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Abstract

We develop a thermodynamically consistent model for the viscoelastic deformation of electrically charged bodies in the Lagrangian frame, incorporating a Jeffreys-type rheology and flow of free charge-carriers within the framework of the General Equations for Non-Equilibrium Reversible–Irreversible Coupling (GENERIC). The formulation couples mechanical, electrostatic, and dissipative effects in a structure that ensures compatibility with the principles of nonequilibrium thermodynamics. Furthermore, in the isothermal and mechanically quasistatic regime, i.e., where inertial effects are neglected, we establish the existence of weak solutions to a reduced version of the governing system. The proof relies on a Galerkin approximation combined with suitable regularizations of the degenerate mobilities of the free charge carriers and of the free energy. A typical application is the description of conductive hydrogels used in biomedicine.

1 Introduction

Many applications in engineering, biomechanics, and physics are concerned with dissipative materials featuring a complex interplay between mechanical deformations and other physical processes. Hydrogels, for example, are soft, water-swollen polymer networks that exhibit complex mechanical and transport behavior, making them prototypical examples of materials where multiphysics processes are strongly coupled. In applications such as soft robotics and bioelectronics, these materials undergo large deformations, requiring a finite-strain description to accurately capture their nonlinear elastic response [Avi+21; Zol+22; Arm+23]. Beyond purely mechanical effects, hydrogels often exhibit diffusive transport of solvents, ions, or other charge carriers, leading to coupled chemo-mechanical or electro-mechanical phenomena [Rou17]. In particular, conductive hydrogels present rich multiphysics challenges: the motion of charge carriers interacts with the polymer network and the solvent, inducing stresses, swelling, and nontrivial feedback on the material's electrical and mechanical properties [ATM20; NSA21]. Hence, developing sound mathematical models based on partial differential equations that integrate finite-strain mechanics with solvent and charge-carrier transport is therefore crucial for both predicting material behavior and designing e.g. efficient hydrogel-based devices.

In this paper, we investigate a model for charged visco-elastic materials at finite strains, which describes the evolution of several coupled variables in the Lagrangian frame. First, we assume that the mechanical response of the medium is given by the deformation χ from the reference configuration, and that its gradient admits the multiplicative decomposition $\nabla\chi = F_{\rm el}F_{\rm in}$ into an elastic and inelastic part. In particular, we take as the mechanical variables the deformation χ and the inelastic strain $F_{\rm in}$, so that the equations can be formulated in the fixed reference domain Ω, and not in a time-dependent intermediate domain as in the finite-strain Poynting–Thomson model studied in [CKS25]. Therein the inelastic (or viscous) strain is assumed to be compatible, i.e., $F_{\rm in} = \nabla\chi_{\rm in}$ for a viscous deformation $\chi_{\rm in}$ that maps the reference configuration Ω to the intermediate configuration $\Omega_{\rm in}(t) := \chi_{\rm in}(t, \Omega)$.

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Furthermore, we consider a (scalar) charge-carrier concentration n, and the electric potential ψ that arises due to the movement of the mobile and fixed charges. We focus on a general description of the viscous material behavior, which results from an interplay of a Kelvin-Voigt-type rheology for the strain $\nabla\chi$ and the inelastic evolution of Maxwell-type for F_{in}, which is often called Jeffreys rheology (see [KR19, Sect. 6.6]). Nonlinear viscoelastic materials with Maxwell rheology and without additional physical effects have been discussed in [KR19, Sect. 9.4.2] and in [RS19] coupled to temperature. Therein, quadratic growth of the (nonsmooth) dissipation potential was assumed.

We consider the evolution in the Lagrangian frame, meaning that the equation for the concentration n is pulled-back to the reference configuration such that its conductivity or mobility tensor depends nonlinearly on the deformation gradient, see also [RS19; MR20]. In addition, we take the physically relevant case of degenerate conductivities into account.

As a first result, we follow [MR25] and derive in Section 2 the evolution equations in a thermodynamically consistent way using the GENERIC framework (General Equation for Non-Equilibrium Reversible-Irreversible Coupling) introduced in [GÖ97; ÖG97]. In particular, we also consider temperature effects leading to an additional evolution equation for a thermal variable (e.g. temperature, entropy or internal energy) with heat sources due to the dissipative processes. We emphasize that in contrast to [MR25], we consider the Lagrangian setting, which is more natural for the description of solids. Moreover, we also provide first analytical results for a simplified model, see below. GENERIC offers a unified formalism for multi-scale and multi-physics systems, splitting dynamics into a reversible part driven by energy (via a Hamiltonian structure) and an irreversible part driven by entropy (via a convex dissipation potential). It enables consistent coupling across scales while ensuring compliance with thermodynamic laws, such as energy conservation and entropy production. GENERIC also guides computational models [SB21], gives rise to stable and efficient time-incremental schemes [JST22], and has recently been used in Physics-Informed-Neural-Networks (PINNs) to enforce thermodynamic constraints [Her+21].

Next, we consider in Section 3 a simplification of the model derived via GENERIC, where temperature and inertia effects are neglected, i.e., we restrict to the isothermal and mechanically quasistatic case. However, we note that the resulting system is still a gradient system for the free energy. We introduce a notion of weak solution, and show that under suitable assumptions (e.g. second-order nonsimple materials) the existence of weak solutions is guaranteed. The proof relies on a Faedo–Galerkin approximation in combination with certain regularizations that are necessary to deal with the degenerate behavior of the mobilities, see also [EG96]. Having shown the existence of approximate solutions, we derive an energy-dissipation balance, from which uniform a priori estimates follow. Finally, we use compactness arguments to pass to the limit and obtain the existence of a weak solution.

We remark that for the poro-visco-elastic model without charge transport studied in [OL24; Oos25], the thermo-mechanical model in [MR20; BFK23], and the finite-strain Poynting–Thomson viscoelastic model in [CKS25], (staggered) time-discrete schemes were used to show existence of solutions. However, in the present setting with additional electrostatic interaction it is not clear how to establish a suitable time-discretization scheme. The main difficulty lies in the structure of the system's underlying free energy functional: Without electrostatic interaction, this energy is (semi-)convex with respect to all variables, while with additional charged species, the energy becomes concave in the electric potential, see also [Rou17; RT20]. Consequently, time-discrete solutions would be obtained either as a saddle-point of an incremental functional, rather than minimizers, or via a staggered scheme. In the first case, to the best of the authors' knowledge, no general existence result for saddle points in the semi-convex setting is available. In the second case, it is not clear how to design a suitable time-discrete scheme that allows for an easy derivation of an energy-dissipation inequality. We therefore follow [RT20] and take the better suited method of Faedo–Galerkin approximations. In particular,

[RT20] studies finite-strain electro-elasticity in a mixed Lagrangian–Eulerian setting. However, inelastic strains are not included in the model and the mobility law for the diffusing charged species is not degenerate. Moreover, the mixed Lagrangian – Eulerian setting necessitates a regularization of the Poisson equation for the electrostatic potential, which is not required in our setting.

The structure of this paper is as follows. In Section 2, we present the thermodynamically consistent modeling of our system within the GENERIC framework. We first introduce the abstract GENERIC framework in Section 2.1 and then illustrate how charged visco-elastic materials can be described within it in Section 2.2. In Section 2.3, we specialize the model to the isothermal case. Section 3 is devoted to the mathematical analysis of the reduced model introduced in Section 2. We begin by stating the assumptions required for the analysis in Section 3.1, followed by the introduction of our notion of weak solution and the main result in Section 3.2. Finally, Section 4 is dedicated to the proof of the main result, namely the existence of weak solutions. In Section 4.1, we introduce the Galerkin scheme. Section 4.2 establishes the necessary a priori estimates, and Section 4.3 concludes with the passage to the limit.

2 GENERIC formulation of visco-elastic materials with free charges at finite-strains in the Lagrangian description

In this section, we systematically derive the equations for charged poro-visco-elastic materials, such as hydrogels, in the Lagrangian frame using the framework of GENERIC. The acronym stands for General Equation for Non-Equilibrium Reversible-Irreversible Coupling and was introduced in [GÖ97; ÖG97]. However, we emphasize that its origins in the metriplectic theory developed by Morrison in [Mor84; Mor86], see also the survey [Mor09]. The GENERIC framework offers a unified formalism for multiscale systems, splitting dynamics into a reversible part driven by energy (via a Poisson bracket) and an irreversible part driven by entropy (via a symmetric dissipative bracket). It enables consistent coupling across scales while ensuring compliance with thermodynamic laws, such as energy conservation and entropy production.

The GENERIC framework in continuum mechanics is a powerful and flexible modeling tool that unifies reversible and irreversible dynamics, making it particularly useful for describing complex materials such as viscoelastic fluids, active gels, and poroelastic tissues, while naturally incorporating thermodynamic consistency and dissipation mechanisms. We refer to [Mie11b; VPE21; Las21; Pel+22; ZPT23; MR25] for a non-exhaustive list of applications. Moreover, let us mention that it was shown recently in [MPZ25] how GENERIC structures can be rigorously derived from coarse-graining Hamiltonian systems coupled to heat baths, revealing the emergence of energy conservation, entropy production, and Onsager dissipation operators.

Notation: In the following, we denote by "$a \cdot b$", "$A : B$", and "$G \,\vdots\, H$" the scalar products between vectors $a, b \in \mathbb{R}^d$, matrices $A, B \in \mathbb{R}^{d\times d}$, and third-order tensors $G, H \in \mathbb{R}^{d\times d\times d}$, respectively.

Given a matrix $A \in \mathbb{R}^{d\times d}$, we write $\mathrm{dev}(A)$ and $\mathrm{sph}(A)$ for the deviatoric and spherical part of A, i.e., $\mathrm{dev}(A) := A - \frac{1}{d}\mathrm{tr}(A)I$ and $\mathrm{sph}(A) := \frac{1}{d}\mathrm{tr}(A)I$. The subset of deviatoric matrices is denoted by $\mathbb{R}^{d\times d}_{\mathrm{dev}}$, i.e., $A \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$ if and only if $\mathrm{sph}(A) = 0$.

2.1 Abstract setting of GENERIC

We consider states q in a state space $\boldsymbol{Q}$, which is either a flat space or a smooth manifold with tangent and cotangent bundle $\mathsf{T}\boldsymbol{Q}$ and $\mathsf{T}^*\boldsymbol{Q}$, respectively. In the GENERIC formalism, the model is determined by energy and entropy functionals $\mathcal{E} : \boldsymbol{Q} \to \mathbb{R}_\infty$ and $\mathcal{S} : \boldsymbol{Q} \to \mathbb{R}_\infty$ as well as geometric operators $\mathbb{J}$ and $\mathbb{K}$ describing the Hamiltonian and the dissipative parts of the evolution of the system in the state $q(t) \in \boldsymbol{Q}$, namely,

$$\dot{q} = \mathbb{J}(q)\mathrm{D}\mathcal{E}(t, q) + \mathbb{K}(q)\mathrm{D}\mathcal{S}(q), \qquad q(0) = q_0. \tag{1}$$

Here, the Poisson operator $\mathbb{J}$ is defined by being skew-symmetric $\mathbb{J}(q)^* = -\mathbb{J}(q)$ and satisfying the Jacobi identity, i.e., $\langle \eta_1, \mathrm{D}\mathbb{J}(q)[\mathbb{J}(q)\eta_2]\eta_3\rangle + \text{cycl. perm.} \equiv 0$ for all $\eta_1, \eta_2, \eta_3 \in \mathsf{T}^*_q\boldsymbol{Q}$. The Onsager operator $\mathbb{K}$ is defined as being symmetric and positive semi-definite, namely $\mathbb{K}(q)^* = \mathbb{K}(q) \geq 0$.

In a more general setting, the operator $\mathbb{K}(q)$ is replaced by the subdifferential of a dissipation potential $\mathcal{R}^*(q, \xi)$. The latter means that $\xi \mapsto \mathcal{R}^*(q, \xi)$ is a lower semicontinuous, convex functional such that $0 = \mathcal{R}^*(q; 0) \leq \mathcal{R}^*(q; \xi)$. In this case, the evolution equation takes the form

$$\dot{q} = \mathbb{J}(q)\mathrm{D}\mathcal{E}(t, q) + \partial_\xi\mathcal{R}^*(q, \mathrm{D}\mathcal{S}(q)), \qquad q(0) = q_0.$$

Here, the $*$ indicates that $\mathcal{R}^*$ has to be understood as dual functional to a primal functional $\mathcal{R}(q, V)$ via the (partial) Legendre transformation, i.e.,

$$\mathcal{R}^*(q, \xi) = \sup\left\{ \langle \xi, V\rangle - \mathcal{R}(q; V) \,\middle|\, V \in \mathsf{T}_q\boldsymbol{Q} \right\}$$

The original setting is recovered in the case that $\mathcal{R}^*(q, \xi) = \frac{1}{2}\langle \xi, \mathbb{K}(q)\xi\rangle$. In addition to the above conditions on $\mathbb{J}$ and $\mathbb{K}$ (or $\mathcal{R}$), we assume that the following non-interaction conditions are satisfied

$$\begin{gathered} \mathbb{J}(q)\mathrm{D}\mathcal{S}(q) \equiv 0 \quad \text{and} \quad \mathbb{K}(q)\mathrm{D}_q\mathcal{E}(q) \equiv 0, \\ \text{or } \mathcal{R}^*(q, \xi + \lambda\mathrm{D}_q\mathcal{E}) = \mathcal{R}^*(q, \xi) \text{ for all } \lambda \in \mathbb{R} \text{ and } \xi \in \mathsf{T}^*_q\boldsymbol{Q}. \end{gathered} \tag{2}$$

By convexity, the latter condition is equivalent to $\mathcal{R}^*(q, \lambda\mathrm{D}\mathcal{E}(t, q)) \equiv 0$ for all $\lambda \in \mathbb{R}$. Crucially, the latter condition gives that $\langle \mathrm{D}\mathcal{E}(t, q), V\rangle = 0$ for all $V \in \partial_\xi\mathcal{R}^*(q, \xi)$.

With the non-interaction conditions in (2), the solutions to (1) automatically satisfy the conservation of the total energy and the positivity of the entropy production, namely

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, q) &= \langle \mathrm{D}_q\mathcal{E}(t, q), \dot{q}\rangle + \partial_t\mathcal{E}(t, q) = \langle \mathrm{D}_q\mathcal{E}(t, q), \mathbb{J}(q)\mathrm{D}_q\mathcal{E}(t, q)\rangle + \partial_t\mathcal{E}(t, q) = \partial_t\mathcal{E}(t, q), \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(q) &= \langle \mathrm{D}\mathcal{S}(q), \dot{q}\rangle = \langle \mathrm{D}\mathcal{S}(q), \mathbb{K}(q)\mathrm{D}\mathcal{S}(q)\rangle \geq 0. \end{aligned}$$

2.2 GENERIC for poro-visco-elasticity

We consider the specific case where the state is given by $q = (m, \chi, F_{\mathsf{in}}, n, \tau)$, where m is the momentum, χ is the deformation, F_{in} is the inelastic strain, n is a concentration, and τ is the thermal variable, i.e., either temperature, entropy density, internal energy density, or thermal part of the internal energy. In particular, we will distinguish between the thermal variable and the other variables by writing $q = (w, \tau)$. The total energy and the entropy read

$$\mathcal{E}(q) = \int_\Omega E(m, \nabla\chi, F_{\mathsf{in}}, n, \tau)\,\mathrm{d}x, \quad \mathcal{S}(q) = \int_\Omega S(\nabla\chi, F_{\mathsf{in}}, n, \tau)\,\mathrm{d}x.$$

We emphasize that it is possible to include additional time-dependent mechanical loadings $\langle \ell(t), \chi \rangle$ in the total energy, see [Mie11b, Sect. 4.3], and surface contributions in $\mathcal{E}$ and $\mathcal{S}$, see [GM13]. The total energy density is the sum of the kinetic energy and the internal energy U, i.e.,

$$E(m, \nabla\chi, F_{\mathsf{in}}, n, \tau) = \frac{|m|^2}{2\rho} + U(\nabla\chi, F_{\mathsf{in}}, n, \tau).$$

As in [Mie11b; MR25], the variable τ is a general scalar thermodynamic variable. We only assume that the absolute temperature is given by

$$\theta = \Theta(\nabla\chi, F_{\mathsf{in}}, n, \tau) := \frac{\partial_\tau U(\nabla\chi, F_{\mathsf{in}}, n, \tau)}{\partial_\tau S(\nabla\chi, F_{\mathsf{in}}, n, \tau)} > 0.$$

For general choices of the thermal variable τ, the free energy is always given by $\Phi(\nabla\chi, F_{\mathsf{in}}, n, \tau) = U(\nabla\chi, F_{\mathsf{in}}, n, \tau) - \Theta(\nabla\chi, F_{\mathsf{in}}, n, \tau)S(\nabla\chi, F_{\mathsf{in}}, n, \tau)$. Let us also note that the following crucial relation holds between the derivatives of the free energy density $\widehat{\Phi}(\nabla\chi, F_{\mathsf{in}}, n, \theta)$ at fixed temperature:

$$\partial_z \widehat{\Phi}(\nabla\chi, F_{\mathsf{in}}, n, \theta)\big|_{\theta=\Theta(\nabla\chi, F_{\mathsf{in}}, n, \tau)} \\ = \partial_z U(\nabla\chi, F_{\mathsf{in}}, n, \tau) - \Theta(\nabla\chi, F_{\mathsf{in}}, n, \tau)\partial_z S(\nabla\chi, F_{\mathsf{in}}, n, \tau), \quad (3)$$

where z represents any of the arguments $F = \nabla\chi, F_{\mathrm{in}}$ or n. To shorten notation, we denote by $\sigma_{\mathrm{el}} = \partial_F U - \Theta\partial_F S$ the elastic stress, by $\sigma_{\mathrm{in}} = \partial_{F_{\mathrm{in}}} U - \Theta\partial_{F_{\mathrm{in}}} S$ the inelastic stress, and by $\mu_n = \partial_n U - \Theta\partial_n S$ the chemical potential associated with n, respectively. We stress that the above relation in (3) for the forces only holds if the free energy is written as a function of the temperature $\tau = \theta$.

Remark 2.1. *We can also consider settings where E and S depend on (higher) derivatives of $\nabla\chi, F_{\mathsf{in}}, n$ without any substantial changes to the theory. In fact, in the mathematical analysis of the isothermal model in Section 3, we assume that Φ depends also on $\mathrm{D}^2\chi$, ∇F_{in}, and ∇n to exploit the higher regularity for additional compactness properties of solutions. However, for a simpler representation, we omit these additional dependencies in this section.*

2.2.1 The reversible part of the evolution

We first focus on the Hamiltonian (or reversible) part of the evolution given in terms of the total energy $\mathcal{E}$ and the symplectic structure $\mathbb{J}$. We define

$$\mathbb{J}(q) = M_{\mathcal{S}}(q)\mathbb{J}_{\mathsf{simple}}M_{\mathcal{S}}(q)^*, \quad \text{where} \quad \langle \eta^{(1)}, \mathbb{J}_{\mathsf{simple}}\eta^{(2)} \rangle = \eta^{(1)}_\chi \cdot \eta^{(2)}_m - \eta^{(2)}_\chi \cdot \eta^{(1)}_m \quad (4)$$

for $\eta^{(i)} = (\eta^{(i)}_m, \eta^{(i)}_\chi, \eta^{(i)}_{F_{\mathsf{in}}}, \eta^{(i)}_n, \eta^{(i)}_\tau)$ being the (energy-)conjugated forces. The operator $M_{\mathcal{S}}(q)$ is given as

$$M_{\mathcal{S}}(q) = \begin{pmatrix} \mathbb{I} & 0 & 0 & 0 & 0 \\ 0 & \mathbb{I} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{I} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{\Delta_\chi S[\square]}{\partial_\tau S} & -\frac{\partial_{F_{\mathsf{in}}} S}{\partial_\tau S} & -\frac{\partial_n S}{\partial_\tau S} & \frac{1}{\partial_\tau S} \end{pmatrix},$$

where we have defined $\Delta_\chi S[v] = \partial_F S : \nabla v$ for $v : \Omega \to \mathbb{R}^d$. Note that in the case that S depends on (higher) derivatives of $\nabla\chi, F_{\mathsf{in}}, n$ (see Remark 2.1), we have to replace $\Delta_\chi S[v]$ by $\Delta_\chi S[v] = \partial_F S : \nabla v + \partial_G S \,\vdots\, \mathrm{D}^2 v$, $\partial_{F_{\mathrm{in}}} S[V]$ by $\Delta_{F_{\mathrm{in}}} S[V] = \partial_{F_{\mathrm{in}}} S : V + \partial_{G_{\mathrm{in}}} S \,\vdots\, \nabla V$, and so on.

We get for $\eta = (\eta_m, \eta_\chi, \eta_{F_{\rm in}}, \eta_n, \eta_\tau)$

$$M_{\mathcal{S}}(q)^*\eta = \begin{pmatrix} \eta_m \\ \eta_\chi + \operatorname{div}\big(\frac{\eta_\tau}{\partial_\tau S}\partial_F S\big) \\ \eta_{F_{\mathsf{in}}} - \frac{\eta_\tau}{\partial_\tau S}\partial_{F_{\rm in}} S \\ \eta_n - \frac{\eta_\tau}{\partial_\tau S}\partial_n S \\ \frac{\eta_\tau}{\partial_\tau S} \end{pmatrix}.$$

We easily check that $M_{\mathcal{S}}^* \mathrm{D}\mathcal{S} = (0, \dots, 0, 1)^\top$, and the first non-interaction condition in (2) is satisfied. Moreover, we compute for the reversible part in the evolution that

$$\mathbb{J}(q)\mathrm{D}\mathcal{E}(q) = \big(\operatorname{div}(\sigma_{\mathsf{el}}), \tfrac{1}{\rho}m, 0, 0, -\tfrac{1}{\partial_\tau S}(\partial_F S : \nabla\dot{\chi})\big)^\top, \tag{5}$$

with $\sigma_{\mathsf{el}} = \partial_F E - \Theta\partial_F S$ being the elastic stress tensor, as defined above. If E and S depended also on $\mathrm{D}^2\chi$, additional hyperstress contributions would appear in the first and last components, see (11a).

Remark 2.2. *Clearly, the operator $\mathbb{J}$ defined in* (4) *is skew-symmetric. It also satisfies the Jacobi identity, since $\mathbb{J}_{simple}$ satisfies it, and the Jacobi identity is invariant under coordinate transformations (see e.g. [Mie06, Lemma 4.5]).*

2.2.2 The irreversible part of the evolution

Following [MR25], the dual dissipation potential $\mathcal{R}^*$ is given in the form

$$\mathcal{R}^*(q;\xi) = \mathcal{R}^*_{\mathsf{simple}}(q, N_{\mathcal{E}}(q)^*\xi), \quad \text{for } \xi \in \mathsf{T}^*_q\boldsymbol{Q} \tag{6}$$

where the operator $N_{\mathcal{E}}$ is constructed such that $N_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{E}(q) \equiv (0,0,0,0,1)^\top$. The second noninteraction condition in (2) then follows from assuming that $\mathcal{R}^*_{\mathsf{simple}}(q, (0,0,0,0,\lambda)^\top) \equiv 0$ for all $\lambda \in \mathbb{R}$.

We assume that $\mathcal{R}^*_{\mathsf{simple}}$ has the additive decomposition into a viscous, a plastic, a diffusion, and a heat conduction contribution, viz.

$$\mathcal{R}^*_{\mathsf{simple}}(q,\mu) = \mathcal{R}^*_{\mathsf{visc}}(q, \boldsymbol{\mu}_m) + \mathcal{R}^*_{\mathsf{plast}}(q, \mu_{F_{\mathsf{in}}}) + \mathcal{R}^*_{\mathsf{diff}}(q, \mu_n) + \mathcal{R}^*_{\mathsf{heat}}(q, \mu_\tau),$$

where we highlight that the placeholder $\boldsymbol{\mu}_m$ is matrix-valued. The operator $N_{\mathcal{E}}(q)$ now has the following form

$$N_{\mathcal{E}}(q) = \begin{pmatrix} -\operatorname{div}(\cdot) & 0 & 0 & 0 & 0 \\ 0 & \mathbb{I} & 0 & 0 & 0 \\ 0 & 0 & F_{\rm in} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{\nabla\partial_m E}{\partial_\tau E}:(\cdot) & -\frac{\Delta_\chi E}{\partial_\tau E} & -\frac{\partial_{F_{\rm in}} E F_{\rm in}^\top}{\partial_\tau E}:(\cdot) & -\frac{\partial_n E}{\partial_\tau E} & \frac{1}{\partial_\tau E} \end{pmatrix},$$

so that we compute

$$N_{\mathcal{E}}(q)^*\xi = \begin{pmatrix} \nabla\xi_m - \frac{\xi_\tau}{\partial_\tau E}\nabla\partial_m E \\ \xi_\chi + \operatorname{div}\big(\frac{\xi_\tau}{\partial_\tau E}\partial_F E\big) \\ \xi_{F_{\mathsf{in}}} F_{\mathsf{in}}^\top - \frac{\xi_\tau}{\partial_\tau E}\partial_{F_{\mathsf{in}}} E F_{\mathsf{in}}^\top \\ \xi_n - \frac{\xi_\tau}{\partial_\tau E}\partial_n E \\ \frac{\xi_\tau}{\partial_\tau E} \end{pmatrix} \quad \Longrightarrow \quad N_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{E}(q) = (0,\dots,0,1)^\top. \tag{7}$$

In particular, we immediately see that the second non-interaction condition in (2) is satisfied if $\mathcal{R}^*_{\text{heat}}(q,\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Indeed, we make the following choices for $\mathcal{R}^*_{\text{visc}}$ and $\mathcal{R}^*_{\text{in}}$:

$$\mathcal{R}^*_{\text{visc}}(q;\boldsymbol{\mu}_m) = \int_\Omega \frac{\Theta(q)}{2}\mathbb{D}(q)\boldsymbol{\mu}_m : \boldsymbol{\mu}_m \,\mathrm{d}x, \quad \mathcal{R}^*_{\text{in}}(q;\mu_{F_{\text{in}}}) = \int_\Omega \zeta^*_{\text{in}}(q; -\Theta(q)\text{dev}(\mu_{F_{\text{in}}}))\,\mathrm{d}x, \tag{8}$$

where $\text{dev}(A) = A - \frac{1}{d}\text{tr}(A)I$ denotes the deviatoric (trace-free) part of the matrix $A \in \mathbb{R}^{d\times d}$, and $\widetilde{\mathbb{D}}(q)$ is a symmetric, positive definite fourth order tensor. The remaining dissipation potentials have the form

$$\mathcal{R}^*_{\text{diff}}(q,\mu_n) = \frac{1}{2}\int_\Omega \mathcal{A}(q)\nabla\mu_n\cdot\nabla\mu_n\,\mathrm{d}x \quad \text{and} \quad \mathcal{R}^*_{\text{heat}}(q,\mu_\tau) = \frac{1}{2}\int_\Omega \mathcal{K}(q)\nabla\mu_\tau\cdot\nabla\mu_\tau\,\mathrm{d}x, \tag{9}$$

for some symmetric, positive definite second order tensors $\mathcal{A}(q)$ and $\mathcal{K}(q)$. With these choices, we arrive at the following irreversible part of the evolution

$$\begin{aligned}
&\partial_\xi\mathcal{R}^*(q;\mathrm{D}_q\mathcal{S}(q)) = N_{\mathcal{E}}(q)\partial_\mu\mathcal{R}^*_{\text{simple}}(q;N_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{S}(q))\\
&= \begin{pmatrix} \text{div}\,(\sigma_{\text{vi}}) \\ 0 \\ -\Theta\text{dev}\big(\partial\zeta^*_{\text{in}}(\text{dev}(\sigma_{\text{in}}F_{\text{in}}^\top))\big)F_{\text{in}} \\ \text{div}\,j_n \\ -\frac{1}{\partial_\tau E}\text{div}\,j_{\text{heat}} + \frac{1}{\partial_\tau E}\sigma_{\text{vi}}{:}\nabla\dot{\chi} - \frac{\Theta\partial_{F_{\text{in}}}E}{\partial_\tau E}{:}\text{dev}\partial\zeta^*_{\text{in}}(\text{dev}(\sigma_{\text{in}})) - \frac{\partial_n E}{\partial_\tau E}\text{div}\,j_n \end{pmatrix}
\end{aligned}$$

where $\sigma_{\text{vi}} = \mathbb{D}(q)\nabla\dot{\chi}$ is the viscous stress, $\sigma_{\text{in}} = \partial_{F_{\text{in}}}E - \Theta\partial_{F_{\text{in}}}S$ is the inelastic stress, the velocity is given via $\dot{\chi} = \partial_m E$, and the flux via $j_n = \mathcal{A}\nabla\frac{\mu_n}{\Theta}$.

2.2.3 The full system of equations

We can now write down the full set of equations for the GENERIC system by combining Section 2.2.1 and Section 2.2.2. Indeed, the abstract evolution equation

$$\dot{q} = \mathbb{J}(q)\mathrm{D}\mathcal{E}(t,q) + \partial_\xi\mathcal{R}^*(q;\mathrm{D}_q\mathcal{S}(q)),$$

leads to the following system of equations:

$$\partial_t(\rho\dot{\chi}) = \text{div}\,(\sigma_{\text{el}} + \mathbb{D}(q)\nabla\dot{\chi}), \tag{10a}$$

$$\dot{F}_{\text{in}}F_{\text{in}}^{-1} = -\Theta\text{dev}\big(\partial\zeta^*_{\text{in}}(\text{dev}(\sigma_{\text{in}}F_{\text{in}}^\top))\big), \tag{10b}$$

$$\dot{n} = \text{div}\,(\mathcal{A}(q)\nabla\frac{\mu_n}{\Theta}), \tag{10c}$$

$$\begin{aligned}\partial_\tau E\dot{\tau} = {}& -\Theta(\partial_F S : \nabla\dot{\chi}) - \text{div}\,\big(\mathcal{K}\nabla\frac{1}{\Theta}\big) + \mathbb{D}(q)\nabla\dot{\chi} : \nabla\dot{\chi}\\ &- \Theta\partial_{F_{\text{in}}}E : \text{dev}\partial\zeta^*_{\text{in}}(\text{dev}(\partial_{F_{\text{in}}}\Phi F_{\text{in}}^\top)) - \partial_n E\text{div}\,j_n.\end{aligned} \tag{10d}$$

Remark 2.3 (Boundary conditions)**.** *Let us highlight that the GENERIC structure can be easily extended to also take chemical reactions of several interacting species with concentrations $n = (n_1,\ldots,n_I)$ or bulk-interface coupling into account. We refer the reader to [GM13; Mie11a] and [TH22] for more details.*

Remark 2.4 (Local balance laws for energy and entropy)**.** *Local balance laws for the (internal) energy and entropy can be derived similarly as in [MR25, Sec. 4.6]. To derive them, one makes for the thermal variable the choice $\tau = e = E(\cdot)$ or $\tau = s = S(\cdot)$, respectively.*

Remark 2.5 (Eulerian formulation)**.** *In [MR25, Sec. 4] a similar system to* (10) *is considered, but in the Eulerian framework. Instead of the variable $F_{\rm in}$, therein the variable $F_e = \nabla\chi F_{\rm in}^{-1}$ is considered. To reformulate our system, we note that by differentiating the relation $F = F_e F_{\rm in}$ we obtain*

$$\dot{F}_e = \dot{F}F^{-1}F_e - F_e\dot{F}_{\rm in}F_{\rm in}^{-1} = LF_e + F_e\Theta\mathrm{dev}\big(\partial\zeta_{\rm in}^*(\mathrm{dev}(\sigma_{\rm in}F_{in}^\top))\big),$$

where we have abbreviated the distortion $L = \dot{F}F^{-1}$. This corresponds to their equation (4.21b), the different minus sign coming from a different convention used.

2.2.4 Adding charges

In conductive hydrogels, the transport of charged species and their electrostatic interaction has to be taken into account [ATM20]. We can additionally include the latter following [Mie11a], where consistent models for the transport of electrons and holes in semiconductor materials were considered. We let n denote the concentration of (for simplicity) one charged species and consider the electrostatic potential $\psi(x) = \Psi[\nabla\chi, \delta_{\sf ch}](x)$ that solves for given deformation gradient $\nabla\chi$ and given charge concentration $\delta_{\sf ch}(x)$ the Poisson equation

$$-\mathrm{div}\,(\varepsilon(\nabla\chi)\nabla\psi) = \delta_{\sf ch} \quad \text{in } \Omega,$$

where $\varepsilon(\nabla\chi) \in \mathbb{R}^{d\times d}$ is the symmetric and positive definite electric permittivity tensor. We assume Robin boundary condition

$$-\varepsilon(\nabla\chi)\nabla\psi\cdot\vec{n} = \kappa(x)(\psi_{\rm ext}(x) - \psi) \quad \text{on } \partial\Omega,$$

where $\kappa(x) \geq 0$ is some given transmission function and $\psi_{\rm ext}(x)$ is an applied external voltage. The total internal energy density is given via $U_{\sf tot} = U + U_{\sf elec}$ where the (nonlocal) electrostatic contribution $U_{\sf elec}$ reads

$$(\nabla\chi, n) \mapsto U_{\rm elec}[\nabla\chi, n] := \frac{\varepsilon(\nabla\chi)}{2}\nabla\psi\cdot\nabla\psi, \quad \psi = \Psi[\nabla\chi, n]$$

with $\delta_{\sf ch} = e_0 n + N$ and $N(x) \in \mathbb{R}$ a fixed charge concentration. The entropy density S remains unchanged since we do not take a dependence of ε on the thermal variable into account. For the functional $\mathcal{E}_{\sf elec}(\chi, n) = \int_\Omega U_{\sf elec}[\nabla\chi, n](x)\,\mathrm{d}x$, we formally obtain the derivatives

$$\frac{\delta\mathcal{E}_{\sf elec}}{\delta n}(\chi, n) = e_0\Psi(\nabla\chi, e_0 n + N), \quad \frac{\delta\mathcal{E}_{\sf elec}}{\delta\chi}(\chi, n) = -\mathrm{div}\,(\sigma_{\sf Maxwell}[\nabla\chi, n]),$$

where the Maxwell stress $\sigma_{\sf Maxwell}[\nabla\chi, n]$ is given as

$$\sigma_{\sf Maxwell}[\nabla\chi, n] = -\frac{1}{2}\partial_F\varepsilon(\nabla\chi)\nabla\psi\cdot\nabla\psi.$$

In particular, if the permittivity has the form $\varepsilon(x, F) = \mathrm{Cof}F^\top\varepsilon_*(x)\mathrm{Cof}F/\det F$, for a constant (Eulerian) permittivity tensor $\varepsilon_*(x) \in \mathbb{R}^{d\times d}$, a calculation shows that

$$\sigma_{\sf Maxwell}[\nabla\chi, n] = \big(\nabla\chi^{-\top}\nabla\psi\otimes\varepsilon_*\nabla\chi^{-\top}\nabla\psi - \frac{1}{2}\nabla\chi^{-\top}\nabla\psi\cdot\varepsilon_*\nabla\chi^{-\top}\nabla\psi\big)\mathrm{Cof}\nabla\chi.$$

Under suitable regularity assumptions, one can show (see e.g. [RT20, Remark 1]) that

$$-\mathrm{div}\,(\sigma_{\sf Maxwell}) = (e_0 n + N)F^{-\top}\nabla\psi,$$

which is also known as the Lorentz force.

The Maxwell stress $\sigma_{\sf Maxwell}$ enters as an additional stress contribution in equation (10a), while the gradient of the electrostatic potential appears additionally as a drift term in the continuity for n in (10c). The quantity $\mu_n = \partial_n U - \Theta\partial_n S + e_0\Psi$ is usually referred to as the electrochemical potential.

2.3 Reduction to isothermal case

In many applications, it is sufficient to consider the isothermal case, where $\tau = \theta \equiv \theta_*$ can be assumed to be a constant. Indeed, we argue as in [Mie11b] and take a coupling to a heat bath fixed to the given temperature $\theta_* > 0$ into account. In the limit of large coupling to the heat bath, the driving functional is given by (see also (3))

$$\mathcal{F}_*(w) = \mathcal{E}(w, \theta_*) - \theta_* \mathcal{S}(w, \theta_*).$$

In this setting, the reduced evolutionary system reads

$$\dot{w} = \mathbb{J}_{\mathsf{simple}}(w) \mathrm{D}\mathcal{F}_*(w) + \partial_{\xi_w} \mathcal{R}_{\mathsf{simple}}\Big(q, \frac{1}{\theta_*} \mathrm{D}\mathcal{F}_*(w)\Big),$$

where we used the non-interaction condition and the special structure of the skew-symmetric operator $\mathbb{J}$ and the dissipation potential $\mathcal{R}^*$ in (4) and (6), respectively. In particular, the system of partial differential equations reduces to (10a) – (10c) with fixed $\Theta = \theta_*$.

3 Mathematical analysis of reduced model

We now turn to the analysis of the isothermal version of the model in (10) including electric charges. We consider a body occupying the reference configuration $\Omega \subset \mathbb{R}^d$, which is assumed to be an open, bounded domain with $C^{1,1}$ boundary. The latter satisfies $\partial\Omega = \Gamma_D \cup \Gamma_N$ (disjoint) such that the Dirichlet part Γ_D has positive surface measure $\int_{\Gamma_D} 1 \,\mathrm{d}S > 0$. We denote by $L^p(\Omega)$, $H^k(\Omega)$, and $W^{k,p}(\Omega)$ the usual Lebesgue and Sobolev spaces with the standard norms. For all $p \geq 1$, we consider the dual exponent $p' = p/(p-1)$ (with $p' = \infty$ if $p = 1$). We will consider deformations χ on Ω that are fixed on the Dirichlet part Γ_D, namely, we introduce the space

$$W^{2,p}_{\mathrm{id}}(\Omega; \mathbb{R}^d) := \{\chi \in W^{2,p}(\Omega; \mathbb{R}^d) \mid \chi|_{\Gamma_D} = \mathrm{id}\}.$$

Similarly, the closed subspace $W^{k,p}_0(\Omega)$ denotes the functions in $W^{k,p}(\Omega)$ with zero trace on Γ_D.

We make the following simplifications and amendments compared to Section 2:

(i) We assume that we are in the isothermal setting, i.e., $\theta \equiv \theta_*$, see Section 2.3.

(ii) We restrict ourselves to the mechanically quasistatic case, i.e., there are no inertial forces such that the term $\rho\ddot{\chi}$ in (10a) can be neglected.

(iii) We include time-dependent mechanical loadings $\ell(t)$ in (10a).

(iv) For a rigorous analysis, we add higher-order regularization terms depending on $\mathrm{D}^2\chi$, ∇F_{in} and ∇n to the free energy. See also Remark 2.1.

(v) We do not take additional constraints on the inelastic strain such as $\det F_{\mathrm{in}} = 1$ into account.

Our model consists of a deformation χ, which admits the multiplicative decomposition of the deformation gradient $F = \nabla\chi = F_{\mathrm{el}} F_{\mathrm{in}}$ into an elastic and inelastic part. Furthermore, we consider a charge carrier concentration n, and an electrostatic potential ψ.

Let $\Phi = \Phi(\nabla\chi, F_{\rm in}, n, \nabla n, \psi, \nabla\psi)$ be a free energy density which gives rise to the elastic stress $\sigma_{\rm el} := \partial_F\Phi$, the inelastic stress $\sigma_{\rm in} := \partial_{F_{\rm in}}\Phi$, and electro-chemical potential $\mu_n := \partial_n\Phi$. Denote by $\mathscr{H} = \mathscr{H}(\mathrm{D}^2\chi, \nabla F_{\rm in})$ a hyperstress potential, which gives rise to the hyperstresses $\mathfrak{h}_{\rm el} = \partial_G\mathscr{H}$ and $\mathfrak{h}_{\rm in} = \partial_{G_{\rm in}}\mathscr{H}$. Finally, we consider a dissipation potential $\zeta = \zeta(\nabla\chi, \nabla\dot\chi, F_{\rm in}, \dot F_{\rm in})$, which gives rise to the viscous stress $\sigma_{\rm vi} = \partial_{\dot F}\zeta$. We denote by $\mathcal{A}$ the conductivity tensor, by ε the permittivity tensor, by e_0 the elementary charge, and by N a fixed doping. The model is now given in the reference domain Ω by the evolutionary equations:

$$-\mathrm{div}\,\big(\sigma_{\rm el}(\nabla\chi F_{\rm in}^{-1}, n, \nabla n, \nabla\psi) + \sigma_{\rm KV}(\nabla\chi, \nabla\dot\chi) - \mathrm{div}\,\mathfrak{h}_{\rm el}(\mathrm{D}^2\chi)\big) = f, \tag{11a}$$

$$\sigma_{\rm in}(\nabla\chi F_{\rm in}^{-1}, F_{\rm in}) - \mathrm{div}\,\mathfrak{h}_{\rm in}(\nabla F_{\rm in}) + \zeta_{\rm in}'(\dot F_{\rm in} F_{\rm in}^{-1})F_{\rm in}^{-\top} = 0, \tag{11b}$$

$$\dot n - \mathrm{div}\,(\mathcal{A}(\nabla\chi, n)\nabla\mu_n) = 0, \tag{11c}$$

$$-\mathrm{div}\,(\varepsilon(\nabla\chi)\nabla\psi) = e_0 n + N, \tag{11d}$$

where f is some external bulk force density.

We complement this system with the boundary conditions

$$\chi = \mathrm{id} \quad \text{and} \quad F_{\rm in} = I \qquad \text{on } \Gamma_D, \tag{12a}$$

$$(\sigma_{\rm el} + \sigma_{\rm vi})\vec{n} - \mathrm{div}_s(\mathfrak{h}_{\rm el}\vec{n}) = g \quad \text{and} \quad \mathfrak{h}_{\rm in}\vec{n} = 0 \qquad \text{on } \Gamma_N, \tag{12b}$$

$$\mathfrak{h}_{\rm el}(\mathrm{D}^2\chi) : (\vec{n}\otimes\vec{n}) = 0 \qquad \text{on } \partial\Omega, \tag{12c}$$

$$\mathcal{A}(\nabla\chi, n)\nabla\mu_n\cdot\vec{n} = 0 \quad \text{and} \quad \nabla n\cdot\vec{n} = 0 \qquad \text{on } \partial\Omega, \tag{12d}$$

$$-\varepsilon(\nabla\chi)\nabla\psi\cdot\vec{n} = \kappa(\psi_{\rm ext} - \psi) \qquad \text{on } \partial\Omega, \tag{12e}$$

where we follow [KR19, Sect. 9.4.1] for the boundary conditions for $F_{\rm in}$. Here, g is a boundary traction; $\psi_{\rm ext}$ is a given external potential; $\kappa \geq 0$ is a boundary permeability; $\vec{n}$ is the unit normal on $\partial\Omega$; and div_s is the surface divergence, defined by $\mathrm{div}_s(\cdot) = \mathrm{tr}(\nabla_s(\cdot))$, i.e., the trace of the surface gradient $\nabla_s v = (I - \vec{n}\otimes\vec{n})\nabla v = \nabla v - \frac{\partial v}{\partial\vec{n}}\vec{n}$. For more details on the surface divergence, we refer to [MR20, Sect. 2]. We remark that instead of the homogeneous Neumann condition for the flux $\mathcal{A}\nabla\mu_n$ in (12d), it is also possible to use a general Robin boundary condition allowing for interaction with the environment, see e.g. [RT20].

Finally, we have the initial conditions

$$\chi(0) = \chi_0, \quad F_{\rm in}(0) = F_{\rm in,0}, \quad n(0) = n_0, \quad \psi(0) = \psi_0. \tag{13}$$

We note that from a physical perspective it is crucial to impose frame- and plastic-indifference of the free energy, the hyperstress potential and the dissipation potential (see e.g. [RS18]). Concretely, this means that for all rotations $R_1, R_2 \in \mathrm{SO}(d)$ the free energy satisfies the invariance $\Phi(R_1F, R_2F_{\rm in}) = \Phi(F, F_{\rm in})$, and for all $\widetilde{R} \in \mathrm{GL}^+(d)$: $\Phi(F\widetilde{R}, F_{\rm in}\widetilde{R}) = \Phi(F, F_{\rm in})$. Similarly, for the hyperstress potential we have for all $R_1, R_2 \in \mathrm{SO}(d)$ that $\mathscr{H}(R_1G, R_2G_{\rm in}) = \mathscr{H}(G, G_{\rm in})$. Finally, the dissipation potential satisfies dynamic frame-indifference with respect to F, i.e., $\zeta(RF, \dot RF + R\dot F, F_{\rm in}, \dot F_{\rm in}) = \zeta(F, \dot F, F_{\rm in}, \dot F_{\rm in})$ for all smooth $t \mapsto R(t) \in \mathrm{SO}(d)$, and plastic indifference with respect to $F_{\rm in}$, i.e., $\zeta(F, \dot F, F_{\rm in}\widetilde{R}, \dot F_{\rm in}\widetilde{R}) = \zeta(F, \dot F, F_{\rm in}, \dot F_{\rm in})$ for all $\widetilde{R} \in \mathrm{GL}^+(d)$. However, for the mathematical analysis these additional properties are neither necessary nor restrictive.

3.1 Assumptions

We state the main assumptions on the constitutive functions and the data of the problem, which are the basis for the subsequent analysis of above system.

For any $R > 0$, denote by F_R the set given by

$$\mathsf{F}_R = \big\{F \in \mathrm{GL}^+(d) \,\big|\, |F| \le R,\ |F^{-1}| \le R,\ \text{and}\ \det F \ge 1/R\big\}.$$

(A1) (i) The conductivity tensor $\mathcal{A} : \mathrm{GL}^+(d) \times (0,1) \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is C^1, and can be decomposed as $\mathcal{A}(F,n) = B(n)\mathcal{A}^F(F)$ with $B : [0,1] \to \mathbb{R}_+$ and $\mathcal{A}^F : \mathrm{GL}^+(d) \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$. For all $R > 0$ there exist constants $C_{\mathcal{A},i} > 0$ $(0 \le i \le 5)$ such that

$$\mathcal{A}^F(F)\xi\cdot\xi \ge C_{\mathcal{A},0}|\xi|^2, \quad |\mathcal{A}^F(F)| \le C_{\mathcal{A},1}, \quad \big|\partial_F\mathcal{A}^F(F)\big| \le C_{\mathcal{A},2} \quad \text{for all} \xi \in \mathbb{R}^d,\ F \in \mathsf{F}_R,$$

and

$$C_{\mathcal{A},3}n(1-n) \le B(n) \le C_{\mathcal{A},4}n(1-n), \quad |B'(n)| \le C_{\mathcal{A},5}(1-2n) \quad \text{for all } n \in [0,1].$$

We extend B to a function defined for all $n \in \mathbb{R}$ by setting $B(n) = 0$ for $n < 0$ and $n > 1$.

(ii) The electric permittivity $\varepsilon : \mathrm{GL}^+(d) \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is C^1. For all $R > 0$ there exist constants $C_{\varepsilon,0}, C_{\varepsilon,1} > 0$ such that

$$\xi \cdot \varepsilon(F)\xi \ge C_{\varepsilon,0}|\xi|^2, \quad |\varepsilon(F)| \le C_{\varepsilon,1} \quad \text{for all } \xi \in \mathbb{R}^d, F \in \mathsf{F}_R.$$

(A2) The free energy density

$$\Phi : \begin{cases} \mathrm{GL}^+(d) \times \mathrm{GL}^+(d) \times [0,1] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d & \to \quad \mathbb{R} \\ (\nabla\chi, F_{\mathrm{in}}, n, \nabla n, \psi, \nabla\psi) & \mapsto \quad \Phi(\nabla\chi, F_{\mathrm{in}}, n, \nabla n, \psi, \nabla\psi) \end{cases}$$

is the sum of a mechanical energy Φ_{mech}, an inelastic energy Φ_{in}, a Biot-type energy Φ_{Biot}, and an electric energy Φ_{elec}:

$$\Phi(\nabla\chi, F_{\mathrm{in}}, n, \nabla n, \psi, \nabla\psi) = \Phi_{\mathrm{mech}}(\nabla\chi F_{\mathrm{in}}^{-1}) + \Phi_{\mathrm{in}}(F_{\mathrm{in}}) + \Phi_{\mathrm{elec}}(\nabla\chi, n, \nabla n, \psi, \nabla\psi) \tag{14}$$

(i) The mechanical energy $\Phi_{\mathrm{mech}}(\nabla\chi F_{\mathrm{in}}^{-1})$ is bounded from below, C^1 on $\mathrm{GL}^+(d)$, frame-indifferent, and satisfies for some $q > \frac{pd}{p-d}$ and constants $C_{\Phi,0}, C_{\Phi,1} > 0$

$$\Phi_{\mathrm{mech}}(F_{\mathrm{el}}) \ge C_{\Phi,0}|F_{\mathrm{el}}| + \frac{C_{\Phi,0}}{(\det F_{\mathrm{el}})^q} - C_{\Phi,1} \quad \text{for all } F_{\mathrm{el}} \in \mathrm{GL}^+(d)$$

(ii) The inelastic energy $\Phi_{\mathrm{in}}(F_{\mathrm{in}})$ is bounded from below, C^1 on $\mathrm{GL}^+(d)$, frame-indifferent, and satisfies for some $q_{\mathrm{in}} > \frac{p_{\mathrm{in}}d}{p-d}$ and constants $\widetilde{C}_{\Phi,0}, \widetilde{C}_{\Phi,1} > 0$

$$\Phi_{\mathrm{in}}(F_{\mathrm{in}}) \ge \widetilde{C}_{\Phi,0}|F_{\mathrm{in}}| + \frac{\widetilde{C}_{\Phi,0}}{(\det F_{\mathrm{in}})^{q_{\mathrm{in}}}} - \widetilde{C}_{\Phi,1}.$$

(iii) The electric energy $\Phi_{\mathrm{elec}}(\nabla\chi, n, \nabla n, \psi, \nabla\psi)$ is given by

$$\Phi_{\mathrm{elec}}(\nabla\chi, n, \nabla n, \psi, \nabla\psi) = (e_0 n + N)\psi - \frac{\varepsilon(\nabla\chi)}{2}\nabla\psi\cdot\nabla\psi + \Phi_n(\nabla\chi, n) + \frac{\gamma}{2}\mathcal{A}^F(\nabla\chi)\nabla n\cdot\nabla n, \tag{15}$$

where e_0 is the elementary charge, $\gamma > 0$ is a capillarity coefficient, and $\mathcal{A}^F$ is defined in Assumption ***(A1)***(i). Moreover, Φ_n can be decomposed as $\Phi_n(F,n) = \Phi_n^1(F,n) + \Phi_n^2(n)$, with $\Phi_n^1 \in C^2(\mathrm{GL}^+(d)\times\mathbb{R};\mathbb{R})$, $\Phi_n^2 \in C^2((0,1);\mathbb{R})$ such that for all $R > 0$ there exist constants $\widetilde{C}_i := \widetilde{C}_{\Phi,R,i} > 0$ $(0 \le i \le 2)$ such that:

$$\begin{aligned} &\left|\partial^2_{Fn}\Phi_n^1(F,n)\right| \le \widetilde{C}_0 \qquad \text{for } F \in \mathsf{F}_R, n \in \mathbb{R}. \\ &\frac{\widetilde{C}_1}{n(1-n)} \le (\Phi_n^2(n))'' \le \frac{\widetilde{C}_2}{n(1-n)} \qquad \text{for } n \in (0,1). \end{aligned} \tag{16}$$

(A3) The hyperstress potential $\mathscr{H} : \mathbb{R}^{d\times d\times d} \times \mathbb{R}^{d\times d\times d} \to \mathbb{R}$ is the sum of an elastic part $\mathscr{H}_{\mathrm{el}} = \mathscr{H}_{\mathrm{el}}(\mathrm{D}^2\chi)$ and an inelastic part $\mathscr{H}_{\mathrm{in}} = \mathscr{H}_{\mathrm{in}}(\nabla F_{\mathrm{in}})$:

(i) The elastic part $\mathscr{H}_{\mathrm{el}}(\mathrm{D}^2\chi)$ is convex, C^1 on $\mathbb{R}^{d\times d\times d}$, and frame-indifferent. Moreover, for some $p \in (d,\infty) \cap [3,\infty)$ and constants $C_{\mathscr{H},i} > 0$ $(0 \le i \le 2)$ we have

$$C_{\mathscr{H},0}|G|^p \le \mathscr{H}_{\mathrm{el}}(G) \le C_{\mathscr{H},1}(1+|G|^p)$$

and

$$\big(\partial_G\mathscr{H}(G) - \partial_G\mathscr{H}(\widetilde{G})\big) \,\vdots\, (G-\widetilde{G}) \ge C_{\mathscr{H},2}|G-\widetilde{G}|^{p_{\mathrm{in}}}. \tag{17}$$

(ii) The inelastic part $\mathscr{H}_{\mathrm{in}}(\nabla F_{\mathrm{in}})$ is convex, C^1 on $\mathbb{R}^{d\times d\times d}$, and frame-indifferent. Moreover, for some $p_{\mathrm{in}} \in (d,\infty) \cap [2,\infty)$ and constants $C_{\mathscr{H},i} > 0$ $(0 \le i \le 2)$ we have

$$C_{\mathscr{H},0}|G_{\mathrm{in}}|^p \le \mathscr{H}_{\mathrm{in}}(G_{\mathrm{in}}) \le C_{\mathscr{H},1}(1+|G_{\mathrm{in}}|^{p_{\mathrm{in}}})$$

and

$$\big(\partial_{G_{\mathrm{in}}}\mathscr{H}_{\mathrm{in}}(G_{\mathrm{in}}) - \partial_{G_{\mathrm{in}}}\mathscr{H}_{\mathrm{in}}(\widetilde{G}_{\mathrm{in}})\big) \,\vdots\, (G_{\mathrm{in}}-\widetilde{G}_{\mathrm{in}}) \ge C_{\mathscr{H},2}|G_{\mathrm{in}}-\widetilde{G}_{\mathrm{in}}|^{p_{\mathrm{in}}}. \tag{18}$$

(A4) The dissipation potential $\zeta : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is the sum of a part $\zeta_{\mathrm{KV}}(\nabla\chi, \nabla\dot{\chi})$ related to the Kelvin–Voigt rheology, and a part $\widetilde{\zeta}_{\mathrm{in}}(F_{\mathrm{in}}, \dot{F}_{\mathrm{in}})$ related to the Maxwell rheology:

(i) For the Kelvin–Voigt part $\zeta_{\mathrm{KV}}(\nabla\chi, \nabla\dot{\chi})$, note that dynamic frame-indifference implies the existence of $\widetilde{\zeta}_{\mathrm{KV}}$ such that $\zeta_{\mathrm{KV}}(F,H) = \widehat{\zeta}_{\mathrm{KV}}(F^\top F, H^\top F + F^\top H)$. For simplicity, we assume that

$$\widehat{\zeta}_{\mathrm{KV}}(\mathsf{C},\dot{\mathsf{C}}) = \frac{1}{2}\dot{\mathsf{C}} : \mathbb{D}(F)\dot{\mathsf{C}},$$

where $\mathbb{D}$ is uniformly positive definite and bounded.

(ii) For the inelastic part $\widetilde{\zeta}_{\mathrm{in}}(F_{\mathrm{in}}, \dot{F}_{\mathrm{in}})$, we note that frame-indifference implies that there exists $\zeta_{\mathrm{in}} : \mathbb{R}^{d\times d} \to [0,\infty)$ such that $\widetilde{\zeta}_{\mathrm{in}}(F_{\mathrm{in}}, \dot{F}_{\mathrm{in}}) = \zeta_{\mathrm{in}}(\dot{F}_{\mathrm{in}}F_{\mathrm{in}}^{-1})$ (see e.g. [KR19, Eqn. (9.4.5)]). We assume that ζ_{in} is C^1, convex, satisfies $\zeta_{\mathrm{in}}(0) = 0$, and that there exist $r \ge 2$ and constants $C_{\zeta,i} > 0$ $(0 \le i \le 2)$ such that

$$C_{\zeta,0}|L|^r \le \zeta_{\mathrm{in}}(L) = C_{\zeta,1}(1+|L|^r) \quad \forall\, L \in \mathbb{R}^{d\times d}. \tag{19}$$

and

$$\big(\zeta'_{\mathrm{in}}(L_1) - \zeta'_{\mathrm{in}}(L_2)\big) : (L_1-L_2) \ge C_{\zeta,2}|L_1-L_2|^r \quad \forall\, L_1, L_2 \in \mathbb{R}^{d\times d}. \tag{20}$$

(A5) (i) The external forces satisfy $f \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))$, $g \in W^{1,\infty}(0,T;L^2(\partial\Omega;\mathbb{R}^d))$. We set

$$\langle \ell(t), \chi\rangle := \int_\Omega f(t)\cdot\chi \,\mathrm{d}x + \int_{\Gamma_N} g(t)\cdot\chi\,\mathrm{d}S$$

such that $\ell \in W^{1,\infty}(0,T;W^{2,p}(\Omega;\mathbb{R}^d)^*)$.

(ii) The boundary permeability $\kappa \in L^\infty(\Omega)$ is nonnegative and strictly positive on a part of the boundary $\partial\Omega$ with positive surface measure, i.e., $\int_{\partial\Omega}\kappa\,\mathrm{d}S \geq \alpha_* > 0$.
The external potential ψ_{ext} satisfies $\psi_{\mathrm{ext}} \in H^1(0,T;L^2(\partial\Omega))$.

(iii) The doping N satisfies $N \in L^\infty(\Omega)$.

(A6) The initial conditions satisfy $\chi_0 \in W^{2,p}_{\mathrm{id}}(\Omega;\mathbb{R}^d)$ with $\det\nabla\chi_0 \geq \rho_0 > 0$ for a.e. $x \in \Omega$; $F_{\mathrm{in},0} \in W^{1,p}_I(\Omega;\mathbb{R}^{d\times d})$ with $\det F_{\mathrm{in},0} = 1$ for a.e. $x \in \Omega$; for some $\delta > 0$, we have $n_0 \in H^1(\Omega)$ with $n_0(x) \in [\delta, 1-\delta]$ for a.e. $x \in \Omega$; $\psi_0 \in H^1(\Omega)$.

Furthermore, we have that $\int_\Omega \Phi(\nabla\chi_0, F_{\mathrm{in},0}, n_0, \nabla n_0, \psi_0, \nabla\psi_0)\,\mathrm{d}x < \infty$.

Remark 3.1 (More general dissipation potentials)**.** *As pointed out in [RS18, Remark 2.6], the isochoric constraint $\det F_{\mathrm{in}} = 1$ is of special interest. In this case, an additional force term $\Lambda_{iso}\mathrm{Cof}\,F_{\mathrm{in}}$ in the inelastic flow rule would appear with a Lagrange multiplier Λ_{iso}. Moreover, the dissipation potential ζ_{in} would be nonsmooth and of the form*

$$\zeta_{\mathrm{in}}(L) = \begin{cases} \widehat{\zeta}_{\mathrm{in}}(L) & \text{for } L \in \mathbb{R}^{d\times d}_{\mathrm{dev}}, \\ +\infty & \text{otherwise.}\end{cases}$$

However, the mathematical treatment of this additional constraint is unclear. Let us point out that ζ_{in} is nonsmooth in 0 in [RS18], which is related to plasticity being an activated effect. Here, we restrict ourselves to the "simpler" case in Assumption **(A4)***(ii) since the focus is on the electromechanical coupling.*

Remark 3.2. *(i) Similar to [Oos25, Lemma 2.1], we can obtain a lower bound for the energy contribution Φ^2_n. Indeed, by integrating* (16) *twice and setting $\Phi^2_n(\frac{1}{2}) = 0$, $(\Phi^2_n)'(\frac{1}{2}) = 0$, we find a constant $C > 0$ such that:*

$$\Phi^2_n(n) \geq C\big(n\log n + (1-n)\log(1-n)\big) + \log(2).$$

Indeed, a standard example for the singular part Φ^2_n of the electric energy in **(A2)***(iv) is*

$$\Phi^2_n(n) = k\big(n\log n + (1-n)\log(1-n)\big)$$

for some constant $k > 0$. Note that the finiteness of $\Phi^2_n(n)$ implies that $n \in [0,1]$. In particular, this choice gives rise to the so-called Blakemore statistical relation between the charge-carrier concentration n and the potentials μ_n and ψ in semiconducting devices (see e.g. [AGL25, Sect. 2.2.]), viz.

$$n = \frac{1}{\exp((\mu_n - e_0\psi)/k) + 1}.$$

(ii) We included a gradient contribution ∇n to the free energy such that the diffusion equation is of fourth order. This is necessary to obtain sufficiently strong a priori estimates to carry out the Galerkin scheme below and to deal with the degenerate conductivity, see also [KR19, Sect. 7.6.2] and Remark 9.6.3 therein. In particular, we could also allow for a free energy, which is only semi-convex with respect to n.

(iii) A prototypical example for the hyperstress potential $\mathscr{H}$ in Assumption **(A3)** *is $\mathscr{H}(G, G_{\rm in}) = \frac{1}{p}|G|^p + \frac{1}{p_{\rm in}}|G_{\rm in}|^{p_{\rm in}}$. This potential satisfies* (17) *and* (18)*, as can be seen from the fact that for any P, $Q \in \mathbb{R}^{d\times d\times d}$ and $p \geq 2$ the following estimate holds*

$$(|P|^{p_{\rm in}-2}P - |Q|^{p_{\rm in}-2}Q) \vdots (P-Q)$$
$$= \frac{1}{2}\frac{|P|^{p_{\rm in}-2} + |Q|^{p_{\rm in}-2}}{|P-Q|^{p_{\rm in}-2}}|P-Q|^{p_{\rm in}} + \frac{1}{2}(|P|^2 - |Q|^2)(|P|^{p_{\rm in}-2} - |Q|^{p_{\rm in}-2}) \geq 2^{2-p_{\rm in}}|P-Q|^{p_{\rm in}}.$$

(iv) Since the momentum balance equation is quasistatic, we can consider a non-quadratic hyperstress regularization with sufficient growth to guarantee the compact embedding $W^{2,p}(\Omega;\mathbb{R}^d) \subset W^{1,\infty}(\Omega;\mathbb{R}^d)$. This is in contrast to the setting in [RS18] with inertia, where the deformation is only in $H^2(\Omega;\mathbb{R}^d)$. Consequentially, the pulled-back conductivity tensor is only allowed to depend on the inelastic strain $F_{\rm in}$. Here, we do not have this restriction.

The structure of the free energy in Assumption ***(A2)*** leads to the following decomposition of the elastic stress tensor $\sigma_{\rm el}$, the inelastic stress tensor $\sigma_{\rm in}$, and the electro-chemical potential μ_n

$$\sigma_{\rm el}(F, F_{\rm in}, n, \nabla n, \nabla\psi) = \Phi'_{\rm mech}(FF_{\rm in}^{-1})F_{\rm in}^{-\top} - \frac{1}{2}\partial_F\varepsilon(F)\nabla\psi\cdot\nabla\psi + \partial_F\Phi_n^1(F,n)$$
$$+ \frac{\gamma}{2}\partial_F\mathcal{A}^F(F)\nabla n\cdot\nabla n, \tag{21}$$
$$\sigma_{\rm in}(F, F_{\rm in}) = -F_{\rm in}^{-\top}F^\top\Phi'_{\rm mech}(FF_{\rm in}^{-1})F_{\rm in}^{-\top} + \Phi'_{\rm in}(F_{\rm in}) \tag{22}$$
$$\mu_n(F, n, \nabla n, \psi) = e_0\psi + \partial_n\Phi_n(F,n) - \operatorname{div}\big(\gamma\mathcal{A}^F(F)\nabla n\big). \tag{23}$$

In particular, $\sigma_{\rm el}$ decomposes into a purely elastic contribution, a Maxwell-type stress, and stresses due to the coupling to the concentration n.

3.2 Main result

To state our main result, namely the existence of weak solutions to the system (11), we first need to introduce a suitable notion of weak solution.

Definition 3.3 (Weak solution)**.** A tuple $(\chi, F_{\rm in}, n, \psi)$ is called a weak solution of the initial-boundary-value problem (11)–(13) in $[0,T]\times\Omega$ if

(i) $\chi \in L^\infty(0,T;W^{2,p}_{\rm id}(\Omega;\mathbb{R}^d)) \cap H^1(0,T;H^1(\Omega;\mathbb{R}^d))$, and $\det\nabla\chi > 0$ for all $x\in\Omega$,

(ii) $F_{\rm in} \in L^\infty(0,T;W_I^{1,p_{\rm in}}(\Omega;\mathbb{R}^{d\times d}))\cap W^{1,r}(0,T;L^r(\Omega;\mathbb{R}^{d\times d}))$, and $\det F_{\rm in} > 0$ for all $x\in\Omega$,

(iii) $n \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*)$, and $0 < n < 1$ for a.e. $x\in\Omega$,

(iv) $\operatorname{div}(\mathcal{A}^F(\nabla\chi)\nabla n) \in L^2((0,T)\times\Omega)$,

(v) $\psi \in L^\infty(0,T;H^1(\Omega))$.

This tuple satisfies the following integral equations (abbreviating $X = (\nabla\chi, \mathrm{D}^2\chi, F_{\rm in}, \nabla F_{\rm in}, n, \nabla n, \psi, \nabla\psi)$):

(A) The weak formulation of the (quasistatic) momentum balance for the deformation χ

$$\int_0^T\int_\Omega \big\{ \big(\partial_F\Phi(X) + \partial_{\dot F}\zeta_{\rm KV}(\nabla\chi,\nabla\dot\chi)\big) : \nabla v + \mathfrak{h}_{\rm el}(\mathrm{D}^2\chi) \,\vdots\, \mathrm{D}^2 v\big\} \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle \ell(t), v\rangle \,\mathrm{d}t = 0, \tag{24a}$$

holds for all test functions $v \in L^2(0,T; W_0^{2,p}(\Omega;\mathbb{R}^d))$.

(B) The weak form of the inelastic flow rule for the inelastic strain $F_{\rm in}$

$$\int_0^T\int_\Omega \big\{ \big(\partial_{F_{\rm in}}\Phi(X) + \zeta_{\rm in}'(\dot F_{\rm in}F_{\rm in}^{-1})F_{\rm in}^{-\top}\big) : w + \mathfrak{h}_{\rm in}(\nabla F_{\rm in}) \,\vdots\, \nabla w\big\} \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{24b}$$

holds for all test functions $w \in L^2(0,T; W_0^{1,p_{\rm in}}(\Omega;\mathbb{R}^{d\times d})) \cap L^r((0,T)\times\Omega;\mathbb{R}^{d\times d})$.

(C) The weak formulation of the diffusion equation for the concentration n

$$\int_0^T \langle \dot n, \phi\rangle \,\mathrm{d}t + \int_0^T\int_\Omega j_n\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{24c}$$

holds for all $\phi \in L^2(0,T;H^1(\Omega))$, where $j_n \in L^2((0,T)\times\Omega;\mathbb{R}^d)$ is the flux

$$j_n := \mathcal{A}(\nabla\chi,n)\nabla\big(-\gamma \mathrm{div}\,(\mathcal{A}^F(\nabla\chi)\nabla n) + \partial_n\Phi_n(\nabla\chi,n) + e_0\psi\big), \tag{24d}$$

which is satisfied in the following weak sense:

$$\int_0^T\int_\Omega j_n\cdot\widetilde\phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \mathcal{A}(\nabla\chi,n)\Big\{\partial^2_{Fn}\Phi_n(\nabla\chi,n)\mathrm{D}^2\chi + \partial^2_{nn}\Phi_n(\nabla\chi,n)\nabla n + e_0\nabla\psi\Big\}\cdot\widetilde\phi \\ + \gamma\mathrm{div}\big(\mathcal{A}^F(\nabla\chi)\nabla n\big)\mathrm{div}\,(\mathcal{A}(\nabla\chi,n)\widetilde\phi)\,\mathrm{d}x\,\mathrm{d}t \tag{24e}$$

for all $\widetilde\phi \in L^2(0,T;H^1(\Omega;\mathbb{R}^d)) \cap L^\infty((0,T)\times\Omega;\mathbb{R}^d)$.

(D) The weak formulation of the Poisson equation for the electrostatic potential ψ

$$\int_0^T\int_\Omega \big\{\varepsilon(\nabla\chi)\nabla\psi\cdot\nabla\omega - (e_0 n + N)\omega\big\}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega}\kappa(\psi-\psi_{\rm ext})\omega\,\mathrm{d}S\,\mathrm{d}t = 0, \tag{24f}$$

holds for all test functions $\omega \in L^2(0,T;H^1(\Omega))$.

Note that sufficiently regular weak solutions indeed satisfy the classical formulation (11) with boundary conditions (12). Furthermore, we note that all the integrals in (24), and in particular (24e), are well-defined. Indeed, for (24e) it follows that the first three integrals are finite using Assumptions ***(A1)***(i) and ***(A2)***(iii). To show that the last integral is finite, we note that $\mathrm{div}\,(\mathcal{A}^F(\nabla\chi)\nabla n) \in L^2((0,T)\times\Omega)$ by definition of weak solution. For the contraction product $V = \mathrm{D}_F\mathcal{A}^F(\nabla\chi)\mathrm{D}^2\chi = \sum_{k,l,i,j=1}^d \frac{\partial \mathcal{A}^F_{kl}}{\partial F_{ij}}\frac{\partial^2\chi_i}{\partial x_j\partial x_k}e_l$, we have

$$\mathrm{div}\,(\mathcal{A}(\nabla\chi,n)\widetilde\phi) = \mathcal{A}(\nabla\chi,n) : \nabla\widetilde\phi + B(n)V\cdot\widetilde\phi + B'(n)\nabla n\cdot\mathcal{A}^F(\nabla\chi)\widetilde\phi \in L^2((0,T)\times\Omega).$$

Thus, for bounded test functions $\widetilde\phi \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ and by Assumption ***(A1)***(i), the last integral in (24e) is also finite.

We can now state the main result.

Theorem 3.4 (Existence of weak solutions)**.** *Let the Assumptions* **(A1)**–**(A6)** *hold. Then, the system* (11)–(13) *possesses at least one weak solution in the sense of Definition 3.3.*

4 Existence of weak solutions

4.1 Faedo–Galerkin scheme

We prove the existence of weak solutions (i.e. Theorem 3.4) using a Faedo–Galerkin scheme, see [KR19, Sect. C.2.4]. In particular, we follow [EG96; RT20] and perform a separate Galerkin approximation for n and μ_n, see below. This is in contrast e.g. to [JL19], where only μ_n is approximated and n is written as a function in μ_n.

We follow [EG96] and regularize the diffusion equation (24c) by modifying the n-dependent part B of the conductivity tensor $\mathcal{A}$ (see ***(A1)***(i)), and the convex part of the free energy Φ^2_n (see ***(A2)***(iii)) so that they are defined for any $n \in \mathbb{R}$. We define $B_\eta : \mathbb{R} \to \mathbb{R}_+$ via

$$B_\eta(n) = \begin{cases} B(\eta) & \text{for } n \le \eta, \\ B(n) & \text{for } \eta < n < 1-\eta, \\ B(1-\eta) & \text{for } n \ge 1-\eta. \end{cases} \tag{25}$$

In particular, note that this implies that $\mathcal{A}_\eta(F,n) := B_\eta(n)\mathcal{A}^F(F)$ is uniformly positive definite (and thus non-degenerate) if $F \in \mathrm{GL}^+(d)$. We define the regularization $\Phi^2_{n,\eta} \in C^2(\mathbb{R};\mathbb{R})$ by setting

$$(\Phi^2_{n,\eta})''(n) = \begin{cases} (\Phi^2_n)''(\eta) & \text{for } n \le \eta, \\ (\Phi^2_n)''(n) & \text{for } \eta < n < 1-\eta, \\ (\Phi^2_n)''(1-\eta) & \text{for } n \ge 1-\eta. \end{cases} \tag{26}$$

We assume that $\Phi^2_{n,\eta}(1/2) = \Phi^2_n(1/2)$ and $(\Phi^2_{n,\eta})'(1/2) = (\Phi^2_n)'(1/2)$. In particular, $\Phi^2_{n,\eta}$ has quadratic growth and $\Phi^2_{n,\eta} \le \Phi^2_n$ holds.

Let us comment on these regularizations. The Galerkin scheme only exhibits local-in-time solutions. In the unregularized case, we cannot guarantee that the Galerkin solutions n_k are bounded away from the singularities of Φ^2_n in $\{0,1\}$ (comp. Assumption ***(A2)***(iii)) for all times $t \in [0,T]$. In particular, the local-in-time solutions cannot be extended via successive prolongation. Moreover, the separate Galerkin approximations n_k and $\mu_{n,k}$ of n and μ_n, respectively, prevents us from directly using the relation $\mu_n = \partial_n\Phi - \gamma\mathrm{div}\,(\mathcal{A}^F(\nabla\chi)\nabla n)$, as was done e.g. in [OL24] in the time-discrete setting. Instead, we only obtain that $\mu_{n,k} = \mathfrak{P}_{Z_k}(\partial_n\Phi(\nabla\chi_k, n_k, \psi_k) - \gamma\mathrm{div}\,(\mathcal{A}^F(\nabla\chi_k)\nabla n_k))$, where $\mathfrak{P}_{Z_k}$ denotes the projection onto the finite-dimensional subspace Z_k, see below. Consequently, substitutions such as

$$\nabla\mu_n = \partial^2_{Fn}\Phi\mathrm{D}^2\chi + \partial^2_{nn}\Phi\nabla n + \partial^2_{\psi n}\Phi\nabla\psi - \nabla(\gamma\mathrm{div}\,(\mathcal{A}^F(\nabla\chi)\nabla n)) \tag{27}$$

are not valid on the Galerkin level. Thus, we cannot argue as in [OL24] to pass to the limit. For this reason, we have to add the additional gradient contribution $\gamma\mathcal{A}^F(\nabla\chi)\nabla n\cdot\nabla n$ to the free energy. We highlight that it is necessary to have here the same tensor $\mathcal{A}^F(\nabla\chi)$ as in the definition of the conductivity tensor $\mathcal{A} = B\mathcal{A}^F$ to derive suitable bounds. We first pass to the limit with respect to the Galerkin parameter k, after which the relation in (27) becomes valid, see Section 4.3. Section 4.4 then deals with the limit passage $\eta \to 0$.

We now introduce the Faedo–Galerkin scheme. We highlight that the treatment of the resulting system of ordinary differential-algebraic equations is not straightforward due to the singular behavior of the (regularized) free energy Φ_η as $\det\nabla\chi \to 0$ and $\det F_{\mathrm{in}} \to 0$, respectively. However, suitable a priori estimates in connection with the Healey–Krömer lemma (see [MR20, Thm. 3.1]) guarantee that solutions do not approach these singularities, comp. [KR19, Section 9.2].

For notational brevity, we drop the regularization parameter η for the solutions of the finite-dimensional problem for the rest of section, i.e., χ_k should be read as $\chi_{k\eta}$, etc. First, we list the relevant finite-dimensional spaces:

$$\begin{aligned} V_{k-1} \subset V_k &:= \mathrm{span}\{v_1,\dots,v_k\}, & \overline{\bigcup_k V_k}^{\|\cdot\|_{W^{2,p}}} &= W^{2,p}_0(\Omega;\mathbb{R}^d),\\ W_{k-1} \subset W_k &:= \mathrm{span}\{w_1,\dots,w_k\}, & \overline{\bigcup_k W_k}^{\|\cdot\|_{W^{1,p_{\rm in}}}} &= W^{1,p_{\rm in}}_0(\Omega;\mathbb{R}^{d\times d}),\\ Z_{k-1} \subset Z_k &:= \mathrm{span}\{z_1,\dots,z_k\}, & \overline{\bigcup_k Z_k}^{\|\cdot\|_{H^1}} &= H^1(\Omega),\\ U_{k-1} \subset U_k &:= \mathrm{span}\{u_1,\dots,u_k\}, & \overline{\bigcup_k U_k}^{\|\cdot\|_{H^1}} &= H^1(\Omega). \end{aligned}$$

Moreover, we assume that the bases $\{z_k\}_k^\infty$ and $\{u_k\}_k^\infty$ are orthonormal with respect to the L^2-scalarproduct.

The Galerkin approximation of the weak solutions to (11) takes the form

$$\begin{aligned} \chi_k(t,x) &= \chi_0(x) + \sum_{l=1}^k \widetilde{\chi}_k^l(t) v_l(x), \quad F_{{\rm in},k}(t,x) = F_{{\rm in},0}(x) + \sum_{l=1}^k \widetilde{F}_{{\rm in},k}^l(t) w_l(x),\\ n_k(t,x) &= n_0(x) + \sum_{l=1}^k \widetilde{n}_k^l(t) z_l(x), \quad \mu_{n,k}(t,x) = \mu_{n,0}(x) + \sum_{l=1}^k \widetilde{\mu}_{n,k}^l(t) z_l(x),\\ \psi_k(t,x) &= \psi_0(x) + \sum_{l=1}^k \widetilde{\psi}_k^l(t) u_l(x), \end{aligned} \tag{28}$$

where χ_0, $F_{{\rm in},0}$, n_0, and ψ_0 are the fixed initial values in Assumption ***(A6)***. It is important that both n_k and $\mu_{n,k}$ use the same finite-dimensional subspaces of $H^1(\Omega)$, which allows us to cross-test the equations for n_k by $\mu_{n,k}$ and vice versa.

We solve the following ODE systems for the coefficient functions
$t \mapsto (\widetilde{\chi}_k^l(t), \widetilde{F}_{{\rm in},k}^l(t), \widetilde{n}_k^l(t), \widetilde{\mu}_{n,k}^l(t), \widetilde{\psi}_k^l(t))$, where $l = 1,\dots,k$:

$$\int_\Omega \big\{ \big(\partial_F \Phi(X_k) + \partial_{\dot F}\zeta_{\rm KV}(\nabla\chi_k, \nabla\dot\chi_k)\big) : \nabla v_l + \mathfrak{h}_{\rm el}(\mathrm{D}^2\chi_k) \,\vdots\, \mathrm{D}^2 v_l \big\}\,\mathrm{d}x - \langle \ell, v_l\rangle = 0, \tag{29a}$$

$$\int_\Omega \big(\partial_{F_{\rm in}}\Phi(X_k) + \zeta_{\rm in}'(\dot F_{{\rm in},k} F_{{\rm in},k}^{-1}) F_{{\rm in},k}^{-\top}\big) : w_l + \mathfrak{h}_{\rm in}(\nabla F_{{\rm in},k}) \,\vdots\, \nabla w_l \,\mathrm{d}x = 0, \tag{29b}$$

$$\int_\Omega \big(\dot n_k z_l + \mathcal{A}_\eta(\nabla\chi_k, n_k)\nabla\mu_{n,k}\cdot\nabla z_l\big)\,\mathrm{d}x = 0, \tag{29c}$$

$$\int_\Omega \big(\mu_{n,k} - \partial_n\Phi_\eta(\nabla\chi_k, n_k, \psi_k)\big) z_l - \gamma\mathcal{A}^F(\nabla\chi_k)\nabla n_k\cdot\nabla z_l \,\mathrm{d}x = 0, \tag{29d}$$

$$\int_\Omega \big(\varepsilon(\nabla\chi_k)\nabla\psi_k\cdot\nabla w_l - (e_0 n_k + N) w_l\big)\,\mathrm{d}x + \int_{\partial\Omega} \kappa(\psi_k - \psi_{\rm ext}) w_l \,\mathrm{d}S = 0. \tag{29e}$$

Lemma 4.1 (Existence of Galerkin solutions)**.** *Let the Assumptions* **(A1)–(A6)** *hold. Then, the Galerkin scheme* (29) *has a solution* $(\chi_{k\eta}, F_{{\rm in},k\eta}, n_{k\eta}, \mu_{n,k\eta}, \psi_{k\eta})$ *on a maximal time interval* $(0,t_*)$*, which is absolutely continuous with respect to* t*.*

Proof. Again, we drop the regularization parameter η throughout the proof for notational convenience. Denote $\widetilde{\chi}(t) = (\widetilde{\chi}_k^1(t), \ldots, \widetilde{\chi}_k^k(t))^\top \in \mathbb{R}^k$, $\widetilde{F}_{\rm in}(t) = (\widetilde{F}^1_{{\rm in},k}(t), \ldots, F^k_{{\rm in},k}(t)) \in \mathbb{R}^k$, etc., and let $\widetilde{X} = (\widetilde{\chi}, \widetilde{F}_{\rm in}, \widetilde{n})$. Then, the Galerkin scheme can be reformulated as a system of ordinary differential-algebraic equations $\frac{\rm d}{{\rm d}t}\widetilde{X} = \Xi_1(t, \widetilde{X})$ and $(\widetilde{\mu}_n, \widetilde{\psi}) = \Xi_2(t, \widetilde{X})$ for some functions $\Xi_1 : [0, T] \times U \to (\mathbb{R}^k)^3$ and $\Xi_2 : [0, T] \times U \to (\mathbb{R}^k)^2$ defined on a subset $U \subset (\mathbb{R}^k)^3$. Both functions Ξ_i $(i = 1, 2)$ are continuous in the variable $\widetilde{X}$, and measurable in t.

We stress that the functions Ξ_i have to be restricted to the subset U to ensure that the equations are well-defined. For example, we need to restrict the coefficients $\widetilde{\chi} \in B_r(0) \subset \mathbb{R}^k$ to some sufficiently small ball to ensure that $\det \nabla\chi_k \geq \rho_r > 0$. Similarly, the coefficients for $\widetilde{F}_{{\rm in},k}$ need to be restricted to sufficiently small balls to ensure that $\det F_{{\rm in},k} \geq \rho_r > 0$.

We now illustrate how to obtain the functions Ξ_1 and Ξ_2. The function Ξ_2 is obtained by considering the equations (29d), (29e). Indeed, the first equation provides a relation $\widetilde{\mu}_{n,k} = \Xi_2^1(\nabla\widetilde{\chi}_k, \widetilde{n}_k, \widetilde{\psi}_k)$, while the second equation provides a relation $\widetilde{\psi}_k = \Xi_2^2(\nabla\widetilde{\chi}_k, \widetilde{n}_k)$.

To illustrate how to obtain the function Ξ_1, we consider the first equation (29a). Define $A_1^k(\widetilde{\chi}) \in$ Lin$(\mathbb{R}^k, \mathbb{R}^k)$ as the linear map given by

$$(A_1^k(\widetilde{\chi})(\dot{\widetilde{\chi}}))_i = \int_\Omega \big(\partial_{\dot{F}}\zeta_{\rm KV}(\nabla\chi_k, \nabla\dot{\chi}_k)\big) : \nabla v_i \,{\rm d}x.$$

By the coercivity of $\zeta_{\rm KV}$ in Assumption ***(A4)***(i), A_1 is invertible. Let A_2^k be the mapping defined by

$$(A_2^k(\widetilde{\chi}, \widetilde{F}_{\rm in}, \widetilde{n}, \widetilde{\psi}))_i = \int_\Omega \big(\partial_F\Phi(\nabla\chi_k, F_{{\rm in},k}, n_k, \nabla n_k, \psi_k, \nabla\psi_k) : \nabla v_i + \mathfrak{h}_{\rm el}({\rm D}^2\chi_k) \vdots {\rm D}^2 v_i \,{\rm d}x.$$

In particular, note that A_2^k is continuous, e.g., with respect to $\widetilde{\chi}$ as long as the coefficients $\widetilde{\chi}$ are in a sufficiently small ball $\widetilde{\chi}(t) \in B_r(0)$ so that $\det \nabla\chi_k \geq \widetilde{r} > 0$. Furthermore, using the relation $\widetilde{\psi}_k = \Xi_2^2(\nabla\widetilde{\chi}_k, \widetilde{n}_k)$, we can rewrite A_2^k so that $A_2^k = A_2^k(\widetilde{X})$. Finally, let $A_3^k(t) \in \mathbb{R}^k$ be the vector given by $(A_3^k(t))_i = \langle \ell(t), v_i\rangle$. Combining everything, we then arrive at an equation of the form $\frac{\rm d}{{\rm d}t}\widetilde{\chi} = \Xi_1^1(t, \widetilde{X})$.

Doing this for the equations (29b) and (29c), we then end up with a system of ordinary differential equations of the form $\frac{\rm d}{{\rm d}t}\widetilde{Q} = \Xi(t, \widetilde{X})$.

Using Carathéodory's theorem (cf. [Rou05, Thm. 1.44]), we can now solve this system of ordinary differential equations to obtain solutions $(\chi_k, F_{{\rm in},k}, n_k, \mu_{n,k}, \psi_k)$ (absolutely continuous with respect to t) to the Galerkin scheme on some maximal time interval $(0, t_*)$.

□

The existence of solutions to the ODE system for all times $t \in [0, T]$ follows from successive prolongation based on the below estimates. Thus, we will consider $t_* = T$ from now on.

4.2 A priori estimates

We first show that the Galerkin solutions obtained in Lemma 4.1 satisfy an energy-dissipation balance. To this end, we abbreviate $X := (\chi, F_{\rm in}, n, \psi)$ and introduce the total (regularized) free energy, the total dissipation functional, and the power of the external forces via

$$\begin{aligned}\mathcal{E}_\eta(t,X) &= \int_\Omega \big\{\Phi_\eta(\nabla\chi, F_{\rm in}, n, \nabla n, \psi, \nabla\psi) + \mathscr{H}(\mathrm{D}^2\chi, \nabla F_{\rm in})\big\}\,\mathrm{d}x \\ &\quad - \int_{\partial\Omega} \frac{\kappa}{2}(\psi - \psi_{\rm ext}(t))^2\,\mathrm{d}S - \langle \ell(t), \chi\rangle, \\ \mathcal{D}_\eta(X, \mu_n, \dot\chi, \dot F_{\rm in}) &= \int_\Omega \big\{\partial_{\dot F}\zeta_{\rm KV}(\nabla\chi, \nabla\dot\chi) : \nabla\dot\chi + \zeta_{\rm in}'(\dot F_{\rm in}F_{\rm in}^{-1}) : \dot F_{\rm in}F_{\rm in}^{-1} \\ &\quad + \mathcal{A}_\eta(\nabla\chi, n)\nabla\mu_n\cdot\nabla\mu_n\big\}\mathrm{d}x, \\ \mathcal{P}(t,\chi,\psi) &= -\langle\dot\ell(t),\chi\rangle + \int_{\partial\Omega}\kappa(\psi-\psi_{\rm ext}(t))\dot\psi_{\rm ext}(t)\,\mathrm{d}S.\end{aligned}$$

Proposition 4.2 (Energy-dissipation balance)**.** *Let $X_{k\eta}$ be a solution of the Galerkin scheme* (29) *established in Lemma 4.1 on the interval $[0,T]$. Then, we have for every $0 < t \le T$*

$$\mathcal{E}_\eta(t, X_{k\eta}(t)) + \int_0^t \mathcal{D}_\eta(X_{k\eta}, \mu_{n,k\eta}, \dot\chi_{k\eta}, \dot F_{{\rm in},k\eta})\,\mathrm{d}s = \mathcal{E}_\eta(0, X_{k\eta}(0)) + \int_0^t \mathcal{P}(t, \chi_{k\eta}, \psi_{k\eta})\,\mathrm{d}s. \tag{30}$$

Proof. For notational convenience, we again drop the regularization parameter η throughout the proof. We test the Galerkin scheme (29) with $\dot\chi_k$, $\dot F_{{\rm in},k}$, $\mu_{n,k}$, $\dot n_k$, and $\dot\psi_k$, respectively. More precisely, we test (29a) with $\dot\chi_k$, (29b) with $\dot F_{{\rm in},k}$, (29c) with $\mu_{n,k}$, and (29e) with $\dot\psi_k$. Then, testing (29d) with $\dot n_k$ gives together with the chain rule after integrating over time

$$\begin{aligned}&\mathcal{E}_\eta(t, X_k(t)) - \mathcal{E}_\eta(0, X_k(0)) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_\eta(s, X_k)\,\mathrm{d}s \\ &= \int_0^t \int_\Omega \Big\{\big\{\partial_F\Phi_\eta(\nabla\chi_k, F_{{\rm in},k}, n_k, \nabla n_k, \psi_k, \nabla\psi_k) : \nabla\dot\chi_k + \mathfrak{h}_{\rm el}(\mathrm{D}^2\chi_k) \,\vdots\, \mathrm{D}^2\dot\chi_k \\ &\quad + \partial_{F_{\rm in}}\Phi_\eta(\nabla\chi_k, F_{{\rm in},k}) : \dot F_{{\rm in},k} + \mathfrak{h}_{\rm in}(\nabla F_{{\rm in},k}) \,\vdots\, \nabla\dot F_{\rm in} \\ &\quad + \mu_{n,k}\dot n_k + (e_0 n_k + N)\dot\psi_k - \varepsilon(\nabla\chi_k)\nabla\psi_k\cdot\nabla\dot\psi_k\big\}\,\mathrm{d}x - \langle\ell, \dot\chi_k\rangle - \langle\dot\ell, \chi_k\rangle \\ &\quad - \int_{\partial\Omega}\kappa(\psi_k - \psi_{\rm ext})(\dot\psi_k - \dot\psi_{\rm ext})\,\mathrm{d}S\Big\}\,\mathrm{d}s \\ &= -\int_0^t\int_\Omega \Big\{\big\{\partial_{\dot F}\zeta_{\rm KV}(\nabla\chi_k, \nabla\dot\chi_k) : \nabla\dot\chi_k + \zeta_{\rm in}'(\dot F_{{\rm in},k}F_{{\rm in},k}^{-1}) : \dot F_{{\rm in},k}F_{{\rm in},k}^{-1} \\ &\quad + \mathcal{A}_\eta(\nabla\chi_k, n_k)\nabla\mu_{n,k}\cdot\nabla\mu_{n,k} + \langle\dot\ell, \chi_k\rangle\Big\}\,\mathrm{d}s + \int_0^t\int_{\partial\Omega}\kappa(\psi_k - \psi_{\rm ext})\dot\psi_{\rm ext}\,\mathrm{d}S\,\mathrm{d}s \\ &= \int_0^t \big\{ - \mathcal{D}_\eta(X_k, \mu_{n,k}, \dot\chi_k, \dot F_{{\rm in},k}) + \mathcal{P}(t, \chi_k, \psi_k)\big\}\,\mathrm{d}s,\end{aligned}$$

which proves the proposition. □

Having established the energy-dissipation balance for the Galerkin approximation, we now aim to derive a priori estimates from it. To this end, we first need to verify that the energy $\mathcal{E}_\eta$ is coercive. However, due to the form of the electrostatic contribution of the energy from (15) this is not immediately clear. For solutions of the Galerkin approximation, we can exploit the following result, which follows from testing the finite-dimensional Poisson equation in (29e) by ψ_k.

Lemma 4.3. *Let $X_{k\eta}$ denote a Galerkin solution obtained in Lemma 4.1. Then, the electrostatic part of the energy (cf. Assumption* **(A2)***(iii))*

$$\mathcal{E}_{\mathrm{elec}}(t,\chi,n,\psi) := \int_\Omega \Phi_{\mathrm{elec}}(\nabla\chi,n,\nabla n,\psi)\,\mathrm{d}x - \int_{\partial\Omega}\frac{\kappa}{2}(\psi-\psi_{\mathrm{ext}}(t))^2\,\mathrm{d}S$$

can be rewritten as

$$\mathcal{E}_{\mathrm{elec}}(t,\chi_k,n_k,\psi_k) = \int_\Omega \frac{\varepsilon(\nabla\chi_k)}{2}\nabla\psi_k\cdot\nabla\psi_k + \Phi_n(\nabla\chi_k,n_k,\nabla n_k)\,\mathrm{d}x + \int_{\partial\Omega}\frac{\kappa}{2}(\psi_k-\psi_{\mathrm{ext}})^2\,\mathrm{d}S. \tag{31}$$

Proof. By testing (29e) with ψ_k, we have

$$\int_\Omega \varepsilon(\nabla\chi_k)\nabla\psi_k\cdot\nabla\psi_k\,\mathrm{d}x + \int_{\partial\Omega}\kappa\psi_k^2\,\mathrm{d}S = \int_\Omega (e_0 n_k + N)\psi_k\,\mathrm{d}x + \int_{\partial\Omega}\kappa\psi_{\mathrm{ext}}\psi_k\,\mathrm{d}S.$$

Substituting this into the formula for Φ_{elec} in (15), we then get the identity in (31). □

Corollary 4.4. *Let $X_{k\eta}$ be a solution of the Galerkin scheme* (29)*. Then, there exists a constant $C>0$ (independent of k, η) such that*

$$\mathcal{E}_\eta(t,X_{k\eta}) \le C \quad \text{and} \quad \int_0^t \mathcal{D}_\eta(X_{k\eta},\mu_{n,k\eta},\dot{\chi}_{k\eta},\dot{F}_{\mathrm{in},k\eta})\,\mathrm{d}s \le C.$$

Proof. We note that $\mathcal{D}$ is bounded from below, and that all boundary contributions from $\mathcal{P}$ can be absorbed in $\mathcal{D}$ using Young's inequality with some sufficiently small ρ. The energy-dissipation balance (30) and Assumption ***(A5)***(i) thus imply that

$$\begin{aligned}\mathcal{E}_\eta(t,X_{k\eta}) &\le C\Big(1+\int_0^t \Big\{\langle\dot{\ell},\chi_{k\eta}\rangle + \int_{\partial\Omega}\dot{\psi}_{\mathrm{ext}}(\psi_{k\eta}-\psi_{\mathrm{ext}})\,\mathrm{d}S\Big\}\,\mathrm{d}s\Big)\\ &\le C\Big(1+\int_0^t \|\chi_{k\eta}\|_{W^{2,p}(\Omega;\mathbb{R}^d)} + \|\psi_{k\eta}-\psi_{\mathrm{ext}}\|^2_{L^2(\Omega)}\,\mathrm{d}s\Big).\end{aligned}$$

Using the Dirichlet boundary condition in (12a) and Poincaré's inequality, we now find

$$\|\chi_{k\eta}\|_{W^{2,p}(\Omega;\mathbb{R}^d)} \le C(1+\|\mathrm{D}^2\chi_{k\eta}\|_{L^p(\Omega;\mathbb{R}^{d\times d\times d})}).$$

The lower bounds in Assumption ***(A2)***, ***(A3)***(i), and Lemma 4.3 then imply that

$$\|\chi_{k\eta}\|^p_{W^{2,p}(\Omega;\mathbb{R}^d)} + \|\psi_{k\eta}-\psi_{\mathrm{ext}}\|^2_{L^2(\partial\Omega)} \le C(1+\mathcal{E}_\eta(t,X_{k\eta}))$$

with a constant C independent of k, and η. Combining everything, we thus obtain

$$\mathcal{E}_\eta(t,X_{k\eta}) \le C\Big(1+\int_0^t \mathcal{E}_\eta(t,X_{k\eta})\,\mathrm{d}s\Big),$$

so that Assumption ***(A6)*** and an application of Grönwall's lemma give the bound $\mathcal{E}_\eta(t,X_{k\eta}(t)) \le C$ for a.a. $t\in(0,T]$ (note that $\Phi_{n,\eta}(n)\le\Phi_n(n)$ for all $n\in\mathbb{R}$). Again using the energy-dissipation balance, we then also find that $\int_0^t \mathcal{D}(X_{k\eta},\mu_{n,k\eta},\dot{\chi}_{k\eta},\dot{F}_{\mathrm{in},k\eta})\,\mathrm{d}s \le C$. □

Lemma 4.5 (A priori estimates I)**.** *Let $X_{k\eta}$ denote a Galerkin solutions obtained in Lemma 4.1 on the time interval $[0,T]$. There exist constants $C>0$, $C_*>0$ (independent of k and η) such that:*

(i) $\|\chi_{k\eta}\|_{L^\infty(0,T;W^{2,p}(\Omega;\mathbb{R}^d))} + \|\dot{\chi}_{k\eta}\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))} \leq C$,

(ii) $\det \nabla\chi_{k\eta} \geq C_*$ *for all* $(t,x) \in [0,T] \times \overline{\Omega}$,

(iii) $\|F_{\mathrm{in},k\eta}\|_{L^\infty(0,T;W^{1,p_{\mathrm{in}}}(\Omega;\mathbb{R}^{d\times d}))} + \|\dot{F}_{\mathrm{in},k\eta}\|_{L^r((0,T)\times\Omega)} \leq C$,

(iv) $\det F_{\mathrm{in},k\eta} \geq C_*$ *for all* $(t,x) \in [0,T] \times \overline{\Omega}$.

Proof. Step 1. Since $\|\chi_{k\eta}(t)\|^p_{W^{2,p}(\Omega;\mathbb{R}^d)} \leq C(1 + \mathcal{E}_\eta(t, X_{k\eta}))$ (using again Lemma 4.3), Corollary 4.4 immediately gives the bound $\|\chi_{k\eta}\|_{L^\infty(0,T;W^{2,p}(\Omega;\mathbb{R}^d))} \leq C$, i.e., the first term of (i) is bounded. Similarly, the lower bound in Assumption ***(A3)***(ii) and Poincaré's inequality give $\|F_{\mathrm{in},k\eta}\|_{L^\infty(0,T;W^{1,p_{\mathrm{in}}}(\Omega;\mathbb{R}^{d\times d}))} \leq C$, i.e., the boundedness of the first term of (iv).

Step 2. Using Assumption ***(A2)***(ii), the uniform boundedness of $\mathcal{E}_\eta$ also implies $\int_\Omega \frac{1}{|\det F_{\mathrm{in},k\eta}(t)|^{q_{\mathrm{in}}}}\,\mathrm{d}x \leq C$ for all $t \in [0,T]$. Thus, an application of the Healey–Krömer lemma as in [MR20, Thm. 3.1] gives a constant $C_* > 0$ such that $\det F_{\mathrm{in},k\eta} \geq C_* > 0$, which is (iv).

Using this lower bound, we can now show the bound in (ii). Indeed, we note that the lower bound for Φ_{el} in Assumption ***(A2)***(i) and the fact that $F_{\mathrm{el}} = F F_{\mathrm{in}}^{-1}$ imply that

$$\int_\Omega \Phi_{\mathrm{mech}}(F_{\mathrm{el}})\,\mathrm{d}x \geq \int_\Omega \frac{C}{|\det F_{\mathrm{el},k\eta}|^q}\,\mathrm{d}x = C\int_\Omega \frac{|\det F_{\mathrm{in},k\eta}|^q}{|\det F_{k\eta}|^q}\,\mathrm{d}x \geq \int_\Omega \frac{C C^q_{*,\eta}}{|\det F_{k\eta}|^q}\,\mathrm{d}x.$$

As before, we can use the uniform bound for $\mathcal{E}_\eta$ from Corollary 4.4, and we obtain $\int_\Omega \frac{1}{|\det\nabla\chi_{k\eta}|^q}\,\mathrm{d}x \leq C$. Thus, the Healey–Krömer lemma can be applied again to obtain a further constant $\overline{C}_* > 0$ such that $\det\nabla\chi_{k\eta} \geq \overline{C}_* > 0$ for all $(t,x) \in (0,T) \times \overline{\Omega}$, which proves (ii).

Step 3. We now prove the second bound of (i). A calculation shows that for our choice of ζ_{KV} in Assumption ***(A4)***(i), we have $\partial_{\dot{F}}\zeta_{\mathrm{KV}}(F,\dot{F}) : \dot{F} = 2\zeta_{\mathrm{KV}}(F,\dot{F})$. The boundedness of $\int_0^t \mathcal{D}\,\mathrm{d}s$ from Corollary 4.4, and the generalized Korn's inequality as in [MR20, Cor. 3.4] now give the uniform bound $\|\dot{\chi}_{k\eta}\|_{L^2(0,T;H^1(\Omega))} \leq C$, proving the second bound of (i).

Step 4. Finally, we prove the second bound of (iii). By Corollary 4.4, it holds that

$$\int_0^T \int_\Omega \zeta'_{\mathrm{in}}(\dot{F}_{\mathrm{in},k\eta} F^{-1}_{\mathrm{in},k\eta}) : \dot{F}_{\mathrm{in},k\eta} F^{-1}_{\mathrm{in},k\eta}\,\mathrm{d}x\,\mathrm{d}t \leq C.$$

Using the convexity of ζ_{in}, and the bounds from Assumption ***(A4)***(ii), it then follows that

$$\big\|\dot{F}_{\mathrm{in},k\eta} F^{-1}_{\mathrm{in},k\eta}\big\|_{L^r((0,T)\times\Omega)} \leq C.$$

In particular, this implies that

$$\|\dot{F}_{\mathrm{in},k\eta}\|_{L^r((0,T)\times\Omega)} \leq \|\dot{F}_{\mathrm{in},k\eta} F^{-1}_{\mathrm{in},k\eta}\|_{L^r((0,T)\times\Omega)} \|F_{\mathrm{in},k\eta}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d}))} \leq C,$$

proving the boundedness of the second term of (iii).

□

We now turn to the a priori estimates for the variables n_k and ψ_k.

Lemma 4.6 (A priori estimates II)**.** *Let $X_{k\eta}$ denote a Galerkin solution obtained in Lemma 4.1 on the time interval $[0,T]$. There exists a constant $C = C_\eta > 0$ (independent of k, but dependent on η), such that:*

(i) $\|n_{k\eta}\|_{L^\infty(0,T;H^1(\Omega))} + \|\dot{n}_{k\eta}\|_{L^2(0,T;H^1(\Omega)^*)} \leq C$

(ii) $\|\mu_{n,k\eta}\|_{L^2(0,T;H^1(\Omega))} \leq C$,

(iii) $\|\psi_{k\eta}\|_{L^\infty(0,T;H^1(\Omega))} + \|\sqrt{\kappa}\psi_{k\eta}\|_{L^2((0,T)\times\partial\Omega)} \leq C.$

Proof. Step 1. The bound $\|n_{k\eta}\|_{L^\infty(0,T;H^1(\Omega))} \leq C$ in (i) follows from Assumption ***(A2)***(iii) and the uniform boundedness of $\mathcal{E}_\eta$ from Corollary 4.4. Similarly, the boundary estimate for $\psi_{k\eta}$ in (iii) follows from Lemma 4.3 and the uniform boundedness of $\mathcal{E}_\eta$.

Step 2. The bound for $\|\mu_{n,k\eta}\|_{L^2(0,T;H^1(\Omega))}$ in (ii) follows directly from the uniform boundedness of $\int_0^T \mathcal{D}_\eta \,\mathrm{d}s$ in Corollary 4.4. In particular, note that this uses that the regularized conductivity tensor $\mathcal{A}_\eta$ is uniformly positive definite (see (25)), so that this bound depends on the regularization parameter η.

Step 3. To obtain the bound for $\dot{n}_{k\eta}$ in (i), we denote by $P_k : L^2(\Omega) \to Z_k \subset L^2(\Omega)$ the projection onto the finite-dimensional Galerkin space Z_k. In particular, using the spectral theorem, the projection can be chosen in such a way that $P_k(H^1) = H^1$, that P_k is orthogonal with respect to the H^1-inner product, and has norm $\|P_k\| = 1$ for all $k \geq 1$.

Then, it follows that for $z \in L^2(0,T;H^1(\Omega))$:

$$\begin{aligned}
\int_0^T \langle \dot{n}_{k\eta}, z\rangle \,\mathrm{d}s = \int_0^T \langle P_k\dot{n}_{k\eta}, z\rangle \,\mathrm{d}s = \int_0^T \langle \dot{n}_{k\eta}, P_k z\rangle \,\mathrm{d}s \\
= -\int_0^T \int_\Omega \big(\mathcal{A}_\eta(\nabla\chi_{k\eta}, n_{k\eta})\nabla\mu_{n,k\eta}\cdot\nabla(P_k z)\big)\,\mathrm{d}x\,\mathrm{d}s \\
\leq \|\mathcal{A}_\eta(\nabla\chi_{k\eta}, n_{k\eta})\|_{L^\infty((0,T)\times\Omega)}\|\nabla\mu_{n,k\eta}\|_{L^2((0,T)\times\Omega)}\|\nabla(P_k z)\|_{L^2((0,T)\times\Omega)} \\
\leq C\|P_k\|_{\mathcal{L}(H^1(\Omega))}\|z\|_{L^2(0,T;H^1(\Omega))} \leq C\|z\|_{L^2(0,T;H^1(\Omega))},
\end{aligned}$$

which proves the uniform bound $\|\dot{n}_{k\eta}\|_{L^2(0,T;H^1(\Omega)^*)} \leq C$.

Step 4. Finally, we show the bound for $\psi_{k\eta}$ in (iii). Using Lemma 4.3 and Assumption ***(A1)***(iii), it follows that $\|\nabla\psi_{k\eta}\|_{L^\infty(0,T;L^2(\Omega))} \leq C$. In fact, by defining an equivalent norm on $H^1(\Omega))$ by $\|(\cdot)\|_\kappa = \big\|\nabla(\cdot)\big\|_{L^2(\Omega)} + \|\sqrt{\kappa}(\cdot)\|_{L^2(\partial\Omega)}$, we see that $\|\psi_{k\eta}\|_{L^\infty(0,T;H^1(\Omega))} \leq C$.

□

4.3 Limit passage $k \to \infty$ for fixed η

We now pass to the limit $k \to \infty$ while keeping the regularization parameter η fixed.

Using the a priori estimates from Lemma 4.5 and 4.6, together with the Aubin–Lions lemma, we can extract (non-relabeled) converging subsequences and some $(\chi_\eta, F_{\mathrm{in},\eta}, n_\eta, \mu_{n,\eta}, \psi_\eta)$ such that the following convergences hold:

(i) $\chi_{k\eta} \overset{w^*}{\rightharpoonup} \chi_\eta$ in $L^\infty(0,T;W^{2,p}(\Omega;\mathbb{R}^d)) \cap H^1(0,T;H^1(\Omega;\mathbb{R}^d))$, and $\chi_{k\eta} \overset{s}{\to} \chi_\eta$ strongly in $C(0,T;C^{1,\lambda}(\Omega;\mathbb{R}^d))$ for all $\lambda \in (0,1-\frac{d}{p})$,

(ii) $F_{\mathrm{in},k\eta} \xrightarrow{w^*} F_{\mathrm{in},\eta}$ in $L^\infty(0,T;W^{1,p_{\mathrm{in}}}(\Omega;\mathbb{R}^{d\times d})) \cap W^{1,r}(0,T;L^r(\Omega;\mathbb{R}^{d\times d}))$, and $F_{\mathrm{in},k\eta} \xrightarrow{s} F_{\mathrm{in},\eta}$ in $C(0,T;C^{0,\widetilde{\lambda}}(\Omega;\mathbb{R}^{d\times d}))$ for all $\widetilde{\lambda} \in (0,1-\frac{d}{p_{\mathrm{in}}})$. In particular, also $F_{\mathrm{in},k\eta}^{-1} \xrightarrow{s} F_{\mathrm{in},\eta}^{-1}$ in $C(0,T;C^{0,\widetilde{\lambda}}(\Omega;\mathbb{R}^{d\times d}))$ since $F_{\mathrm{in}}^{-1} = \frac{\mathrm{Cof} F_{\mathrm{in}}^\top}{\det F_{\mathrm{in}}}$,

(iii) $n_{k\eta} \xrightarrow{w^*} n_\eta$ in $L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*)$, and $n_{k\eta} \xrightarrow{s} n_\eta$ in $C(0,T;L^2(\Omega))$,

(iv) $\mu_{n,k\eta} \xrightarrow{w} \mu_{n,\eta}$ in $L^2(0,T;H^1(\Omega))$,

(v) $\psi_{k\eta} \xrightarrow{w^*} \psi_\eta$ in $L^\infty(0,T;H^1(\Omega)) \cap L^2((0,T)\times\partial\Omega)$.

Lemma 4.7 (Properties of limits of solutions)**.** *(i) The electro-chemical potential $\mu_{n,\eta} \in L^2(0,T;H^1(\Omega))$ satisfies*

$$\mu_{n,\eta} = e_0\psi + \partial_n \Phi_{n,\eta}(\nabla\chi_\eta, n_\eta) - \gamma \mathrm{div}\,(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta) \tag{32}$$

for almost every $(t,x) \in [0,T]\times\Omega$. Moreover, $\mathrm{div}\,(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta) \in L^2(0,T;H^1(\Omega))$.

(ii) The electric potential ψ_η satisfies the Poisson equation, i.e.,

$$\int_0^T\!\!\int_\Omega \varepsilon(\nabla\chi_\eta)\nabla\psi_\eta\cdot\nabla\omega\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_{\partial\Omega} \kappa(\psi_\eta - \psi_{\mathrm{ext}})\omega\,\mathrm{d}S\,\mathrm{d}t = \int_0^T\!\!\int_\Omega (e_0 n_\eta + N)\omega\,\mathrm{d}x\,\mathrm{d}t \tag{33}$$

for $\omega \in L^2(0,T;H^1(\Omega))$.

Proof. (i) The formula (32) for $\mu_{n,\eta}$ follows by passing to the limit in (29d), using the strong convergences of $\nabla\chi_{k,\eta} \xrightarrow{s} \nabla\chi_\eta$ in $C(0,T;C^{0,\lambda}(\Omega;\mathbb{R}^{d\times d}))$, $n_{k\eta} \xrightarrow{s} n_\eta$ in $L^2((0,T)\times\Omega)$, and the weak convergences of $\mu_{n,k\eta} \xrightarrow{w} \mu_\eta$ in $L^2(0,T;H^1(\Omega))$, $\psi_{k\eta} \xrightarrow{w^*} \psi_\eta$ in $L^\infty(0,T;H^1(\Omega))$, and $\nabla n_{k\eta} \xrightarrow{w^*} \nabla n_\eta$ in $L^\infty(0,T;L^2(\Omega))$.

The regularity for $\gamma\mathrm{div}\,(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta)$ follows from the fact that $\gamma\mathrm{div}\,(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta) = e_0\psi + \partial_n\Phi_{n,\eta}(\nabla\chi_\eta, n_\eta) - \mu_{n,\eta}$ and all the terms on the right-hand side are in $L^2(0,T;H^1(\Omega))$.

(ii) This follows directly from passing to the limit in (29e), and using the given strong and weak convergences. In particular, we use the strong convergence $\nabla\chi_{k\eta} \xrightarrow{s} \nabla\chi_\eta$ in $C(0,T;C^{0,\lambda}(\Omega;\mathbb{R}^{d\times d}))$, and the weak convergences $\psi_{k\eta} \xrightarrow{w} \psi_\eta$ in $L^2(0,T;H^1(\Omega))$ and $n_{k\eta} \xrightarrow{w} n_\eta$ in $L^2((0,T)\times\Omega)$.

□

Lemma 4.8 (Improved convergence)**.** *We have the additional improved convergences:*

(i) $\nabla F_{\mathrm{in},k\eta} \xrightarrow{s} \nabla F_{\mathrm{in},\eta}$ *in* $L^{p_{\mathrm{in}}}((0,T)\times\Omega;\mathbb{R}^{d\times d\times d})$,

(ii) $\mathrm{div}\,(\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})) \xrightarrow{w} \mathrm{div}\,(\mathfrak{h}(\nabla F_{\mathrm{in},\eta}))$ *in* $L^{r'}((0,T)\times\Omega)$,

(iii) $\dot{F}_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1} \xrightarrow{s} \dot{F}_{\mathrm{in},\eta}F_{\mathrm{in},\eta}^{-1}$ *in* $L^r((0,T)\times\Omega)$,

(iv) $\nabla n_{k\eta} \xrightarrow{s} \nabla n_\eta$ *in* $L^2([0,T]\times\Omega)$*, and*

(v) $\psi_{k\eta} \xrightarrow{s} \psi_\eta$ *in* $L^2(0,T;H^1(\Omega))$.

Proof. Part (i). Using the strong monotonicity (18) from Assumption ***(A3)***(ii) it follows that

$$\begin{aligned}\lim_{k\to\infty}\|\nabla F_{\mathrm{in},k\eta}-\nabla F_{\mathrm{in},\eta}\|^{p_{\mathrm{in}}}_{L^{p_{\mathrm{in}}}((0,T)\times\Omega;\mathbb{R}^{d\times d\times d})} &= \lim_{k\to\infty}\int_0^T\int_\Omega|\nabla F_{\mathrm{in},k\eta}-\nabla F_{\mathrm{in},\eta}|^{p_{\mathrm{in}}}\,\mathrm{d}x\,\mathrm{d}s\\ &\le C\lim_{k\to\infty}\int_0^T\int_\Omega(\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})-\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},\eta}))\vdots(\nabla F_{\mathrm{in},k\eta}-\nabla F_{\mathrm{in},\eta})\,\mathrm{d}x\,\mathrm{d}s\\ &= C\lim_{k\to\infty}\int_0^T\int_\Omega\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})\vdots(\nabla F_{\mathrm{in},k\eta}-\nabla F_{\mathrm{in},\eta})\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

where the last line follows from the weak* convergence $\nabla F_{\mathrm{in},k\eta}\xrightarrow{w^*}\nabla F_{\mathrm{in},\eta}$ in $L^\infty(0,T;L^{p_{\mathrm{in}}}(\Omega;\mathbb{R}^{d\times d\times d}))$. It is now important to note that $F_{\mathrm{in},k\eta}$ is a valid test function, but $F_{\mathrm{in},\eta}$ is not a valid test function for (29b) since it does not lie in the finite-dimensional Galerkin subspace. So, let $\widetilde{F}_{\mathrm{in},k}$ be a function in this finite-dimensional Galerkin subspace such that $\widetilde{F}_{\mathrm{in},k}\xrightarrow{s}F_{\mathrm{in},\eta}$ in $L^\infty(0,T;W^{1,p_{\mathrm{in}}}(\Omega;\mathbb{R}^{d\times d}))\cap H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$. Then, since $\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})$ is bounded in $L^\infty(0,T;L^{p'_{\mathrm{in}}}(\Omega;\mathbb{R}^{d\times d\times d}))$ it follows that

$$\lim_{k\to\infty}\|\nabla F_{\mathrm{in},k\eta}-\nabla F_{\mathrm{in},\eta}\|^{p_{\mathrm{in}}}_{L^{p_{\mathrm{in}}}((0,T)\times\Omega;\mathbb{R}^{d\times d\times d})} \le C\lim_{k\to\infty}\int_0^T\int_\Omega\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})\vdots(\nabla F_{\mathrm{in},k\eta}-\nabla\widetilde{F}_{\mathrm{in},k})\,\mathrm{d}x\,\mathrm{d}s.$$

Since $F_{\mathrm{in},k\eta}-\widetilde{F}_{\mathrm{in},k}$ is a valid test function for (29b), we thus obtain

$$\begin{aligned}&\lim_{k\to\infty}\|\nabla F_{\mathrm{in},k\eta}-\nabla F_{\mathrm{in},\eta}\|^{p_{\mathrm{in}}}_{L^{p_{\mathrm{in}}}((0,T)\times\Omega;\mathbb{R}^{d\times d\times d})}\\ &\le C\lim_{k\to\infty}\int_0^T\int_\Omega\big(\partial_{F_{\mathrm{in}}}\Phi_\eta(\nabla\chi_{k\eta}F^{-1}_{\mathrm{in},k\eta},F_{\mathrm{in},k\eta})+\zeta'_{\mathrm{in}}(\dot{F}_{\mathrm{in},k\eta}F^{-1}_{\mathrm{in},k\eta})F^{-\top}_{\mathrm{in},k\eta}\big):(F_{\mathrm{in},k\eta}-\widetilde{F}_{\mathrm{in},k})\,\mathrm{d}x\,\mathrm{d}s\\ &\le C\lim_{k\to\infty}\|F_{\mathrm{in},k\eta}-\widetilde{F}_{\mathrm{in},k}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d}))}\\ &\quad\times\int_0^T\int_\Omega\big|\partial_{F_{\mathrm{in}}}\Phi_\eta(\nabla\chi_{k\eta}F^{-1}_{\mathrm{in},k\eta},F_{\mathrm{in},k\eta})+\zeta'_{\mathrm{in}}(\dot{F}_{\mathrm{in},k\eta}F^{-1}_{\mathrm{in},k\eta})F^{-\top}_{\mathrm{in},k\eta}\big|\,\mathrm{d}x\,\mathrm{d}s=0,\end{aligned}$$

where we have used that $F_{\mathrm{in},k\eta}\xrightarrow{s}F_{\mathrm{in},\eta}$ in $C(0,T;C^{0,\widetilde{\lambda}}(\Omega;\mathbb{R}^{d\times d}))$ for all $\widetilde{\lambda}\in(0,1-\frac{d}{p})$. Thus, we have that $\nabla F_{\mathrm{in},k\eta}\xrightarrow{s}\nabla F_{\mathrm{in},\eta}$ in $L^{p_{\mathrm{in}}}((0,T)\times\Omega;\mathbb{R}^{d\times d\times d})$.

Part (ii). We follow [RS19, Prop. 4.1&4.2]. Define a seminorm $|\cdot|_k$ on $L^{r'}((0,T)\times\Omega)$ by setting

$$|\xi|_k:=\Big\{\int_0^T\int_\Omega\xi:w\,\mathrm{d}x\,\mathrm{d}s\;\Big|\;\|w\|_{L^r((0,T)\times\Omega;\mathbb{R}^{d\times d})}\le1,\ w(t)\in W_k\text{ for a.a. }t\in[0,T]\Big\}.$$

Since $\mathrm{div}\,(\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta}))$ satisfies the equation

$$\int_0^T\int_\Omega\mathrm{div}\,(\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})):w\,\mathrm{d}x\,\mathrm{d}s=\int_0^T\int_\Omega\big(\sigma_{\mathrm{in}}(\nabla\chi_{k\eta},F_{\mathrm{in},k\eta})+\zeta'_{\mathrm{in}}(\dot{F}_{\mathrm{in},k\eta}F^{-\top}_{\mathrm{in},k\eta})\big):w\,\mathrm{d}x\,\mathrm{d}s$$

for any test function $w\in L^r(0,T;W_k)$ taking values in the Galerkin space W_k, it follows that $\big|\mathrm{div}\,(\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta}))\big|_l\le C$ for all $k\ge l$ and a constant C independent of k,l. By considering the Hahn–Banach extension of this linear bounded functional, we can extend the seminorm to the whole space $L^{r'}((0,T)\times\Omega)$. In particular, this full norm is still bounded by the same constant C. Taking subsequences, and using the strong convergence $\nabla F_{\mathrm{in},k\eta} \xrightarrow{s} \nabla F_{\mathrm{in},\eta}$ in $L^{p_{\mathrm{in}}}((0,T)\times\Omega;\mathbb{R}^{d\times d\times d})$ from (i), it then follows that $\operatorname{div}(\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})) \xrightarrow{w} \operatorname{div}(\mathfrak{h}(\nabla F_{\mathrm{in},\eta}))$ in $L^{r'}((0,T)\times\Omega)$.

Part (iii). Using the strong monotonicity of ζ_{in}' (see (20) in Assumption ***(A4)***(ii)), we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\|\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1}-\dot F_{\mathrm{in},\eta}F_{\mathrm{in},\eta}^{-1}\|_{L^r((0,T)\times\Omega)}^r=\lim_{k\to\infty}\int_0^T\int_\Omega\big|\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1}-\dot F_{\mathrm{in},\eta}F_{\mathrm{in},\eta}^{-1}\big|^r\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\lim_{k\to\infty}\int_0^T\int_\Omega(\zeta_{\mathrm{in}}'(\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1})-\zeta_{\mathrm{in}}'(\dot F_{\mathrm{in},\eta}F_{\mathrm{in},\eta}^{-1})):(\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1}-\dot F_{\mathrm{in},\eta}F_{\mathrm{in},\eta}^{-1})\,\mathrm{d}x\,\mathrm{d}s\\
&=C\lim_{k\to\infty}\int_0^T\int_\Omega\zeta_{\mathrm{in}}'(\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1}):(\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1}-\dot F_{\mathrm{in},\eta}F_{\mathrm{in},\eta}^{-1})\,\mathrm{d}x\,\mathrm{d}s:=I,
\end{aligned}$$

where the last equality follows from the weak convergence $\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1} \xrightarrow{w} \dot F_{\mathrm{in},\eta}F_{\mathrm{in},\eta}^{-1}$ in $L^r((0,T)\times\Omega)$. As before, we note that $F_{\mathrm{in},\eta}$ is not a valid test function for (29b) since it does not lie in the finite-dimensional Galerkin subspace. So, let $\widetilde F_{\mathrm{in},k}$ be a function in this finite-dimensional Galerkin subspace such that $\widetilde F_{\mathrm{in},k} \xrightarrow{s} F_{\mathrm{in},\eta}$ in $L^\infty(0,T;W^{1,p_{\mathrm{in}}}(\Omega;\mathbb{R}^{d\times d}))\cap W^{1,r}(0,T;L^r(\Omega;\mathbb{R}^{d\times d}))$. We split the integral I into three parts:

$$\begin{aligned}
I=C\lim_{k\to\infty}\int_0^T\int_\Omega&\Big\{\zeta_{\mathrm{in}}'(\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1}):(\dot F_{\mathrm{in},k\eta}-\dot{\widetilde F}_{\mathrm{in},k})F_{\mathrm{in},k\eta}^{-1}\\
&+\zeta_{\mathrm{in}}'(\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1}):(\dot{\widetilde F}_{\mathrm{in},k}-\dot F_{\mathrm{in},\eta})F_{\mathrm{in},k\eta}^{-1}+\zeta_{\mathrm{in}}'(\dot F_{\mathrm{in},k\eta}F_{\mathrm{in},k\eta}^{-1}):\dot F_{\mathrm{in},\eta}(F_{\mathrm{in},k\eta}^{-1}-F_{\mathrm{in},\eta}^{-1})\Big\}\,\mathrm{d}x\,\mathrm{d}s\\
=:{}&I_1+I_2+I_3.
\end{aligned}$$

Using the strong convergence $\dot{\widetilde F}_{\mathrm{in},k} \xrightarrow{s} \dot F_{\mathrm{in},\eta}$ in $L^r((0,T)\times\Omega;\mathbb{R}^{d\times d})$, we obtain $I_2=0$. Similarly, using the strong convergence $F_{\mathrm{in},k\eta}^{-1} \xrightarrow{s} F_{\mathrm{in},\eta}^{-1}$ in $C(0,T;C^{0,\widetilde\lambda}(\Omega;\mathbb{R}^{d\times d}))$ it follows that $I_3=0$. To show that $I_1=0$, we use that $\dot F_{\mathrm{in},k\eta}-\dot{\widetilde F}_{\mathrm{in},k}$ is a valid test function for (29b), so that

$$I_1=-\lim_{k\to\infty}\int_0^T\int_\Omega\sigma_{\mathrm{in}}(\nabla\chi_{k\eta},F_{\mathrm{in},k\eta}):(\dot F_{\mathrm{in},k\eta}-\dot{\widetilde F}_{\mathrm{in},k})+\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})\,\vdots\,\nabla(\dot F_{\mathrm{in},k\eta}-\dot{\widetilde F}_{\mathrm{in},k})\,\mathrm{d}x\,\mathrm{d}s$$

In particular, note that the term $\nabla\dot F_{\mathrm{in},k\eta}$ is well-defined on the Galerkin level (but not for the limit $k\to\infty$). Since $\nabla\chi_{k\eta} \xrightarrow{s} \nabla\chi_\eta$ converges strongly in $C(0,T;C^{0,\lambda}(\Omega;\mathbb{R}^{d\times d}))$ and $F_{\mathrm{in},k\eta} \xrightarrow{s} F_{\mathrm{in},\eta}$ in $C(0,T;C^{0,\widetilde\lambda}(\Omega;\mathbb{R}^{d\times d}))$, the weak convergence $\dot F_{\mathrm{in},k\eta} \xrightarrow{w} \dot F_{\mathrm{in},\eta}$ implies that the first term with the inelastic stress is zero. For the second term with the hyperstress, we use the chain rule and integration by parts to write

$$\begin{aligned}
&-\int_0^T\int_\Omega\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})\,\vdots\,\nabla(\dot F_{\mathrm{in},k\eta}-\dot{\widetilde F}_{\mathrm{in},k})\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_\Omega\mathscr{H}_{\mathrm{in}}(\nabla F_{\mathrm{in},\eta}(0))-\mathscr{H}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta}(T))\,\mathrm{d}x-\int_0^T\int_\Omega\operatorname{div}(\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta}))\dot{\widetilde F}_{\mathrm{in},k}\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

Using the weak convergence $\operatorname{div}(\mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},k\eta})) \xrightarrow{w} \operatorname{div}(\mathfrak{h}(\nabla F_{\mathrm{in},\eta}))$ in $L^{r'}((0,T)\times\Omega)$ from (ii), it then follows that

$$
\begin{aligned}
&-\int_0^T\int_\Omega \mathfrak{h}_{\rm in}(\nabla F_{{\rm in},k\eta})\vdots\nabla(\dot{F}_{{\rm in},k\eta}-\dot{\tilde{F}}_{{\rm in},k})\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_\Omega \mathscr{H}_{\rm in}(\nabla F_{{\rm in},\eta}(0))-\mathscr{H}_{\rm in}(\nabla F_{{\rm in},k\eta}(T))\,\mathrm{d}x-\int_0^T\int_\Omega {\rm div}\,(\mathfrak{h}_{\rm in}(\nabla F_{{\rm in},\eta}))\dot{F}_{{\rm in},\eta}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_\Omega \mathscr{H}_{\rm in}(\nabla F_{{\rm in},\eta}(0))-\mathscr{H}_{\rm in}(\nabla F_{{\rm in},k\eta}(T))\,\mathrm{d}x-\int_\Omega \mathscr{H}_{\rm in}(\nabla F_{{\rm in},\eta}(T))-\mathscr{H}_{\rm in}(\nabla F_{{\rm in},\eta}(0)\,\mathrm{d}x,
\end{aligned}
$$

where the last equality follows from a chain rule as in [Ste08, Prop. 2.2]. Passing to the $\liminf$ it then follows that

$$
-\lim_{k\to\infty}\int_0^T\int_\Omega \mathfrak{h}_{\rm in}(\nabla F_{{\rm in},k\eta})\vdots\nabla(\dot{F}_{{\rm in},k\eta}-\dot{\tilde{F}}_{{\rm in},k})\,\mathrm{d}x\,\mathrm{d}s=0,
$$

and thus $I_1=0$. In conclusion, we have shown that $\dot{F}_{{\rm in},k\eta}F_{{\rm in},k\eta}^{-1}\overset{s}{\to}\dot{F}_{{\rm in},\eta}F_{{\rm in},\eta}^{-1}$ in $L^r((0,T)\times\Omega)$.

Part (iv). Due to Lemma 4.5(ii) and the resulting uniform ellipticity of $\mathcal{A}^F$ (see Assumption ***(A1)***(ii)), there exists a constant $C>0$ (independent of k) such that we can estimate

$$
\begin{aligned}
C\|\nabla(n_{k\eta}-n_\eta)\|^2_{L^2([0,T]\times\Omega)}&\le\int_0^T\int_\Omega \mathcal{A}^F(\nabla\chi_{k\eta})\nabla(n_{k\eta}-n_\eta)\cdot\nabla(n_{k\eta}-n_\eta)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_\Omega\Big\{\mathcal{A}^F(\nabla\chi_{k\eta})\nabla(n_\eta-2n_{k\eta})\cdot\nabla n_\eta\\
&\qquad+\big(\mu_{n,k\eta}-e_0\psi_{k\eta}-\partial_n\Phi_{n,\eta}(\nabla\chi_{k\eta},n_{k\eta})\big)n_{k\eta}\Big\}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where we used equation (29d). Due to the strong convergences of $\nabla\chi_{k,\eta}\overset{s}{\to}\nabla\chi_\eta$ in $C(0,T;C^{0,\lambda}(\Omega;\mathbb{R}^{d\times d}))$, $n_{k\eta}\overset{s}{\to}n_\eta$ in $L^2((0,T)\times\Omega)$, and the weak convergences of $\mu_{n,k\eta}\overset{w}{\rightharpoonup}\mu_\eta$ in $L^2(0,T;H^1(\Omega))$, $\psi_{k\eta}\overset{w^*}{\rightharpoonup}\psi_\eta$ in $L^\infty(0,T;H^1(\Omega))$, and $\nabla n_{k\eta}\overset{w^*}{\rightharpoonup}\nabla n_\eta$ in $L^\infty(0,T;L^2(\Omega))$, we can pass to the limit $k\to\infty$ on the right-hand side. In particular, using (32), it follows that the right-hand side converges to zero as $k\to\infty$, which proves the result.

Part (v). Recall the equivalent norm $\|(\cdot)\|_\kappa=\big\|\nabla(\cdot)\big\|_{L^2(\Omega)}+\|\sqrt{\kappa}(\cdot)\|_{L^2(\partial\Omega)}$ from the proof of Lemma 4.6. Using the uniform ellipticity of the permittivity tensor $\varepsilon(\nabla\chi_{k\eta})$, it follows that

$$
\begin{aligned}
&\lim_{k\to\infty}\|\psi_{k\eta}-\psi_\eta\|_\kappa^2\\
&\le C\lim_{k\to\infty}\int_0^T\int_\Omega\varepsilon(\nabla\chi_{k\eta})(\nabla\psi_{k\eta}-\nabla\psi_\eta)\cdot(\nabla\psi_{k\eta}-\nabla\psi_\eta)\,\mathrm{d}x\,\mathrm{d}s+\int_0^T\int_{\partial\Omega}\kappa(\psi_{k\eta}-\psi_\eta)^2\,\mathrm{d}S\,\mathrm{d}s\\
&=C\lim_{k\to\infty}\int_0^T\int_\Omega\varepsilon(\nabla\chi_{k\eta})\nabla\psi_{k\eta}\cdot\nabla\psi_{k\eta}-\varepsilon(\nabla\chi_\eta)\nabla\psi_\eta\cdot\nabla\psi_\eta\,\mathrm{d}x\,\mathrm{d}s+\int_0^T\int_{\partial\Omega}\kappa(\psi_{k\eta}^2-\psi_\eta^2)\,\mathrm{d}S\,\mathrm{d}s,
\end{aligned}
$$

where we have used the strong convergence $\nabla\chi_{k\eta}\overset{s}{\to}\nabla\chi_\eta$ in $C(0,T;C^{0,\lambda}(\Omega;\mathbb{R}^{d\times d}))$ and the weak* convergence $\psi_{k\eta}\overset{w^*}{\rightharpoonup}\psi_\eta$ in $L^\infty(0,T;H^1(\Omega))\cap L^2((0,T)\times\partial\Omega)$. Since $\psi_{k\eta}$ satisfies the Poisson equation (29e), it then follows that

$$
\begin{aligned}
\lim_{k\to\infty}\|\psi_{k\eta}-\psi_\eta\|_\kappa^2=C\lim_{k\to\infty}\int_0^T\int_\Omega e_0(n_{k\eta}+N)\psi_{k\eta}\,\mathrm{d}x\,\mathrm{d}s-\varepsilon(\nabla\chi_\eta)\nabla\psi_\eta\cdot\nabla\psi_\eta\,\mathrm{d}x\,\mathrm{d}s\\
+\int_0^T\int_{\partial\Omega}\kappa(\psi_{k\eta}\psi_{\rm ext}-\psi_\eta^2)\,\mathrm{d}S\,\mathrm{d}s.
\end{aligned}
$$

Using the strong convergence of $n_{k\eta} \xrightarrow{s} n_\eta$ in $L^2((0,T)\times\Omega)$ and the weak* convergence of $\psi_{k\eta} \xrightarrow{w^*} \psi_\eta$ in $L^\infty(0,T;L^2(\Omega)) \cap L^2((0,T)\times\partial\Omega)$, and that ψ_η satisfies the Poisson equation by Lemma 4.7(ii), we then obtain

$$\lim_{k\to\infty} \|\psi_{k\eta}-\psi_\eta\|_\kappa^2 = C \lim_{k\to\infty} \int_0^T \int_\Omega e_0(n_\eta{+}N)\psi_\eta \,\mathrm{d}x\,\mathrm{d}s - \varepsilon(\nabla\chi_\eta)\nabla\psi_\eta\cdot\nabla\psi_\eta\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^T\int_{\partial\Omega} \kappa(\psi_\eta\psi_{\mathrm{ext}}{-}\psi_\eta^2)\,\mathrm{d}S\,\mathrm{d}s = 0,$$

proving the statement. □

Using these convergences, we now pass to the limit $k\to\infty$, while keeping η fixed, in the equations (29). We prove that we converge to the system:

$$\int_0^T\int_\Omega \big(\sigma_{\mathrm{el}}(\nabla\chi_\eta, F_{\mathrm{in},\eta}, n_\eta, \nabla n_\eta, \nabla\psi_\eta) + \partial_{\dot F}\zeta_{\mathrm{KV}}(\nabla\chi_\eta, \nabla\dot\chi_\eta)\big) : \nabla v \\ + \mathfrak{h}_{\mathrm{el}}(\mathrm{D}^2\chi_\eta) \,\vdots\, \mathrm{D}^2 v\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle \ell(t), v\rangle\,\mathrm{d}t = 0, \tag{34a}$$

$$\int_0^T\int_\Omega \big(\sigma_{\mathrm{in}}(\nabla\chi_\eta, F_{\mathrm{in},\eta}) + \zeta_{\mathrm{in}}'(\dot F_{\mathrm{in},\eta}F_{\mathrm{in},\eta}^{-1})F_{\mathrm{in},\eta}^{-\top}\big) : w + \mathfrak{h}_{\mathrm{in}}(\nabla F_{\mathrm{in},\eta}) \,\vdots\, \nabla w\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{34b}$$

$$\int_0^T \langle \dot n_\eta, \phi\rangle\,\mathrm{d}t + \int_0^T\int_\Omega \mathcal{A}_\eta(\nabla\chi_\eta, n_\eta)\nabla\mu_{n,\eta}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{34c}$$

$$\int_0^T\int_\Omega \big(\mu_{n,\eta} - \partial_n\Phi_\eta(\nabla\chi_\eta, n_\eta, \psi_\eta)\big)\phi - \gamma\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{34d}$$

$$\int_0^T\int_\Omega \varepsilon(\nabla\chi_\eta)\nabla\psi_\eta\cdot\nabla\omega - (e_0 n_\eta{+}N)\omega\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega}\kappa(\psi_\eta - \psi_{\mathrm{ext}})\omega\,\mathrm{d}S\,\mathrm{d}t = 0, \tag{34e}$$

where $v \in L^\infty(0,T;W_0^{2,p}(\Omega;\mathbb{R}^d))$, $w \in L^\infty(0,T;W_0^{1,p_{\mathrm{in}}}(\Omega;\mathbb{R}^{d\times d}))$, $\phi \in L^2(0,T;H^1(\Omega))$ and $\omega \in L^2(0,T;H^1(\Omega))$.

Lemma 4.9 (Limit passage $k\to\infty$ for fixed η)**.** *There exist subsequences (not relabeled) such that the Galerkin solution* $(\chi_{k\eta}, F_{\mathrm{in},k\eta}, n_{k\eta}, \psi_{k\eta})$ *obtained in Lemma 4.1 converges to a solution* $(\chi_\eta, F_{\mathrm{in},\eta}, n_\eta, \psi_\eta)$*, which satisfies the equations* (34).

Proof. We pass to the limit $k\to\infty$, while keeping the regularization parameter η fixed in the Galerkin scheme (29).

Step 1. Limit in mechanical equation (29a)*:* The limit passage in the first term follows easily using the given strong convergences and the decomposition (21). Indeed, we have

$$\int_0^T\int_\Omega \partial_F\Phi_\eta(\nabla\chi_{k\eta}, F_{\mathrm{in},k\eta}, n_{k\eta}, \nabla n_{k\eta}, \psi_{k\eta}, \nabla\psi_{k\eta}) : \nabla v\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\int_\Omega \Big\{\Phi_{\mathrm{mech}}'(\nabla\chi_{k\eta}F_{\mathrm{in},k\eta}^{-1})F_{\mathrm{in},k\eta}^{-1} - \frac{1}{2}\partial_F\varepsilon(\nabla\chi_{k\eta})\nabla\psi_{k\eta}\cdot\nabla\psi_{k\eta} \\ + \partial_F\Phi_n^1(\nabla\chi_{k\eta}, n_{k\eta}) + \frac{\gamma}{2}\partial_F\mathcal{A}^F(\nabla\chi_{k\eta})\nabla n_{k\eta}\cdot\nabla n_{k\eta}\Big\} : \nabla v\,\mathrm{d}x\,\mathrm{d}t,$$

and $\nabla\chi_{k\eta} \xrightarrow{s} \nabla\chi_\eta$ converges strongly in $C(0,T;C^{0,\lambda}(\Omega;\mathbb{R}^{d\times d}))$, $F_{\mathrm{in},k\eta}^{-1} \xrightarrow{s} F_{\mathrm{in},\eta}^{-1}$ in $C(0,T;C^{0,\widetilde\lambda}(\Omega;\mathbb{R}^{d\times d}))$, $\nabla\psi_{k\eta} \xrightarrow{s} \nabla\psi_\eta$ in $L^2((0,T)\times\Omega)$, and $n_{k\eta} \xrightarrow{s} n_\eta$ in $L^2(0,T;H^1(\Omega))$.

For the viscous stresses, we note that $\partial_{\dot F}\zeta_{\rm KV}(F,\dot F)$ is linear with respect to $\dot F$, so that the weak convergence of $\nabla\dot\chi_{k\eta}$ suffices here. Finally, the limit passage in the hyperstress term follows using Minty's trick applied to the functional $\mathscr{A} : W^{2,p}(\Omega;\mathbb{R}^d) \to [0,\infty]$ defined by

$$\mathscr{A}(\chi) = \begin{cases} \int_0^T \int_\Omega \mathscr{H}(\mathrm{D}^2\chi)\,\mathrm{d}x\,\mathrm{d}t & \text{if } \chi \in L^\infty(0,T;W^{2,p}(\Omega;\mathbb{R}^d)), \\ +\infty & \text{otherwise.} \end{cases}$$

For more details, we refer to e.g. the proof of [MR20, Prop. 5.1].

Step 2. Limit of inelastic flow rate equation (29b)*:* The limit passage in the first term with $\sigma_{\rm in}$ follows easily using the decomposition (22) and the convergences of $\nabla\chi_{k\eta} \xrightarrow{s} \nabla\chi_\eta$ in $C(0,T;C^{0,\lambda}(\Omega;\mathbb{R}^{d\times d}))$, and $F_{{\rm in},k\eta} \xrightarrow{s} F_{{\rm in},\eta}$, $F_{{\rm in},k\eta}^{-1} \xrightarrow{s} F_{{\rm in},\eta}^{-1}$ in $C(0,T;C^{0,\widetilde\lambda}(\Omega;\mathbb{R}^{d\times d}))$. The limit passage in the second term with $\zeta_{\rm in}'$ follows directly from the strong convergence of $\dot F_{{\rm in},k\eta}F_{{\rm in},k\eta}^{-1} \xrightarrow{s} \dot F_{{\rm in},\eta}F_{{\rm in},\eta}^{-1}$ in $L^r((0,T)\times\Omega)$ from Lemma 4.8(iii). Finally, the limit passage in the last term with the hyperstress $\mathfrak{h}_{\rm in}$ follows from the strong convergence of $\nabla F_{{\rm in},k\eta} \xrightarrow{s} \nabla F_{{\rm in},\eta}$ in $L^{p_{\rm in}}((0,T)\times\Omega;\mathbb{R}^{d\times d\times d})$ from Lemma 4.8(i).

Step 3. Limit in diffusion equation (34c)*:* The limit passage follows directly from the given weak and strong convergences. Indeed, we use the fact that $|\mathcal{A}_\eta(F,n)| \le C_R$ for all $F \in \mathsf{F}_R$ with some $R>0$ and $n\in\mathbb{R}$, the strong convergences of $\nabla\chi_{k\eta} \xrightarrow{s} \nabla\chi_\eta$ in $C(0,T;C^{0,\lambda}(\Omega;\mathbb{R}^{d\times d}))$ and $n_{k\eta} \xrightarrow{s} n_\eta$ in $L^2(0,T;H^1(\Omega))$, and the weak convergence $\mu_{n,k\eta} \xrightarrow{w} \mu_{n,\eta}$ in $L^2(0,T;H^1(\Omega))$.

Step 4. Limit of (34d)*:* This has been done in Lemma 4.7(ii). In particular, note that the limit $\mu_{n,\eta}$ satisfies (32).

Step 5. Limit in Poisson equation (34e)*:* This has been done in Lemma 4.7(iii). □

4.4 Limit passage $\eta \to 0$

It remains to pass to the limit $\eta \to 0$, where η is the regularization parameter appearing in the definition of B_η and $\Phi_{n,\eta}$ in (25) and (26), respectively. First, we show an energy-dissipation inequality similar to (30). Recall the definitions of $\mathcal{E}_\eta$, $\mathcal{D}_\eta$ and $\mathcal{P}$ from Proposition 4.2.

Lemma 4.10 (Energy-dissipation inequality)**.** *The limit solution $(\chi_\eta, F_{{\rm in},\eta}, n_\eta, \psi_\eta)$ satisfies the energy-dissipation inequality*

$$\mathcal{E}_\eta(t,X_\eta(t)) + \int_0^t \mathcal{D}_\eta(X_\eta,\mu_{n,\eta},\dot\chi_\eta,\dot F_{{\rm in},\eta})\,\mathrm{d}s \le \mathcal{E}_\eta(0,X_\eta(0)) + \int_0^t \mathcal{P}(t,\chi_\eta,\psi_\eta)\,\mathrm{d}s. \tag{35}$$

Consequently,

$$\mathcal{E}_\eta(t,X_\eta) \le C \quad \textit{and} \quad \int_0^t \mathcal{D}_\eta(X_\eta,\mu_{n,\eta},\dot\chi_\eta,\dot F_{{\rm in},\eta})\,\mathrm{d}s \le C.$$

Proof. The energy-dissipation inequality (35) follows by passing taking the $\liminf$ of the the energy-dissipation balance (30) and letting $k\to\infty$. Using the given weak and strong convergences, and standard lower semicontinuity arguments as in [FL07, Thm. 7.5], we then arrive at (35).

The bounds for $\mathcal{E}_\eta$ and $\int_0^t \mathcal{D}_\eta\,\mathrm{d}s$ now follow similarly to Corollary 4.4. □

The estimates for χ, $F_{\rm in}$ and ψ follow analogously to Lemma's 4.5 and 4.6. However, the estimates for n are different. In particular, we are now in the position to exploit the relation for $\mu_{n,\eta}$ as in Lemma 4.7(ii). Using a similar strategy as in [EG96, Lemma 2], we now obtain the following estimates.

Lemma 4.11 (A priori estimates). *There exists a constant $C > 0$ (independent of η), such that:*

(i) $\|n_\eta\|_{L^\infty(0,T;H^1(\Omega))} + \big\|\mathrm{div}\,(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta)\big\|^2_{L^2((0,T)\times\Omega)} \leq C,$

(ii) $\int_0^T \int_\Omega (n_\eta^-)^2 + ((n_\eta - 1)^+)^2 \leq C\eta,$

(iii) $\|j_{n,\eta}\|_{L^2((0,T)\times\Omega)} \leq C$*, where* $j_{n,\eta} = \mathcal{A}_\eta(\nabla\chi_\eta, n_\eta)\nabla\mu_{n,\eta},$

(iv) $\|\dot{n}_\eta\|_{L^2(0,T;H^1(\Omega)^*)} \leq C.$

Proof. Step 1. The bound $\|n_\eta\|_{L^\infty(0,T;H^1(\Omega))} \leq C$ in (i) follows directly from Assumption ***(A2)***(iii) and the uniform boundedness of $\mathcal{E}_\eta$.

Step 2. To obtain the second bound of (i), we define the function $\xi_\eta : \mathbb{R} \to \mathbb{R}$ by setting $\xi_\eta''(n) = (B_\eta(n))^{-1}$ and $\xi_\eta'(\frac{1}{2}) = 0$, $\xi_\eta(\frac{1}{2}) = 0$. In particular, we note that ξ_η'' is bounded, so that $\xi_\eta'(n_\eta) \in L^2(0,T;H^1(\Omega))$ is a valid test function for (34c). In particular, we have $\nabla\xi_\eta'(n_\eta) = \xi_\eta''(n_\eta)\nabla n_\eta = B_\eta(n_\eta)^{-1}\nabla n_\eta$. Then, using a chain rule for convex functionals as in [EG96, Proof of Lemma 2] we obtain

$$\begin{aligned}\int_\Omega \xi_\eta(n_\eta(T)) - \xi_\eta(n_\eta(0))\,\mathrm{d}x &= \int_0^T \langle \xi'(n_\eta), \dot{n}_\eta\rangle\,\mathrm{d}s\\ &= -\int_0^T \int_\Omega \mathcal{A}_\eta(\nabla\chi_\eta, n_\eta)\nabla\mu_{n,\eta}\cdot\nabla\xi'(n_\eta)\,\mathrm{d}x\,\mathrm{d}s. \qquad (36)\end{aligned}$$

We now use Lemma 4.7(ii) to write

$$\begin{aligned}&\int_0^T\int_\Omega \mathcal{A}_\eta(\nabla\chi_\eta, n_\eta)\nabla\mu_{n,\eta}\cdot\nabla\xi'(n_\eta)\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^T\int_\Omega \Big\{\mathcal{A}_\eta(\nabla\chi_\eta, n_\eta)\big(\partial^2_{Fn}\Phi^1_n(\nabla\chi_\eta, n_\eta)\mathrm{D}^2\chi_\eta + \big(\partial^2_{nn}\Phi^1_n(\nabla\chi_\eta, n_\eta) + (\Phi^2_{n,\eta})''(n_\eta)\big)\nabla n_\eta + e_0\nabla\psi_\eta\big)\\ &\qquad\cdot\xi_\eta''(n_\eta)\nabla n_\eta\Big\}\,\mathrm{d}x\,\mathrm{d}s - \int_0^T\int_\Omega \mathcal{A}_\eta(\nabla\chi_\eta, n_\eta)\gamma\nabla\big(\mathrm{div}\,(\mathcal{A}^F(\nabla\chi_\eta)n_\eta\big)\cdot\xi_\eta''(n_\eta)\nabla n_\eta\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^T\int_\Omega \Big\{\mathcal{A}^F(\nabla\chi_\eta)\big(\partial^2_{Fn}\Phi^1_n(\nabla\chi_\eta, n_\eta)\mathrm{D}^2\chi_\eta + \big(\partial^2_{nn}\Phi^1_n(\nabla\chi_\eta, n_\eta) + (\Phi^2_{n,\eta})''(n_\eta)\big)\nabla n_\eta\\ &\qquad + e_0\nabla\psi_\eta\big)\cdot\nabla n_\eta\Big\}\,\mathrm{d}x\,\mathrm{d}s + \gamma\int_0^T\int_\Omega \big|\mathrm{div}\,(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta)\big|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Next, we recall that by Assumption ***(A1)***(ii) and the respective estimates for $\nabla\chi_\eta$, the estimate $|\mathcal{A}^F(\nabla\chi_\eta)| \leq C$ holds. Moreover, Assumption ***(A2)***(iii) leads to the estimates $|\partial^2_{Fn}\Phi(\nabla\chi_\eta, n_\eta)| \leq C$ and $|\partial^2_{nn}\Phi^1_n(\nabla\chi_\eta, n_\eta)| \leq C$. Using the bounds for χ_η in $L^\infty(0,T;W^{2,p}(\Omega;\mathbb{R}^d)$, n_η in $L^\infty(0,T;H^1(\Omega))$ and ψ_η in $L^\infty(0,T;H^1(\Omega))$ it then follows that

$$\begin{aligned}&\int_0^T\int_\Omega \mathcal{A}_\eta(\nabla\chi_\eta, n_\eta)\nabla\mu_{n,\eta}\cdot\nabla\xi'(n_\eta)\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad\geq \int_0^T\int_\Omega (\Phi^2_{n,\eta})''(n_\eta)|\nabla n_\eta|^2\,\mathrm{d}x\,\mathrm{d}s + \big\|\mathrm{div}\,(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta)\big\|^2_{L^2((0,T)\times\Omega)} - C\end{aligned}$$

Combining this with (36) we obtain

$$\sup_{t\in[0,T]}\int_\Omega \xi_\eta(n_\eta(t))\,\mathrm{d}x + \int_0^T\int_\Omega (\Phi^2_{n,\eta})''(n_\eta)|\nabla n_\eta|^2\,\mathrm{d}x\,\mathrm{d}s + \big\|\mathrm{div}\,(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta)\big\|^2_{L^2((0,T)\times\Omega)} \leq C, \qquad (37)$$

showing the second bound of (i).

Step 3. Next, using the definition ξ_η, we get for $z<0$ (and sufficiently small η):

$$\xi_\eta(z) = \xi_\eta(\eta) + \xi_\eta'(\eta)(z-\eta)^2 + \frac{1}{2}\xi_\eta''(\eta)(z-\eta)^2 \geq \frac{1}{2}\frac{1}{\eta(1-\eta)}(z-\eta)^2 \geq \frac{C}{\eta}z^2.$$

In particular, using (37), we see that $\int_\Omega (n_\eta^-)^2\,\mathrm{d}x \leq C\eta$ for almost all $t\in[0,T]$. Similarly, we can now do an expansion around $n=1-\eta$ to obtain $\int_\Omega ((n_\eta-1)^+)^2\,\mathrm{d}x \leq C\eta$. This shows (ii).

Step 4. We note that

$$\begin{aligned}&\|\mathcal{A}_\eta(\nabla\chi_\eta, c_\eta)\nabla\mu_{n,\eta}\|_{L^2((0,T)\times\Omega)}\\ &\qquad\leq \big\|(\mathcal{A}_\eta(\nabla\chi_\eta,n_\eta))^{1/2}\big\|_{L^\infty((0,T)\times\Omega)}\big\|(\mathcal{A}_\eta(\nabla\chi_\eta,n_\eta))^{1/2}\nabla\mu_{n,\eta}\big\|_{L^2((0,T)\times\Omega)} \leq C,\end{aligned}$$

where we have used that the conductivity $\mathcal{A}_\eta$ is uniformly bounded, and that $\big\|(\mathcal{A}_\eta(\nabla\chi_\eta,n_\eta))^{1/2}\nabla\mu_{n,\eta}\big\|_{L^2((0,T)\times\Omega)}$ is uniformly bounded by the energy-dissipation inequality (35). This shows the bound in (iii). Finally, the bound in (iv) easily follows by testing (34c) with a function $\phi\in L^2(0,T;H^1(\Omega))$. □

Using the a priori estimates, together with the Aubin–Lions lemma, we can extract (non-relabeled) converging subsequences and some $(\chi, F_{\mathrm{in}}, n, \psi, \mu_n)$ such that:

(i) $\chi_\eta \xrightarrow{w^*} \chi$ in $L^\infty(0,T;W^{2,p}(\Omega;\mathbb{R}^d))\cap H^1(0,T;H^1(\Omega;\mathbb{R}^d))$, and $\chi_\eta \xrightarrow{s} \chi$ strongly in $C(0,T;C^{1,\lambda}(\Omega;\mathbb{R}^d))$ for all $\lambda\in(0,1-\frac{d}{p})$,

(ii) $F_{\mathrm{in},\eta} \xrightarrow{w^*} F_{\mathrm{in}}$ in $L^\infty(0,T;W^{1,p_{\mathrm{in}}}(\Omega;\mathbb{R}^{d\times d}))\cap W^{1,r}(0,T;L^r(\Omega;\mathbb{R}^{d\times d}))$, and $F_{\mathrm{in},\eta}\xrightarrow{s} F_{\mathrm{in}}$ in $C(0,T;C^{0,\widetilde{\lambda}}(\Omega;\mathbb{R}^{d\times d}))$ for all $\widetilde{\lambda}\in(0,1-\frac{d}{p_{\mathrm{in}}})$. In particular, it also holds that $F_{\mathrm{in},\eta}^{-1}\xrightarrow{s} F_{\mathrm{in}}^{-1}$ in $C(0,T;C^{0,\widetilde{\lambda}}(\Omega;\mathbb{R}^{d\times d}))$ since $F_{\mathrm{in}}^{-1} = \frac{\mathrm{Cof}\,F_{\mathrm{in}}^\top}{\det F_{\mathrm{in}}}$,

(iii) $n_\eta \xrightarrow{w^*} n$ in $L^\infty(0,T;H^1(\Omega))\cap H^1(0,T;H^1(\Omega)^*)$, and $n_\eta\xrightarrow{s} n$ in $C(0,T;L^2(\Omega))$,

(iv) $\psi_\eta \xrightarrow{w^*} \psi$ in $L^\infty(0,T;H^1(\Omega))\cap L^2((0,T)\times\partial\Omega)$.

Lemma 4.12 (Improved convergence)**.** *We have the additional improved convergences:*

(i) $\mathrm{D}^2\chi_\eta \xrightarrow{s} \mathrm{D}^2\chi$ *in* $L^p((0,T)\times\Omega;\mathbb{R}^{d\times d\times d})$,

(ii) $\psi_\eta\xrightarrow{s}\psi$ *in* $L^2(0,T;H^1(\Omega))$,

(iii) $\operatorname{div}\big(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta\big) \xrightarrow{w} \operatorname{div}(\mathcal{A}^F(\nabla\chi)\nabla n)$ *in* $L^2((0,T)\times\Omega)$,

(iv) $\nabla n_\eta \xrightarrow{s} \nabla n$ *in* $L^2((0,T)\times\Omega)$.

Proof. The proof of (i) is similar to the proof of Lemma 4.8(i), now using the strong monotonicity in (17), and is therefore omitted. In particular, note that it is now not necessary to approximate the limit solution in a finite-dimensional subspace, since we are no longer in the Galerkin setting. The other statements are also similar to Lemma 4.8, therefore the proofs are omitted.

□

Lemma 4.13 (Limit passage $\eta \to 0$). *There exist subsequences (not relabeled) such that the solution $(\chi_\eta, F_{\text{in},\eta}, n_\eta, \psi_\eta)$ obtained in Lemma 4.9 converges to a solution $(\chi, F_{\text{in}}, n, \psi)$ of the system* (11)–(13) *in the sense of Definition 3.3.*

Proof. We note that the limit passage $\eta \to 0$ in the equations for χ_η, F_{in} and ψ_η is unchanged from the proof of Lemma 4.9, and therefore omitted.

It remains to pass to the limit in the diffusion equation for n_η, i.e., in (34c). First, by letting $\eta \to 0$ in Lemma 4.11(ii), we obtain $0 < n < 1$ for a.e. $(t,x) \in [0,T] \times \Omega$. The limit passage in the first term of (34c) follows directly from the weak convergence of $\dot{n}_\eta$ in $L^2(0,T;H^1(\Omega)^*)$. Using Lemma 4.11(iii), we see that the second integral with the flux also converges. However, it remains to identify this limit. To this end, we use integration by parts to rewrite the integral as

$$
\begin{aligned}
&\mathcal{A}_\eta(F_\eta,n_\eta)\nabla\big\{\partial_n\Phi_{\text{elec},\eta}(F_\eta,n_\eta,\psi_\eta) - \gamma\text{div}\left(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta\right)\big\}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}s\\
&= \int_0^T\int_\Omega \mathcal{A}_\eta(F_\eta,n_\eta)\Big\{\partial^2_{Fn}\Phi^1_n(\nabla\chi_\eta,n_\eta)\mathrm{D}^2\chi_\eta\cdot\nabla\phi + \left(\partial^2_{nn}\Phi^1_n(\nabla\chi_\eta,n_\eta) + (\Phi^2_{n,\eta})''(n_\eta)\right)\nabla n_\eta\cdot\nabla\phi\\
&\quad + e_0\nabla\psi_\eta\cdot\nabla\phi\Big\} + \gamma\mathcal{A}_\eta(F_\eta,n_\eta)\,\text{div}\left(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta\right)\mathrm{D}^2\phi\\
&\quad + \gamma B'_\eta(n_\eta)\nabla n_\eta\mathcal{A}^F(F_\eta)\,\text{div}\left(\mathcal{A}^F(\nabla\chi)\nabla n_\eta\right)\nabla\phi\\
&\quad + \gamma B_\eta(n_\eta)\left(\mathrm{D}_F\mathcal{A}^F(F_\eta)\right)\mathrm{D}^2\chi_\eta\,\text{div}\left(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta\right)\nabla\phi\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

The limit passage now follows using the given strong and weak convergences, together with the fact that $\mathcal{A}_\eta(F_\eta,n_\eta) \to \mathcal{A}(F,n)$ and $\mathrm{D}_F\mathcal{A}_\eta(F_\eta,n_\eta) \to \mathrm{D}_F\mathcal{A}(F,n)$ for a.e. $(t,x) \in [0,T]\times\Omega$. In particular, we use the strong convergences of $F_\eta \overset{s}{\to} F$ in $C(0,T;C^{0,\widetilde{\lambda}}(\Omega;\mathbb{R}^{d\times d}))$, $n_\eta \overset{s}{\to} n$ in $L^2(0,T;H^1(\Omega))$, $\psi_\eta \overset{s}{\to} \psi$ in $L^2(0.T;H^1(\Omega))$, and the weak convergence of $\text{div}\left(\mathcal{A}^F(\nabla\chi_\eta)\nabla n_\eta\right) \overset{w}{\rightharpoonup} \text{div}\left(\mathcal{A}^F(\nabla\chi)\nabla n\right)$ in $L^2((0,T)\times\Omega)$.

Only the limit passage in the fifth term (i.e., the term with $B'(n_\eta)$) is not immediate since B'_η is not continuous in $n=0$ and $n=1$. However, by decomposing the domain Ω as

$$\Omega = \{n<0\}\cup\{n=0\}\cup\{0<n<1\}\cup\{n=1\}\cup\{n>1\},$$

the limit passage follows by noting that on $\{n{=}0\}$ or $\{n{=}1\}$ we have $\nabla n = 0$ almost everywhere. Indeed, $B'_\eta(n_\eta)\nabla n_\eta \overset{s}{\to} B(n)$ in $L^2((0,T)\times\Omega)$ since on $\{n=0\}$ or $\{n=1\}$ it follows that

$$
\begin{aligned}
\int_0^T\int_{\{n(x)\in\{0,1\}\}}|B'_\eta(n_\eta)\nabla n_\eta - B'(n)\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s &= \int_0^T\int_{\{n(x)\in\{0,1\}\}}|B'_\eta(n_\eta)\nabla n_\eta|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\int_0^T\int_{\{n(x)\in\{0,1\}\}}|\nabla n_\eta|^2\,\mathrm{d}x\,\mathrm{d}s \to C\int_0^T\int_{\{n(x)\in\{0,1\}\}}|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s = 0.
\end{aligned}
$$

□

This concludes the proof of existence of solutions, i.e., Theorem 3.4.

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