (v_{k+1}, S_{k+1}, \varphi_{k+1}, \mu_{k+1})$ is a weak solution to (4.2c)-(4.2f) if and only if

$$\mathscr{L}_k(w) - \mathscr{F}_k(w) = 0. \tag{4.32}$$

Properties of $\mathscr{L}_k$. Now we want to prove the invertibility of the operator $\mathscr{L}_k$. For the first entry, we can derive the invertibility and continuity of the inverse with help of the Lax-–Milgram theorem. To show the invertibility, for all $\tilde{f} \in (H^1_{0,\mathrm{div}}(\Omega))'$, we want to prove the existence of a unique $v \in H^1_{0,\mathrm{div}}(\Omega)$ such that $-\mathrm{div}(2\nu(\varphi_k)(\nabla v)_{\mathsf{sym}}) - f_{k+1} = \tilde{f}$. Since $f_{k+1} \in H^{-1}(\Omega) \subseteq (H^1_{0,\mathrm{div}}(\Omega))'$. There holds $\bar{f} := \tilde{f} + f_{k+1} \in (H^1_{0,\mathrm{div}}(\Omega))'$. Observing that the operator $-\mathrm{div}(2\nu(\varphi_k)\cdot) : H^1_{0,\mathrm{div}}(\Omega) \to (H^1_{0,\mathrm{div}}(\Omega))'$ induces a continuous, coercive bilinear form, the Lax–Milgram theorem yields the invertibility and continuity of the inverse. For the second entry, notice that it can be viewed as the sum of a maximal monotone operator and the duality map. Hence, we can conclude the invertibility by Minty—Browder Theorem see [8, Theorem 2.2]. To see the continuity of the inverse, let $F_n = -\gamma\Delta S_n + \xi_n^k + \gamma S_n$, $F = -\gamma\Delta S + \xi^k + \gamma S$ and $F_n \to F$ in $(H^1(\Omega))'$. For all $n \in \mathbb{N}$, observe that

$$\begin{aligned}\gamma\|S_n - S\|^2_{H^1} &\le \gamma\langle S_n - S, S_n - S\rangle_{H^1} + \langle \xi_n^k - \xi^k, S_n - S\rangle_{H^1_{\mathrm{sym,Tr}}} \\ &= \langle F_n - F, S_n - S\rangle_{H^1_{\mathrm{sym,Tr}}} \le \frac{1}{2\gamma}\|F_n - F\|^2_{(H^1_{\mathrm{sym,Tr}})'} + \frac{\gamma}{2}\|S_n - S\|^2_{H^1}.\end{aligned}$$

By rearranging terms, one can see that the inverse operator is continuous. For the third entry, let us consider the following elliptic equation

$$\begin{cases} -\Delta u + \int_\Omega u\,\mathrm{d}x = g & \text{in } \Omega, \\ \vec{n}\cdot\nabla u|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.33}$$

where $g \in L^2(\Omega)$ is a given function. The invertibility of the operator represented by the third entry is equivalent to the existence of a unique weak solution $u \in H^2_{\vec{n}}(\Omega) := \{u \in H^2(\Omega) : \vec{n} \cdot \nabla u|_{\partial\Omega} = 0\}$ for any given $f \in L^2(\Omega)$ and this can be guaranteed by [26, Chapter 2]. Moreover, one can also derive that

$$\|\mu\|_{H^2} \leq C \left(\|\mu\|_{H^1} + \|g\|_{L^2}\right). \tag{4.34}$$

This gives the continuity of the inverse operator. For the last component of $\mathscr{L}_k$, notice that $\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}$ is a maximal monotone operator. Again, by Minty—Browder Theorem, we have the invertibility. Moreover, we want to derive the continuity of the inverse operator. To do so, we interpret the inverse operator as a mapping $L^2(\Omega) \to H^{2-s}(\Omega)$ for arbitrary $0 < s < 1/4$. Let $F_k = u_k + \mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(u_k)$ and $F = u + \mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(u)$ be given. Assume $F_k \to F$ in $L^2(\Omega)$, then

$$\begin{aligned} \|u_k - u\|_{L^2}^2 + \|\nabla u_k - \nabla u\|_{L^2}^2 &\leq \|u_k - u\|_{L^2}^2 + \langle \mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(u_k) - \mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(u), u_k - u\rangle_{L^2} \\ &\leq \|u_k + \mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(u_k) - u - \mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(u)\|_{L^2} \cdot \|u_k - u\|_{L^2} \\ &\leq \frac{1}{2}\|F_k - F\|_{L^2}^2 + \frac{1}{2}\|u_k - u\|_{L^2}^2. \end{aligned}$$

This shows that $u_k \to u$ in $H^1(\Omega)$. Besides, due to (2.20), $(u_k)_k$ is bounded in $H^2(\Omega)$. Then, by interpolation, we have an inequality of the form

$$\|u_k - u\|_{H^{2-s}} \leq C\|u_k - u\|_{H^2}^{1-s}\|u_k - u\|_{H^1}^s,$$

which implies that $u_k \to u$ in $H^{2-s}(\Omega)$.

Altogether, we now have the invertibility of $\mathscr{L}_k : X \to Y$ and write the inverse operator as $\mathscr{L}_k^{-1} : Y \to X$. But for the continuity and even compactness of the inverse operator, we need to introduce two refined Banach spaces:

$$\tilde{X} := H^1_{0,\mathrm{div}}(\Omega) \times H^1_{\mathrm{sym,Tr}}(\Omega) \times H^{2-s}(\Omega) \times H^2_{\vec{n}}(\Omega), \tag{4.35a}$$

$$\tilde{Y} := L^{\frac{3}{2}}(\Omega)^3 \times L^{\frac{3}{2}}(\Omega)^{3\times 3} \times W^{1,\frac{3}{2}}(\Omega) \times H^1(\Omega). \tag{4.35b}$$

where $0 < s < 1/4$. From above arguments, we know that $\mathscr{L}_k^{-1} : Y \to \tilde{X}$ is continuous. Since $\tilde{Y} \hookrightarrow\hookrightarrow Y$, the restriction $\mathscr{L}_k^{-1} : \tilde{Y} \to \tilde{X}$ is compact.

Properties of $\mathscr{F}_k$. Now, let us consider the operator $\mathscr{F}_k$. We want to derive that $\mathscr{F}_k : \tilde{X} \to \tilde{Y}$ is continuous and that it maps bounded sets to bounded sets. To this end, let $(v, S, \varphi, \mu) \in \tilde{X}$, and we deduce the following estimates for the different components of $\mathscr{F}_k$:
We first discuss the terms in the first line of $\mathscr{F}_k$. Since $v \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\|\rho v\|_{L^{\frac{3}{2}}} \leq \|\rho\|_{L^2}\|v\|_{L^6} \leq C\|v\|_{H^1}\left(\|\varphi\|_{L^2} + 1\right).$$

Notice that div$(\rho_k v \otimes v)$ contains terms of the form $\rho_k(\partial_{x_l} v_i)v_j$ and $(\partial_{x_l}\rho_k)v_i v_j$ for $i, j, l = 1, \ldots, 3$. Besides, we have $\rho_k \in L^\infty(\Omega) \cap H^2(\Omega)$. Hence, $\partial_{x_l}\rho_k, v_i \in H^1(\Omega) \hookrightarrow L^6(\Omega)$. Hence, we obtain

$$\begin{aligned} \|\rho_k(\partial_{x_l} v_i)v_j\|_{L^{\frac{3}{2}}} &\leq \|\rho_k\|_{L^\infty}\|\partial_{x_l} v_i\|_{L^2}\|v_j\|_{L^6}, \\ \|(\partial_{x_l}\rho_k)v_i v_j\|_{L^{\frac{3}{2}}} &\leq C\|\partial_{x_l}\rho_k\|_{L^6}\|v_i\|_{L^6}\|v_j\|_{L^6}, \end{aligned}$$

and thus,

$$\|\mathrm{div}(\rho_k v \otimes v)\|_{L^{\frac{3}{2}}} \leq C_k\|v\|_{H^1}^2.$$

Since $\mu \in H^2(\Omega)$ and since $\varphi_k \in H^2(\Omega)$ implies that $\partial_{x_l}\varphi_k \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, we also obtain

$$\|\mu \nabla \varphi_k\|_{L^{\frac{3}{2}}} \leq \|\nabla \varphi_k\|_{L^6} \|\mu\|_{L^2} = C_k \|\mu\|_{L^2}.$$

Note that $\text{div}(J)v = \text{div}(\frac{\rho_2-\rho_1}{2}\nabla\mu)v$ consists of terms of the form $\frac{\rho_2-\rho_1}{2}(\partial_{x_i}\partial_{x_i}\mu)v_l$. Moreover, we have $\mu \in H^2(\Omega)$ and $v_l \in H^1(\Omega) \hookrightarrow L^6(\Omega)$. Thus, we obtain

$$\|\text{div}(J)v\|_{L^{\frac{3}{2}}} \leq C\|\mu\|_{H^2}\|v\|_{L^6} \leq C\|\mu\|_{H^2}\|v\|_{H^1}.$$

Similarly, $J \cdot \nabla v$ has terms of the form $\frac{\rho_2-\rho_1}{2}\partial_{x_i}\mu \partial_{x_j} v$. Since $\partial_{x_i}\mu \in H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $v \in H^1(\Omega)$, we obtain

$$\|J\nabla v\|_{L^{\frac{3}{2}}} \leq C\|\nabla\mu\|_{H^1}\|\nabla v\|_{L^2} \leq C\|\mu\|_{H^2}\|\nabla v\|_{H^1}.$$

Observe that $\text{div}(\eta(\varphi_k)S)$ contains terms of the form that $\eta'(\varphi_k)\partial_{x_l}\varphi_k S_{ij}$ and $\eta(\varphi_k)\partial_{x_l}S_{ij}$. Since $\partial_{x_l}\varphi_k \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, $S \in H^1(\Omega)$ and $|\eta(\varphi_k)|$, $|\eta'(\varphi)|$ are bounded by assumption (2.8), we obtain

$$\begin{aligned}
&\|\eta'(\varphi_k)\partial_{x_l}\varphi_k S_{ij}\|_{L^{\frac{3}{2}}} \leq C\|\partial_{x_l}\varphi_k\|_{L^6}\|S_{ij}\|_{L^2},\\
&\|\eta(\varphi_k)\partial_{x_l}S_{ij}\|_{L^{\frac{3}{2}}} \leq C\|\partial_{x_l}S_{ij}\|_{L^2},
\end{aligned}$$

and therefore also

$$\|\text{div}(\eta(\varphi_k)S)\|_{L^{\frac{3}{2}}} \leq C_k\|S\|_{H^1}.$$

This finishes the estimates for the terms in the first line of $\mathscr{F}_k$ and we turn to the terms in the second line. Since $S \in H^1(\Omega)$, we directly obtain

$$\|S\|_{L^{\frac{3}{2}}} \leq C\|S\|_{H^1}.$$

Thanks to $v \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, we find

$$\|v \cdot \nabla S\|_{L^{\frac{3}{2}}} \leq C\|v\|_{L^6}\|\nabla S\|_{L^2} \leq C\|v\|_{H^1}\|S\|_{H^1}.$$

Moreover, due to $S \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\|S(\nabla v)_{\text{skw}} - (\nabla v)_{\text{skw}}S\|_{L^{\frac{3}{2}}} \leq C\|\nabla v\|_{L^2}\|S\|_{L^6} \leq C\|v\|_{H^1}\|S\|_{H^1}.$$

Since $|\eta|$ is bounded by (2.8) and $v \in H^1(\Omega)$, we also get

$$\|\eta(\varphi_k)(\nabla v)_{\text{sym}}\|_{L^{\frac{3}{2}}} \leq C\|(\nabla v)_{\text{sym}}\|_{L^2} \leq C\|v\|_{H^1}.$$

Now, we discuss the estimates for the terms in the third line of $\mathscr{F}_k$. By Hölder inequality, we directly have

$$\|\varphi\|_{W^{1,\frac{3}{2}}} \leq C\|\varphi\|_{H^1}.$$

Also notice that, since $\nabla\varphi_k \in H^1(\Omega)$ and $v \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\|v \cdot \nabla\varphi_k\|_{L^{\frac{3}{2}}} \leq \|\nabla\varphi_k\|_{L^2}\|v\|_{L^6} \leq C_k\|v\|_{H^1}.$$

Moreover, The derivative of $v \cdot \nabla\varphi_k$ consists of terms of the form $\partial_{x_i} v_j \partial_{x_l}\varphi_k$ and $v_j\partial_{x_i}\partial_{x_l}\varphi_k$. Since $\partial_{x_i} v_j \in L^2(\Omega)$, $\partial_{x_l}\varphi_k \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, $v_j \in H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $\partial_{x_i}\partial_{x_l}\varphi_k \in L^2(\Omega)$, we arrive at

$$\begin{aligned}\|\partial_{x_i} v_j \partial_{x_l}\varphi_k\|_{L^{\frac{3}{2}}} &\le \|\partial_{x_i} v_j\|_{L^2}\|\partial_{x_l}\varphi_k\|_{L^6},\\ \|v_j\partial_{x_i}\partial_{x_l}\varphi_k\|_{L^{\frac{3}{2}}} &\le \|v_j\|_{L^6}\|\partial_{x_i}\partial_{x_l}\varphi_k\|_{L^2}.\end{aligned}$$

Therefore, we also conclude

$$\|v\cdot\nabla\varphi_k\|_{W^{1,\frac{3}{2}}} \le C_k\|v\|_{H^1},$$

where C_k is a positive constant depending on k. Since $\mu \in H^2(\Omega)$, we obtain

$$\|\int_\Omega \mu \,\mathrm{d}x\|_{W^{1,\frac{3}{2}}} \le C\left|\int_\Omega \mu\,\mathrm{d}x\right| \le C\|\mu\|_{L^2}.$$

For the last line of $\mathscr{F}_k$, the estimate is direct, since

$$\|\varphi\|_{H^1} = \|\varphi\|_{H^1} \text{ and } \|\mu\|_{H^1} \le \|\mu\|_{H^2}.$$

Altogether, from the above discussion, we can see $\mathscr{F}_k : \tilde{X} \to \tilde{Y}$ is continuous. Moreover, for $w = (v, S, \varphi, \mu)$ bounded in $\tilde{X}$, also $\mathscr{F}_k(w)$ is bounded in $\tilde{Y}$ i.e. $\mathscr{F}_k$ maps bounded sets to bounded sets.

Definition of the operator $\mathscr{K}_k : \tilde{Y} \to \tilde{Y}$. Recall (4.32). In order to apply Schaefer's fixed point theorem,see [21, Chapter 9.2.2, Theorem 2] for details, we need to introduce a new operator $\mathscr{K}_k$ whose image space and preimage space coincide. To this end, using $\tilde{Y}$ from (4.35b), we define the operator as follows:

$$\mathscr{K}_k : \tilde{Y} \to \tilde{Y},\, u \mapsto \mathscr{F}_k \circ \mathscr{L}_k^{-1}(u), \tag{4.36}$$

which is feasible by the invertibility of $\mathscr{L}_k$. With the help of this operator, we can rewrite (4.32) as

$$u - \mathscr{K}_k(u) = 0 \iff u = \mathscr{K}_k(u), \tag{4.37}$$

where $u = \mathscr{L}_k(w)$ for $w \in \tilde{X}$. Since we already showed that $\mathscr{L}_k^{-1}$ is compact and $\mathscr{F}_k$ is continuous, then $\mathscr{K}_k$ is also continuous and compact on $\tilde{Y}$.

Boundedness of $\mathscr{K}_k$ in $\tilde{X}$. In order to apply Schaefer's fixed point theorem, it remains to show that

$$\left\{u \in \tilde{Y} : u = \lambda\mathscr{K}_k(u) \text{ for some } 0 \le \lambda \le 1\right\} \tag{4.38}$$

is bounded. To this end, let $u \in \tilde{Y}$ and $0 \le \lambda \le 1$ satisfy $u = \lambda\mathscr{K}_k(u)$. Again by the invertibility of $\mathscr{L}_k$, we find $w = \mathscr{L}_k^{-1}(u)$ satisfying

$$\mathscr{L}_k(w) - \lambda\mathscr{F}_k(w) = 0.$$

By the definition of the operator $\mathscr{L}_k$ from (4.27) and of $\mathscr{F}_k$ from (4.31), we arrive at the following weak formulations

$$\begin{aligned}&\int_\Omega 2\nu(\varphi_k)(\nabla v)_{\mathsf{sym}} : (\nabla\Phi)_{\mathsf{sym}}\,\mathrm{d}x + \lambda\int_\Omega \frac{\rho v - \rho_k v_k}{h}\cdot\Phi - (\rho_k v\otimes v) : \nabla\Phi\,\mathrm{d}x\\ &+ \lambda\int_\Omega (\mathsf{div}(J) + \frac{\rho-\rho_k}{h} - v\cdot\nabla\rho_k)\frac{v}{2}\cdot\Phi + (J\cdot\nabla v)\cdot\Phi + \eta(\varphi_k)S : \nabla\Phi\,\mathrm{d}x\\ &= \lambda\int_\Omega (\mu\nabla\varphi_k)\cdot\Phi\,\mathrm{d}x + \langle f_{k+1}, \Phi\rangle_{H^1},\end{aligned} \tag{4.39a}$$

for all $\Phi \in C^\infty_{0,\mathrm{div}}(\Omega)$.

$$\begin{aligned}
&\int_\Omega \gamma \nabla S \,\vdots\, \nabla \Psi + \gamma S : \Psi \,\mathrm{d}x + \langle \xi^k, \Psi \rangle_{H^1_{\mathrm{sym,Tr}}} \\
&+ \lambda \int_\Omega \frac{S - S_k}{h} : \Psi + (v \cdot \nabla S) : \Psi \; + (S(\nabla v)_{\mathsf{skw}} - (\nabla v)_{\mathsf{skw}} S) : \Psi \,\mathrm{d}x \\
&= \lambda \int_\Omega \eta(\varphi_k)(\nabla v)_{\mathsf{sym}} : \Psi + \gamma S : \Psi \,\mathrm{d}x \,,
\end{aligned} \tag{4.39b}$$

for all $\Psi \in C^\infty_{\mathrm{sym,Tr}}(\bar{\Omega})$.

$$\lambda \frac{\varphi - \varphi_k}{h} + \lambda v \cdot \nabla \varphi_k - \lambda \int_\Omega \mu \,\mathrm{d}x = \Delta \mu - \int_\Omega \mu \,\mathrm{d}x \tag{4.39c}$$

as well as

$$\varphi + \mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi) = \lambda \varphi + \lambda \mu + \lambda \kappa \frac{\varphi + \varphi_k}{2}. \tag{4.39d}$$

Now due to the bounds deduced above and by using a density argument, we conclude that $\Phi = v$ and $\Psi = S$ are admissible test functions for (4.39a) and (4.39b). Moreover, testing (4.39c) with μ and (4.39d) with $\frac{1}{h}(\varphi - \varphi_k)$ and integrating in space gives

$$\begin{aligned}
&\int_\Omega 2\nu(\varphi_k) |(\nabla v)_{\mathsf{sym}}|^2 \,\mathrm{d}x + \frac{\lambda}{h} \int_\Omega \rho \frac{|v|^2}{2} - \rho_k \frac{|v_k|^2}{2} + \rho_k \frac{|v - v_k|^2}{2} \,\mathrm{d}x \\
&= \lambda \int_\Omega \mu (\nabla \varphi_k \cdot v) \,\mathrm{d}x - \lambda \int_\Omega \eta(\varphi_k) S : (\nabla v)_{\mathsf{sym}} \,\mathrm{d}x + \langle f_{k+1}, v \rangle_{H^1},
\end{aligned} \tag{4.40a}$$

and

$$\begin{aligned}
&\int_\Omega \gamma |\nabla S|^2 \,\mathrm{d}x + \langle \xi^k, S \rangle_{H^1_{\mathrm{sym,Tr}}} + \int_\Omega \gamma |S|^2 \,\mathrm{d}x + \frac{\lambda}{h} \int_\Omega \frac{|S|^2}{2} - \frac{|S_k|^2}{2} + \frac{|S - S_k|^2}{2} \,\mathrm{d}x \\
&= \lambda \int_\Omega \eta(\varphi_k)(\nabla v)_{\mathsf{sym}} : S + \gamma |S|^2 \,\mathrm{d}x \,,
\end{aligned} \tag{4.40b}$$

and

$$\frac{\lambda}{h} \int_\Omega (\varphi - \varphi_k) \mu \,\mathrm{d}x + \lambda \int_\Omega (v \cdot \nabla \varphi) \mu \,\mathrm{d}x - (\lambda - 1) \left| \int_\Omega \mu \,\mathrm{d}x \right|^2 + \int_\Omega |\nabla \mu|^2 \,\mathrm{d}x = 0, \tag{4.40c}$$

as well as

$$\begin{aligned}
&\frac{1}{h} \int_\Omega \varphi(\varphi - \varphi_k) \,\mathrm{d}x + \frac{1}{h} \int_\Omega \nabla \varphi \cdot (\nabla \varphi - \nabla \varphi_k) \,\mathrm{d}x + \frac{1}{h} \int_\Omega W_\kappa'(\varphi)(\varphi - \varphi_k) \,\mathrm{d}x \\
&= \frac{\lambda}{h} \int_\Omega \varphi(\varphi - \varphi_k) - \frac{\lambda}{h} \int_\Omega \mu(\varphi - \varphi_k) \,\mathrm{d}x + \frac{\lambda \kappa}{h} \int_\Omega \frac{\varphi^2 - \varphi_k^2}{2} \,\mathrm{d}x \,.
\end{aligned} \tag{4.40d}$$

Summing up (4.40a)-(4.40d), we obtain that

$$\begin{aligned}
0 = &\int_\Omega 2\nu(\varphi_k) |(\nabla v)_{\mathsf{sym}}|^2 \,\mathrm{d}x - \langle f_{k+1}, v \rangle_{H^1} + \int_\Omega \gamma |\nabla S|^2 \,\mathrm{d}x + \langle \xi^k, S \rangle_{H^1_{\mathrm{sym,Tr}}} + \int_\Omega |\nabla \mu|^2 \,\mathrm{d}x \\
&+ \frac{\lambda}{h} \int_\Omega \rho \frac{|v|^2}{2} - \rho_k \frac{|v_k|^2}{2} \,\mathrm{d}x + \frac{\lambda}{h} \int_\Omega \rho_k \frac{|v - v_k|^2}{2} \,\mathrm{d}x + \frac{\lambda}{h} \int_\Omega \frac{|S|^2}{2} - \frac{|S_k|^2}{2} \,\mathrm{d}x + \frac{\lambda}{h} \int_\Omega \frac{|S - S_k|^2}{2} \,\mathrm{d}x \\
&+ (1 - \lambda) \int_\Omega \gamma |S|^2 \,\mathrm{d}x + (1 - \lambda) \left| \int_\Omega \mu \,\mathrm{d}x \right|^2 + \frac{1 - \lambda}{h} \int_\Omega \frac{\varphi^2}{2} - \frac{\varphi_k^2}{2} \,\mathrm{d}x + \frac{1 - \lambda}{h} \int_\Omega \frac{|\varphi - \varphi_k|^2}{2} \,\mathrm{d}x \\
&+ \frac{1}{h} \int_\Omega \frac{|\nabla \varphi|^2}{2} - \frac{|\varphi_k|^2}{2} \,\mathrm{d}x + \frac{1}{h} \int_\Omega \frac{|\varphi - \varphi_k|^2}{2} \,\mathrm{d}x + \frac{1}{h} \int_\Omega W_\kappa'(\varphi)(\varphi - \varphi_k) \,\mathrm{d}x - \frac{\lambda}{h} \int_\Omega \kappa \frac{\varphi^2 - \varphi_k^2}{2} \,\mathrm{d}x \,.
\end{aligned}$$

Recall that $\lambda \in [0,1]$ and, by convexity, there holds

$$\int_\Omega W_\kappa'(\varphi)(\varphi - \varphi_k)\,\mathrm{d}x \geq \int_\Omega W_\kappa(\varphi) - W_\kappa(\varphi_k)\,\mathrm{d}x\,.$$

This leads to the estimate

$$\begin{aligned}
&\int_\Omega 2\nu(\varphi_k)|(\nabla v)_{\text{sym}}|^2\,\mathrm{d}x - \langle f_{k+1}, v\rangle_{H^1} + \int_\Omega \gamma|\nabla S|^2\,\mathrm{d}x + \langle \xi^k, S\rangle_{H^1_{\mathrm{sym,Tr}}} + \int_\Omega |\nabla\mu|^2\,\mathrm{d}x \\
&+ \frac{\lambda}{h}\int_\Omega \rho\frac{|v|^2}{2} - \rho_k\frac{|v_k|^2}{2}\,\mathrm{d}x + \frac{\lambda}{h}\int_\Omega \frac{|S|^2}{2} - \frac{|S_k|^2}{2}\,\mathrm{d}x + (1-\lambda)\int_\Omega \gamma|S|^2\,\mathrm{d}x + (1-\lambda)\left|\int_\Omega \mu\,\mathrm{d}x\right|^2 \\
&+ \frac{1-\lambda}{h}\int_\Omega \frac{|\varphi|^2}{2} - \frac{|\varphi_k|^2}{2}\,\mathrm{d}x + \frac{1}{h}\int_\Omega \frac{|\nabla\varphi|^2}{2} - \frac{|\nabla\varphi_k|^2}{2}\,\mathrm{d}x + \frac{1}{h}\int_\Omega W_\kappa(\varphi) - W_\kappa(\varphi_k)\,\mathrm{d}x \\
&- \frac{\lambda}{h}\int_\Omega \kappa\frac{\varphi^2 - \varphi_k^2}{2}\,\mathrm{d}x \leq 0.
\end{aligned}$$

Furthermore, rearranging terms results in

$$\begin{aligned}
&\int_\Omega 2\nu(\varphi_k)|(\nabla v)_{\text{sym}}|^2\,\mathrm{d}x - \langle f_{k+1}, v\rangle_{H^1} + \int_\Omega \gamma|\nabla S|^2\,\mathrm{d}x + \langle \xi^k, S\rangle_{H^1_{\mathrm{sym,Tr}}} + \int_\Omega |\nabla\mu|^2\,\mathrm{d}x \\
&+ \frac{\lambda}{h}\int_\Omega \frac{|S|^2}{2}\,\mathrm{d}x + (1-\lambda)\int_\Omega \gamma|S|^2\,\mathrm{d}x + (1-\lambda)\left|\int_\Omega \mu\,\mathrm{d}x\right|^2 + \frac{1}{h}\int_\Omega \frac{|\nabla\varphi|^2}{2}\,\mathrm{d}x + \frac{1}{h}\int_\Omega W(\varphi)\,\mathrm{d}x \\
&\leq \frac{\lambda}{h}\int_\Omega \rho_k\frac{|v_k|^2}{2}\,\mathrm{d}x + \frac{\lambda}{h}\int_\Omega \frac{|S_k|^2}{2}\,\mathrm{d}x + \frac{1-\lambda}{h}\int_\Omega \frac{|\varphi_k|^2}{2}\,\mathrm{d}x + \frac{1}{h}\int_\Omega \frac{|\nabla\varphi_k|}{2}\,\mathrm{d}x + \frac{1}{h}\int_\Omega W_\kappa(\varphi_k)\,\mathrm{d}x\,.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&h\int_\Omega 2\nu(\varphi_k)|(\nabla v)_{\text{sym}}|^2\,\mathrm{d}x + h\int_\Omega \gamma|\nabla S|^2\,\mathrm{d}x + h\langle \xi^k, S\rangle_{H^1_{\mathrm{sym,Tr}}} + h\int_\Omega |\nabla\mu|^2\,\mathrm{d}x + \int_\Omega \frac{|\nabla\varphi|^2}{2}\,\mathrm{d}x \\
&+ \lambda\int_\Omega \frac{|S|^2}{2}\,\mathrm{d}x + (1-\lambda)h\int_\Omega \gamma|S|^2\,\mathrm{d}x + h(1-\lambda)\left|\int_\Omega \mu\,\mathrm{d}x\right|^2 + \int_\Omega W(\varphi)\,\mathrm{d}x \\
&\leq C_k + h\langle f_{k+1}, v\rangle_{H^1},
\end{aligned}$$

where C_k is a positive constant depending on k. Notice that by the invertibility of $\mathscr{L}_k$, $w = (v, S, \varphi, \mu) = \mathscr{L}_k^{-1}(u) \in X$ so that $\varphi \in \mathrm{dom}(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa})$. Hence $\varphi \in [-1,1]$. Moreover, due to the continuity of W on $[-1,1]$, $W(\varphi)$ is bounded both from above and from below. Therefore, it can be absorbed by the constant C_k on the right-hand side, i.e.,

$$\begin{aligned}
&h\int_\Omega 2\nu(\varphi_k)|(\nabla v)_{\text{sym}}|^2\,\mathrm{d}x + h\int_\Omega \gamma|\nabla S|^2\,\mathrm{d}x + h\langle \xi^k, S\rangle_{H^1_{\mathrm{sym,Tr}}} + h\int_\Omega |\nabla\mu|^2\,\mathrm{d}x + \int_\Omega \frac{|\nabla\varphi|^2}{2}\,\mathrm{d}x \\
&+ \lambda\int_\Omega \frac{|S|^2}{2}\,\mathrm{d}x + h(1-\lambda)\int_\Omega \gamma|S|^2\,\mathrm{d}x + h(1-\lambda)\left|\int_\Omega \mu\,\mathrm{d}x\right|^2 \leq C_k + \langle f_{k+1}, v\rangle_{H^1}.
\end{aligned}$$

Notice that for $\lambda \in [0, \frac{1}{2}]$, the term $h(1-\lambda)\int_\Omega \gamma|S|^2\,\mathrm{d}x$ gives an estimate on $\int_\Omega |S|^2\,\mathrm{d}x$. Similarly, for $\lambda \in (\frac{1}{2}, 1]$, the term $\lambda\int_\Omega \frac{|S|^2}{2}\,\mathrm{d}x$ also gives an estimate on $\int_\Omega |S|^2\,\mathrm{d}x$. Moreover, since $\gamma > 0$ and ν is bounded from below by a positive constant by (2.8), we finally arrive at

$$\begin{aligned}
&\int_\Omega |\nabla v|^2\,\mathrm{d}x + \int_\Omega |\nabla S|^2\,\mathrm{d}x + \int_\Omega |S|^2\,\mathrm{d}x + \langle \xi^k, S\rangle_{H^1_{\mathrm{sym,Tr}}} + \int_\Omega |\nabla\mu|^2\,\mathrm{d}x \\
&+ \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + (1-\lambda)\left|\int_\Omega \mu\,\mathrm{d}x\right|^2 \leq C_k + \langle f_{k+1}, v\rangle_{H^1}.
\end{aligned}$$

Combined with $\varphi \in [-1,1]$ and Poincaré's inequality, we obtain

$$\|v\|_{H^1} + \|S\|_{H^1} + \widetilde{\mathcal{P}}(\varphi_k; S) + \|\varphi\|_{H^1} + \|\nabla\mu\|_{L^2} + \sqrt{1-\lambda}\left|\int_\Omega \mu \,\mathrm{d}x\right| \le C_k. \tag{4.41}$$

It remains to find an H^2-estimate for μ. By (4.34), this is equivalent to finding an H^1-estimate. Since we already have an estimate on $\|\nabla\mu\|_{L^2}$, it remains to find an L^2-estimate for μ. Again by Poincaré's inequality, we have

$$\|\mu\|_{L^2} \le \|\nabla\mu\|_{L^2} + \left|\int_\Omega \mu \,\mathrm{d}x\right|.$$

Hence, it is sufficient to find a bound for $\left|\int_\Omega \mu \,\mathrm{d}x\right|$. To this end, first consider $\lambda \in [0, \frac{1}{2})$. Then we directly have

$$\frac{\sqrt{2}}{2}\left|\int_\Omega \mu \,\mathrm{d}x\right| \le \sqrt{1-\lambda}\left|\int_\Omega \mu \,\mathrm{d}x\right| \le C_k.$$

For $\lambda \in [\frac{1}{2}, 1]$, $\frac{1}{2}\left|\int_\Omega \mu \,\mathrm{d}x\right| \le \lambda\left|\int_\Omega \mu \,\mathrm{d}x\right|$. Repeating the argument as in proving (4.6) to equation (4.40d) and notice $\|\nabla\mu\|_{L^2}$ and $\|\nabla\varphi\|_{L^2}$ are bounded by C_k due to (4.41), we can get

$$\left|\int_\Omega \mu \,\mathrm{d}x\right| \le C_k.$$

Therefore, (4.41) can be improved to

$$\|v\|_{H^1} + \|S\|_{H^1} + \widetilde{\mathcal{P}}(\varphi_k; S) + \|\varphi\|_{H^1} + \|\mu\|_{H^2} \le C_k. \tag{4.42}$$

Moreover, from (4.39d), we have the additional estimate

$$\|\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi)\|_{L^2} \le (\lambda+1)\|\varphi\|_{L^2} + \lambda\|\mu\|_{L^2} + \frac{\lambda\kappa}{2}\left(\|\varphi\|_{L^2} + \|\varphi_k\|_{L^2}\right) \le C_k.$$

Altogether, we have the following estimate for $w = (v, S, \varphi, \mu)$

$$\|w\|_{\tilde{X}} + \|\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi)\|_{L^2} \le C_k,$$

with $\tilde{X}$ from (4.35a).

Boundedness of $\mathscr{K}_k$ in $\tilde{Y}$. It remains to show that $u = \mathscr{L}_k(w)$ is bounded in $\tilde{Y}$. Recall that $\mathscr{F}_k$ maps bounded sets in $\tilde{X}$ to bounded sets in $\tilde{Y}$ and that $u = \lambda\mathscr{F}_k(w)$. Therefore,

$$\|u\|_{\tilde{Y}} = \|\lambda\mathscr{F}_k(w)\|_{\tilde{Y}} \le C_k\left(\|w\|_{\tilde{X}} + 1\right) \le \tilde{C}_k.$$

where $\tilde{C}_k$ is a positive constant depending on k.

Now, since all assumptions in Schaefer's fixed point theorem have been satisfied, the existence of a fixed point of the operator $\mathscr{K}_k$ can be guaranteed. Hence, there exists a weak solution to (4.2c)-(4.2f). □

4.2 Existence of weak solutions

In Section 4.1, we have constructed a sequence of approximating solutions $(v_h, S_h, \varphi_h, \mu_h)_h$. In order to get a time-continuous solution, we perform the limit passage $h \to 0$. We follow the ideas of [2, Theorem 3.4] where the existence of weak solutions for the Cahn–Hilliard–Navier–Stokes system was shown. In our case here, we have to deal with the extra stress tensor which will need some more attention.

Theorem 4.4. Let $\gamma > 0$. Let v_0, S_0 and φ_0 satisfy Assumption 2.3. Let Assumption 2.4 be satisfied and let $\mathcal{P}$ fulfill Assumption 2.2. Then, there exists a weak solution (v, S, φ, μ) of (2.23) in the sense of Definition 2.7. Moreover, the phase-field variable φ takes values in $(-1, 1)$ a.e. in $\Omega \times (0, T)$.

Proof. **Step 1: Existence of discrete solutions at each time step $k = 0, 1 \ldots, N$.** We would like to apply Lemma 4.3 in order to conclude the existence of a weak solution $(v_k, S_k, \varphi_k, \mu_k)_k$ at each time step. For this, notice that the initial datum φ_0 is required to be in $H^2(\Omega)$. Therefore, we have to approximate φ_0 by functions in $H^2(\Omega)$. To this end, consider the following parabolic problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in}\, \Omega \times (0, T), \\ u(0) = \varphi_0 & \text{in}\, \Omega, \\ \vec{n} \cdot \nabla u = 0 & \text{on}\, \partial\Omega \times (0, T). \end{cases}$$

By the regularity theorem of parabolic equations e.g. [21, Chapter 7.1.3], there exists a solution $u \in L^2(0, T; H^2(\Omega))$, hence we can set $\varphi_0^N := u|_{t=\frac{1}{N}}$. Moreover, the maximum principle gives that $\left|\varphi_0^N\right| \le 1$ and we also have that

$$\varphi_0^N \to \varphi_0 \text{ in } H^1(\Omega) \tag{4.43}$$

(For details, see [21, Chapter 7.1]). Set the time-step size to be $h = \frac{T}{N}$ ($h = \frac{1}{N}$ when $T = \infty$) and $t_k = kh$ for all $k = 0, 1, \ldots, N$. Now, we can apply Lemma 4.3 with initial data (v_0, S_0, φ_0^N) to deduce the existence of approximating solutions $(v_k, S_k, \varphi_k, \mu_k)$ where $k = 1, \ldots, N$.

Step 2: Interpolation in time and weak formulation for the interpolations. Define $F^N(t)$ on $[0, T)$ to be the piecewise constant interpolation i.e.

$$F^N(t) := F_k \text{ for } t \in ((k-1)h, kh] \text{ and } F(0) = F_0, \tag{4.44a}$$

where $k \in \mathbb{N}$ and $F \in \{v, S, \varphi, \mu, \xi\}$. Also set

$$\rho^N := \frac{1}{2}(\rho_2 + \rho_1) + \frac{1}{2}(\rho_2 - \rho_1)\varphi^N. \tag{4.44b}$$

Moreover, we define

$$\partial_{t,h}^+ F^N(t) := \frac{1}{h}(F^N(t+h) - F^N(t)), \tag{4.44c}$$

$$\partial_{t,h}^- F^N(t) := \frac{1}{h}(F^N(t) - F^N(t-h)), \tag{4.44d}$$

$$F_h^N(t) := F^N(t-h). \tag{4.44e}$$

Notice that the approximating problem should be tested by static test functions, but, in the limit, we aim for a weak formulation which also involves time. For this purpose, for $\Phi \in C_{0,\mathrm{div}}^\infty(\Omega \times [0, T))$, we define the interpolations $\tilde{\Phi}^{k+1} = \int_{kh}^{(k+1)h} \Phi \,\mathrm{d}t$ and use $\tilde{\Phi}^{k+1}$ as a the test function in (4.2c). Sum over $k \in \{0, 1, \dots, N\}$, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_{t,h}^-(\rho^N v^N) \cdot \Phi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega (\rho_h^N v^N \otimes v^N) : \nabla\Phi \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega 2\nu(\varphi_h^N)(\nabla v^N)_{\mathsf{sym}} : (\nabla\Phi)_{\mathsf{sym}} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \eta(\varphi_h^N) S^N : (\nabla\Phi)_{\mathsf{sym}} \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^T \int_\Omega (v^N \otimes J^N) : \nabla\Phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mu^N \nabla\varphi_h^N \cdot \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle f^N, \Phi\rangle_{H^1} \,\mathrm{d}t\,,
\end{aligned} \tag{4.45a}$$

for all $\Phi \in C_{0,\mathrm{div}}^\infty(\Omega \times [0,T))$. Analogously, for all $\zeta \in C_0^\infty([0,T); C^1(\bar{\Omega}))$, we approximate ζ in the same way as above to get $\tilde{\zeta}^{k+1}$. Hence, testing (4.2e) with $\tilde{\zeta}^{k+1}$ and summing over $k \in \{0, 1, \dots, N\}$ gives

$$\int_0^T \int_\Omega \partial_{t,h}^-(\varphi^N)\zeta \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega v^N \varphi_h^N \cdot \nabla\zeta \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \nabla\mu^N \cdot \nabla\zeta \,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.45b}$$

and from (4.2f), we have

$$\mu^N + \frac{\kappa}{2}(\varphi^N + \varphi_h^N) = -\Delta\varphi^N + W_0'(\varphi^N), \tag{4.45c}$$

which holds almost everywhere in $\Omega \times (0,T)$.

Step 3: Energy-dissipation estimate and uniform a priori bound. For all $N \in \mathbb{N}$, for all $t \in (t_k, t_{k+1})$ and for all $k \in \{0, 1, \dots, N\}$, we define $E^N(t)$ to be the piecewise linear interpolation of total energy $\mathcal{E}_{\mathrm{tot}}(v_k, S_k, \varphi_k)$ i.e.

$$E^N(t) := \frac{(k+1)h - t}{h}\mathcal{E}_{\mathrm{tot}}(v_k, S_k, \varphi_k) + \frac{t - kh}{h}\mathcal{E}_{\mathrm{tot}}(v_{k+1}, S_{k+1}, \varphi_{k+1}), \tag{4.46}$$

and define $D^N(t)$ to be the piecewise constant dissipation for all $t \in (t_k, t_{k+1})$, i.e.,

$$D^N(t) := \int_\Omega 2\nu(\varphi_h^N)\big|(\nabla v^N)_{\mathsf{sym}}\big|^2 + \gamma\big|\nabla S^N\big|^2 + \big|\nabla\mu^N\big|^2 \,\mathrm{d}x - \langle f^N, v^N\rangle_{H^1} + \mathcal{P}(\varphi_h^N; S^N). \tag{4.47}$$

From (4.17), we can directly see for all $t \in (t_k, t_{k+1})$ that

$$-\frac{\mathrm{d}}{\mathrm{d}t}E^N(t) = \frac{E_{\mathrm{tot}}(v_k, S_k, \varphi_k) - E_{\mathrm{tot}}(v_{k+1}, S_{k+1}, \varphi_{k+1})}{h} \geq D^N(t). \tag{4.48}$$

Integrating (4.48) in each time interval (t_k, t_{k+1}) and summing over $k \in \{0, 1, \dots, N\}$ yields that

$$\begin{aligned}
\mathcal{E}_{\mathrm{tot}}(v_0, S_0, \varphi_0^N) \geq &\int_0^t \int_\Omega 2\nu(\varphi_h^N)\big|(\nabla v^N)_{\mathsf{sym}}\big|^2 + \gamma\big|\nabla S^N\big|^2 + \big|\nabla\mu^N\big|^2 \,\mathrm{d}x - \langle f^N, v^N\rangle_{H^1} \,\mathrm{d}\tau \\
&+ \int_0^t \mathcal{P}(\varphi_h^N; S^N) \,\mathrm{d}\tau + E^N(t),
\end{aligned} \tag{4.49}$$

for all $t \in [0,T)$. Since $\mathcal{E}_{\mathrm{tot}}(v_0, S_0, \varphi_0^N)$ is finite and by Assumption 2.3 that $f \in L_{\mathrm{loc}}^2([0,T); H^{-1}(\Omega)^3)$, and using (2.8), we deduce the following uniform bounds with respect to $N \in \mathbb{N}$:

$$(v^N)_N \text{ is bounded in } L^2(0,T'; H_{0,\mathrm{div}}^1(\Omega)) \text{ and in } L^\infty(0,T'; L_{\mathrm{div}}^2(\Omega)), \tag{4.50a}$$

$$(S^N)_N \text{ is bounded in } L^2(0,T';H^1_{\mathrm{sym,Tr}}(\Omega)) \text{ and in } L^\infty(0,T';L^2_{\mathrm{sym,Tr}}(\Omega)), \tag{4.50b}$$
$$(\varphi^N)_N \text{ is bounded in } L^\infty(0,T';H^1(\Omega)), \tag{4.50c}$$
$$(\nabla\mu^N)_N \text{ is bounded in } L^2(0,T';(L^2(\Omega))^3). \tag{4.50d}$$

for all $0 < T' < T$. Moreover, from the uniform bounds (4.50c) on $(\nabla\varphi^N)_N$ and (4.50d) on $(\nabla\mu^N)_N$, we can deduce the following estimate on $(\mu^N)_N$ with the aid of (4.6)

$$\int_0^{T'} \left| \int_\Omega \mu^N \,\mathrm{d}x \right| \mathrm{d}t \le C \cdot T' \text{ for all } 0 < T' < T, \tag{4.51}$$

where $C > 0$ is a constant.

Step 4: Immediate convergence results. By a classical diagonalization argument, we can extract a not relabeled subsequence and a limit quadruplet (v, S, φ, μ) such that for all $0 < T' < T$:

$$v^N \rightharpoonup v \text{ in } L^2(0,T';H^1(\Omega)), \tag{4.52a}$$
$$v^N \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T';L^2_{\mathrm{div}}(\Omega)), \tag{4.52b}$$
$$S^N \rightharpoonup S \text{ in } L^2(0,T';H^1(\Omega)), \tag{4.52c}$$
$$S^N \overset{*}{\rightharpoonup} S \text{ in } L^\infty(0,T';L^2_{\mathrm{sym,Tr}}(\Omega)), \tag{4.52d}$$
$$\varphi^N \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty(0,T';H^1(\Omega)), \tag{4.52e}$$
$$\mu^N \rightharpoonup \mu \text{ in } L^2(0,T';H^1(\Omega)). \tag{4.52f}$$

Step 5: Improved convergence results. Now we derive additional strong convergence results with the help of the Aubin-Lions lemma, see [39, Chapter 7.3] for details. This will require estimates on the time derivatives. To this end, we define the piecewise linear interpolations, i.e.,

$$\tilde{F}^N(t) := \frac{(k+1)h - t}{h} F_k + \frac{t - kh}{h} F_{k+1} \tag{4.53}$$

for $t \in [t_k, t_{k+1}]$ and $F \in \{\rho v, \varphi\}$; if $F = \rho v$, then $\tilde{F}^N := \widetilde{\rho v}^N$ with the piecewise linear interpolation of ρv and $F_k = \rho_k v_k$ with the solutions ρ_k, v_k at step k. By definition, $\partial_t \tilde{F}^N = \partial^-_{t,h} F^N$ and the bounds on the piecewise constant interpolations can be transferred to the piecewise linear interpolations, i.e., we have

$$(\widetilde{\rho v}^N)_N \text{ is bounded in } L^2(0,T;W_0^{1,\frac{3}{2}}(\Omega)) \text{ and in } L^\infty(0,T;L^2(\Omega)), \tag{4.54a}$$
$$(\tilde{\varphi}^N)_N \text{ is bounded in } L^\infty(0,T;H^1(\Omega)). \tag{4.54b}$$

Moreover, we can estimate the difference between the piecewise constant interpolation and the piecewise linear interpolation pointwise, i.e., we have

$$\left| \tilde{F}^N(t,x) - F^N(t,x) \right| \le h \left| \partial_t \tilde{F}^N(t,x) \right| \text{ almost everywhere in } \Omega \times (0,T). \tag{4.55}$$

We first derive a uniform bound on $(\partial_t \tilde{\varphi}^N)_N$ from (4.45b). Since $(v^N \varphi_h^N)_N$ and $(\nabla\mu^N)_N$ are bounded in $L^2(0,T;L^2(\Omega))$, the test function ζ can be chosen with $\nabla\zeta \in L^2(0,T;L^2(\Omega))$. Hence, $\zeta \in L^2(0,T;H^1(\Omega))$ is sufficient by comparison in (4.45b). This implies that $(\partial_t \tilde{\varphi}^N)_N$ is bounded in

$L^2(0,T;H^{-1}(\Omega))$. Besides, $(\tilde{\varphi}^N)_N$ is bounded in $L^\infty(0,T;H^1(\Omega))$ from (4.54b). Therefore, by Aubin-Lions Lemma, we get the strong convergence

$$\tilde{\varphi}^N \to \tilde{\varphi} \text{ in } L^2(0,T';L^2(\Omega))$$

as $N \to \infty$, for all $0 < T' < T$ and some $\tilde{\varphi} \in L^\infty(0,T;L^2(\Omega))$. The bound on $(\partial_t\tilde{\varphi}^N)_N$ in $L^2(0,T;H^{-1}(\Omega))$ and (4.55) give us that

$$\tilde{\varphi}^N - \varphi^N \to 0 \text{ in } L^2(0,T';H^{-1}(\Omega)).$$

as $N \to \infty$. Furthermore, we obtain

$$\varphi^N \to \tilde{\varphi} \text{ in } L^2(0,T';H^{-1}(\Omega))$$

as $N \to \infty$, for all $0 < T' < T$, which implies that $\varphi = \tilde{\varphi}$. Besides, by interpolation of Bochner spaces, see [11, Theorem 5.1.2] for details, the strong convergence of $(\varphi^N)_N$ in $L^2(0,T';H^{-1}(\Omega))$ and its bound in $L^2(0,T';H^1(\Omega))$ imply the strong convergence

$$\varphi^N \to \varphi \text{ in } L^2(0,T';L^2(\Omega)). \tag{4.56}$$

Next, we want to improve the strong convergence of $(\varphi^N)_N$ by interpolation inequalities and find convergence result for $(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi^N))_N$. By Lemma 4.2, we obtain that $(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi^N))_N$ is bounded in $L^2(0,T';L^2(\Omega))$ for all $0 < T' < T$. Moreover, in (4.45c), the right-hand side is actually $(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi^N))_N$, while the left-hand side weakly converges to $\mu + \kappa\varphi$ in $L^2(0,T';L^2(\Omega))$. Hence, we have

$$\begin{aligned}\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi^N) = -\Delta\varphi^N + W_\kappa'(\varphi^N) = \mu^N + \frac{\kappa}{2}(\varphi^N + \varphi_h^N)&\\ \rightharpoonup \mu + \kappa\varphi \text{ in } L^2(0,T';L^2(\Omega)).&\end{aligned}$$

This also implies that

$$\|\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi^N)\|_{L^2(0,T';L^2(\Omega))} \le C \tag{4.57}$$

for all $N \in \mathbb{N}$. Furthermore, by (2.20), we have

$$\|\varphi^N\|_{H^2}^2 + \|W_\kappa'(\varphi^N)\|_{L^2}^2 + \int_\Omega W_\kappa''(\varphi^N)\left|\nabla\varphi^N\right|^2 \mathrm{d}x \le C\left(\|\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi^N)\|_{L^2}^2 + \|\varphi^N\|_{L^2}^2 + 1\right),$$

The boundedness of $(\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi^N))_N$ together with the bounds on $(\varphi^N)_N$ in $L^2(0,T';L^2(\Omega))$ implies that

$$(\varphi^N)_N \text{ is bounded in } L^2(0,T';H^2(\Omega)). \tag{4.58}$$

Hence, by interpolation of Bochner spaces and the strong convergence of $(\varphi^N)_N$ in $L^2(0,T';L^2(\Omega))$ from (4.56), we obtain

$$\varphi^N \to \varphi \text{ in } L^2(0,T';H^1(\Omega)). \tag{4.59}$$

Since $(\rho^N)_N$ depends in an affine linear way on $(\varphi^N)_N$ by formula (2.7), we have the same convergence result for $(\rho^N)_N$.

Now, we derive a uniform bound on $(\partial_t(\widetilde{\rho v}^N))_N$ from (4.45a). For this, we argue by comparison in (4.45a) and therefore, we now derive uniform estimates for all the other terms in (4.45a). In particular, for all $0 < T' < T$, we have the following estimates:
Notice that $\rho_h^N v^N \otimes v^N$ contains terms of the form $\rho_h^N v_i^N v_j^N$. Since $(v^N)_N$ is bounded both in

$L^\infty(0,T';L^2(\Omega))$ by (4.50a) and $L^2(0,T';H^1(\Omega)) \hookrightarrow L^2(0,T';L^6(\Omega))$ and $(\rho_h^N)_N$ is bounded in $L^\infty(\Omega\times(0,T'))$, we obtain

$$\begin{aligned}\|\rho_h^N v^N\otimes v^N\|_{L^2(0,T';L^{\frac32}(\Omega))} &\le C\|\rho_h^N\|_{L^\infty(\Omega\times(0,T'))}\|v^N\|_{L^\infty(0,T';L^2(\Omega))}\|v^N\|_{L^2(0,T';L^6(\Omega))}\\ &\le C\|\rho^N\|_{L^\infty(\Omega\times(0,T'))}\|v^N\|_{L^\infty(0,T';L^2(\Omega))}\|v^N\|_{L^2(0,T';H^1(\Omega))},\end{aligned}$$

where we used that $\|\rho_h^N\|_{L^\infty(\Omega\times(0,T'))} = \|\rho^N\|_{L^\infty(\Omega\times(0,T'))}$.
Since $(v^N)_N$ is bounded in $L^2(0,T';H^1(\Omega))$ by (4.50a) and $(\nu(\varphi_h^N))_N$ is bounded from above by a positive constant thanks to (2.8), we obtain

$$\|2\nu(\varphi_h^N)(\nabla v^N)_{\text{sym}}\|_{L^2(0,T';L^2(\Omega))} \le C\|v^N\|_{L^2(0,T';H^1(\Omega))}.$$

Since $(S^N)_N$ is bounded in $L^2(0,T';H^1(\Omega))$ by (4.50b) and $(\eta(\varphi_h^N))_N$ is bounded from above by a positive constant again by (2.8), we have

$$\|\eta(\varphi_h^N)S^N\|_{L^2(0,T';L^2(\Omega))} \le C\|S^N\|_{L^2(0,T';H^1(\Omega))}.$$

Notice that $v^N\nabla\mu^N$ consists of terms of the form $v_i^N\partial_{x_j}\mu^N$. Since $(\nabla\mu^N)_N$ is bounded in $L^2(0,T';L^2(\Omega))$ by (4.50d) and $(v^N)_N$ is bounded in $L^\infty(0,T';L^2(\Omega))$ and in $L^2(0,T';H^1(\Omega)) \hookrightarrow L^2(0,T';L^6(\Omega))$ by (4.50a), we find

$$\|v_i^N\partial_{x_j}\mu^N\|_{L^2(0,T';L^1(\Omega))} \le C\|v_i^N\|_{L^\infty(0,T';L^2(\Omega))}\|\partial_{x_j}\mu^N\|_{L^2(0,T';L^2(\Omega))}$$

and

$$\|v_i^N\partial_{x_j}\mu^N\|_{L^1(0,T';L^{\frac32}(\Omega))} \le C\|v_i^N\|_{L^2(0,T';L^6(\Omega))}\|\partial_{x_j}\mu^N\|_{L^2(0,T';L^2(\Omega))},$$

which implies that $(v_i^N\partial_{x_j}\mu^N)_N$ is bounded in $L^2(0,T';L^1(\Omega))$ and in $L^1(0,T';L^{\frac32}(\Omega))$. Now, by interpolation of Bochner spaces, we obtain here

$$\Big(L^2(0,T';L^1(\Omega)), L^1(0,T';L^{\frac32}(\Omega))\Big)_\theta = L^{\frac87}(0,T';L^{\frac43}(\Omega)),$$

with $\theta = \frac34$. This implies

$$\|v^N\nabla\mu^N\|_{L^{\frac87}(0,T';L^{\frac43}(\Omega))} \le C\|v^N\nabla\mu^N\|^{1-\theta}_{L^2(0,T';L^1(\Omega))}\|v^N\nabla\mu^N\|^{\theta}_{L^1(0,T';L^{\frac32}(\Omega))}.$$

Since $(\mu^N)_N$ is bounded in $L^2(0,T';H^1(\Omega)) \hookrightarrow L^2(0,T';L^6(\Omega))$ by (4.50d) and (4.51). Moreover, $(\nabla\varphi_h^N)_N$ is bounded in $L^\infty(0,T';L^2(\Omega))$. Thus, we obtain

$$\begin{aligned}\|\mu^N\nabla\varphi_h^N\|_{L^2(0,T';L^{\frac32}(\Omega))} &\le C\|\mu^N\|_{L^2(0,T';L^6(\Omega))}\|\nabla\varphi_h^N\|_{L^\infty(0,T';L^2(\Omega))}\\ &\le C\|\mu^N\|_{L^2(0,T';H^1(\Omega))}\|\varphi^N\|_{L^\infty(0,T';H^1(\Omega))},\end{aligned}$$

where we again used $\|\nabla\varphi_h^N\|_{L^\infty(0,T';L^2(\Omega))} = \|\nabla\varphi^N\|_{L^\infty(0,T';L^2(\Omega))}$.

Altogether, all these terms are bounded in $L^{\frac87}(0,T';L^{\frac43}(\Omega))$, so we can choose test function Φ in (4.45a) such that

$$\Phi, \nabla\Phi \in \Big(L^{\frac87}(0,T';L^{\frac43}(\Omega))\Big)' = L^8(0,T';L^4(\Omega)).$$

Hence, choosing test functions $\Phi \in L^8(0,T';W^{1,4}(\Omega))$ is sufficient. This implies that $(\partial_t(\widetilde{\rho v}^N))_N$ is bounded in $L^{\frac87}(0,T';W^{-1,4}(\Omega))$. Besides, $(\widetilde{\rho v}^N)_N$ is bounded in $L^2(0,T';W^{1,\frac32}(\Omega))$ from (4.54a). Therefore, by the Aubin-Lions Lemma, we obtain the strong convergence

$$\widetilde{\rho v}^N \to \widetilde{\rho v} \text{ in } L^2(0,T';L^2(\Omega)),$$

as $N \to \infty$ for some $\widetilde{\rho v} \in L^\infty(0,T';L^2(\Omega)) \cap L^2(0,T';W^{1,\frac{3}{2}}(\Omega))$. Moreover, (4.55) and the bound on $(\partial_t(\widetilde{\rho v}^N))_N$ in $L^{\frac{8}{7}}(0,T';W^{-1,4}(\Omega))$ obtained above imply that

$$\widetilde{\rho v}^N - \rho^N v^N \to 0 \text{ in } L^{\frac{8}{7}}(0,T';W^{-1,4}(\Omega)).$$

as $N \to \infty$. Thus, we obtain

$$\rho^N v^N \to \widetilde{\rho v} \text{ in } L^{\frac{8}{7}}(0,T';W^{-1,4}(\Omega)),$$

as $N \to \infty$ which implies $\widetilde{\rho v} = \rho v$. Moreover, by interpolation of Bochner spaces, the strong convergence of $(\rho^N v^N)_N$ in $L^{\frac{8}{7}}(0,T';W^{-1,4}(\Omega))$ and the bound of $(\rho^N v^N)_N$ in $L^2(0,T';W^{1,\frac{3}{2}}(\Omega))$ imply the strong convergence in $L^{\frac{16}{11}}(0,T';L^{\frac{24}{11}}(\Omega))$ due to the choice of $\theta = \frac{1}{2}$ for the interpolation. By continuous embedding $L^{\frac{16}{11}}(0,T';L^{\frac{24}{11}}(\Omega)) \hookrightarrow L^{\frac{16}{11}}(0,T';L^2(\Omega))$, we get the strong convergence in $L^{\frac{16}{11}}(0,T';L^2(\Omega))$. By repeating this argument with $L^{\frac{16}{11}}(0,T';L^2(\Omega))$ and $L^\infty(0,T';L^2(\Omega))$, we arrive at the strong convergence

$$\rho^N v^N \to \rho v \text{ in } L^2(0,T';L^2(\Omega)). \tag{4.60}$$

Now, we want to obtain the strong convergence of $(v^N)_N$ from (4.60). To this end, first notice that

$$\begin{aligned}
&\left| \int_0^{T'} \int_\Omega \rho^N |v^N|^2 \,\mathrm{d}x\,\mathrm{d}t - \int_0^{T'} \int_\Omega \rho |v|^2 \,\mathrm{d}x\,\mathrm{d}t \right| \\
\leq &\left| \int_0^{T'} \int_\Omega (\rho^N v^N - \rho v) v^N \,\mathrm{d}x\,\mathrm{d}t \right| + \left| \int_0^{T'} \int_\Omega \rho v (v^N - v) \,\mathrm{d}x\,\mathrm{d}t \right|,
\end{aligned}$$

where the first term in the last line tends to 0 due to the strong convergence of $(\rho^N v^N)_N$ in $L^2(0,T';L^2(\Omega))$ by (4.60) and the second term tends to 0, thanks to the weak convergence of $(v^N)_N$ in $L^2(0,T';L^2(\Omega))$ by (4.52b). Besides, in combination with $(\rho^N)^{\frac{1}{2}} v^N \rightharpoonup (\rho)^{\frac{1}{2}} v$ in $L^2(0,T';L^2(\Omega))$, we obtain that $(\rho^N)^{\frac{1}{2}} v^N \to (\rho)^{\frac{1}{2}} v$ in $L^2(0,T';L^2(\Omega))$. Since $(\rho^N)_N \to \rho$ almost everywhere in $\Omega \times (0,T)$ and $0 < \rho_1 \leq \rho^N \leq \rho_2$, we obtain

$$v^N = (\rho^N)^{-\frac{1}{2}} (\rho^N)^{\frac{1}{2}} v^N \to v \text{ in } L^2(0,T';L^2(\Omega)). \tag{4.61}$$

Step 6: Limit passage in the weak formulation (4.45a)**-**(4.45c)**.** Next we want to pass (4.45a)-(4.45c) to the limit $N \to \infty$. Notice that for all divergence-free test functions Φ, we have the following relation

$$\begin{aligned}
\int_0^T \int_\Omega \mu^N \nabla \varphi_h^N \cdot \Phi \,\mathrm{d}x\,\mathrm{d}t &= - \int_0^T \int_\Omega \nabla \mu^N \varphi_h^N \cdot \Phi \,\mathrm{d}x\,\mathrm{d}t \\
&\to - \int_0^T \int_\Omega \nabla \mu \varphi \cdot \Phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mu \nabla \varphi \cdot \Phi \,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

Then the limit passage of (4.45a)-(4.45b) follows from the convergence results (4.43), (4.52), (4.59)-(4.61). To pass to the limit in (4.45c), we use (4.57) and the fact that $\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}$ is a maximal monotone operator and apply [42, Proposition IV.1.6]. Hence, we conclude (2.25e).

Step 7: The partial energy estimates. For all $N \in \mathbb{N}$, for all $t \in (t_k, t_{k+1})$ and for all $k \in \{0, 1, \dots, N\}$, define $E_{\mathrm{k}}^N(t)$ to be the piecewise linear interpolation of the kinetic energy, i.e.,

$$E_{\mathrm{k}}^N(t) := \frac{(k+1)h - t}{h}\mathcal{E}_{\mathrm{k}}(\varphi_k, v_k) + \frac{t - kh}{h}\mathcal{E}_{\mathrm{k}}(\varphi_{k+1}, v_{k+1}) \tag{4.62}$$

and define $D_{\mathrm{s}}^N(t)$ to be the piecewise constant dissipation as

$$\begin{aligned} D_{\mathrm{s}}^N(t) := \int_\Omega 2\nu(\varphi_h^N)\big|(\nabla v^N)_{\mathsf{sym}}\big|^2 \,\mathrm{d}x + \int_\Omega \eta(\varphi_h^N) S^N : \nabla v^N \,\mathrm{d}x \\ - \int_\Omega \mu^N (\nabla\varphi_h^N \cdot v^N)\,\mathrm{d}x - \langle f^N, v^N\rangle_{H^1}. \end{aligned}$$

for all $t \in [t_k, t_{k+1})$. Then from (4.21), we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{k}}^N(t) = -\frac{1}{h}\mathcal{E}_{\mathrm{k}}(\varphi_k, v_k) + \frac{1}{h}\mathcal{E}_{\mathrm{k}}(v_{k+1}) \geq D_{\mathrm{s}}^N(t). \tag{4.63}$$

Now, we multiply both sides of (4.63) by arbitrary $\phi \in \tilde{C}([0, T'])$, integrate over time, and use integration by parts. This results in

$$\mathcal{E}_k(\varphi_0, v_0) \geq -\int_0^{T'} \phi'(t)E_{\mathrm{k}}^N(t)\,\mathrm{d}t + \int_0^{T'} \phi(t)D_{\mathrm{s}}^N(t)\,\mathrm{d}t\,. \tag{4.64}$$

By the strong convergence of $(v^N)_N$ in $L^2(0, T'; L^2(\Omega))$ from (4.61) and the strong convergence of $(\varphi^N)_N$ in $L^2(0, T'; H^1(\Omega))$ from (4.59), we derive

$$\lim_{N\to\infty} -\int_0^{T'} \phi'(t)E_{\mathrm{k}}^N(t)\,\mathrm{d}t \to -\int_0^{T'} \phi'(t)\mathcal{E}_{\mathrm{k}}(\varphi(t), v(t))\,\mathrm{d}t\,. \tag{4.65}$$

Next, by the weak convergence of $(v^N)_N$ in $L^2(0, T'; H^1(\Omega))$ from (4.52a), the strong convergence of $(\varphi^N)_N$ in $L^2(0, T'; H^1(\Omega))$ from (4.59) and assumption (2.8), we obtain

$$\int_0^{T'} \phi \int_\Omega 2\nu(\varphi)\big|(\nabla v)_{\mathsf{sym}}\big|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \liminf_{N\to\infty} \int_0^{T'} \phi \int_\Omega 2\nu(\varphi_h^N)\big|(\nabla v^N)_{\mathsf{sym}}\big|^2 \,\mathrm{d}x\,\mathrm{d}s\,. \tag{4.66}$$

Moreover, by the strong convergence of $(v^N)_N$ in $L^2(0, T'; L^2(\Omega))$ from (4.61), the strong convergence of $(\varphi^N)_N$ in $L^2(0, T'; H^1(\Omega))$ from (4.59), the weak convergence of $(S^N)_N$ in $L^2(0, T'; H^1(\Omega))$ from (4.52c) and assumption (2.8), we derive

$$\begin{aligned} &\lim_{N\to\infty} \int_0^t \phi \int_\Omega \eta(\varphi_h^N) S^N : \nabla v^N \,\mathrm{d}x\,\mathrm{d}s \\ &= \lim_{N\to\infty} \left(-\int_0^t \phi \int_\Omega \eta'(\varphi_h^N) v^N \otimes \nabla\varphi_h^N : S^N \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \phi \int_\Omega \eta(\varphi_h^N)\mathsf{div}(S^N) \cdot v^N \,\mathrm{d}x\,\mathrm{d}s \right) \\ &= -\int_0^t \phi \int_\Omega \eta'(\varphi) v \otimes \nabla\varphi : S \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \phi \int_\Omega \eta(\varphi)\mathsf{div}(S) \cdot v \,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t \phi \int_\Omega \eta(\varphi) S : \nabla v \,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{4.67}$$

with the help of integration by parts and Assumption (2.8). Now, notice that we have the embedding $L^2(0, T'; H^1(\Omega)) \hookrightarrow L^2(0, T'; L^6(\Omega))$, so we have the weak convergence of $(\mu^N)_N$ in

$L^2(0,T';L^6(\Omega))$ from (4.52f). Now, we use the interpolation of Bochner spaces to improve the strong convergence of $(v^N)_N$ and $(\nabla\varphi^N)_N$. By choosing $\theta=\frac{2}{3}$, the strong convergence of $(v^N)_N$ in $L^2(0,T';L^2(\Omega))$ from (4.61) and bound of $(v^N)_N$ in $L^\infty(0,T';L^2(\Omega))$ from (4.50a) result in the strong convergence of $(v^N)_N$ in $L^6(0,T',L^2(\Omega))$. Then by choosing $\theta=\frac{1}{4}$ in the interpolation of Bochner spaces, the strong convergence of $(v^N)_N$ in $L^6(0,T',L^2(\Omega))$ and the bound of $(v^N)_N$ in $L^2(0,T';H^1(\Omega))\hookrightarrow L^2(0,T';L^6(\Omega))$ from (4.50a) yields the strong convergence of $(v^N)_N$ in $L^4(0,T';L^{\frac{12}{5}}(\Omega))$. A similar argument gives us the strong convergence of $(\nabla\varphi^N)_N$ in $L^4(0,T';L^{\frac{12}{5}}(\Omega))$. Therefore, by the weak-strong-strong convergence, we obtain

$$\lim_{N\to\infty}\int_0^{T'}\phi\int_\Omega\mu^N(\nabla\varphi_h^N\cdot v^N)\,\mathrm{d}x\,\mathrm{d}t=\int_0^{T'}\phi\int_\Omega\mu(\nabla\varphi\cdot v)\,\mathrm{d}x\,\mathrm{d}t \tag{4.68}$$

By the weak convergence of $(v^N)_N$ in $L^2(0,T';H^1(\Omega))$ from (4.52a), we have

$$\lim_{N\to+\infty}\int_0^t\phi\langle f^N,v^N\rangle_{H^1}\,\mathrm{d}s=\int_0^t\phi\langle f,v\rangle_{H^1}\,\mathrm{d}s\,. \tag{4.69}$$

Inserting (4.65)-(4.69) into (4.64) gives (2.25b). Similarly, we also prove (2.25f).

Step 8: The evolutionary variational inequality for the internal stress. Now, we show the evolutionary variational inequality for the internal stress. To this end, we derive

$$\begin{aligned}\mathcal{E}_\mathrm{e}(S_0)\geq&-\int_0^{T'}\phi'\mathcal{E}_e(S^N)\,\mathrm{d}t+\int_0^{T'}\phi\int_\Omega\gamma\left|\nabla S^N\right|^2\mathrm{d}x\,\mathrm{d}t\\&+\int_0^{T'}\phi\langle\xi^N,S^N\rangle_{H^1_{\mathrm{sym,Tr}}}\,\mathrm{d}t-\int_0^{T'}\phi\int_\Omega\eta(\varphi_h^N)S^N:(\nabla v^N)_{\mathsf{sym}}\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{4.70}$$

from (4.23) by using the same argument as for (4.64). Here, $\phi\in\tilde{C}([0,T'])$ is arbitrary. Then, for all $\tilde{S}\in C^\infty_{0,\mathrm{sym,Tr}}(\Omega\times[0,T))$, we choose $\tilde{S}^{k+1}=-\int_{kh}^{(k+1)h}\phi\tilde{S}\,\mathrm{d}t$ to be the test function in (4.2d) and sum over $k\in\{0,1\dots,N\}$ to derive

$$\begin{aligned}&-\int_0^{T'}\int_\Omega\partial^-_{t,h}(S^N):\phi\tilde{S}\,\mathrm{d}x\,\mathrm{d}t-\int_0^{T'}\phi\int_\Omega(v^N\cdot\nabla S^N):\tilde{S}\,\mathrm{d}x\,\mathrm{d}t\\&-\int_0^{T'}\phi\int_\Omega(S^N(\nabla v^N)_{\mathsf{skw}}-(\nabla v^N)_{\mathsf{skw}}S^N):\tilde{S}\,\mathrm{d}x\,\mathrm{d}t-\int_0^{T'}\phi\langle\xi^N:\tilde{S}\rangle_{H^1_{\mathrm{sym,Tr}}}\,\mathrm{d}t\\&-\int_0^{T'}\phi\int_\Omega\gamma\nabla S^N:\nabla\tilde{S}\,\mathrm{d}x\,\mathrm{d}t=-\int_0^{T'}\phi\int_\Omega\eta(\varphi_h^N)(\nabla v^N)_{\mathsf{sym}}:\tilde{S}\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Notice that

$$\langle\xi^N,S^N-\tilde{S}\rangle_{H^1_{\mathrm{sym,Tr}}}\geq\widetilde{\mathcal{P}}(\varphi_h^N;S^N)-\widetilde{\mathcal{P}}(\varphi_h^N;\tilde{S})=\mathcal{P}(\varphi_h^N;S^N)-\mathcal{P}(\varphi_h^N;\tilde{S}),$$

since $\xi^N\in\partial\widetilde{\mathcal{P}}(\varphi_h^N;S^N)$ and by (4.1). Then, by summing (4.70) and (4.71), we obtain

$$\begin{aligned}&-\int_0^{T'}\phi'\int_\Omega\frac{1}{2}\left|S^N\right|^2\mathrm{d}x\,\mathrm{d}t-\int_0^{T'}\int_\Omega\partial^-_{t,h}(S^N):\phi\tilde{S}\,\mathrm{d}x\,\mathrm{d}t\\&-\int_0^{T'}\phi\int_\Omega(v^N\cdot\nabla S^N):\tilde{S}+(S^N(\nabla v^N)_{\mathsf{skw}}-(\nabla v^N)_{\mathsf{skw}}S^N):\tilde{S}\,\mathrm{d}x\,\mathrm{d}t\\&+\int_0^{T'}\phi\left(\mathcal{P}(\varphi_h^N;S^N)-\mathcal{P}(\varphi_h^N;\tilde{S})\right)\mathrm{d}x\,\mathrm{d}t+\int_0^{T'}\phi\int_\Omega\gamma\nabla S^N\,\vdots\,(\nabla S^N-\nabla\tilde{S})\,\mathrm{d}x\,\mathrm{d}t\\&-\int_0^{T'}\phi\int_\Omega\eta(\varphi_h^N)(\nabla v^N)_{\mathsf{sym}}:(S^N-\tilde{S})\,\mathrm{d}x\,\mathrm{d}t\leq\int_\Omega\frac{1}{2}|S_0|^2\,\mathrm{d}x\,.\end{aligned} \tag{4.71}$$

Now, we carry out the limit passage in (4.71). Since $\phi' \le 0$, the functional

$$S \mapsto -\int_0^{T'} \phi' \int_\Omega \frac{1}{2}|S(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{4.72}$$

is convex and continuous on $L^2(0,T';L^2(\Omega))$. Therefore, this functional is weakly lower semicontinuous on $L^2(0,T';L^2(\Omega))$, which implies

$$-\int_0^{T'} \phi' \int_\Omega \frac{1}{2}|S(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \le \liminf_{N\to\infty} \left(-\int_0^{T'} \phi' \int_\Omega \frac{1}{2}\left|S^N(t)\right|^2 \mathrm{d}x\,\mathrm{d}t \right), \tag{4.73}$$

thanks to the weak* convergence of $(S^N)_N$ in $L^\infty(0,T';L^2(\Omega))$ from (4.52d). Moreover, by the weak convergence of $(S^N)_N$ in $L^2(0,T';H^1(\Omega))$ from (4.52c) and with the help of an integration by parts for difference quotients, we deduce

$$\begin{aligned} & -\int_0^{T'} \int_\Omega \partial_{t,h}^-(S^N) : \phi\tilde{S} \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_{-h}^0 \int_\Omega S_0 : \frac{1}{h}\phi(t+h)\tilde{S}(t+h)\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'} \int_\Omega S^N : \partial_{t,h}^+(\phi\tilde{S})\,\mathrm{d}x\,\mathrm{d}t \\ &\to -\int_\Omega S_0 : \tilde{S}(0)\,\mathrm{d}x + \int_0^{T'} \int_\Omega S : \partial_t(\phi\tilde{S})\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{4.74}$$

for all $\tilde{S} \in C^\infty_{0,\mathrm{sym},\mathrm{Tr}}(\Omega \times [0,T))$. Next, by the weak convergence of $(S^N)_N$ in $L^2(0,T';H^1(\Omega))$ from (4.52c), we have

$$\int_0^{T'} \phi \int_\Omega -\nabla S^N \vdots \nabla\tilde{S}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^{T'} \phi \int_\Omega -\nabla S^N \vdots \nabla\tilde{S}\,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.75}$$

as well as

$$\int_0^{T'} \phi \int_\Omega \nabla S \vdots \nabla S\,\mathrm{d}x\,\mathrm{d}t \le \liminf_{N\to\infty} \left(\int_0^{T'} \phi \int_\Omega \nabla S^N \vdots \nabla S^N\,\mathrm{d}x\,\mathrm{d}t \right), \tag{4.76}$$

where we have used the same argument as in proving (4.73) for the last inequality.
To see the limit passage regarding $\mathcal{P}(\varphi_h^N; S^N)$, consider the functional

$$(\varphi, S) \mapsto \int_0^{T'} \phi(t)\mathcal{P}(\varphi(t);S(t))\,\mathrm{d}t = \int_0^{T'} \int_\Omega \phi(t)P(x,\varphi(t,x),S(t,x))\,\mathrm{d}x\,\mathrm{d}t\,.$$

By the weak* convergence of $(S^N)_N$ in $L^\infty(0,T';L^2(\Omega))$ from (4.52c) and strong convergence of $(\varphi^N)_N$ in $L^2(0,T';H^1(\Omega))$ from (4.59) and applying [29, Theorem 3], we arrive at

$$\int_0^{T'} \phi\mathcal{P}(\varphi;S)\,\mathrm{d}t \le \liminf_{N\to\infty} \int_0^{T'} \phi\mathcal{P}(\varphi_h^N;S^N)\,\mathrm{d}t\,. \tag{4.77}$$

By the continuity of the mapping $\varphi \mapsto P(\varphi;S)$ for all $S \in L^2_{\mathrm{sym},\mathrm{Tr}}(\Omega)$ and the strong convergence of $(\varphi^N)_N$ in $L^2(0,T';H^1(\Omega))$ from (4.59), we also have

$$-\int_0^{T'} \phi\mathcal{P}(\varphi;\tilde{S})\,\mathrm{d}t \le \liminf_{N\to\infty} \left(-\int_0^{T'} \phi\mathcal{P}(\varphi_h^N;\tilde{S})\,\mathrm{d}t \right), \tag{4.78}$$

with the help of Fatou's lemma and the fact that the integrand is bounded from below. Notice that by the weak convergence of $(S^N)_N$ in $L^2(0,T';H^1(\Omega))$ from (4.52c), the strong convergence of $(v^N)_N$ in $L^2(0,T';L^2(\Omega))$ from (4.61), the strong convergence of $(\varphi^N)_N$ in $L^2(0,T';H^1(\Omega))$ from (4.59) and assumption (2.8), we obtain

$$\begin{aligned} &\int_0^{T'} \phi \int_\Omega (v^N \cdot \nabla S^N) : \tilde{S} + (S^N (\nabla v^N)_{\mathsf{skw}} - (\nabla v^N)_{\mathsf{skw}} S^N) : \tilde{S} \,\mathrm{d}x\,\mathrm{d}t \\ \to &\int_0^{T'} \phi \int_\Omega (v \cdot \nabla S) : \tilde{S} + (S (\nabla v)_{\mathsf{skw}} - (\nabla v)_{\mathsf{skw}} S) : \tilde{S} \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.79}$$

as well as

$$\int_0^{T'} \phi \int_\Omega \eta(\varphi_h^N)(\nabla v^N)_{\mathsf{sym}} : (S^N - \tilde{S}) \,\mathrm{d}x\,\mathrm{d}t \to \int_0^{T'} \phi \int_\Omega \eta(\varphi)(\nabla v)_{\mathsf{sym}} : (S - \tilde{S}) \,\mathrm{d}x\,\mathrm{d}t \tag{4.80}$$

with the help of an integration by parts. Inserting (4.73)-(4.80) into (4.71) and adding both sides

$$-\int_0^{T'} \phi' \int_\Omega \frac{1}{2}\Big|\tilde{S}(t)\Big|^2 \,\mathrm{d}x\,\mathrm{d}t - \int_0^{T'} \phi \int_\Omega \partial_t \tilde{S} : \tilde{S} \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \frac{1}{2}\Big|\tilde{S}(0)\Big|^2 \,\mathrm{d}x\,,$$

we obtain

$$\begin{aligned} &-\int_0^{T'} \phi' \int_\Omega \frac{1}{2}\Big|S(t) - \tilde{S}(t)\Big|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^{T'} \phi \int_\Omega \partial_t \tilde{S} : (S - \tilde{S}) - v \cdot \nabla S : \tilde{S} - (S(\nabla v)_{\mathsf{skw}} - (\nabla v)_{\mathsf{skw}} S) : \tilde{S} \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^{T'} \phi \Big(\mathcal{P}(\varphi; S) - \mathcal{P}(\varphi; \tilde{S})\Big) \,\mathrm{d}t \\ &+ \int_0^{T'} \phi \int_\Omega \gamma \nabla S : \nabla (S - \tilde{S}) - \eta(\varphi)(\nabla v)_{\mathsf{sym}} : (S - \tilde{S}) \,\mathrm{d}x\,\mathrm{d}t \\ \le &\frac{1}{2}\|S_0 - \tilde{S}(0)\|_{L^2}^2 \end{aligned} \tag{4.81}$$

Finally, by applying Lemma 3.5 to (4.81), we arrive at (2.25c).

Step 9: φ takes value in $(-1,1)$ a.e. in $\Omega \times (0,T)$. By the total energy-dissipation estimate (2.26), one can see that

$$\int_\Omega W(\varphi(t)) \,\mathrm{d}x \le C$$

for a.e. $t \in (0,T)$. With the aid of (2.9), this implies that $\varphi \in [-1,1]$ a.e. in $\Omega \times (0,T)$. Now, we show further that $\varphi \in (-1,1)$ a.e. in $\Omega \times (0,T)$. To see this, recall that by (4.50d) and (4.51), μ is bounded in $L^2(0,T';L^2(\Omega))$ for a.e. $0 < T' < T$. Besides, we have

$$\begin{aligned} \|W'(\varphi)\|_{L^2(0,T';L^2(\Omega))} &= \|\mu + \Delta\varphi\|_{L^2(0,T';L^2(\Omega))} \\ &\le \|\mu\|_{L^2(0,T';L^2(\Omega))} + \|\Delta\varphi\|_{L^2(0,T';L^2(\Omega))}. \end{aligned}$$

Since φ is bounded in $L^2(0,T';H^2(\Omega))$ by (4.58), $W'(\varphi)$ is bounded in $L^2(0,T';L^2(\Omega))$. Thanks to (2.9), we derive that $\varphi \neq -1, 1$ a.e. in $\Omega \times (0,T')$. Since it holds true for a.e. $0 < T' < T$, we conclude that $\varphi \in (-1,1)$ a.e. in $\Omega \times (0,T)$. This proves the assertion. □

Remark 4.5. By testing (2.25e) with $\varphi - \varphi_\Omega$ and repeating the argument as in Lemma 4.2, we obtain an estimate

$$\|W'_\kappa(\varphi)\|_{L^2} + \left|\int_\Omega \mu \,\mathrm{d}x\right| \le C\left(\|\nabla\mu\|_{L^2} + \|\nabla\varphi\|_{L^2}^2 + 1\right)$$
$$\|\mathrm{D}\mathcal{E}_{\mathrm{pf},\kappa}(\varphi)\|_{L^2} \le C\left(\|\mu\|_{L^2} + 1\right)$$

Remark 4.6. By Proposition 3.6, one can see that a weak solution obtained from Theorem 4.4 is also a dissipative solution for any regularity weight $\mathcal{K}$.

5 Global existence result for the non-regularized two-phase system $\gamma = 0$

In Section 4, we have shown that, for all $\gamma > 0$, there exists a dissipative solution (v, S, φ, μ). Now, we will prove the existence of a dissipative solution for the case $\gamma = 0$ by carrying out the limit passage $\gamma \to 0$ in this notion of solution.

Theorem 5.1. Let $\gamma = 0$. Let v_0, S_0 and φ_0 satisfy Assumption 2.3. Let Assumption 2.4 be satisfied, and let $\mathcal{P}$ satisfy Assumption 2.2. Then there exists a dissipative solution (v, S, φ, μ) of type $\mathcal{K}$ to the limit system (2.23) with $\gamma = 0$ in the sense of Definition 3.3, where the regularity weight is given by

$$\mathcal{K}(\tilde{S}) = \frac{k_\Omega^2}{\nu_1}\|\tilde{S}\|_{L^\infty}^2 \tag{5.1}$$

for all $(\tilde{v}, \tilde{S}, \tilde{\varphi}) \in \mathfrak{T}$. Here, the constant $k_\Omega > 0$ is the constant from Korn's inequality and ν_1 is the constant from Assumption (2.8).

Proof. For $\gamma > 0$, there exists a dissipative solution $(v_\gamma, S_\gamma.\varphi_\gamma, \mu_\gamma)$ of type $\mathcal{K}$ thanks to Theorem 4.4 and Remark 4.6. Now, we want to take the limit $\gamma \to 0$. From Theorem 4.4, we have the energy inequality

$$\mathcal{E}_{\mathrm{tot}}(v_\gamma(t), S_\gamma(t), \varphi_\gamma(t)) + \int_0^t \mathcal{D}_{\mathrm{tot}}(v_\gamma, S_\gamma, \varphi_\gamma, \mu_\gamma)\,\mathrm{d}\tau \le \mathcal{E}_{\mathrm{tot}}(v_0, S_0, \varphi_0) + \int_0^t \langle f, v_\gamma\rangle_{H^1}\,\mathrm{d}\tau\,. \tag{5.2}$$

Step 1: Compactness. This estimate gives us the following $\gamma-$uniform bounds:

$$(v_\gamma)_\gamma \text{ is bounded in } L^2(0,T';H^1_{0,\mathrm{div}}(\Omega)) \text{ and in } L^\infty(0,T';L^2_{\mathrm{div}}(\Omega)), \tag{5.3a}$$
$$(S_\gamma)_\gamma \text{ is bounded in } L^\infty(0,T';L^2_{\mathrm{sym,Tr}}(\Omega)), \tag{5.3b}$$
$$(\varphi_\gamma)_\gamma \text{ is bounded in } L^\infty(0,T';H^1(\Omega)), \tag{5.3c}$$
$$(\nabla\mu_\gamma)_\gamma \text{ is bounded in } L^2(0,T';(L^2(\Omega))^3), \tag{5.3d}$$

for a.e. $0 < T' < T$. Moreover, from Remark 4.5 and the uniform bounds (5.3c) on $\nabla\varphi_\gamma$ and (5.3d) on $\nabla\mu_\gamma$, we deduce the following estimate on $(\mu_\gamma)_\gamma$

$$\int_0^{T'} \left|\int_\Omega \mu_\gamma \,\mathrm{d}x\right| \mathrm{d}t \le C\cdot T' \text{ for a.e. } 0 < T' < T, \tag{5.4}$$

where $C > 0$ is a constant. Then, by a classical diagonalization argument, we can extract a not relabeled subsequence and a limit quadruplet (v, S, φ, μ) such that

$$v_\gamma \rightharpoonup v \text{ in } L^2(0,T';H^1(\Omega)), \tag{5.5a}$$
$$v_\gamma \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T';L^2_{\mathrm{div}}(\Omega)), \tag{5.5b}$$
$$S_\gamma \overset{*}{\rightharpoonup} S \text{ in } L^\infty(0,T';L^2_{\mathrm{sym,Tr}}(\Omega)), \tag{5.5c}$$
$$\varphi_\gamma \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty(0,T';H^1(\Omega)), \tag{5.5d}$$
$$\mu_\gamma \rightharpoonup \mu \text{ in } L^2(0,T';H^1(\Omega)), \tag{5.5e}$$

for a.e. $0 < T' < T$. Moreover, thanks to (2.20), we also know that $(\varphi_\gamma)_\gamma$ is bounded in $L^2(0,T';H^2(\Omega))$. This yields

$$\varphi_\gamma \rightharpoonup \varphi \text{ in } L^2(0,T';H^2(\Omega)). \tag{5.5f}$$

Besides, by repeating arguments in the proof of Theorem 4.4, we conclude the following strong convergence results for a.e. $0 < T' < T$

$$\varphi_\gamma \to \varphi \text{ in } L^2(0,T';H^1(\Omega)), \tag{5.6a}$$
$$v_\gamma \to v \text{ in } L^2(0,T';L^2(\Omega)). \tag{5.6b}$$

Step 2: Limit passage $\gamma \to 0$ in the relative energy-dissipation estimate. By Lemma 3.5, we can transform the relative energy-dissipation estimate (3.16b) into the following weak form

$$\begin{aligned}
&-\int_0^{T'} \phi'\left(\mathcal{R}(v_\gamma, S_\gamma, \varphi_\gamma|\tilde v, \tilde S, \tilde\varphi\right)\mathrm{d}s \\
&\quad+\int_0^{T'} \phi\Big(\left\langle \mathcal{A}_\gamma(\tilde v, \tilde S, \tilde\varphi), \begin{pmatrix} v_\gamma - \tilde v \\ S_\gamma - \tilde S \\ -\Delta(\varphi_\gamma - \tilde\varphi) + W''(\tilde\varphi)(\varphi_\gamma - \tilde\varphi) + \kappa(\varphi_\gamma - \tilde\varphi) - \frac{\rho_2-\rho_1}{2}(v_\gamma - \tilde v)\tilde v \end{pmatrix}\right\rangle_{\mathbb{Y}} \\
&\quad+\mathcal{P}(\varphi_\gamma; S_\gamma) - \mathcal{P}(\varphi_\gamma; \tilde S) + \mathcal{W}^{(\mathcal{K})}_\gamma(v_\gamma, S_\gamma, \varphi_\gamma|\tilde v, \tilde S, \tilde\varphi)\Big)\exp\left(\int_s^{T'} \mathcal{K}(\tilde v, \tilde S, \tilde\varphi)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\le \mathcal{R}(v_0, S_0, \varphi_0|\tilde v(0), \tilde S(0), \tilde\varphi(0))\exp\left(\int_0^{T'} \mathcal{K}(\tilde v, \tilde S, \tilde\varphi)\,\mathrm{d}s\right)
\end{aligned} \tag{5.7}$$

for all $\phi \in \tilde C([0,t])$.

First notice that, for the limit passage regarding the term $\mathcal{P}(\varphi_\gamma; S_\gamma) - \mathcal{P}(\varphi_\gamma; \tilde S)$ can be shown in a similar way as deriving (4.77) and (4.78).

Now, we investigate the limit passage in the terms involving operator $\mathcal{A}_\gamma$. To this end, from (3.9b)-(3.9d), one can notice that the terms involving $(v_\gamma)_\gamma$, $(\nabla v_\gamma)_\gamma$, $(S_\gamma)_\gamma$, $(\varphi_\gamma)_\gamma$, $(\nabla\varphi_\gamma)_\gamma$ or $(\Delta\varphi_\gamma)_\gamma$ depend on these quantities only linearly, so that we can pass to limit by weak convergence results (5.5a)-(5.5f). The only exception is the term

$$\int_\Omega \gamma\nabla\tilde S(t) \,\vdots\, \nabla(S_\gamma(t) - \tilde S(t))\,\mathrm{d}x\,,$$

because we do not have any weak convergence result for $(\nabla S_\gamma)_\gamma$. Instead, notice that

$$
\begin{aligned}
&\int_0^{T'}\int_\Omega \gamma\nabla\tilde{S}(t) \vdots \nabla(S_\gamma(t)-\tilde{S}(t))\,\mathrm{d}x \\
\le&\sqrt{\gamma}\left(\int_0^{T'}\|\nabla\tilde{S}\|_{L^2}^2\,\mathrm{d}s\right)^{\frac12}\left(\gamma\int_0^{T'}\|\nabla S_\gamma\|_{L^2}^2\,\mathrm{d}s\right)^{\frac12}+\gamma\|\nabla\tilde{S}\|^2_{L^2(0,T';L^2(\Omega))},
\end{aligned}
\tag{5.8}
$$

where $\gamma\|\nabla S_\gamma\|_{L^2}^2 \le C$ uniformly in γ thanks to (2.26). Thus, the right-hand side of (5.8) tends 0 as $\gamma \to 0$.

Now, we consider the limit passage of the terms in $\mathcal{R}(v_\gamma, S_\gamma, \varphi_\gamma|\tilde{v}, \tilde{S}, \tilde{\varphi})$. By the strong convergence of $(v_\gamma)_\gamma$ in $L^2(0,T';L^2(\Omega))$ from (5.6b) and the strong convergence of $(\varphi_\gamma)_\gamma$ in $L^2(0,T';H^1(\Omega))$ from (5.6a), we obtain

$$
\lim_{\gamma\to0}\left(-\int_0^{T'}\phi'\int_\Omega\rho_\gamma\frac{|v_\gamma-\tilde{v}|^2}{2}\,\mathrm{d}x\,\mathrm{d}s\right)=-\int_0^{T'}\phi'\int_\Omega\rho\frac{|v-\tilde{v}|^2}{2}\,\mathrm{d}x\,\mathrm{d}s\,. \tag{5.9}
$$

Since $\phi'\le 0$, $\tilde{\varphi}\in C_0^\infty(\Omega\times[0,T))$ and $\tilde{\varphi}\in(-1,1)$, the functional

$$
(S,\varphi)\mapsto -\int_0^{T'}\phi'\int_\Omega\frac{\left|S-\tilde{S}\right|^2}{2}+\frac{|\nabla\varphi-\nabla\tilde{\varphi}|^2}{2}-W'(\tilde{\varphi})(\varphi-\tilde{\varphi})+\kappa|\varphi-\tilde{\varphi}|^2\,\mathrm{d}x\,\mathrm{d}s
$$

is convex and continuous on $L^2(0,T';L^2_{\mathrm{sym,Tr}}(\Omega))\times L^2(0,T';H^1(\Omega))$. Therefore, it is weakly lower semicontinuous on $L^2(0,T';L^2_{\mathrm{sym,Tr}}(\Omega))\times L^2(0,T';H^1(\Omega))$. Hence, by the weak*-convergence of $(S_\gamma)_\gamma$ in $L^\infty(0,T';L^2_{\mathrm{sym,Tr}}(\Omega))$ from (5.5c) and weak*-convergence of $(\varphi_\gamma)_\gamma$ in $L^\infty(0,T';H^1(\Omega))$ from (5.5d), we deduce

$$
\begin{aligned}
&-\int_0^{T'}\phi'\int_\Omega\frac{\left|S-\tilde{S}\right|^2}{2}+\frac{|\nabla\varphi-\nabla\tilde{\varphi}|^2}{2}-W'(\tilde{\varphi})(\varphi-\tilde{\varphi})+\kappa|\varphi-\tilde{\varphi}|^2\,\mathrm{d}x\,\mathrm{d}s\\
\le&\liminf_{\gamma\to0}\left(-\int_0^{T'}\phi'\int_\Omega\frac{\left|S_\gamma-\tilde{S}\right|^2}{2}+\frac{|\nabla\varphi_\gamma-\nabla\tilde{\varphi}|^2}{2}-W'(\tilde{\varphi})(\varphi_\gamma-\tilde{\varphi})+\kappa|\varphi_\gamma-\tilde{\varphi}|^2\,\mathrm{d}x\,\mathrm{d}s\right).
\end{aligned}
\tag{5.10}
$$

Moreover, since $\varphi_\gamma\in(-1,1)$ a.e. in $\Omega\times(0,T')$, by the continuity of W, we have $|W(\varphi_\gamma)|\le C$ a.e. in $\Omega\times(0,T')$ for all $\gamma>0$. Hence, with the help of the dominated convergence theorem, we derive

$$
\lim_{\gamma\to0}-\int_0^{T'}\phi'\int_\Omega W(\varphi_\gamma)\,\mathrm{d}x\,\mathrm{d}s=-\int_0^{T'}\phi'\int_\Omega W(\varphi)\,\mathrm{d}x\,\mathrm{d}s\,. \tag{5.11}
$$

Summing up (5.9)-(5.11) yields

$$
-\int_0^{T'}\phi'\mathcal{R}(v,S,\varphi|\tilde{v},\tilde{S},\tilde{\varphi})\,\mathrm{d}s\le\liminf_{\gamma\to0}\left(-\int_0^{T'}\phi'\mathcal{R}(v_\gamma,S_\gamma,\varphi_\gamma|\tilde{v},\tilde{S},\tilde{\varphi})\,\mathrm{d}s\right). \tag{5.12}
$$

Now, we turn to the limit passage in $\mathcal{W}_\gamma^{(\mathcal{K})}(v_\gamma,S_\gamma,\varphi_\gamma|\tilde{v},\tilde{S},\tilde{\varphi})$. For this, we recall from (3.14) that

$$
\mathcal{W}_\gamma^{(\mathcal{K})}(v,S,\varphi|\tilde{v},\tilde{S},\tilde{\varphi})=\mathcal{W}_0^{(\mathcal{K})}(v,S,\varphi|\tilde{v},\tilde{S},\tilde{\varphi})+\mathcal{D}_{\mathrm{sd},\gamma}(S-\tilde{S})\,,
$$

and we split $\mathcal{W}_0^{(\mathcal{K})}(v, S, \varphi|\tilde{v}, \tilde{S}, \tilde{\varphi})$ into two parts as follows:

$$\mathcal{W}_0^{(\mathcal{K})}(v_\gamma, S_\gamma, \varphi_\gamma|\tilde{v}, \tilde{S}, \tilde{\varphi}) = \mathfrak{Q}(v_\gamma, S_\gamma, \varphi_\gamma, \mu_\gamma|\tilde{v}, \tilde{S}, \tilde{\varphi}, \tilde{\mu}) + \mathfrak{R}(v_\gamma, S_\gamma, \varphi_\gamma, \mu_\gamma|\tilde{v}, \tilde{S}, \tilde{\varphi}, \tilde{\mu}), \tag{5.13}$$

where

$$\begin{aligned}\mathfrak{Q}(v, S, \varphi|\tilde{v}, \tilde{S}, \tilde{\varphi}) :=& \int_\Omega 2\nu_1 |(\nabla v)_{\text{sym}} - (\nabla \tilde{v})_{\text{sym}}|^2 + \frac{1}{2}|\nabla\mu - \nabla\tilde{\mu}|^2 + \frac{1}{2}|\nabla\mu|^2 \,\mathrm{d}x \\ & - \int_\Omega 2(S - \tilde{S})(\nabla v - \nabla\tilde{v})_{\text{skw}} : \tilde{S} \,\mathrm{d}x \\ & + \mathcal{K}(\tilde{v}, \tilde{S}, \tilde{\varphi}) \int_\Omega \frac{\left|S - \tilde{S}\right|^2}{2} + \frac{|\nabla\varphi - \nabla\tilde{\varphi}|^2}{2} + \kappa|\varphi - \tilde{\varphi}|^2 \,\mathrm{d}x\,,\end{aligned} \tag{5.14}$$

collects all the quadratic terms with respect to (v, S, φ, μ), and where

$$\begin{aligned}\mathfrak{R}(v, S, \varphi|\tilde{v}, \tilde{S}, \tilde{\varphi}) :=& \int_\Omega 2\nu(\varphi)|(\nabla v)_{\text{sym}} - (\nabla\tilde{v})_{\text{sym}}|^2 - 2\nu_1|(\nabla v)_{\text{sym}} - (\nabla\tilde{v})_{\text{sym}}|^2 \,\mathrm{d}x \\ & + \int_\Omega 2(\nu(\varphi) - \nu(\tilde{\varphi}))(\nabla\tilde{v})_{\text{sym}} : (\nabla v - \nabla\tilde{v}) - \frac{1}{2}|\nabla\tilde{\mu}|^2 \,\mathrm{d}x \\ & + \int_\Omega \Delta\tilde{\mu}\,(-\Delta(\varphi - \tilde{\varphi}) + W''(\tilde{\varphi})(\varphi - \tilde{\varphi}))\,\mathrm{d}x \\ & + \int_\Omega (v - \tilde{v}) \otimes (\rho v - \tilde{\rho}\tilde{v} + J - \tilde{J}) : \nabla\tilde{v} + (\rho - \tilde{\rho})(v - \tilde{v}) \cdot \partial_t\tilde{v} \,\mathrm{d}x \\ & - \int_\Omega (\eta(\varphi) - \eta(\tilde{\varphi}))(S - \tilde{S}) : \nabla\tilde{v} + (\eta(\varphi) - \eta(\tilde{\varphi}))\tilde{S} : (\nabla v - \nabla\tilde{v}) \,\mathrm{d}x \\ & - \int_\Omega (S - \tilde{S}) \otimes (v - \tilde{v}) \,\vdots\, \nabla\tilde{S} \,\mathrm{d}x \\ & - \int_\Omega \tilde{\mu}(\nabla\varphi - \nabla\tilde{\varphi}) \cdot (v - \tilde{v}) - (\nabla\varphi - \nabla\tilde{\varphi}) \otimes (\nabla\varphi - \nabla\tilde{\varphi}) : \nabla\tilde{v} \,\mathrm{d}x \\ & + \int_\Omega \kappa(\nabla\mu - \nabla\tilde{\mu}) \cdot (\nabla\varphi - \nabla\tilde{\varphi}) + \kappa(v - \tilde{v}) \cdot \nabla\tilde{\varphi}(\varphi - \tilde{\varphi}) \,\mathrm{d}x \\ & + \mathcal{K}(\tilde{v}, \tilde{S}, \tilde{\varphi}) \int_\Omega \rho\frac{|v - \tilde{v}|^2}{2} + W(\varphi) - W(\tilde{\varphi}) - W'(\tilde{\varphi})(\varphi - \tilde{\varphi}) \,\mathrm{d}x\end{aligned} \tag{5.15}$$

collects all the remaining terms.

Next, we discuss the limit passage for the quadratic terms in $\mathfrak{Q}$. Recall that the non-negativity of a quadratic form implies convexity, see [12, Proposition 3.71] for details. In order to show that $\mathfrak{Q}(v, S, \varphi|\tilde{v}, \tilde{S}, \tilde{\varphi})$ is non-negative, we make the following estimates:

$$\begin{aligned}\mathfrak{Q}(v, S, \varphi, \mu|\tilde{v}, \tilde{S}, \tilde{\varphi}, \tilde{\mu}) \geq& \int_\Omega 2\nu_1|(\nabla v)_{\text{sym}} - (\nabla\tilde{v})_{\text{sym}}|^2 + \frac{1}{2}|\nabla\mu - \nabla\tilde{\mu}|^2 + \frac{1}{2}|\nabla\mu|^2 \,\mathrm{d}x \\ & + \mathcal{K}(\tilde{v}, \tilde{S}, \tilde{\varphi}) \int_\Omega \frac{\left|S - \tilde{S}\right|^2}{2} + \frac{|\nabla\varphi - \nabla\tilde{\varphi}|^2}{2} + \kappa|\varphi - \tilde{\varphi}|^2 + |v - \tilde{v}|^2 \,\mathrm{d}x \\ & - \left|\int_\Omega 2(S - \tilde{S})(\nabla v - \nabla\tilde{v})_{\text{skw}} : \tilde{S} \,\mathrm{d}x\right|\end{aligned} \tag{5.16}$$

Next, we estimate the negative terms in (5.16) using Hölder's inequality, Korn's inequality and Young inequality:

$$\begin{aligned}\int_\Omega 2(S-\tilde S)(\nabla v-\nabla\tilde v)_{\mathsf{skw}}:\tilde S\,\mathrm{d}x \le& 2\|\tilde S\|_{L^\infty}\left(\int_\Omega\left|S-\tilde S\right|^2\mathrm{d}x\right)^{\frac12}\left(\int_\Omega|\nabla v-\nabla\tilde v|^2\,\mathrm{d}x\right)^{\frac12}\\ \le& 2k_\Omega\|\tilde S\|_{L^\infty}\left(\int_\Omega\left|S-\tilde S\right|^2\mathrm{d}x\right)^{\frac12}\left(\int_\Omega|(\nabla v)_{\mathsf{sym}}-(\nabla\tilde v)_{\mathsf{sym}}|^2\,\mathrm{d}x\right)^{\frac12}\\ \le& \frac{k_\Omega^2}{\nu_1}\|\tilde S\|_{L^\infty}^2\int_\Omega\frac{\left|S-\tilde S\right|^2}{2}\,\mathrm{d}x+2\nu_1\int_\Omega|(\nabla v)_{\mathsf{sym}}-(\nabla\tilde v)_{\mathsf{sym}}|^2\,\mathrm{d}x\,.\end{aligned} \tag{5.17}$$

By making use of the bounds (5.16) and (5.17), one can see that $\mathfrak{Q}$ is non-negative and therefore convex. Besides, notice that $\mathfrak{Q}$ is also a continuous functional on the space $H^1_{0,\mathrm{div}}(\Omega)\times L^2_{\mathrm{sym},\mathrm{Tr}}(\Omega)\times H^1(\Omega)\times H^1(\Omega)$, so $\mathfrak{Q}$ is weakly lower semicontinuous on this space. This allows us to pass to the limit $\gamma\to 0$, i.e.,

$$\begin{aligned}&\int_0^{T'}\phi\mathfrak{Q}(v,S,\varphi,\mu|\tilde v,\tilde S,\tilde\varphi,\tilde\mu)\exp\left(\int_s^{T'}\mathcal{K}(\tilde v,\tilde S,\tilde\varphi)\,\mathrm{d}\tau\right)\mathrm{d}s\\ \le&\liminf_{\gamma\to0}\int_0^{T'}\phi\mathfrak{Q}(v_\gamma,S_\gamma,\varphi_\gamma,\mu_\gamma|\tilde v,\tilde S,\tilde\varphi,\tilde\mu)\exp\left(\int_s^{T'}\mathcal{K}(\tilde v,\tilde S,\tilde\varphi)\,\mathrm{d}\tau\right)\mathrm{d}s\,.\end{aligned}$$

Now, we turn to the limit passage $\gamma\to0$ in $\mathfrak{R}$. Since $2\nu(\varphi_\gamma)-2\nu_1\ge0$ by (2.8), then by the weak convergence of $((\nabla v_\gamma)_{\mathsf{sym}})_\gamma$ in $L^2(0,T';L^2(\Omega))$ from (5.5a) and the convergence of $(\varphi_\gamma)_\gamma$ almost everywhere in $\Omega\times(0,T')$ obtained from (5.6a), we deduce

$$\begin{aligned}&\int_0^{T'}\phi\int_\Omega(2\nu(\varphi)-2\nu_1)|(\nabla v)_{\mathsf{sym}}-(\nabla\tilde v)_{\mathsf{sym}}|^2\,\mathrm{d}x\exp\left(\int_s^{T'}\mathcal{K}(\tilde v,\tilde S,\tilde\varphi)\,\mathrm{d}\tau\right)\mathrm{d}s\\ \le&\liminf_{\gamma\to0}\int_0^{T'}\phi\int_\Omega(2\nu(\varphi_\gamma)-2\nu_1)|(\nabla v_\gamma)_{\mathsf{sym}}-(\nabla\tilde v)_{\mathsf{sym}}|^2\,\mathrm{d}x\exp\left(\int_s^{T'}\mathcal{K}(\tilde v,\tilde S,\tilde\varphi)\,\mathrm{d}\tau\right)\mathrm{d}s\,.\end{aligned}$$

The limit passage of the remaining part of $\mathfrak{R}$ is a direct consequence of the convergence results (5.5a)-(5.6b) with the aid of (2.8). We exemplarily discuss one term and note that the remaining terms can be handled with in a similar way. We estimate

$$\begin{aligned}&\left|\int_0^{T'}\int_\Omega((v_\gamma-\tilde v)\otimes(\rho_\gamma v_\gamma-\tilde\rho\tilde v)-(v-\tilde v)\otimes(\rho v-\tilde\rho\tilde v)):\nabla\tilde v\,\mathrm{d}x\,\mathrm{d}s\right|\\ \le&\|\nabla\tilde v\|_{L^\infty(\Omega\times(0,T'))}\int_0^{T'}\int_\Omega|(v_\gamma-\tilde v)\otimes(\rho_\gamma v_\gamma-\tilde\rho\tilde v)-(v-\tilde v)\otimes(\rho v-\tilde\rho\tilde v)|\,\mathrm{d}x\,\mathrm{d}s\,,\end{aligned}$$

where we have used that $\tilde v\in C^\infty_{0,\mathrm{div}}(\Omega\times[0,T))$. The term on the right-hand side tends to zero, thanks to the strong convergence of $(v_\gamma)_\gamma$ in $L^2(0,T';L^2(\Omega))$ from (5.6b) and the strong convergence of $(\varphi_\gamma)_\gamma$ in $L^2(0,T';H^1(\Omega))$ from (5.6a).

In total, taking the limit $\gamma\to0$ in (5.7) implies (3.16b) with $\gamma=0$. Therefore, (v,S,φ,μ) is a dissipative solution to system (2.23) with $\gamma=0$. □

Remark 5.2. By Proposition 3.7 and Proposition 3.8, one can see that a dissipative solution (v, S, φ, μ) obtained from Theorem 5.1 satisfies weak formulation (2.25a) and (2.25d). Moreover, (v, S, φ, μ) also satisfies (2.25e) by using the same argument as in the proof of Theorem 4.4. In addition, φ takes value $(-1, 1)$ a.e. in $\Omega \times (0, T)$.

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