

Structure versus regularity of set-valued maps in convex generalized Nash equilibrium problems in Banach spaces

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ABSTRACT. A generalized Nash equilibrium problem (GNEP) in Banach space consists of $N > 1$ optimal control problems with couplings in both the objective functions and, most importantly, in the feasible sets. We address the existence of equilibria for convex GNEPs in Banach space. We show that the standard assumption of lower semicontinuity of the set-valued constraint maps – foundational in the current literature on GNEPs – can be replaced by graph convexity or the so-called Knaster–Kuratowski–Mazurkiewicz (KKM) property. Lower semicontinuity is often essential for obtaining upper semicontinuity of best response maps, crucial for the existence theory based on Kakutani–Fan fixed-point arguments. However, in function spaces or PDE-constrained settings, verifying lower semicontinuity becomes much more challenging (even in convex cases), whereas graph convexity, for example, is often straightforward to check. Our results unify several existence theorems in the literature and clarify the structural role of constraint maps. We also extend Rosen’s uniqueness condition to Banach spaces using a multiplier bias framework. Additionally, we present a geometric counterpart to our analytic framework using preference maps. This geometric is intended as a complement to, rather than a replacement for, the analytic theory developed in the main body of the paper.

1. INTRODUCTION

Game theory provides a mathematical framework for studying strategic interactions among decision makers. Games can be cooperative, if enforceable agreements are allowed, or noncooperative [17], and since the seminal work of von Neumann and Morgenstern [39] and Nash’s foundational equilibrium concepts [28, 27], equilibrium analysis has become a central paradigm in economics, engineering, and the social sciences. A *generalized Nash equilibrium problem* (GNEP) extends the classical setting by allowing each player’s feasible set to depend on the strategies of others. This formulation captures shared or coupled constraints, which arise naturally in models such as electricity markets [21, 23], coupled hydrogen and electricity markets [9], environmental management [7], and resource allocation [37].

Most classical results on GNEPs are finite-dimensional. However, important applications such as the coupling energy markets to physical transport of energy carriers (see, e.g., [9] and the references therein) lead to GNEPs with partial differential equations (PDEs) modeling the transport. But infinite-dimensional formulations, particularly those constrained by PDEs, remain less developed due to compactness and regularity issues. While some works analyze PDE-constrained Nash equilibrium problems via optimality systems [36, 35, 31, 32, 6], genuine generalized frameworks for multiple players have been addressed mainly under jointly convex assumptions [20, 19, 14, 15, 18, 24]. These studies typically require lower semicontinuity of the constraint maps to invoke Kakutani–Fan-type fixed-point theorems. While this is a standard assumption in abstract analysis, it is in general not very often verifiable in function space settings.

This paper revisits the existence theory for convex GNEPs in Banach spaces to clarify the trade-off between structural assumptions, such as convexity and regularity of set-valued constraint maps. We show that equilibrium existence can be ensured under *graph-convexity* or the *Knaster–Kuratowski–Mazurkiewicz (KKM)* property without assuming lower semicontinuity. These geometric conditions are often much easier to verify in infinite-dimensional or PDE-constrained contexts and thus broaden the class of problems where equilibrium existence can be rigorously established. This relaxes one of the most restrictive assumptions in the standard theory while retaining the classical framework as a special case.

Beyond existence, we study several structural subclasses of convex games. We show that Rosen's uniqueness condition for variational equilibria [34] extends naturally to Banach spaces, and introduce the notion of *multiplier bias* as a mechanism for equilibrium selection. We also discuss jointly convex and potential games as key examples illustrating the trade-off between structural assumptions and regularity requirements.

Compared with the existing literature on convex GNEPs and QVIs [10, 16, 30, 8], the novelty is not in a single new fixed-point theorem, but in the way structural assumptions on the constraint maps and the objective functionals can be traded against regularity assumptions, with an eye on PDE-constrained applications. The paper thus complements, rather than overlaps with, the jointly convex theory in [20, 24] and with equilibrium results in function spaces [25].

1.1. Outline. Section 2 introduces the mathematical structure of generalized games in Banach spaces. Section 3 revisits classical existence results and highlights the role of lower semicontinuity in the proofs based on the Kakutani's fixed-point theorem. Section 4 discusses the trade-off between structure and regularity, focusing on jointly convex and potential games. Section 5 establishes equilibrium existence under graph-convexity and KKM assumptions. Section 6 extends Rosen's uniqueness result and interprets it as a selection mechanism through multiplier bias. Section 7 then presents the geometric counterpart of our setting and discusses when and how the main analytic ideas extend to preference maps.

2. GENERALIZED FINITE GAMES IN BANACH SPACES

Let $N > 1$ be a fixed natural number and $I = \{1, \dots, N\}$ be a set of indices representing the players or agents participating in the game.

The strategy set of each player is a fixed Banach space X_i , likely of infinite dimension, with its topological dual denoted by X_i^* . Representing their private (or individual) constraints, each player observes a nonempty, closed and convex subset $X_i^{\text{ad}} \subset X_i$. We write $X := X_1 \times \dots \times X_N$, analogously for X^* , and $X^{\text{ad}} := X_1^{\text{ad}} \times \dots \times X_N^{\text{ad}}$ to denote the full strategy space, its topological dual, and the admissible strategy set, respectively. In game theory, it is common to use the notation x_{-i} to denote the bundle of $N - 1$ strategies containing all strategies of a given bundle $x \in X$ *except* that of player i . Moreover, we write $x = (x_i, x_{-i})$ when emphasis on player i 's strategy is needed, but without changing the original order of the components of x . Consequently, we let $X_{-i} := X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_N$ (and the corresponding X_{-i}^{ad}) and say that $x_{-i} \in X_{-i}$. In earlier literature on Nash games the bundle x_{-i} is sometimes referred to as an *i-incomplete combination* [17].

To complete the description of a generalized game, two ingredients are still needed. First, a family of objectives $\mathcal{J} = \{\mathcal{J}_i\}_{i=1}^N$ with each function \mathcal{J}_i being a real-valued function defined on the open set $O := O_1 \times \dots \times O_N$ where each O_i is an open set of X_i containing X_i^{ad} . And second, a family of set-valued constraint maps $\{\mathbf{X}_i\}_{i=1}^N$ with each \mathbf{X}_i defined from O_{-i} to O_i . Then $\mathbf{X}_i(x_{-i})$ is the set of *admissible* strategies which player i can take given the other players decisions x_{-i} . Finally, we denote by \mathbf{X} the set-valued map $\mathbf{X}_1 \times \dots \times \mathbf{X}_N$.

Definition 2.1 (GNEP). *A generalized Nash equilibrium problem (GNEP), denoted by $G = (\mathbf{X}, \mathcal{J})$ is a family of N coupled constrained optimization problems of the form*

$$(1) \quad \begin{cases} \text{Given } x_{-i} \in X_{-i}^{\text{ad}}; \\ \text{Minimize } \mathcal{J}_i(x_i, x_{-i}) \text{ subject to } x_i \in X_i^{\text{ad}} \cap \mathbf{X}_i(x_{-i}). \end{cases}$$

If $\mathbf{X}_i(x_{-i}) = O_i$ for all $x_{-i} \in X_{-i}$ and all $i \in I$, then a GNEP reduces to a so called Nash equilibrium problem (NEP) as dependence on x_{-i} only occurs in the objectives, but not the constraints.

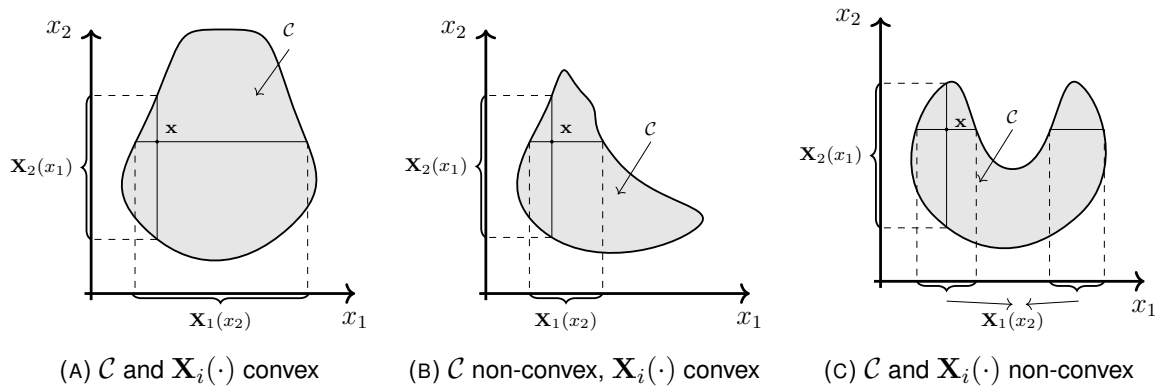


FIGURE 1. Convexity of \mathcal{C} versus convexity of $\mathbf{X}_i(x_{-i})$ for a two-player game at $x = (x_1, x_2)$.

For future reference, we denote by \mathcal{D}_i the *domain* of the map \mathbf{X}_i , i.e.,

$$(2) \quad \mathcal{D}_i := \{x_{-i} \in X_{-i}^{\text{ad}}; \mathbf{X}_i(x_{-i}) \neq \emptyset\}.$$

The simplest situation in which generalized games appear is when a constraint is *shared* among players. This is the case when, for example, a *regulatory* individual or agency determines that the strategy bundle x must belong to a certain *closed* set $\mathcal{C} \subset X$. This leads to non-constant constraint maps

$$(3) \quad \mathbf{X}_i(x_{-i}) = \{x_i \in X_i; x = (x_i, x_{-i}) \in \mathcal{C}\} = \pi_i(\mathcal{C}),$$

where π_i denotes the projection onto X_i . In Figure 1 we illustrate three different situations regarding the convexity of \mathcal{C} and $\mathbf{X}_i(\cdot)$.

We are interested in the concept of *Nash equilibrium* [28, 27] as the notion of *solution* for a generalized game. A game is at a Nash equilibrium if no player can (or have an incentive to) deviate from their current strategy.

Definition 2.2 (GNE). We say that $\bar{x} \in \mathbf{X}(\bar{x})$ is a *generalized Nash equilibrium (GNE)* of a game $G = (\mathbf{X}, \mathcal{J})$ provided

$$(4) \quad \mathcal{J}_i(\bar{x}) = \min_{x_i \in X_i^{\text{ad}} \cap \mathbf{X}_i(\bar{x}_{-i})} \mathcal{J}_i(x_i, \bar{x}_{-i})$$

for all $i \in I$. We denote the set of generalized Nash equilibria by $\mathbf{E}(G)$.

With a very simple finite dimensional example we illustrate that constant versus non-constant constraint maps may change the set of equilibria drastically.

Example 2.3. Suppose two companies $I = \{1, 2\}$ choose quantities x_i to produce and sell a homogeneous product on the same market. The selling price is given by the linear inverse demand function

$$P(x_1, x_2) = \eta - p(x_1 + x_2), \quad \eta, p > 0,$$

and the unit production cost of firm i satisfies $0 < c_i < \eta$. The loss of firm i is

$$J_i(x_1, x_2) = -P(x_1, x_2)x_i + c_i x_i = -x_i(\eta - c_i - p(x_1 + x_2)).$$

Let $X_i = \mathbb{R}$ and $X_i^{\text{ad}} = \mathbb{R}_+$ for $i = 1, 2$. If the constraint maps are constant, $\mathbf{X}_i(\cdot) \equiv X_i^{\text{ad}}$, each firm solves

$$\min_{x_i \geq 0} J_i(x_i, x_{-i}),$$

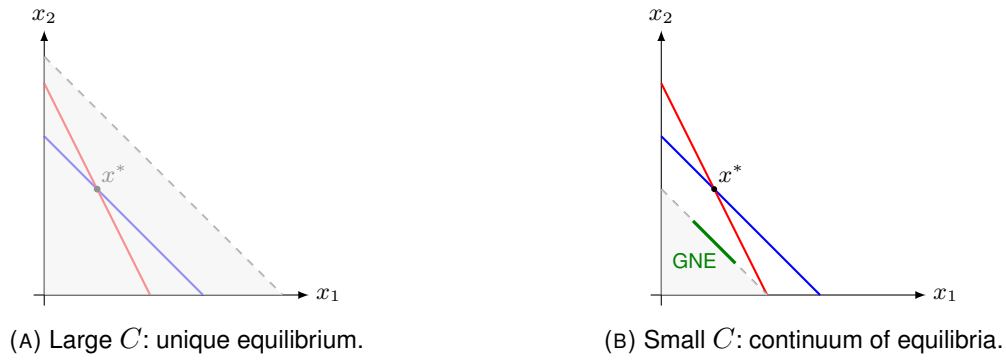


FIGURE 2. In the (x_1, x_2) -plane, player 1's best response is $x_1 = \Phi_1(x_2)$ (blue) and player 2's best response is $x_2 = \Phi_2(x_1)$ (red). For large production capacity (left), the intersection x^* is feasible and unique. For a small production capacity (right), x^* becomes infeasible and a continuum of GNE lie on the boundary of the feasible set

and the (unconstrained) best response of player i is

$$\Phi_i(x_{-i}) = \frac{\eta - c_i - px_{-i}}{2p}.$$

The Nash equilibrium (x_1^*, x_2^*) is uniquely determined as

$$x_1^* = \Phi_1(x_2^*), \quad x_2^* = \Phi_2(x_1^*).$$

Now introduce a shared capacity constraint

$$x_1 + x_2 \leq C$$

for some constant $C > 0$. Then the feasible set of player i becomes $\mathbf{X}_i(x_{-i}) = [0, C - x_{-i}]$. Thus, each best response is the truncation of Φ_i to $[0, C - x_{-i}]$. If C is large enough so that the unconstrained equilibrium (x_1^*, x_2^*) satisfies $x_1^* + x_2^* < C$, the equilibrium is unique and unchanged. However, if C is small, the intersection of the unconstrained best-response curves lies outside the feasible region. In that regime, the equilibrium set becomes a full segment on the boundary $\{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = C\}$, and the game admits infinitely many (indeed uncountably many) generalized Nash equilibria. For an illustration, see Figure 2. \square

In addition to non-uniqueness, the example above points out that in the general case we cannot expect that a single factor will be enough to guarantee that equilibria exist. In games with nonsmooth objectives, the issue of compactness becomes essential. The next example can be found in [25].

Example 2.4. Consider a two-player game $I = \{1, 2\}$ in which each player chooses a continuous nonnegative function on $[0, 1]$. For $i = 1, 2$, let

$$X_i = \{x \in C([0, 1]) : x(t) \geq 0 \text{ for all } t \in [0, 1]\}, \quad X_i^{\text{ad}} = \{x_i \in X_i : \|x_i\|_{C([0, 1])} \leq 1\},$$

where $\|x\|_{C([0, 1])} = \sup_{t \in [0, 1]} |x(t)|$ denotes the uniform norm. Hence, each strategy is an entire function rather than a finite-dimensional vector.

The feasible set of player i depends on the opponent's strategy x_{-i} through

$$\mathbf{X}_i(x_{-i}) = \{x_i \in X_i^{\text{ad}} : x_i \neq 0, x_i(t) + x_{-i}(t) \leq 1 \text{ for all } t \in [0, 1]\}.$$

The payoff (or reward) functionals are

$$\mathcal{J}_i(x_1, x_2) = \begin{cases} \int_0^1 tx_1(t)dt, & i = 1, \\ \int_0^1 (1-t)x_2(t)dt, & i = 2. \end{cases}$$

Because each player benefits from increasing their function wherever possible, the constraints $x_i + x_{-i} \leq 1$ and $x_i \neq 0$ prevent any pair (x_1, x_2) from being mutually optimal. Hence, this game admits no generalized Nash equilibrium. \square

Notice that X_i^{ad} in the example above is not compact. Moreover, in infinite dimensions, even compactness can only provide *hope* that an equilibrium will exist, see [25] for more examples. In particular, the natural topology in which state equations of PDE-constrained games are well posed is usually only weak or weak-star, hence lower semicontinuity of the induced constraint maps with respect to that topology is often impractical. This motivates the move to structural conditions on the graph, as developed later in Section 5.

Definition 2.5 (Convex GNEP). *We say that the a generalized game $G = (\mathbf{X}, \mathcal{J})$ is convex if, for all $i \in I$, the functions \mathbf{X}_i are convex-valued and the functions $\mathcal{J}_i(\cdot, x_{-i})$ are convex.*

3. REGULARITY IN CLASSICAL EXISTENCE THEORY

This section recalls the standard Kakutani approach to existence of equilibria. The goal is twofold: (i) to pinpoint exactly where lower semicontinuity of the constraint maps enters the argument, and (ii) to set up the notation used later when we replace lower semicontinuity by graph-convexity or the KKM property.

Let X, Y be Banach spaces. The graph of a set-valued map $F : X \rightarrow 2^Y$ is defined as $\mathbf{Gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$. Further, F is upper semicontinuous at x_0 if for any open subset V of Y such that $F(x_0) \subset V$ there exists an open subset U of X containing x_0 and such that $F(x) \subset V$ for all $x \in U$. F is lower semicontinuous at x_0 if for any open subset V of Y such that $F(x_0) \cap V \neq \emptyset$ there exists an open set U of X containing x_0 and such that $F(x) \cap V \neq \emptyset$ for all $x \in U$. Equivalently, F is lower semicontinuous at x_0 if for any $y_0 \in F(x_0)$ and any sequence x_n in X converging to x_0 there exists a sequence y_n in Y converging to y_0 and such that $y_n \in F(x_n)$. For more details, see [4].

Assumption 3.1 (Topology and regularity). *For each $i \in I$,*

- (i) X_i^{ad} is nonempty, compact and convex;
- (ii) the map $\mathbf{X}_i : X_{-i} \rightarrow 2^{X_i}$ is convex-valued and has closed graph.
- (ii') $\mathbf{X}_i(x_{-i}) \subset X_i^{\text{ad}}$ for all $x_{-i} \in X_{-i}$.

The assumption above plays a pivotal role in the study of convex GNEPs. Established methodologies, such as the K. Fan inequalities [13] and various formulations of Kakutani's theorem [4], rely heavily on this assumption. It is worth mentioning that (ii') is not actually necessary for any of the results discussed here, but we make it just so there is no need to write $X_i^{\text{ad}} \cap \mathbf{X}_i(x_{-i})$ everywhere.

On the other hand, the assumption below fixes the structure of the objective functions considered in this paper. For further discussion, see [3, 25] for a justification of such conditions in the context of economics, for example.

Assumption 3.2 (Objective functions). *For each $i \in I$*

- (i) the function $\mathcal{J}_i : X \rightarrow \mathbb{R}$ is continuous;
- (ii) the function $\mathcal{J}_i(\cdot, x_{-i}) : X_i \rightarrow \mathbb{R}$ is convex for each x_{-i} .

While not the simplest, perhaps the most classical existence result for GNEPs follows from Kakutani's fixed-point theorem.

Theorem 3.3 (Kakutani's fixed-point theorem). *Let K be a compact convex subset of a Hausdorff locally convex space and let $F : K \rightarrow 2^K$ be upper semicontinuous with nonempty, convex, and closed values. Then F has a fixed point.*

By sketching the proof of the theorem below, we aim at outlining the standard argument showing how Kakutani's theorem implies the existence of a Nash equilibrium for a general GNEP and why lower semicontinuity of the constraint maps appears as a crucial assumption.

Theorem 3.4. *Assume that, for every $i \in I$, the constraint map $\mathbf{X}_i : X_{-i}^{\text{ad}} \rightarrow 2^{X_i^{\text{ad}}}$ is lower semicontinuous and that Assumption 3.2 holds. Then the generalized Nash equilibrium problem $G = (\mathbf{X}, \mathcal{J})$ admits at least one equilibrium.*

Proof. Equilibria of G correspond to fixed points of the *best response map*

$$\mathbf{B}(x) = \prod_{i=1}^N \mathbf{B}_i(x_{-i}), \quad \mathbf{B}_i(x_{-i}) = \underset{x_i \in \mathbf{X}_i(x_{-i})}{\operatorname{argmin}} \mathcal{J}_i(x_i, x_{-i}).$$

By Assumption 3.1(ii), each \mathbf{X}_i is closed-valued and the set $\mathbf{X}_i(x_{-i})$ is nonempty, convex, and compact for all $x_{-i} \in X_{-i}$. Assumption 3.2 ensures that every $\mathbf{B}_i(x_{-i})$ is nonempty; hence $\mathbf{B}(x)$ is well defined and nonempty-valued.

To invoke Kakutani's fixed-point theorem, we must verify that $\mathbf{B} : X^{\text{ad}} \rightarrow 2^{X^{\text{ad}}}$ is upper semicontinuous and has nonempty, convex, and compact values. The latter properties follow directly from the assumptions above. We therefore focus on upper semicontinuity.

By [4, Theorem 6.2.5], \mathbf{B} is upper semicontinuous if and only if, for every closed set $C \subset X^{\text{ad}}$, the inverse image $\mathbf{B}^{-1}(C) = \{x \in X^{\text{ad}} : \mathbf{B}(x) \cap C \neq \emptyset\}$ is closed.

Let $(x^n) \subset X^{\text{ad}}$ satisfy $x^n \rightarrow x$ and $\mathbf{B}(x^n) \cap C \neq \emptyset$ for all n . Choose $y^n \in \mathbf{B}(x^n) \cap C$. Since C is compact, a subsequence (not relabeled) satisfies $y^n \rightarrow y \in C$. Closedness of the graph of \mathbf{X} implies $y \in \mathbf{X}(x)$. To show $y \in \mathbf{B}(x)$, we must verify that each y_i minimizes $\mathcal{J}_i(\cdot, x_{-i})$ over $\mathbf{X}_i(x_{-i})$. This is where the lower semicontinuity of \mathbf{X}_i is essential.

Indeed, since $x^n \rightarrow x$, for every $w_i \in \mathbf{X}_i(x_{-i})$ there exists a sequence $w_i^n \in \mathbf{X}_i(x_{-i}^n)$ with $w_i^n \rightarrow w_i$. Then, for each $i \in I$,

$$\mathcal{J}_i(y_i, x_{-i}) = \lim_{n \rightarrow \infty} \mathcal{J}_i(y_i^n, x_{-i}^n) \leq \lim_{n \rightarrow \infty} \mathcal{J}_i(w_i^n, x_{-i}^n) = \mathcal{J}_i(w_i, x_{-i}),$$

where the equalities follow from continuity of \mathcal{J}_i and the inequality holds because y_i^n minimizes $\mathcal{J}_i(\cdot, x_{-i}^n)$ over $\mathbf{X}_i(x_{-i}^n)$. Hence $y_i \in \mathbf{B}_i(x_{-i})$ for all i , so $y \in \mathbf{B}(x)$. Consequently, $\mathbf{B}^{-1}(C)$ is closed, proving that \mathbf{B} is upper semicontinuous.

By Theorem 3.3, \mathbf{B} admits a fixed point, which is an equilibrium of G . □ □

The argument above makes clear how the lower semicontinuity of \mathbf{X}_i enters the proof and why it is crucial for the closedness of the best-response graph. While standard in abstract fixed-point theory, this assumption is rarely easy to verify in practical settings such as PDE-constrained or function space games, see [5, 2, 10, 11]. This limitation partly explains the popularity of models where the coupling constraints have the structured form (3), for which the lower semicontinuity issue can be bypassed by exploiting properties of the Nikaido-Isoda function, as discussed next.

4. STRUCTURE VERSUS REGULARITY IN THE EXISTENCE THEORY

This section serves as a bridge between the classical Kakutani approach and the structural conditions in Section 5. We first recall the Nikaido-Isoda framework and the notion of variational equilibrium

in jointly convex and potential games. We then explain how these structural assumptions can be viewed as ways to replace lower semicontinuity of the constraint maps by convexity and potential-type properties.

In this section, we clarify how structural assumptions (such as convexity, shared constraints, or potential structure) can be traded against regularity of the constraint maps in existence results for convex GNEPs.

4.1. Jointly convex games, and potential games. The Nikaido-Isoda function $\Psi : X \times X \rightarrow \mathbb{R}$ is defined as

$$(5) \quad \Psi(x, y) = \sum_{i=1}^N \mathcal{J}_i(y_i, x_{-i}).$$

Under Assumption 3.2, Ψ is continuous and, for each $x \in X$, the map $\Psi(x, \cdot)$ is convex. The next lemma is central to the state of the art in the theory of GNEPs, therefore we present a short proof.

Lemma 4.1. *A strategy bundle $\bar{x} \in X^{\text{ad}}$ is a Nash equilibrium of the game $G = (\mathbf{X}, \mathcal{J})$ if and only if $\bar{x} \in \mathbf{X}(\bar{x})$ and $\Psi(\bar{x}, \bar{x}) \leq \Psi(\bar{x}, y)$, for all $y \in \mathbf{X}(\bar{x})$.*

Proof. Proof. Let $\bar{x} \in X^{\text{ad}}$ be a Nash equilibrium. By definition of equilibrium, \bar{x} is a fixed point of \mathbf{X} . Moreover, for each i and each $y_i \in \mathbf{X}_i(\bar{x}_{-i})$ we have $\mathcal{J}_i(\bar{x}) \leq \mathcal{J}_i(y_i, \bar{x}_{-i})$, and summing over i yields $\Psi(\bar{x}, \bar{x}) \leq \Psi(\bar{x}, y)$ for all $y \in \mathbf{X}(\bar{x})$.

Conversely, assume that \bar{x} is a fixed point of \mathbf{X} and that $\Psi(\bar{x}, \cdot)$ reaches its global minimum at \bar{x} . If \bar{x} is not a Nash equilibrium, then for some $i \in I$, there exists $y_i \in \mathbf{X}_i(\bar{x}_{-i})$ such that $\mathcal{J}_i(\bar{x}) > \mathcal{J}_i(y_i, \bar{x}_{-i})$. Noticing that $(y_i, \bar{x}_{-i}) \in \mathbf{X}(\bar{x})$, the result follows because

$$\Psi(\bar{x}, \bar{x}) - \Psi(\bar{x}, (y_i, \bar{x}_{-i})) = \mathcal{J}_i(\bar{x}) - \mathcal{J}_i(y_i, \bar{x}_{-i}) > 0,$$

which contradicts minimality of \bar{x} . □ □

In other words, \bar{x} is a GNE if and only if it solves the minimization problem

$$(6) \quad \min \Psi(\bar{x}, y) \text{ s.t. } y \in \mathbf{X}(\bar{x}).$$

Among other consequences, the above lemma allows one to characterize Nash equilibria – under extra regularity of the objective functions – as the solution of a quasi-variational inequality (QVI).

Lemma 4.2. *Assume that, for each $i \in I$ and each $x_{-i} \in X_{-i}$, the function $\mathcal{J}_i(\cdot, x_{-i})$ is continuously Gâteaux-differentiable with bounded derivative $\partial_i \mathcal{J}_i(\cdot, x_{-i}) : X_i \rightarrow X_i^*$. Hence \bar{x} is a GNE if, and only if, $\bar{x} \in \mathbf{X}(\bar{x})$ and*

$$(7) \quad \sum_{i=1}^N \langle \partial_i \mathcal{J}_i(\bar{x}), y_i - \bar{x}_i \rangle_{X_i^*, X_i} \geq 0$$

for all $y \in \mathbf{X}(\bar{x})$.

Remark 4.3. *The differentiability assumption in Lemma 4.2 can be relaxed. If each $\mathcal{J}_i(\cdot, x_{-i})$ is convex and lower semicontinuous, the equilibrium condition remains valid in the following subdifferential form: $\bar{x} \in \mathbf{X}(\bar{x})$ is a GNE if and only if*

$$\sum_{i=1}^N \langle \xi_i, y_i - \bar{x}_i \rangle_{X_i^*, X_i} \geq 0, \quad \text{for all } y \in \mathbf{X}(\bar{x}),$$

for some $\xi_i \in \partial_i \mathcal{J}_i(\bar{x})$, where ∂_i denotes the convex subdifferential with respect to x_i . When $\mathcal{J}_i(\cdot, x_{-i})$ is continuously Gâteaux-differentiable, $\partial_i \mathcal{J}_i(\bar{x}) = \{\partial_i \mathcal{J}_i(\bar{x})\}$ and the statement reduces to (7).

Under the assumptions of Theorem 3.4, the quasi-variational inequality (QVI) (7) also admits solutions [8][Theorem 5.2]. However, as reported in [10, 16, 30], solution methods for QVIs are scarce and typically require strong assumptions (for instance contractivity) on the problem structure. In both formulations, lower semicontinuity of the constraint maps remains a key requirement.

For each $x \in \mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_N$, Assumption 3.1 ensures that $\mathbf{X}(x)$ is nonempty, convex, and compact. Combined with continuity and convexity of $\Psi(x, \cdot)$, this implies that $\Psi(x, \cdot)$ admits a minimizer on $\mathbf{X}(x)$. Hence, the *merit* function $\Phi : \mathcal{D} \rightarrow \mathbb{R}$,

$$\Phi(x) = \min_{y \in \mathbf{X}(x)} \Psi(x, y)$$

is well defined. In some papers (see [20, 10]) the Nikaido–Isoda function and the merit function are defined as

$$\tilde{\Psi}(x, y) = \Psi(x, x) - \Psi(x, y), \quad \tilde{\Phi}(x) = \Psi(x, x) - \Phi(x),$$

so that equilibria correspond to roots of $\tilde{\Phi}$. Here we keep Nikaido–Isoda’s original definition [29].

When each \mathbf{X}_i is given by (3) for some $\mathcal{C} \subset X$, the set of fixed points of \mathbf{X} coincides with \mathcal{C} . This observation is important because every generalized Nash equilibrium must satisfy $\bar{x} \in \mathbf{X}(\bar{x})$, and therefore $\bar{x} \in \mathcal{C}$. In particular, in Lemma 4.1 and Lemma 4.2 the condition $\bar{x} \in \mathbf{X}(\bar{x})$ can be replaced by $\bar{x} \in \mathcal{C}$.

However, replacing all occurrences of $\mathbf{X}(\bar{x})$ by \mathcal{C} would in general remove many equilibria, because $\mathbf{X}(\bar{x})$ depends on \bar{x} while \mathcal{C} does not. This effect is clearly visible in the two-company Cournot game of Example 2.3. For concreteness, let us fix $\eta = 4, p = 1, c_1 = 1, c_2 = 3/2$, and the shared capacity constraint

$$x_1 + x_2 \leq C,$$

so that

$$\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 \leq C\}.$$

If $C \geq 11/6$, then the unique unconstrained Nash equilibrium lies in \mathcal{C} , so the generalized and the shared-constraint formulations coincide. If $C < 11/6$, the intersection of the unconstrained best-response curves lies outside \mathcal{C} and one obtains a whole segment of generalized Nash equilibria on the boundary $\{x_1 + x_2 = C\}$, as illustrated in Figure 2.

To see what happens when one replaces the pointwise feasible sets $\mathbf{X}_i(x_{-i})$ by the shared constraint \mathcal{C} in the Nikaido–Isoda formulation, consider the case $C = 1$. The associated Nikaido–Isoda function is

$$\Psi(x, y) = J_1(y_1, x_2) + J_2(x_1, y_2) = -y_1(3 - y_1 - x_2) - y_2\left(\frac{5}{2} - x_1 - y_2\right),$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If we now enforce the shared-constraint formulation by solving

$$(8) \quad \min_{y \in \mathcal{C}} \Psi(\bar{x}, y) \quad \text{for } \bar{x} \in \mathcal{C},$$

then a direct Karush-Kuhn-Tucker (KKT) computation (or a variational-inequality argument) shows that

$$\Psi(\bar{x}, \bar{x}) = \min_{y \in \mathcal{C}} \Psi(\bar{x}, y), \quad \bar{x} \in \mathcal{C}$$

holds if and only if $\bar{x} = (3/4, 1/4)$. Thus, for $C = 1$ the formulation in (8) yields one equilibrium, whereas with the original formulation (6), the game admits a continuum of equilibria along the boundary segment $x_1 + x_2 = 1$.

Although, as this example shows, many equilibria are lost when $\mathbf{X}(\bar{x})$ is simply replaced by \mathcal{C} , the restricted notion is widely used because it simplifies analysis and computation and often provides a natural selection mechanism, which is crucial from a numerical viewpoint.

Equilibria obtained under this simplification are called *variational equilibria*.

Definition 4.4 (Variational equilibrium). *A strategy bundle $\bar{x} \in \mathcal{C}$ is called a variational (or sometimes normalized) Nash equilibrium for a game $G = (\mathbf{X}, \mathcal{J})$ provided*

(i) *it solves the minimization problem*

$$(9) \quad \min \Psi(\bar{x}, y) \text{ s.t. } y \in \mathcal{C};$$

or equivalently (under C^1 -regularity of the objectives):

(ii) *the inequality*

$$(10) \quad \sum_{i=1}^N \langle \partial_i \mathcal{J}_i(\bar{x}), y_i - \bar{x}_i \rangle_{X_i^*, X_i} \geq 0$$

holds for all $y \in \mathcal{C}$.

Remark 4.5. *For simplicity of the statements we have implicitly assumed that $\mathcal{C} \subset X^{\text{ad}}$. If not, then we need to replace \mathcal{C} with $X^{\text{ad}} \cap \mathcal{C}$ in the theorem above.*

Obviously variational equilibria are Nash equilibria. The converse, as our example above shows, is not true and there are multiple other examples in the literature showing this fact, see for instance [10, 16]. The most important games in which the constraint maps can be written as (3), are the *jointly convex* and *potential* games [20, 10, 8].

Definition 4.6. *A jointly convex game is a game whose constraint maps are defined by (3) and \mathcal{C} is, in addition to closed, convex.*

Due to convexity, existence of variational equilibria follows, in our setting, if \mathcal{C} is compact. For completeness we include the statement and sketch of the proof below.

Theorem 4.7. *Let \mathcal{C} be a compact and convex subset of X such that $X^{\text{ad}} \cap \mathcal{C} \neq \emptyset$ and assume the structure (3) holds for the constraint maps. Then, every generalized game whose corresponding Nikaido–Isoda function $\Psi(\cdot, y)$ is upper semicontinuous for each $y \in \mathcal{C}$ has a variational equilibrium.*

Proof. Proof. It follows directly from the Ky-Fan theorem [22, Theorem 1.1], which guarantees the existence of $\bar{x} \in \mathcal{C}$ such that

$$(11) \quad \sup_{y \in \mathcal{C}} (\Psi(\bar{x}, \bar{x}) - \Psi(\bar{x}, y)) \leq 0.$$

This completes the proof. □ □

In case \mathcal{C} is not convex, existence can still be proved given more structure on the objective functions (or the preference maps). This leads to so-called *potential* games.

Definition 4.8 (Potential game). *Assume that the constraint maps are as in (3) and that \mathcal{C} is closed. A generalized game is a potential game if there exists a continuous function $\mathcal{G} : X \rightarrow \mathbb{R}$, known as the potential function, such that for all i , all x_{-i} and all $y_i, z_i \in \mathbf{X}_i(x_{-i})$ the condition*

$$\mathcal{J}_i(y_i, x_{-i}) - \mathcal{J}_i(z_i, x_{-i}) > 0$$

implies

$$\mathcal{G}(y_i, x_{-i}) - \mathcal{G}(z_i, x_{-i}) \geq g(\mathcal{J}_i(y_i, x_{-i}) - \mathcal{J}_i(z_i, x_{-i})),$$

where $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, known as the forcing function, is such that $g(t_k) \rightarrow 0$ implies $t_k \rightarrow 0$.

Theorem 4.9. *A potential game with X^{ad} compact and the forcing function g continuous has an equilibrium.*

Proof. Proof. The proof is divided into two steps. First, consider the optimization problem

$$(12) \quad \min_{x \in X^{\text{ad}} \cap \mathcal{C}} \mathcal{G}(x).$$

Since X^{ad} is compact and \mathcal{C} is closed, it follows that $X^{\text{ad}} \cap \mathcal{C}$ is compact. Since \mathcal{G} is continuous, $\mathcal{G}(X^{\text{ad}} \cap \mathcal{C}) \subset \mathbb{R}$ is also compact. Hence there exists at least one minimizer $\hat{x} \in X^{\text{ad}} \cap \mathcal{C}$ of \mathcal{G} . The second step is then to prove that if \bar{x} is an optimal solution of (12), it is also a Nash equilibrium. Suppose not, then there exists at least one $i \in I$ and at least one $y_i \in \mathbf{X}_i(\bar{x}_{-i})$ such that $\mathcal{J}_i(\bar{x}) > \mathcal{J}_i(y_i, \bar{x}_{-i})$. Hence, by definition of potential game we have

$$\mathcal{G}(\bar{x}) \geq \mathcal{G}(y_i, \bar{x}_{-i}) + g(\mathcal{J}_i(\bar{x}) - \mathcal{J}_i(y_i, \bar{x}_{-i})),$$

and by properties of g it follows that $\mathcal{G}(\bar{x}) > \mathcal{G}(y_i, \bar{x}_{-i})$, a contradiction. \square \square

Remark 4.10. Observe that lower semicontinuity of \mathcal{G} is enough for the theorem above to be true, although the proof would be slightly different.

Thus, finding a potential function \mathcal{G} with a continuous forcing function g is a sufficient condition to guarantee the existence of a Nash equilibrium in a potential game. As in the jointly convex case, the set of equilibria of a potential game is in general a proper subset of the full set of Nash equilibria. Every jointly convex game is, in particular, a potential game with \mathcal{G} given by the Nikaido–Isoda function. The main existence results of this paper rely on the same philosophy: we replace strong regularity assumptions on the constraint maps by additional convexity or potential-type structure.

5. POTENTIAL LACK OF LOWER SEMICONTINUITY OF CONSTRAINT MAPS

In this section we show for convex games that lower semicontinuity of the constraint maps is not needed for equilibrium existence once additional geometric structure is available. This allows us to cover games with significantly more complex and heterogeneous constraints, such as generalized PDE-constrained games where each player's state equation depends on forecasts of incomplete or partially available information, only.

Definition 5.1. Let X be a nonempty subset of a topological vector space L . We say that a set valued map $F : X \rightarrow 2^L$ is graph-convex if its graph is convex.

Our first theorem below shows that graph-convexity (in place of lower semicontinuity) is also sufficient for existence of equilibria.

Theorem 5.2. Assume that \mathbf{X} is graph-convex and that $\text{Fix}(\mathbf{X})$, the set of feasible fixed points of \mathbf{X} , i.e.,

$$\text{Fix}(\mathbf{X}) := \{x \in X^{\text{ad}}; x \in \mathbf{X}(x)\}$$

is nonempty. Then the game $G = (\mathbf{X}, \mathcal{J})$ has an equilibrium.

It is important to emphasize here that graph-convexity does not *replace* lower semicontinuity as a weaker condition in general. In fact, it is well known that graph-convexity implies lower semicontinuity in the interior of the domain. A proof of this fact can be found in [33, Theorem 5.9(b)] in finite dimensions, and the same proof can be adapted to the Banach space case without much work. However, in infinite dimensions interiority is delicate, as we illustrate in the example below.

Example 5.3. Let $I = \{1, 2\}$ and $X_1 = X_2 = L^2(0, 1)$. Set $X := X_1 \times X_2$ and define the closed subspace

$$C := \left\{ u \in L^2(0, 1) : \int_0^1 u(x) dx = 0 \right\}.$$

Since $C = \ker \ell$ for the continuous, surjective linear functional

$$\ell(u) := \int_0^1 u(x) dx,$$

the space C has codimension 1 in $L^2(0, 1)$ and hence empty interior in the strong L^2 topology.

Define the closed convex set

$$K := \{v \in L^2(0, 1) : v \geq 0 \text{ a.e., } \|v\|_{L^2} \leq 1\},$$

where “a.e.” stands for “almost everywhere” (in the sense of Lebesgue), and fixed feasible sets

$$X_i^{\text{ad}} = X_i = L^2(0, 1), \quad i = 1, 2.$$

Finally, the player-specific constraint maps $\mathbf{X}_i : X_{-i} \rightarrow 2^{X_i}$ given by

$$\mathbf{X}_i(x_{-i}) := \begin{cases} K, & \int_0^1 x_{-i}(x) dx = 0, \\ \emptyset, & \int_0^1 x_{-i}(x) dx \neq 0, \end{cases}$$

Thus each player's feasible set depends on the other player's strategy. The joint constraint map $\mathbf{X}(x_1, x_2) := X_1(x_1, x_2) \times X_2(x_1, x_2)$ is then given by

$$\mathbf{X}(x_1, x_2) = \begin{cases} K_1 \times K_2, & x_1 \in C, x_2 \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

(One may take, for instance, $\mathcal{J}_i \equiv 0$ such that the objectives play no role in this example.)

The graph of the joint constraint map is

$$\text{Gr}(\mathbf{X}) = \{(x, y) \in X \times X : x_1 \in C, x_2 \in C, y_1 \in K_1, y_2 \in K_2\} = (C \times C) \times (K_1 \times K_2),$$

which is convex because C , K_1 , and K_2 are convex, respectively. The domain of \mathbf{X} is

$$\text{dom } \mathbf{X} = \{x \in X : \mathbf{X}(x) \neq \emptyset\} = \{(x_1, x_2) \in X : x_1 \in C, x_2 \in C\} = C \times C,$$

which has empty interior in X .

Lower semicontinuity of \mathbf{X} fails at every $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \text{dom } \mathbf{X}$. Fix any $\bar{y} \in \mathbf{X}(\bar{x}) = K_1 \times K_2$. Since C is a proper closed subspace of $L^2(0, 1)$, we can choose sequences

$$x_1^n \rightarrow \bar{x}_1 \text{ with } x_1^n \notin C, \quad x_2^n \rightarrow \bar{x}_2 \text{ with } x_2^n \notin C.$$

Set $x^n := (x_1^n, x_2^n)$. Then $x^n \rightarrow \bar{x}$ in X , but by construction $\mathbf{X}(x^n) = \emptyset$ for all n . Hence there is no sequence $y^n \in \mathbf{X}(x^n)$ converging to \bar{y} , i.e., lower semicontinuity of \mathbf{X} fails at \bar{x} .

This example is a genuine two-player generalized Nash game whose joint constraint map has a convex graph, but a domain with empty interior in the strong topology, and it illustrates how lower semicontinuity can fail in such “thin” situations. \square

It follows from Lemma 4.1 that a bundle \bar{x} is a GNE for a game if and only if

$$(13) \quad \bar{x} \in \{\hat{y} \in \mathbf{X}(\bar{x}) \cap \text{Fix}(\mathbf{X}) : \Psi(\bar{x}, \hat{y}) = \Phi(\bar{x})\}.$$

This characterization enables us to prove Theorem 5.2 in a rather elementary way; it extends the classical argument of Nikaido and Isoda [29] to the present setting.

Proof. Proof of Theorem 5.2 Suppose that a GNE does not exist. By Lemma 4.1 and (13) this means, in particular, that for each $x \in \text{Fix}(\mathbf{X})$ there exists $y_x \in \mathbf{X}(x) \cap \text{Fix}(\mathbf{X})$ such that $\Psi(x, x) > \Psi(x, y_x)$. Fixing y_x , it follows from continuity of Ψ that the set $\Psi_x := \{z \in X : \Psi(z, z) > \Psi(z, y_x)\}$ is nonempty (in fact it contains x) and open in X .

Now, notice that the closedness of the graph of each map \mathbf{X}_i (see Assumption 3.1(b)) implies that the set $\text{Fix}(\mathbf{X})$ is closed. In fact, let (x^n) be a sequence in $\text{Fix}(\mathbf{X})$ such that $x^n \rightarrow x \in X$. We have $x_i^n \in \mathbf{X}_i(x_{-i}^n)$, i.e., $x^n \in \text{Gr}(\mathbf{X}_i)$ for each n and each i . By closedness of the graph, it follows that $x \in \text{Gr}(\mathbf{X}_i)$ for each i , hence $x \in \text{Fix}(\mathbf{X})$. Therefore, as a closed subset of X^{ad} , which is compact, $\text{Fix}(\mathbf{X})$ is also compact. Hence, the trivial inclusion

$$\text{Fix}(\mathbf{X}) \subset \bigcup_{x \in \text{Fix}(\mathbf{X})} \Psi_x$$

implies that there exist $x_1, \dots, x_r \in \text{Fix}(\mathbf{X})$ (with $r \in \mathbb{N}$) such that $\text{Fix}(\mathbf{X}) \subset \bigcup_{i=1}^r \Psi_{x_i}$. This implies, in particular, that

$$\Psi(x, x) > \min_{1 \leq i \leq r} \Psi(x, y_{x_i}) \quad \text{for all } x \in \text{Fix}(\mathbf{X}).$$

Define, for each $1 \leq i \leq r$, the (continuous) function $g_i : X \rightarrow \mathbb{R}$ as

$$g_i(x) = -\min\{\Psi(x, y_{x_i}) - \Psi(x, x), 0\}$$

and notice that $g_i(x) \geq 0$ with $\sum_i g_i(x) > 0$ for all $x \in \text{Fix}(\mathbf{X})$. Hence the function G defined by

$$G(x) = \frac{\sum_{i=1}^r g_i(x) y_{x_i}}{\sum_{i=1}^r g_i(x)}$$

maps $\text{Fix}(\mathbf{X})$ into $A = \text{co}(\{y_{x_1}, \dots, y_{x_r}\})$, the convex hull of the set $\{y_{x_1}, \dots, y_{x_r}\}$, and, in particular, A to itself. The set $\text{Fix}(\mathbf{X})$ is convex and contains each y_{x_i} , so the convex hull A is a nonempty convex subset of $\text{Fix}(\mathbf{X})$. Since the y_{x_i} span a finite-dimensional subspace of X , the set A is compact in the induced topology. The map G is continuous on $\text{Fix}(\mathbf{X})$ (and hence on A) because it is built from finitely many continuous functions g_i and finitely many fixed points y_{x_i} . Therefore $G : A \rightarrow A$ is a continuous self-map of a nonempty compact convex subset of a finite-dimensional space. Thus, Brouwer's fixed point theorem applies. It follows that G has a fixed point $\hat{x} \in A$. For indices i such that $g_i(\hat{x}) > 0$ we have $\Psi(\hat{x}, y_{x_i}) < \Psi(\hat{x}, \hat{x})$ and this, combined with convexity of $\Psi(\hat{x}, \cdot)$, yields

$$\Psi(\hat{x}, \hat{x}) = \Psi(\hat{x}, G(\hat{x})) \leq \sum_{i=1}^r \frac{g_i(\hat{x})}{\sum_{j=1}^r g_j(\hat{x})} \Psi(\hat{x}, y_{x_i}) < \Psi(\hat{x}, \hat{x}),$$

which is a contradiction. Hence a Nash equilibrium must exist. \square \square

It is worth noting that in [20, 24] the existence of variational Nash equilibria was established by using Kakutani's fixed-point theorem and a K. Fan inequality, respectively. In this sense, the argument above provides an alternative route to existence of (variational) equilibria in the case of jointly convex games. From a structural viewpoint, the key difference is that here we work directly on the fixed-point set of the joint constraint map and exploit convexity of its graph, instead of building an auxiliary best-response map whose upper semicontinuity must be checked.

The example below, inspired by the viscosity-regularized spot market system of [20, Section 5.2], illustrates how equilibrium existence based on graph-convexity applies even when the feasible sets are not jointly convex.

Example 5.4 (PDE-constrained spot market). *Let $\Omega = (0, 1)$, $T > 0$, and $Q := \Omega \times (0, T)$. For each player $i \in I = \{1, \dots, N\}$, the decision variable is $u_i \in X_i := L^2(0, T; L^2(\Omega))$ with simple box constraints $0 \leq u_i(x, t) \leq \bar{u}_i$ a.e. in Q for given $\bar{u}_i \in \mathbb{R}_+$. Each strategy bundle $u = (u_1, \dots, u_N)$ produces a state $y \in Y := L^2(0, T; H_0^1(\Omega))$ that satisfies the linear PDE*

$$(14) \quad y_t - \Delta y = \sum_{i=1}^N u_i \quad \text{in } Q, \quad y = 0 \text{ on } \{0, 1\} \times (0, T), \quad y(\cdot, 0) = 0.$$

The solution operator $S : X \rightarrow Y$ associated with (14) is linear and continuous. Note that above $L^2(\Omega)$, $H_0^1(\Omega)$ denote the usual Lebesgue and Sobolev spaces [1] and $L^2(0, T; L^2(\Omega))$ as well as $L^2(0, T; H_0^1(\Omega))$ are the associated Bochner spaces [40].

Each player i enforces the shared constraint set by an individual condition that can be interpreted as a personal buffer or uncertainty allowance. For example, assuming player i 's state requirement set (given a competition bundle u_{-i}) is the closed convex set $\mathcal{K}_i(u_{-i})$, we define his or her feasible set as the convex set

$$\mathbf{X}_i(u_{-i}) := \{u_i \in X_i^{\text{ad}} : y = S(u) \in \mathcal{K}_i(u_{-i})\}.$$

Hence the product mapping $\mathbf{X} := \mathbf{X}_1 \times \cdots \times \mathbf{X}_N$ is graph-convex on X^{ad} . Hence, as long as $\text{Fix}(\mathbf{X}) \neq \emptyset$ (which is a standard assumption to make) one can easily prove the existence of equilibrium as long as the objective functions satisfy Assumption 3.2. \square

If \mathbf{X} is not graph-convex, but is a KKM map, then Nash equilibria also exist.

Definition 5.5. Let X be a nonempty subset of a topological vector space L . We say that a set-valued function $F : X \rightarrow 2^L$ is a “Knaster-Kuratowski-Mazurkiewicz” (**KKM**) map if for each finite subset $\{x_1, x_2, \dots, x_n\}$ of X we have

$$(15) \quad \text{co}(\{x_1, x_2, \dots, x_n\}) \subset \bigcup_{i=1}^n F(x_i).$$

Notice that KKM maps are not necessarily lower semicontinuous even if X is finite dimensional. Consider for example $F : [0, 1] \rightarrow 2^{\mathbb{R}}$ given by $F(0) = [0, 1]$ and $F(x) = [0, x]$ if $x \in (0, 1]$. Then F is closed and convex-valued, has a closed graph and is a KKM map. However, it fails to be lower semicontinuous.

Moreover, there are KKM maps which are not graph-convex. For example $F : [0, 1] \rightarrow 2^{\mathbb{R}}$ given by $F(x) = [0, x]$ if $x \in [0, 1/2]$ and $F(x) = [x, 1]$ if $x \in (1/2, 1]$ is a KKM map which is not graph-convex.

Among the many properties of KKM maps (see for instance [4] and references therein) is the fact that $x \in F(x)$ for all $x \in X$, i.e., F is nonempty-valued and all vectors in X are fixed points.

Theorem 5.6. Assume that the function $\mathbf{X} : X^{\text{ad}} \rightarrow 2^{X^{\text{ad}}}$ is a KKM map. Then the game generating such a map has an equilibrium.

Proof. Proof. The proof is analogous to the proof of Theorem 5.2. The only detail that changes is that before we used the graph-convexity of \mathbf{X} to show that convex combinations of fixed points were fixed points. In this case, since every point in X^{ad} is a fixed point, this step is not necessary. \square \square

6. MULTIPLIER BIAS AND UNIQUENESS OF AN EQUILIBRIUM

The previous sections focused on existence. In many applications, however, the set of equilibria is large and may contain continua of points, as in Example 2.3. This section revisits Rosen's diagonal strict convexity condition in our Banach space setting. The main message is twofold: (i) suitable structural conditions on a weighted pseudogradient single out a *unique* variational equilibrium, and (ii) changing the weights $r \in (\mathbb{R}_+^*)^N$ can be interpreted as introducing a *multiplier bias* that selects one equilibrium among many. We keep the objective functionals fixed and bias only the way players react to marginal costs.

Let $G = (\mathbf{X}, \mathcal{J})$ be a jointly convex game in which the family of objective functionals satisfies Assumption 3.2. Consider the jointly convex game $G_r = (\mathbf{X}, r \cdot \mathcal{J})$ for some $r \in \mathbb{R}^N$. Here, ' \cdot ' is the

componentwise multiplication of vectors. A well-known fact [34] is that if $r \in (\mathbb{R}_+^*)^N$ (i.e., $r_i > 0$ for all $i \in I$), then

$$(16) \quad \mathbf{E}(G_r) = \mathbf{E}(G),$$

where $\mathbf{E}(\cdot)$ denotes the set of equilibria. This means that the set of Nash equilibria is unaffected by positive rescaling of the objective functions. An interesting aspect, however, is that except in the orthogonal case, i.e., when \mathcal{C} has the form $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_N$, we have $\mathbf{VE}(G_r) \neq \mathbf{VE}(G)$, where $\mathbf{VE}(\cdot)$ is the set of variational equilibria. We show below, extending the results of [34] to infinite dimensions, that an r -dependent structural condition on G_r ensures that $\mathbf{VE}(G_r)$ is a singleton. Hence, any numerical method designed to compute a variational equilibrium will converge to the same one. Since $\mathbf{VE}(G_r) \subset \mathbf{E}(G_r) = \mathbf{E}(G)$ and $r \in (\mathbb{R}_+^*)^N$ must be chosen so that the structural condition holds, we are still computing a Nash equilibrium of the original game.

Assume that the objective functionals $\{\mathcal{J}_i\}_{i=1}^N$ are continuously differentiable, with derivatives $\partial_i \mathcal{J}_i(\cdot, x_{-i}) : X_i \rightarrow X_i^*$. Given a vector $r \in (\mathbb{R}_+^*)^N$, we define (formally) the *pseudogradient* of $r \cdot \mathcal{J}$ at $x \in X$ in the direction $h \in X$ as

$$(17) \quad d(x, r)h = \begin{bmatrix} r_1 \partial_1 \mathcal{J}_1(x) h_1 \\ \vdots \\ r_N \partial_N \mathcal{J}_N(x) h_N \end{bmatrix}.$$

Definition 6.1. We say that the game G_r is *diagonally strictly convex* (with respect to the shared constraint \mathcal{C}) for a given $r \in (\mathbb{R}_+^*)^N$ if for all $x, y \in \mathcal{C}$,

$$(18) \quad d(x, r)(y - x) + d(y, r)(x - y) < 0.$$

Remark 6.2. In smoother problems, one can verify whether G_r is diagonally strictly convex on \mathcal{C} by checking the positive definiteness of the operator $D(x, r) + D(x, r)^*$ for all $x \in \mathcal{C}$, where $D(x, r)$ denotes the Jacobian of the map $x \mapsto d(x, r)$. Here, the superscript $*$ indicates the adjoint operator.

We begin with the non-generalized situation, i.e., we assume that $\mathbf{X}(x) \equiv X^{\text{ad}}$ even if \mathbf{X}_i has the form (3). This corresponds to the case in which \mathcal{C} is a Cartesian product. The result below extends [34, Theorem 2] to the infinite-dimensional setting. In its proof we need the normal cone to \mathcal{C} at some $x \in \mathcal{C}$ which is given by

$$N_{\mathcal{C}}(x) = \{\mu_x \in X^* : \mu_x(y - x) \leq 0 \forall y \in \mathcal{C}\}.$$

Proposition 6.3. Assume that $\mathbf{X}(x) \equiv X^{\text{ad}} = \mathcal{C}$ and that the family of objective functions satisfies Assumption 3.2. If G_r is diagonally strictly convex on \mathcal{C} for some $r \in \mathbb{R}_+^N$, then $\mathbf{VE}(G)$ is a singleton.

Proof. Proof. It follows from Lemma 4.1 and from convex optimization that if $\bar{x} \in \mathcal{C}$ is a Nash equilibrium, then there exists $\mu_{\bar{x}} \in N_{\mathcal{C}}(\bar{x})$ such that

$$\nabla \Psi(\bar{x}, \bar{x}) + \mu_{\bar{x}} = 0 \quad \text{in } X^*.$$

Suppose that $\bar{x}, \bar{y} \in \mathcal{C}$ are two distinct Nash equilibria. Testing the equation above by $r \cdot (\bar{y} - \bar{x})$ (in the \bar{x} case) and by $r \cdot (\bar{x} - \bar{y})$ (in the \bar{y} case), and adding the resulting equations, we obtain $\alpha + \beta = 0$, where

$$\alpha = d(\bar{x}, r)(\bar{y} - \bar{x}) + d(\bar{y}, r)(\bar{x} - \bar{y}), \quad \beta = \mu_{\bar{x}}(r \cdot (\bar{y} - \bar{x})) + \mu_{\bar{y}}(r \cdot (\bar{x} - \bar{y})).$$

Since $r \in \mathbb{R}_+^N$, it follows from the definition of the normal cone and from the fact that $X^{\text{ad}} = \mathcal{C}$ that $\beta \leq 0$, and hence $\alpha \geq 0$, which contradicts diagonal strict convexity. Therefore, $\bar{x} = \bar{y}$. $\square \quad \square$

We now return to the case in which \mathbf{X} is non-constant but still given by (3).

Proposition 6.4. *Assume that \mathbf{X} has the structure (3) and that the family of objective functions satisfies Assumption 3.2. If G_r is diagonally strictly convex on \mathcal{C} for all $r \in \mathcal{R}$, where $\mathcal{R} \subset (\mathbb{R}_+^*)^N$ is nonempty and convex, then for each $r \in \mathcal{R}$ the set $\mathbf{VE}(G_r) \subset \mathbf{E}(G)$ is a singleton.*

It follows from Proposition 6.3 and Proposition 6.4 that when the constraint map is constant, the variational equilibrium is unique and independent of r . However, in the non-constant case, although the scaling vector r cannot alter the set of Nash equilibria, it may affect the set of variational equilibria – a feature of practical interest given their computational advantages over the full equilibrium set. The next proposition quantifies this effect.

Proposition 6.5. *Suppose that G_r is diagonally strictly convex for all $r \in \mathcal{R}$. Let $r, s \in \mathcal{R} \subset (\mathbb{R}_+^*)^N$ satisfy $r_j > s_j$ for some $j \in I$ and $r_i = s_i$ for all $i \neq j$, and let x^r, x^s denote the corresponding variational equilibria. Then the directional derivative of \mathcal{J}_j at x^r in the direction $x_j^r - x_j^s$ is negative.*

We do not include the proofs of Proposition 6.4 and Proposition 6.5 because the arguments follow the same structure as Proposition 6.3. The corresponding finite-dimensional proofs are found in [34].

Diagonal strict convexity also implies a strong stability property, naturally formulated in terms of an associated ODE-type dynamical system that generates a trajectory of strategies. This dynamical perspective has served as a foundation for algorithms computing variational equilibria and offers a complementary viewpoint to modern numerical methods for variational inequalities; see, for example, [10, 11] and references therein. Beyond its algorithmic relevance, this interpretation provides a useful conceptual and historical connection between equilibrium theory and dynamical systems.

7. GEOMETRIC FORMS OF GENERALIZED GAMES

Geometric formulations of games, which are based on preference relations rather than objective functionals, are less common in the generalized Nash framework, especially in infinite dimensions. Most existing theory has been developed in analytic form, where each player solves an optimization problem and geometric structures appear only implicitly through fixed-point arguments. In this section, we present a geometric counterpart to our setting. Geometric games are interesting in their own right: they separate the modeling of preferences from numerical optimization, can be more natural in applications where utilities are only partially specified, and provide an alternative route to existence results and structural subclasses such as potential games. Our aim here is not to build a complete geometric theory matching all analytic results, but to show that many of the structural insights developed above extend naturally to preference maps and to indicate which additional tools would be needed for a fully parallel treatment.

Geometrically, the goal of each player is modeled by a preference map $\mathcal{P}_i : X \rightarrow 2^{X_i}$ defined in this way: for profile $x \in X$, $\mathcal{P}_i(x)$ contains those strategies $\hat{x}_i \in X_i$ that player i strictly prefers over x_i when the opponents' strategies x_{-i} are fixed. Setting $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_N$, we write $G = (\mathbf{X}, \mathcal{P})$ for a generalized game given in geometric form and refer to it as a *geometric game*.

The notion of equilibrium has a natural formulation in this setting.

Definition 7.1 (Geometric GNE). *We say that $\bar{x} \in \mathbf{X}(\bar{x})$ is a geometric generalized Nash equilibrium of the game $G = (\mathbf{X}, \mathcal{P})$ if*

$$(19) \quad \mathcal{P}_i(\bar{x}) \cap \mathbf{X}_i(\bar{x}_{-i}) = \emptyset,$$

for all $i \in I$.

In words, at a geometric equilibrium no player has a strictly preferred feasible alternative strategy. The regularity and convexity properties of the preference maps, needed to obtain results similar to the ones for analytic games, are listed below.

Assumption 7.2 (Preference maps). *For each $i \in I$,*

- (i) *the graph of $\mathcal{P}_i : X \rightarrow 2^{X_i}$ is open in $X \times X_i$;*
- (ii) *for each fixed $\hat{x}_{-i} \in X_{-i}$, the set $\mathcal{P}_i(x_i, \hat{x}_{-i})$ is nonempty and convex for all $x_i \in X_i$;*
- (iii) *for each fixed $\hat{x}_{-i} \in X_{-i}$ and all $x_i \in X_i$, we have $x_i \in \overline{\mathcal{P}_i(x_i, \hat{x}_{-i})} \setminus \mathcal{P}_i(x_i, \hat{x}_{-i})$.*

Recall that, for the analytic case, we investigated the role of lower semicontinuity in applying Kakutani's fixed-point theorem. In the geometric case, the situation is different: even the construction of a best-response map relies on lower semicontinuity of the constraint maps. We briefly recall the classical existence proof for the geometric case.

Theorem 7.3. *For each $i \in I$, assume that the constraint map $\mathbf{X}_i : X_{-i}^{\text{ad}} \rightarrow 2^{X_i^{\text{ad}}}$ is lower semicontinuous. If X is a separable Banach space and Assumption 7.2 holds, then the geometric GNEP $G = (\mathbf{X}, \mathcal{P})$ has an equilibrium.*

Proof. From the definition above, x is a geometric generalized Nash equilibrium precisely when $x \in \mathbf{X}(x)$ and

$$\mathcal{P}_i(x) \cap \mathbf{X}_i(x_{-i}) = \emptyset \quad \text{for all } i \in I.$$

For each $i \in I$, define

$$\mathcal{C}_i(x) = \mathcal{P}_i(x) \cap \mathbf{X}_i(x_{-i}), \quad \text{and} \quad F_i = \{x \in X^{\text{ad}} : \mathcal{C}_i(x) \neq \emptyset\}.$$

The set F_i consists of strategy bundles that *cannot* be equilibria, since for such an x , player i has at least one strictly preferred feasible alternative. Under Assumption 7.2 and the standing assumptions on \mathbf{X}_i , the restricted maps $\mathcal{C}_i|_{F_i}$ are lower semicontinuous; see [26, 38]. Moreover, because each X_i^{ad} is compact, these maps admit a continuous selection [26]. That is, there exist continuous functions

$$f_i : F_i \rightarrow X_i^{\text{ad}} \quad \text{such that} \quad f_i(x) \in \mathcal{C}_i(x) \quad \text{for all } x \in F_i.$$

We now define, for each $i \in I$,

$$\mathfrak{B}_i(x) = \begin{cases} f_i(x), & \text{if } x \in F_i, \\ \mathbf{X}_i(x_{-i}), & \text{if } x \notin F_i, \end{cases}$$

and introduce the (set-valued) best-response map

$$\mathfrak{B} : X^{\text{ad}} \rightarrow 2^{X^{\text{ad}}}, \quad \text{with } \mathfrak{B}(x) = \prod_{i=1}^N \mathfrak{B}_i(x).$$

By construction, $\mathfrak{B}(x)$ is closed, convex, and compact-valued. Moreover, fixed points of \mathfrak{B} coincide with geometric Nash equilibria. Indeed, suppose \bar{x} is a fixed point. If $\bar{x} \in F_i$ for some i , then

$$\bar{x}_i = f_i(\bar{x}) \in \mathcal{C}_i(\bar{x}) \subset \mathcal{P}_i(\bar{x}),$$

which contradicts the equilibrium condition (19). Thus a fixed point cannot belong to any F_i , and therefore satisfies $\mathcal{P}_i(\bar{x}) \cap \mathbf{X}_i(\bar{x}_{-i}) = \emptyset$ for all i .

To apply Kakutani's fixed-point theorem, we must verify that \mathfrak{B} is upper semicontinuous. It suffices to check upper semicontinuity of each component \mathfrak{B}_i separately [12, Lemma 3, p. 124]. Furthermore, by [41, Lemma 6.1, p. 241], it is enough to show that each constraint map

$$\mathbf{X}_i : X_{-i}^{\text{ad}} \rightarrow 2^{X_i^{\text{ad}}}$$

is upper semicontinuous. This follows from the fact that \mathbf{X}_i has closed graph and that X_i^{ad} is compact: one verifies that $(\mathbf{X}_i)^{-1}(C)$ is closed for every closed $C \subset X_i^{\text{ad}}$.

All conditions required by Kakutani's fixed-point theorem are therefore satisfied, so \mathfrak{B} admits a fixed point in X^{ad} . As argued above, such a point is a geometric generalized Nash equilibrium. \square \square

The theorem above is the geometric counterpart of Theorem 3.4 for analytic games and highlights the role of lower semicontinuity of the constraint maps in the geometric setting. Just as in the analytic case, we introduce a Nikaido–Isoda type map $\Upsilon : X \times X \rightarrow 2^X$ by

$$(20) \quad \Upsilon(x, y) = \prod_{i=1}^N \mathcal{P}_i(y_i, x_{-i}).$$

The next result is the geometric counterpart of Lemma 4.1.

Lemma 7.4. *Let $G = (\mathbf{X}, \mathcal{P})$ be a geometric game. A strategy bundle $\bar{x} \in X^{\text{ad}}$ is a (geometric) Nash equilibrium of G if and only if*

$$\bar{x} \in \mathbf{X}(\bar{x}) \cap \overline{\Upsilon(\bar{x}, y)} \quad \text{for all } y \in \mathbf{X}(\bar{x}).$$

Proof. Proof. Necessity follows directly from the definition of an equilibrium and from the construction of Υ . Indeed, if \bar{x} is a Nash equilibrium, then no player strictly prefers any other feasible strategy, so \bar{x} must lie in the closure of all sets $\Upsilon(\bar{x}, y)$ for $y \in \mathbf{X}(\bar{x})$.

For sufficiency, assume that $\bar{x} \in \mathbf{X}(\bar{x})$ and that

$$\bar{x} \in \overline{\Upsilon(\bar{x}, y)} \quad \text{for all } y \in \mathbf{X}(\bar{x}).$$

Suppose, towards a contradiction, that \bar{x} is not a Nash equilibrium. Then there exist $i \in I$ and $x_i \in \mathbf{X}_i(\bar{x}_{-i})$ such that $x_i \in \mathcal{P}_i(\bar{x})$; that is, player i strictly prefers x_i to \bar{x}_i . This implies $\bar{x}_i \notin \overline{\mathcal{P}_i(x_i, \bar{x}_{-i})}$. But since $(x_i, \bar{x}_{-i}) \in \mathbf{X}(\bar{x})$, the assumption $\bar{x} \in \overline{\Upsilon(\bar{x}, (x_i, \bar{x}_{-i}))}$ forces $\bar{x}_i \in \overline{\mathcal{P}_i(x_i, \bar{x}_{-i})}$, and this is a contradiction. Therefore, \bar{x} must be a Nash equilibrium. \square \square

The notion of a variational equilibrium can also be formulated in geometric terms. We keep the name *variational* to stress the parallel to the analytic case, even though no variational inequality is written explicitly.

Definition 7.5 (Variational equilibrium). *Suppose that \mathbf{X}_i is given by (3). A strategy bundle $\bar{x} \in \mathcal{C}$ is called a variational Nash equilibrium of a geometric game $G = (\mathbf{X}, \mathcal{P})$ if*

$$\bar{x} \in \mathcal{C} \cap \overline{\Upsilon(\bar{x}, y)} \quad \text{for all } y \in \mathcal{C}.$$

The following characterization in terms of the preference map \mathcal{P} is often more convenient for existence arguments; see Theorem 7.7.

Lemma 7.6. *Let $G = (\mathbf{X}, \mathcal{P})$ be a geometric game and $\mathcal{C} \subset X$ closed. A strategy bundle $\bar{x} \in \mathcal{C}$ is a variational equilibrium if and only if*

$$\mathcal{P}(\bar{x}) \cap \mathcal{C} = \emptyset.$$

Proof. Proof. By definition, $\bar{x} \in \mathcal{C}$ is a variational equilibrium if no strategy in \mathcal{C} is strictly preferred to \bar{x} by all players, that is, if there is no $y \in \mathcal{C}$ with $y \in \mathcal{P}(\bar{x})$. This is equivalent to the condition $\mathcal{P}(\bar{x}) \cap \mathcal{C} = \emptyset$. \square \square

Theorem 7.7. *Let \mathcal{C} be a compact and convex subset of X such that $X^{\text{ad}} \cap \mathcal{C} \neq \emptyset$ and assume that the structure (3) holds for the constraint maps. If X is a separable Banach space, then every generalized geometric game has a variational equilibrium.*

Proof. Proof. The proof follows the same general idea as Theorem 7.3, so we only recall the main steps. Define

$$F = \{x \in X^{\text{ad}} \cap \mathcal{C} : \mathcal{P}(x) \cap \mathcal{C} \neq \emptyset\}, \quad F(x) := \mathcal{P}(x) \cap \mathcal{C}.$$

On F , the map $F(\cdot)$ admits a continuous selection f . We then define $\mathfrak{B}(x) = f(x)$ for $x \in F$ and $\mathfrak{B}(x) = \mathcal{C}$ for $x \in X^{\text{ad}} \setminus F$, and show that \mathfrak{B} has a fixed point in X^{ad} which cannot belong to F . This fixed point is a variational equilibrium. \square \square

Potential structures can also be introduced at the geometric level. The next definition parallels Definition 4.8.

Definition 7.8 (Potential game). *Assume that the constraint maps are of the form (3), that $C \subset X$ is closed, and that X is a separable Banach space. A geometric game $G = (\mathbf{X}, \mathcal{P})$ is called a potential game if there exists a continuous function $\mathcal{G} : X \rightarrow \mathbb{R}$ and a function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:*

- (i) $g(t_k) \rightarrow 0$ implies $t_k \rightarrow 0$;
- (ii) for all $i \in I$, all $x_{-i} \in X_{-i}$, and all $y_i, z_i \in \mathbf{X}_i(x_{-i})$, the relation

$$z_i \in \mathcal{P}_i(y_i, x_{-i})$$

implies

$$\mathcal{G}(y_i, x_{-i}) - \mathcal{G}(z_i, x_{-i}) \geq g(\|y_i - z_i\|_{X_i}).$$

The function \mathcal{G} is called a potential for the geometric game G .

The results in this section show that many analytic notions (Nikaido–Isoda maps, variational equilibria, potential games) have natural geometric counterparts. One could ask whether the graph-convex and KKM-based existence results from Section 5 also extend to the geometric setting. The authors believe that this should be possible, but such an extension would require:

- a careful analysis of how graph-convexity or the KKM property for \mathbf{X} interacts with the open-graph assumptions on the preference maps \mathcal{P}_i ;
- refined fixed-point theorems for set-valued maps on infinite-dimensional spaces that combine geometric properties of the feasible sets and of the preference correspondences;
- an adaptation of the selection arguments used in Theorems 7.3 and 7.7 to situations where lower semicontinuity is not available.

Developing this additional machinery would significantly increase the technical length of the paper. For this reason, we chose to restrict the geometric part to a parallel of the lower semicontinuity-based theory and to highlight, rather than fully resolve, the questions that arise when one attempts to replace lower semicontinuity by purely geometric assumptions.

8. CONCLUSIONS AND OUTLOOK

This work revisited the existence theory for convex generalized Nash equilibrium problems in Banach spaces. We showed that equilibrium existence does not require lower semicontinuity of the constraint maps, a classical but restrictive assumption, and that graph-convexity or the KKM property are sufficient. These conditions are geometric in nature and easier to verify in PDE-constrained and infinite-dimensional games.

On the geometric side, we showed that many analytic constructions have direct counterparts in terms of preference maps and that variational equilibria and potential structures can also be formulated without explicit objective functionals.

Future research may address quantitative stability of equilibria under perturbations of the constraint maps, algorithmic schemes exploiting graph-convex or KKM structures, and extensions to nonconvex or stochastic settings. An additional direction suggested by the geometric formulation is to investigate mixed analytic–geometric models in which some players are described by objective functionals and others by preference correspondences, a situation that naturally arises in multi-agent systems with heterogeneous information or incomplete preference specification.

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