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Abstract

In this paper, we study a phase field model for a tumor growth model of Cahn–Hilliard type in which the often assumed parabolic relaxation of the chemical potential is replaced by a hyperbolic one. We show that the resulting initial-boundary value problem is well posed and that its solutions depend continuously on two given functions: one appearing in the mass balance equation and one in the nutrient equation, representing, respectively, sources of drugs (e.g. chemotherapy) and antiangiogenic therapy. We also discuss regularity properties of the solutions. Moreover, in the case of a constant proliferation function, we rigorously analyze the asymptotic behavior as the coefficient of the inertial term tends to zero, establishing convergence to the corresponding viscous Cahn–Hilliard tumor growth model. Our results apply to a broad class of double-well potentials, including nonsmooth ones.

1 Introduction

Let $\alpha > 0,\ \tau > 0$, and let $\Omega \subset \mathbb{R}^3$ denote some open and bounded domain having a smooth boundary $\Gamma = \partial\Omega$ with outward normal $\mathbf{n}$ and corresponding outward normal derivative $\partial_{\mathbf{n}}$. Moreover, we fix some final time $T > 0$ and introduce for every $t \in (0, T]$ the sets $Q_t := \Omega \times (0, t)$ and $\Sigma_t := \Gamma \times (0, t)$, where we put, for the sake of brevity, $Q := Q_T$ and $\Sigma := \Sigma_T$. We then consider the following initial-boundary vaue problem:

$$\alpha\partial_{tt}\mu + \partial_t\varphi - \Delta\mu = P(\varphi)(\sigma + \chi(1-\varphi) - \mu) - \mathbb{h}(\varphi)u_1 \qquad \text{in } Q\,, \tag{1.1}$$

$$\tau\partial_t\varphi - \Delta\varphi + F'(\varphi) = \mu + \chi\,\sigma \qquad \text{in } Q\,, \tag{1.2}$$

$$\partial_t\sigma - \Delta\sigma = -\chi\Delta\varphi - P(\varphi)(\sigma + \chi(1-\varphi) - \mu) + u_2 \qquad \text{in } Q\,, \tag{1.3}$$

$$\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\sigma = 0 \qquad \text{on } \Sigma\,, \tag{1.4}$$

$$\mu(0) = \mu_0, \quad \partial_t\mu(0) = \mu_0', \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \qquad \text{in } \Omega\,. \tag{1.5}$$

The system (1.1)–(1.5) constitutes a simplified and relaxed version of the four-species thermodynamically consistent model for tumor growth originally proposed by Hawkins-Daruud et al. in [34] that additionally includes chemotactic terms. Let us briefly review the role of the occurring symbols. The primary variables φ, μ, σ denote the phase field, the associated chemical potential, and the nutrient concentration, respectively. Furthermore, we stress that the additional term $\alpha\partial_{tt}\mu$ is a hyperbolic regularization of equation (1.1), whereas the term $\tau\partial_t\varphi$ is the viscosity contribution to the Cahn–Hilliard equation. The key idea behind these regularizations originates from the fact that their presence allows us to take into account more general potentials F whose derivatives F', which play the role of a thermodynamic driving force in the model, may be singular and possibly nonregular.

The nonlinearity P denotes a proliferation function, whereas the positive constant χ represents the chemotactic sensitivity. The terms containing $P(\varphi)$ in the system (1.1)–(1.5) model tumor cell proliferation. In (1.1), the factor $P(\varphi)$ modulates the source of the chemical potential according to the local tumor density, coupling proliferation with nutrient availability (σ) and chemotaxis ($\chi(1-\varphi)$). In the third equation (1.3), the same term with minus sign accounts for nutrient consumption by proliferating cells. Thus, $P(\varphi)$ provides a natural coupling between the evolution of the phase field and the nutrient, reflecting the interplay between growth and resource uptake.

About equation (1.2), as is common in phase-field models, the function F is assumed to have a double-well structure. Typical examples include the regular, logarithmic, and double-obstacle potentials, which are respectively defined by

$$F_{\rm reg}(r) = \frac{1}{4}\left(1-r^2\right)^2, \qquad r \in \mathbb{R}, \tag{1.6}$$

$$F_{\rm log}(r) = \begin{cases} (1+r)\,\ln(1+r) + (1-r)\,\ln(1-r) - k_1 r^2, & r \in (-1,1), \\ 2\ln(2) - k_1, & r \in \{-1,1\}, \\ +\infty, & r \notin [-1,1], \end{cases} \tag{1.7}$$

$$F_{\rm obs}(r) = \begin{cases} k_2(1-r^2), & r \in [-1,1], \\ +\infty, & r \notin [-1,1], \end{cases} \tag{1.8}$$

where $k_1 > 1$ and $k_2 > 0$, so that both $F_{\rm log}$ and $F_{\rm obs}$ are nonconvex. All these potentials can be written as the sum of a convex, lower-semicontinuous function (the main part) and a concave quadratic perturbation. Note that $F_{\rm log}$ is particularly relevant in applications, since $F'_{\rm log}(r)$ becomes unbounded as $r \to \pm 1$. Moreover, in the case of $F_{\rm obs}$, the second equation (1.2) must be interpreted as a differential inclusion, where the derivative of the convex part of $F_{\rm obs}(\varphi)$ is understood in the sense of subdifferentials.

In the above model equations, there are two functions that may serve as distributed controls acting in the phase and nutrient equations, respectively. The control variable u_1, which is nonlinearly coupled to the state variable φ in the phase equation (1.1), models the application of a cytotoxic drug into the system; it is multiplied by a truncation function $\mathbb{h}(\cdot)$ in order to have the action only in the spatial region where the tumor cells are located. For instance, it can be assumed that $\mathbb{h}(-1)=0, \mathbb{h}(1)=1, \mathbb{h}(\varphi)$ is in between if $-1<\varphi<1$; see [29, 32, 35, 36] for some insights on possible choices of $\mathbb{h}$. On the other hand, the control u_2 can model the supply of antiangiogenic therapies aiming at reducing the tumor vascularization (cf. [12] and [7, 8] for similar control terms in models of prostate tumor growth).

Let us briefly recall the results already present in the literature on this class of models, which has been first introduced in [34] in case $\alpha=\tau=0$. As far as well-posedness is concerned, the above model has already been deeply investigated in the case $\alpha=\tau=\chi=0$ (cf. [5, 9–11, 25]). Moreover, many variants of this model were considered and similar results were proven, see, for instance, [16, 22, 29, 30, 32]. In fact, a large body of literature is devoted to diffuse-interface and Cahn–Hilliard-type models for tumor growth. Foundational modelling contributions are due to Cristini, Lowengrub, Wise and collaborators [19, 20, 47], of course enlightening the numerical and asymptotic investigations in [34, 35]. Rigorous analytical studies began with [25], later extended to multi-species mixtures and interactions in [21, 26, 27], to Brinkman-type or Darcy-type couplings in [1, 22], and to nonlocal and degenerate settings in [28]. Several works addressed chemotaxis, active transport, mechanical effects, or additional biological mechanisms, establishing well-posedness and qualitative properties for a range of tumour-growth systems [9, 29–33, 38]. Fractional and viscous variants of Cahn–Hilliard tumor-growth models, together with asymptotics, vanishing viscosities and vanishing relaxation limits, have been examined in [10, 11, 14, 15]. Additional results on long-time behaviour and stability can be found in [5, 48]. Many of these analyses rely on convexity and compactness tools.

The optimal control of tumor-growth phase-field systems constitutes another important research direction. Early works on boundary and distributed control appeared in [12, 31]. Control strategies incorporating chemotaxis, active transport, variable mobilities, and Keller–Segel dynamics were developed in [2, 13, 22, 36, 46]. Further contributions included the optimal control theory and advanced optimality conditions in [23, 24], as well as refined analyses of treatment-time optimization and related asymptotics in [41–43, 49]. Singular logarithmic and double-obstacle potentials were addressed in [39, 40], whereas sparse controls and second-order conditions were studied in [6, 45]. Well-posedness, regularity, and asymptotic behavior for models relevant to control applications and including chemotaxis were developed in [16]. These results collectively provide a rigorous framework for the design and optimization of therapeutic strategies governed by diffuse–interface tumor-growth models. The authors of this paper intend to undertake a detailed analysis of the distributed control problems associated with the two controls u_1 and u_2 in a subsequent work.

Concerning the hyperbolic relaxation of the chemical potential in the viscous Cahn–Hilliard equation (uncoupled from the nutrient and without mass sources), we refer to the recent contributions [17, 18], which inspired the present work. In [17], well-posedness, continuous dependence, and regularity results were established, along with an analysis of the asymptotic behavior as the relaxation parameter α tends to 0. A related optimal control problem was studied in [18].

We now briefly outline the contents of the present paper. In Section 2, we first prove the existence of a (weak) solution to the system (1.1)–(1.5), together with a continuous dependence result of the solutions on the controls u_1 and u_2: all this is precisely stated in Theorem 2.2. Section 3 is devoted to establishing regularity results, see Theorem 3.1, which lead to the existence of a strong solution to (1.1)–(1.5) in a very general framework for the potentials, covering all the cases in (1.6)–(1.8). Finally, Section 4 addresses the asymptotic limit as $\alpha \searrow 0$ in the particular — but still relevant — case when the proliferation function $P(\varphi)$ is constant, providing a detailed proof of convergence to the corresponding system with $\alpha = 0$ (see Theorem 4.1). Moreover, we are able to prove an estimate of the difference of solutions in suitable norms with a precise rate of convergence (cf. Theorem 4.3).

2 General setting and well-posedness

In this section, we introduce the general setting of our problem and state well-posedness results for the state system (1.1)–(1.5). To begin with, for a Banach space X we denote by $\|\cdot\|_X$ the norm in the space X or in a power thereof, and by X^* its dual space. The only exception from this rule applies to the norms of the L^p spaces and of their powers, which we often denote by $\|\cdot\|_p$, for $1 \le p \le \infty$. As usual, for Banach spaces X and Y that are contained in the same topological vector space, we introduce the linear space $X \cap Y$ which becomes a Banach space when endowed with its natural norm $\|u\|_{X\cap Y} := \|u\|_X + \|u\|_Y$, for $u \in X \cap Y$. Moreover, we introduce the spaces

$$H := L^2(\Omega)\,, \quad V := H^1(\Omega)\,, \quad W := \{v \in H^2(\Omega) : \ \partial_{\mathbf{n}} v = 0 \text{ on } \Gamma\}. \tag{2.1}$$

Furthermore, by $(\,\cdot\,,\,\cdot\,)$, $\|\,\cdot\,\|$, and $\langle\cdot,\cdot\rangle$, we denote the standard inner product and related norm in H, as well as the dual product between V and its dual V^*. We then have the dense and compact embeddings $V \subset H \subset V^*$, with the standard identification $\langle v, w\rangle = (v, w)$ for $v \in H$ and $w \in V$.

Throughout the paper, we make repeated use of Hölder's inequality, of the elementary Young's inequality

$$ab \le \delta|a|^2 + \frac{1}{4\delta}|b|^2 \quad \forall\, a, b \in \mathbb{R}, \quad \forall\, \delta > 0, \tag{2.2}$$

as well as of the continuity of the embeddings $H^1(\Omega) \subset L^p(\Omega)$ for $1 \le p \le 6$ and $H^2(\Omega) \subset C^0(\overline{\Omega})$. Notice that the latter embedding is also compact, while this holds true for the former embeddings only if $p < 6$. We also introduce for $s \in (0, T]$ and elements $w \in L^1(0, T; L^1(\Omega))$ the notation

$$(1 * w)(s) := \int_0^s w(\cdot, s')\, ds'\,, \tag{2.3}$$

that is, $(1*w)(s) \in L^1(\Omega)$ is the function that assigns to $x \in \Omega$ the value $\int_0^s w(x, s')\, ds'$. Obviously, it holds that

$$|(1 * w)(s)| \;\le\; (1 * |w|)(s)] \quad \text{for a.e. } s \in (0, T)\,. \tag{2.4}$$

Moreover, for $1 \le p,\, p' \le \infty$ conjugate exponents and functions $v \in L^1(0, T; L^p(\Omega))$ and $w \in L^1(0, T; L^{p'}(\Omega))$ we have that

$$\left| \int_\Omega v(s)(1 * w)(s) \right| \;\le\; \|v(s)\|_p\, \|w\|_{L^1(0,s;L^{p'}(\Omega))} \quad \text{for a.e. } s \in (0, T)\,. \tag{2.5}$$

Finally, let us introduce a convention that will be tacitly employed throughout the paper: the symbol C is used to indicate every constant that depends only on the structural data of the problem (such as T, Ω, α or τ, the shape of the nonlinearities, and the norms of the involved functions), so that its meaning may change from line to line. If a parameter δ enters the computation, then the symbol C_δ denotes constants that additionally depend on δ. On the contrary, precise constants that we refer to are denoted in a different way.

We now provide assumptions on the data of the problem.

(A1) α, τ and χ are positive constants.

(A2) $F = F_1 + F_2$ satisfies: $F_1 : \mathbb{R} \to [0, +\infty]$ is convex and lower semicontinuous with $F_1(0) = 0$, while $F_2 \in C^2(\mathbb{R})$ has a Lipschitz continuous derivative F_2'.

(A3) $P \in C^0(\mathbb{R})$ is nonnegative, bounded, and Lipschitz continuous.

(A4) $\mathbb{h} \in C^0(\mathbb{R})$ is nonnegative, bounded, and Lipschitz continuous.

Let us note that all of the potentials (1.6)–(1.8) are admitted. In fact, the assumption **(A2)** implies that the subdifferential ∂F_1 of F_1 is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with effective domain $D(\partial F_1) \subset D(F_1)$, and, since F_1 attains its minimum value 0 at 0, it turns out that $0 \in D(\partial F_1)$ and $0 \in \partial F_1(0)$. We also observe that the assumptions on F_2 imply that F_2 grows at most quadratically, that is, there are constants $\widehat{c}_1, \widehat{c}_2$ such that

$$|F_2(r)| \le \widehat{c}_1 + \widehat{c}_2 r^2 \quad \forall\, r \in \mathbb{R}. \tag{2.6}$$

Moreover, we introduce the following notation: for $r \in D(\partial F_1)$, we denote by $(\partial F_1)^\circ(r)$ the minimal section of $\partial F_1(r)$, that is, the element of $\partial F_1(r)$ having minimal modulus. Finally, we extend the notations F_1, ∂F_1, $D(\partial F_1)$, and $(\partial F_1)^\circ$ to the corresponding functionals and operators induced on L^2 spaces.

Now, in the general setting of **(A1)–(A4)**, we are able to provide a well-posedness result for the system (1.1)–(1.5). First, we introduce our notion of a solution to (1.1)–(1.5).

Definition 2.1. *A quadruple $(\mu, \varphi, \xi, \sigma)$ is called a solution to the initial-boundary value problem* (1.1)–(1.5) *if*

$$\mu \in H^2(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \tag{2.7}$$

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \cap C^0(\overline{Q}), \tag{2.8}$$

$$\sigma \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{2.9}$$

$$\xi \in L^\infty(0,T;H), \tag{2.10}$$

and if $(\mu, \varphi, \xi, \sigma)$ satisfies

$$\langle \alpha \partial_{tt}\mu, v\rangle + \int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega P(\varphi)(\sigma + \chi(1-\varphi) - \mu)v - \int_\Omega \mathbb{h}(\varphi)u_1 v$$
$$\text{for every } v \in V \text{ and a.e. in } (0,T), \tag{2.11}$$

$$\tau\partial_t\varphi - \Delta\varphi + \xi + F_2'(\varphi) = \mu + \chi\,\sigma, \quad \xi \in \partial F_1(\varphi), \ \text{ a.e. in } Q, \tag{2.12}$$

$$\partial_t\sigma - \Delta\sigma = -\chi\Delta\varphi - P(\varphi)(\sigma + \chi(1-\varphi) - \mu) + u_2 \ \text{ a.e. in } Q, \tag{2.13}$$

as well as

$$\mu(0) = \mu_0, \quad \partial_t\mu(0) = \mu_0', \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \text{a.e. in } \Omega. \tag{2.14}$$

It is worth noting that the homogeneous Neumann boundary conditions (1.4) are considered in the conditions (2.8) and (2.9) for φ and σ (cf. the definition of the space W) and incorporated in the variational equality (2.11) for μ, when using the form $\int_\Omega \nabla\mu\cdot\nabla v$. Notice also that the initial conditions (2.14) make sense, since (2.8) and (2.9) imply that $\varphi, \sigma \in C^0([0,T];V)$, while, owing to (2.7), it turns out that $\mu \in C^1([0,T];V^*) \cap C^0([0,T];H)$, and, consequently, $\partial_t\mu$ is at least weakly continuous from $[0,T]$ to H.

We have the following result.

Theorem 2.2 (Well-posedness)**.** *Assume that* **(A1)–(A4)** *hold and let the initial data satisfy*

$$\mu_0 \in V, \quad \mu_0' \in H, \quad \sigma_0 \in V, \quad \varphi_0 \in W \cap D(\partial F_1)^\circ$$
$$\text{with} \quad F_1(\varphi_0) \in L^1(\Omega), \quad (\partial F_1)^\circ(\varphi_0) \in H. \tag{2.15}$$

Moreover, suppose that

$$(u_1, u_2) \in L^2(Q) \times L^2(Q). \tag{2.16}$$

Then there exists at least one solution $(\mu, \varphi, \xi, \sigma)$ in the sense of Definition 2.1. Moreover, if

$$(u_1, u_2) \in L^2(0,T;L^3(\Omega)) \times L^2(Q) \tag{2.17}$$

in addition to (2.16)*, then the solution is unique. Furthermore, let* $(\mu_i,\varphi_i,\xi_i,\sigma_i)$*,* $i=1,2$*, be two solutions to* (1.1)–(1.5) *associated with the data* $(u_1^i,u_2^i)\in L^2(0,T;L^3(\Omega))\times L^2(Q)$*,* $i=1,2$*. Then there exists a constant* $K_1>0$*, which depends only on the data of the system, such that*

$$\begin{aligned}&\|\mu_1-\mu_2\|_{L^\infty(0,T;H)}+\|1*(\mu_1-\mu_2)\|_{L^\infty(0,T;V)}\\&\quad+\|\varphi_1-\varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}+\|\sigma_1-\sigma_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}\\&\leq K_1\Big(\|u_1^1-u_1^2\|_{L^2(0,T;H)}+\|u_2^1-u_2^2\|_{L^2(0,T;H)}\Big).\end{aligned}\tag{2.18}$$

Before entering the proof, let us remark that the above result is very general and includes also the cases of singular and nonsmooth potentials such as the double obstacle potential defined in (1.8). We also note that the assumption $F_1(\varphi_0)\in L^1(\Omega)$ stated in (2.15) actually follows from other requirements on φ_0 thanks to the subdifferential property

$$\int_\Omega F_1(\varphi_0)\leq\int_\Omega F_1(v)+((\partial F_1)^\circ(\varphi_0),\varphi_0-v)\quad\text{for every } v\in H.$$

About the explicit dependencies of the constant K_1 in (2.18), we invite the reader to follow the proof of the estimate given below.

Proof. The existence proof is rather standard, since similar arguments have already been used in previous contributions. Hence, for that part, we proceed rather formally, just employing the Yosida approximation of ∂F_1 for our estimates without recurring to finite-dimensional approximation techniques like the Faedo–Galerkin scheme. Hence, we introduce the Yosida regularization of ∂F_1. For $\varepsilon>0$, let $F_{1,\varepsilon}$ denote the Moreau–Yosida approximation of F_1 at the level ε. It is well known (see, e.g., [4]) that the following conditions are satisfied:

$$0\leq F_{1,\varepsilon}(r)\leq F_1(r)\quad\text{for all } r\in\mathbb{R}\,;\tag{2.19}$$

$$F_{1,\varepsilon}'\text{ is Lipschitz continuous on }\mathbb{R}$$
$$\text{with Lipschitz constant } 1/\varepsilon,\text{ and } F_{1,\varepsilon}'(0)=0\,;\tag{2.20}$$

$$|F_{1,\varepsilon}'(r)|\leq|(\partial F_1)^\circ(r)|\quad\text{and}\quad\lim_{\varepsilon\searrow 0}F_{1,\varepsilon}'(r)=(\partial F_1)^\circ(r)\quad\text{for all } r\in D(\partial F_1)\,.\tag{2.21}$$

We now study the $\varepsilon-$approximating problem that results from the system (2.11)–(2.14) if $(\partial F_1)(r)$ is replaced by $F_{1,\varepsilon}'$ and the inclusion in (2.12) reduces to an equality. Namely, we argue on

$$\tau\partial_t\varphi-\Delta\varphi+F_{1,\varepsilon}'(\varphi)+F_2'(\varphi)=\mu+\chi\,\sigma\quad\text{a.e. in } Q.\tag{2.22}$$

The existence of a solution to the $\varepsilon-$approximating problem thus obtained can be shown by means of a Faedo–Galerkin approximation using the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ and eigenfunctions $\{e_j\}_{j\in\mathbb{N}}$ of the eigenvalue problem $-\Delta e_j=\lambda_j e_j$ in Ω, $\partial_{\mathbf{n}}e_j=0$ on Γ. In order not to overload the exposition, we avoid here to write the Faedo–Galerkin system explicitly. Instead, we just provide the relevant a priori estimates by performing the estimations directly on the solution to the ε-approximating system. Notice that the following estimates, while being only formal for the ε-approximating system, are fully justified on the level of the Faedo–Galerkin approximations. For the sake of simplicity, we still denote by (μ,φ,ξ,σ), with $\xi=F_{1,\varepsilon}'(\varphi)$, the solution to the $\varepsilon-$approximating system in place of $(\mu_\varepsilon,\varphi_\varepsilon,\xi_\varepsilon,\sigma_\varepsilon)$; the correct notation will be reintroduced at the end of each estimate. Before entering the estimates, we note that it follows from (2.15) that $\varphi_0\in C^0(\overline{\Omega})$, and we conclude from the assumption **(A2)** that $F_2(\varphi_0)\in L^1(\Omega)$ and $F_2'(\varphi_0)\in H$.

FIRST ESTIMATE: Let $t \in (0,T]$ be arbitrary. First, we test (2.11) by $\partial_t\mu$ and integrate over $[0,t]$. Using **(A3)**, **(A4)**, (2.15) and Young's inequality, we then obtain that

$$\frac{\alpha}{2}\|\partial_t\mu(t)\|^2 + \int_{Q_t} \partial_t\varphi\partial_t\mu + \frac{1}{2}\int_\Omega |\nabla\mu(t)|^2$$
$$= \frac{\alpha}{2}\|\mu_0'\|^2 + \frac{1}{2}\int_\Omega |\nabla\mu_0|^2 + \int_{Q_t} \big[P(\varphi)(\sigma + (1-\chi)\varphi - \mu) - \mathbb{h}(\varphi)u_1\big]\partial_t\mu\,, \tag{2.23}$$

where the last integral can be estimated as follows:

$$\int_{Q_t} \big[P(\varphi)(\sigma + (1-\chi)\varphi - \mu) - \mathbb{h}(\varphi)u_1\big]\partial_t\mu$$
$$\le C\int_{Q_t} \big(|\sigma|^2 + |\varphi|^2 + |\mu|^2 + |u_1|^2 + |\partial_t\mu|^2\big)\,. \tag{2.24}$$

Next, we differentiate (2.22) with respect to t, test by $\partial_t\varphi$, and integrate over Q_t. We then find the identity

$$\frac{\tau}{2}\|\partial_t\varphi(t)\|^2 + \int_{Q_t} |\nabla\partial_t\varphi|^2 + \int_{Q_t} F_{1,\varepsilon}''(\varphi)|\partial_t\varphi|^2$$
$$= \frac{\tau}{2}\|\partial_t\varphi(0)\|^2 - \int_{Q_t} F_2''(\varphi)|\partial_t\varphi|^2 + \int_{Q_t} \partial_t\mu\partial_t\varphi + \chi\int_{Q_t} \partial_t\sigma\partial_t\varphi\,. \tag{2.25}$$

Now, writing (2.22) at $t=0$, we see that

$$\partial_t\varphi(0) = \tau^{-1}\big(\Delta\varphi_0 - F_{1,\varepsilon}'(\varphi_0) - F_2'(\varphi_0) + \mu_0 + \chi\sigma_0\big),$$

and we infer from (2.15), (2.21) and **(A2)** that $\|\partial_t\varphi(0)\|$ is uniformly bounded. Moreover, we have $F_{1,\varepsilon}''(\varphi) \ge 0$ so that the last term on the left-hand side is nonnegative, while $F_2''(\varphi)$ is bounded. Hence, applying Young's inequality to the fourth term on the right-hand side and then adding the inequalities (2.23) and (2.25), thus cancelling the terms involving $\partial_t\varphi\partial_t\mu$, we arrive at

$$\frac{\alpha}{2}\|\partial_t\mu(t)\|^2 + \frac{1}{2}\int_\Omega |\nabla\mu(t)|^2 + \frac{\tau}{2}\|\partial_t\varphi(t)\|^2 + \int_{Q_t} |\nabla\partial_t\varphi|^2$$
$$\le C + C\int_{Q_t} \big(|\mu|^2 + |\varphi|^2 + |\sigma|^2 + |\partial_t\mu|^2 + |\partial_t\varphi|^2\big) + \frac{1}{16}\int_{Q_t} |\partial_t\sigma|^2\,. \tag{2.26}$$

Next, we test (2.13) by $\partial_t\sigma$, integrate over $(0,t)$, and add to both sides the expression $\frac{1}{2}\|\sigma(t)\|^2 - \frac{1}{2}\|\sigma_0\|^2 = \int_{Q_t} \sigma\partial_t\sigma$. We then find that

$$\int_{Q_t} |\partial_t\sigma|^2 + \frac{1}{2}\|\sigma(t)\|_V^2 = \frac{1}{2}\|\sigma_0\|_V^2 + \chi\int_{Q_t} \nabla\varphi\cdot\nabla\partial_t\sigma$$
$$+ \int_{Q_t} \big[-P(\varphi)(\sigma + (1-\chi)\varphi - \mu) + \sigma + u_2\big]\partial_t\sigma\,.$$

Now observe that, integrating by parts in time and applying Young's inequality,

$$\chi\int_{Q_t} \nabla\varphi\cdot\nabla\partial_t\sigma = \chi\int_\Omega \nabla\varphi(t)\cdot\nabla\sigma(t) - \chi\int_\Omega \nabla\varphi_0\cdot\nabla\sigma_0 - \chi\int_{Q_t} \nabla\partial_t\varphi\cdot\nabla\sigma$$
$$\le C + \frac{1}{4}\int_\Omega |\nabla\sigma(t)|^2 + \chi^2\int_\Omega |\nabla\varphi(t)|^2 + \frac{1}{2}\int_{Q_t} |\nabla\partial_t\varphi|^2 + \frac{\chi^2}{2}\int_{Q_t} |\nabla\sigma|^2\,.$$

Hence, also applying Young's inequality to the last term on the right-hand side of the penultimate identity, we can infer that

$$\frac{1}{2}\int_{Q_t}|\partial_t\sigma|^2+\frac{1}{4}\|\sigma(t)\|_V^2\le C+\chi^2\int_\Omega|\nabla\varphi(t)|^2 \\ +\frac{1}{2}\int_{Q_t}|\nabla\partial_t\varphi|^2+C\int_{Q_t}\left(|\nabla\sigma|^2+|\sigma|^2+|\varphi|^2+|\mu|^2+|u_2|^2\right). \tag{2.27}$$

Finally, we test (2.22) by $\partial_t\varphi$, integrate over Q_t, and add to both sides the expression $\frac{1}{2}\|\varphi(t)\|^2-\frac{1}{2}\|\varphi_0\|^2=\int_{Q_t}\varphi\partial_t\varphi$. In view of **(A2)** and (2.19), and taking the quadratic growth of F_2 into account, we obtain that

$$\begin{aligned}&\tau\int_{Q_t}|\partial_t\varphi|^2+\frac{1}{2}\|\varphi(t)\|_V^2+\int_\Omega F_{1,\varepsilon}(\varphi(t))\\&=\frac{1}{2}\|\varphi_0\|_V^2+\int_\Omega\big(F_{1,\varepsilon}(\varphi_0)+F_2(\varphi_0)\big)-\int_\Omega F_2(\varphi(t))+\int_{Q_t}(\mu+\chi\sigma+\varphi)\partial_t\varphi\\&\le C+\int_\Omega F(\varphi_0)+C\left(1+\|\varphi_0\|^2+\int_{Q_t}\varphi\partial_t\varphi\right)+\int_{Q_t}(\mu+\chi\sigma+\varphi)\partial_t\varphi\\&\le C+C\int_{Q_t}\left(|\mu|^2+|\varphi|^2+|\sigma|^2+|\partial_t\varphi|^2\right),\end{aligned} \tag{2.28}$$

where we have used Young's inequality and (2.15) as well. Note moreover that the third term on first line of (2.28) is nonnegative (cf. (2.19)).

At this point, we multiply (2.28) by $4\chi^2$ and add the resulting inequality to the sum of the inequalities (2.26) and (2.27). We infer that

$$\begin{aligned}&\frac{\alpha}{2}\|\partial_t\mu(t)\|^2+\frac{1}{2}\int_\Omega|\nabla\mu(t)|^2+\frac{\tau}{2}\|\partial_t\varphi(t)\|^2+\frac{\chi^2}{2}\|\varphi(t)\|_V^2\\&\quad+4\tau\chi^2\int_{Q_t}|\partial_t\varphi|^2+\frac{1}{2}\int_{Q_t}|\nabla\partial_t\varphi|^2+\frac{1}{4}\int_{Q_t}|\partial_t\sigma|^2+\frac{1}{4}\|\sigma(t)\|_V^2\\&\le C+C\int_{Q_t}\left(|\mu|^2+|\varphi|^2+|\sigma|^2+|\nabla\sigma|^2+|\partial_t\mu|^2+|\partial_t\varphi|^2\right).\end{aligned} \tag{2.29}$$

Now, note that the term $C\int_{Q_t}|\mu|^2$ on the right-hand side can be estimated using the identity $\mu(s)=\mu_0+\int_0^s\partial_t\mu$, $s\in(0,t)$, so that $C\int_{Q_t}|\mu|^2\le C+C\int_{Q_t}|\partial_t\mu|^2$. It is then easily seen that the inequality thus obtained from (2.29) admits the application of Gronwall's lemma, and we finally can deduce that

$$\|\mu_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)}+\|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \\ +\|\sigma_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\le C. \tag{2.30}$$

SECOND ESTIMATE: Next, we multiply (2.22), written for a fixed $t\in(0,T]$, by $-\Delta\varphi(t)$ and integrate over Ω. This yields the identity

$$\|\Delta\varphi(t)\|^2+\int_\Omega F_{1,\varepsilon}''(\varphi(t))|\nabla\varphi(t)|^2=\int_\Omega(\tau\partial_t\varphi+F_2'(\varphi)-\chi\sigma-\mu)(t)\Delta\varphi(t)$$

and, using Young's inequality, the monotonicity of $F'_{1,\varepsilon}$, the bound in (2.30) which implies an $L^\infty(0,T;H)$-bound for $(\tau\partial_t\varphi + F'_2(\varphi) - \chi\sigma - \mu)$, and the elliptic regularity theory, we plainly deduce that

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;W)} \le C\,. \tag{2.31}$$

But then, by comparison in equation (2.22), we also realize that

$$\|F'_{1,\varepsilon}(\varphi_\varepsilon)\|_{L^\infty(0,T;H)} \le C\,, \tag{2.32}$$

while comparison in (2.13), along with (2.30) and elliptic regularity again, yields that

$$\|\sigma_\varepsilon\|_{L^2(0,T;W)} \le C\,. \tag{2.33}$$

Finally, it follows from a comparison of terms in (2.11) that

$$\|\mu_\varepsilon\|_{H^2(0,T;V^*)} \le C\,. \tag{2.34}$$

PASSAGE TO THE LIMIT AS $\varepsilon \searrow 0$: Now, let for every $\varepsilon > 0$ the triple $(\mu_\varepsilon, \varphi_\varepsilon, \sigma_\varepsilon)$ be a solution to the problem (2.11), (2.22), (2.13), (2.14) with the regularity (2.7)–(2.9). Observe that the constants C occurring in the proof of the estimates (2.30)–(2.34) are all independent of ε. Hence it follows from standard weak and weak-star compactness results that there are functions $\mu, \varphi, \sigma, \xi$ such that

$$\mu_\varepsilon \to \mu \quad \text{weakly star in } H^2(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \tag{2.35}$$
$$\varphi_\varepsilon \to \varphi \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.36}$$
$$\sigma_\varepsilon \to \sigma \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{2.37}$$
$$F'_{1,\varepsilon}(\varphi_\varepsilon) \to \xi \quad \text{weakly star in } L^\infty(0,T;H), \tag{2.38}$$

as $\varepsilon \searrow 0$, possibly along a subsequence. By virtue of (2.36) and the Aubin–Lions–Simon lemma (see, e.g., [44, Sect. 8, Cor. 4]), as W is compactly embedded into $C^0(\overline{\Omega})$, we deduce that

$$\varphi_\varepsilon \to \varphi \text{ strongly in } C^0(\overline{Q}),$$

whence, by Lipschitz continuity, also

$$P(\varphi_\varepsilon) \to P(\varphi), \quad \mathbb{h}(\varphi_\varepsilon) \to \mathbb{h}(\varphi), \quad F'_2(\varphi_\varepsilon) \to F'_2(\varphi), \quad \text{all strongly in } C(\overline{Q}).$$

On the other hand, by the same tool, we have that

$$\mu_\varepsilon \to \mu \quad \text{strongly in } C^1([0,T];V^*) \cap C^0([0,T];H),$$
$$\sigma_\varepsilon \to \sigma \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V),$$

as consequences of (2.35) and (2.37).

We then can pass to the limit in the respective variational equality (2.11) and equation (2.13), in particular, for the product term $P(\varphi_\varepsilon)(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon)$. This is also possible in (2.22) in order to obtain (2.12) in the limit: indeed, the inclusion in (2.12) results as a consequence of (2.38) and the maximal monotonicity of ∂F_1, since we can apply, e.g., [3, Prop. 2.2, p. 38]. Finally, the initial conditions (2.14) can be readily obtained by observing that we have at least strong convergence in $C^0([0,T];V^*)$ for all of the variables $\mu_\varepsilon, \partial_t\mu_\varepsilon, \varphi_\varepsilon, \sigma_\varepsilon$. With this, the existence part of the proof is complete.

UNIQUENESS AND CONTINUOUS DEPENDENCE: We now show the continuous dependence estimate (2.18), which implies the uniqueness of the solution, in particular. To this end, let $(u_1^i, u_2^i) \in L^\infty(Q) \times L^2(Q)$, $i = 1, 2$, be given, and let $(\mu_i, \varphi_i, \xi_i, \sigma_i)$, $i = 1, 2$, be associated solutions in the sense of Definition 2.1. We then introduce the abbreviating notation

$$\begin{aligned}&\mu := \mu_1 - \mu_2, \quad \varphi := \varphi_1 - \varphi_2, \quad \xi := \xi_1 - \xi_2, \quad \sigma := \sigma_1 - \sigma_2,\\ &u_1 := u_1^1 - u_1^2, \quad u_2 := u_2^1 - u_2^2.\end{aligned}$$

We then see that the differences $(\mu, \varphi, \xi, \sigma)$ satisfy the identities

$$\begin{aligned}&\langle \alpha \partial_{tt}\mu, v\rangle + \int_\Omega \partial_t\varphi v + \int_\Omega \nabla\mu\cdot\nabla v = \int_\Omega (P(\varphi_1) - P(\varphi_2))(\sigma_1 + (1-\chi)\varphi_1 - \mu_1)v\\ &\quad + \int_\Omega P(\varphi_2)(\sigma + (1-\chi)\varphi + \mu)v - \int_\Omega (\mathbb{h}(\varphi_1) - \mathbb{h}(\varphi_2))u_1^1 v - \int_\Omega \mathbb{h}(\varphi_2)u_1 v\\ &\quad \text{for every } v \in V \text{ and a.e. in } (0,T), \end{aligned} \tag{2.39}$$

$$\begin{aligned}&\tau\partial_t\varphi - \varphi + \xi + F_2'(\varphi_1) - F_2'(\varphi_2) = \mu + \chi\,\sigma,\\ &\quad \xi_i \in \partial F_1(\varphi_i),\ i = 1, 2, \text{ a.e. in } Q, \end{aligned} \tag{2.40}$$

$$\begin{aligned}\partial_t\sigma - \Delta\sigma = {}& -\chi\Delta\varphi - (P(\varphi_1) - P(\varphi_2))(\sigma_1 + (1-\chi)\varphi_1 - \mu_1)\\ & - P(\varphi_2)(\sigma + (1-\chi)\varphi - \mu) + u_2 \quad \text{a.e. in } Q, \end{aligned} \tag{2.41}$$

$$\mu(0) = 0, \quad \partial_t\mu(0) = 0, \quad \varphi(0) = 0, \quad \sigma(0) = 0, \quad \text{a.e. in } \Omega. \tag{2.42}$$

Now recall that $\varphi_i \in C^0(\overline{Q})$, $i = 1, 2$. Hence, there is some constant $L > 0$, which only depends on $R := \max\{\|\varphi_1\|_{C^0(\overline{Q})}, \|\varphi_2\|_{C^0(\overline{Q})}\}$, such that

$$|P(\varphi_1) - P(\varphi_2)| + |\mathbb{h}(\varphi_1) - \mathbb{h}(\varphi_2)| + |F_2'(\varphi_1) - F_2'(\varphi_2)| \le L|\varphi| \quad \text{a.e. in } Q. \tag{2.43}$$

Let $t \in (0, T]$ be arbitrary. We multiply (2.40) by φ and integrate over Q_t to obtain from (2.43) and Young's inequality that

$$\begin{aligned}&\frac{\tau}{2}\|\varphi(t)\|^2 + \int_{Q_t} |\nabla\varphi|^2 + \int_{Q_t} \xi\varphi\\ &= \int_{Q_t} \big(\chi\sigma - (F_2'(\varphi_1) - F_2'(\varphi_2))\big)\varphi + \int_{Q_t} \mu\varphi\\ &\le C\int_{Q_t} (|\sigma|^2 + |\varphi|^2) + \int_{Q_t} \mu\varphi\,. \end{aligned} \tag{2.44}$$

We observe that the last term on the left-hand side of (2.44) is nonnegative on account of the monotonicity of ∂F_1.

Next, let $M > 0$ denote a constant that will be specified later. We multiply (2.41) by $M\sigma$ and integrate over Q_t. Owing to the boundedness of P, and invoking (2.43), we then obtain the estimate

$$\begin{aligned}&\frac{M}{2}\|\sigma(t)\|^2 + M\int_{Q_t} |\nabla\sigma|^2\\ &\le M\chi\int_{Q_t} \nabla\varphi\cdot\nabla\sigma + ML\int_{Q_t} |\varphi|\,\big(|\mu_1| + |\varphi_1| + |\sigma_1|\big)|\sigma|\\ &\quad + MC\int_{Q_t} \big(|\mu|^2 + |\varphi|^2 + |\sigma|^2\big) + M\int_{Q_t} |u_2|^2\,, \end{aligned} \tag{2.45}$$

where we have applied Young's inequality to the last two terms on the right-hand side. Besides, we observe that

$$M\chi \int_{Q_t} \nabla\varphi \cdot \nabla\sigma \;\le\; \frac{M}{2}\int_{Q_t} |\nabla\sigma|^2 + \frac{M\chi^2}{2}\int_{Q_t} |\nabla\varphi|^2 . \tag{2.46}$$

Finally, we recall that, thanks to the continuity of the embedding $V \subset L^4(\Omega)$ and the regularity properties (2.7)–(2.9), the functions $\mu_1, \varphi_1, \sigma_1$ are all bounded in $L^\infty(0,T;L^4(\Omega))$. Therefore, we can infer that the second term on the right-hand side of (2.45), which we denote by I, can be estimated as follows:

$$\begin{aligned} |I| \;&\le\; MC\int_0^t \|\varphi(s)\|_4 \left(\|\mu_1(s)\|_4 + \|\varphi_1(s)\|_4 + \|\sigma_1(s)\|_4\right) \|\sigma(s)\|\, ds \\ &\le\; MC\int_0^t \|\varphi(s)\|_V \, \|\sigma(s)\|\, ds \\ &\le\; MC\int_{Q_t} |\varphi|^2 \;+\; \frac{M\chi^2}{2}\int_{Q_t} |\nabla\varphi|^2 \;+ MC\int_{Q_t} |\sigma|^2 . \end{aligned} \tag{2.47}$$

Hence, combining (2.45)–(2.47), we have shown the estimate

$$\begin{aligned} &\frac{M}{2}\|\sigma(t)\|^2 + \frac{M}{2}\int_{Q_t} |\nabla\sigma|^2 \\ &\le\; M\chi^2 \int_{Q_t} |\nabla\varphi|^2 + MC\int_{Q_t} \left(|\mu|^2 + |\varphi|^2 + |\sigma|^2 + |u_2|^2\right) . \end{aligned} \tag{2.48}$$

It remains to treat the identity (2.39) which we integrate with respect to time over $[0,s]$ for $s \in (0,t]$. Then we insert $v = \mu(s)$ and integrate over $[0,t]$, arriving at the identity

$$\begin{aligned} &\frac{\alpha}{2}\|\mu(t)\|^2 + \frac{1}{2}\int_\Omega |\nabla(1*\mu)(t)|^2 + \int_{Q_t} \mu\varphi \\ &= \int_{Q_t} \left[1 * \left((P(\varphi_1) - P(\varphi_2))(\sigma_1 + (1-\chi)\varphi_1 - \mu_1)\right)\right] \mu \\ &\quad + \int_{Q_t} \left[1 * \left(P(\varphi_2)(\sigma + (1-\chi)\varphi - \mu)\right)\right] \mu - \int_{Q_t} \left[1 * \left((\mathbb{h}(\varphi_1) - \mathbb{h}(\varphi_2))\, u_1^1\right)\right] \mu \\ &\quad - \int_{Q_t} \left[1 * \left(\mathbb{h}(\varphi_2) u_1\right)\right] \mu \;\; =: I_1 + I_2 + I_3 + I_4 , \end{aligned} \tag{2.49}$$

with obvious meaning, where we have used the notation introduced in (2.3). We estimate the terms on the right-hand side of (2.49) individually, where we make repeated use of the Hölder and Young inequalities, the estimates (2.4) and (2.5) with $p = 2$, and the continuous embedding $V \subset L^q(\Omega)$ for $1 \le q \le 6$. At first, by the Hölder and Young inequalites, as $u_1^1 \in L^2(0,T;L^3(\Omega))$ it is readily seen that

$$\begin{aligned} I_3 \;&\le\; L\int_0^t \|\mu(s)\| \, \|\, |\varphi|\, |u_1^1|\, \|_{L^1(0,s;H)}\, ds \\ &\le\; L\int_0^t \|\mu(s)\| \int_0^s \|u_1^1(s')\|_3 \|\varphi(s')\|_6 \, ds'\, ds \\ &\le\; C\|u_1^1\|_{L^2(0,T;L^3(\Omega))} \int_0^t \|\mu(s)\|\, \|\varphi\|_{L^2(0,s;V)}\, ds \\ &\le\; \frac{1}{4}\int_{Q_t} |\nabla\varphi|^2 + \frac{1}{4}\int_{Q_t} |\varphi|^2 + C\int_0^t \|\mu(s)\|^2\, ds . \end{aligned} \tag{2.50}$$

Similar reasoning, using the boundedness of both P and $\mathbb{h}$, yields

$$
\begin{aligned}
I_2 + I_4 \;&\leq\; C \int_0^t \|\mu(s)\| \, \| \, |\sigma| + |\varphi| + |\mu| + |u_1| \, \|_{L^1(0,s;H)} \, ds \\
&\leq\; C \int_0^t \|\mu(s)\| \, \| \, |\sigma| + |\varphi| + |\mu| + |u_1| \, \|_{L^2(0,t;H)} \, ds \\
&\leq\; C \int_{Q_t} \bigl(|\mu|^2 + |\sigma|^2 + |\varphi|^2 + |u_1|^2\bigr) \,. \qquad (2.51)
\end{aligned}
$$

Finally, using (2.5), (2.43), as well as Hölder's and Young's inequalities, and invoking the fact that $\sigma_1, \varphi_1, \mu_1 \in L^\infty(0,T;L^4(\Omega))$, we find the chain of inequalities

$$
\begin{aligned}
I_1 \;&\leq\; C \int_0^t \|\mu(s)\| \, \| \, |\varphi|(|\sigma_1| + |\varphi_1| + |\mu_1|) \|_{L^1(0,s;H)} \, ds \\
&\leq\; C \int_0^t \|\mu(s)\| \int_0^s \|(|\sigma_1| + |\varphi_1| + |\mu_1|)(s')\|_4 \|\varphi(s')\|_4 \, ds' \, ds \\
&\leq\; C \, \| \, |\mu_1| + |\varphi_1| + |\sigma_1| \, \|_{L^\infty(0,T;L^4(\Omega))} \int_0^t \|\mu(s)\| \, \|\varphi\|_{L^2(0,s;V)} \, ds \\
&\leq\; \frac{1}{4} \int_{Q_t} |\nabla\varphi|^2 + \frac{1}{4} \int_{Q_t} |\varphi|^2 + C \int_0^t \|\mu(s)\|^2 \, ds \,. \qquad (2.52)
\end{aligned}
$$

Combining the estimates (2.49)–(2.52), we thus have shown the estimate

$$
\begin{aligned}
&\frac{\alpha}{2} \|\mu(t)\|^2 + \frac{1}{2} \int_\Omega |\nabla(1 * \mu)(t)|^2 + \int_{Q_t} \mu\varphi \\
&\quad\leq\; \frac{1}{2} \int_{Q_t} |\nabla\varphi|^2 + C \int_{Q_t} \bigl(|\mu|^2 + |\sigma|^2 + |\varphi|^2 + |u_1|^2\bigr). \qquad (2.53)
\end{aligned}
$$

At this point, we add the inequalities (2.44), (2.48) and (2.53), obtaining the estimate

$$
\begin{aligned}
&\frac{\tau}{2} \|\varphi(t)\|^2 + \frac{M}{2} \|\sigma(t)\|^2 + \frac{\alpha}{2} \|\mu(t)\|^2 + \frac{1}{2} \int_\Omega |\nabla(1 * \mu)(t)|^2 \\
&\quad + \frac{1}{2}\bigl(1 - 2M\chi^2\bigr) \int_{Q_t} |\nabla\varphi|^2 + \frac{M}{2} \int_{Q_t} |\nabla\sigma|^2 \\
&\leq\; C_M \int_{Q_t} \bigl(|\mu|^2 + |\sigma|^2 + |\varphi|^2 + |u_1|^2 + |u_2|^2\bigr) \,. \qquad (2.54)
\end{aligned}
$$

Now, we make the choice $M := 1/\bigl(4\chi^2\bigr)$. Then the inequality (2.18) follows from an application of Gronwall's lemma. As a consequence, if $u_1^1 = u_2^1$ and $u_1^2 = u_2^2$, then $\mu_1 = \mu_2$, $\varphi_1 = \varphi_2$ and $\sigma_1 = \sigma_2$. But then, by (2.40), also $\xi_1 = \xi_2$. That is, the solution is unique. The assertion is thus completely proved. $\square$

3 Regularity properties

The next theorem provides a regularity result in the case of a general potential F satisfying **(A2)**, but under more regular initial data and sources (u_1, u_2) with respect to the well-posedness result in Theorem 2.2.

Theorem 3.1 (Regularity)**.** *Assume that* **(A1)**–**(A4)** *hold, and let the initial data satisfy* (2.15) *as well as the additional assumptions*

$$\mu_0 \in W, \quad \mu_0' \in V, \quad \sigma_0 \in L^\infty(\Omega), \quad (\partial F_1)^\circ(\varphi_0) \in L^\infty(\Omega)\,. \tag{3.1}$$

Moreover, suppose that

$$(u_1, u_2) \in L^2(0,T;V) \times L^\infty(0,T;H). \tag{3.2}$$

Then the solution $(\mu, \varphi, \xi, \sigma)$ *to* (1.1)–(1.5) *in the sense of Definition 2.1 enjoys the further regularity properties*

$$\sigma \in L^\infty(Q), \quad \mu \in H^1(0,T;V) \cap L^\infty(0,T;W), \quad \xi \in L^\infty(Q)\,. \tag{3.3}$$

Proof. First, observe that condition (3.2) implies (2.17) by Sobolev embeddings. Therefore, Theorem 2.2 already ensures the existence of a unique solution. Consequently, as in the proof of the existence result, we proceed in a formal manner, employing the Yosida approximation of ∂F_1 in our estimates, without resorting to finite-dimensional approximation techniques.

As in the previous proof, and for ease of notation, we continue to denote by $(\mu, \varphi, \xi, \sigma)$ the solution of the ε–approximating system, where $\xi = F_{1,\varepsilon}'(\varphi)$, throughout the computations below. We will revert to the notation $(\mu_\varepsilon, \varphi_\varepsilon, \xi_\varepsilon, \sigma_\varepsilon)$ at the end of each estimate.

FIRST ESTIMATE. From the boundedness properties in (2.30) and (2.31), along with (3.2), we infer that the right-hand side of (2.13) is uniformly bounded in $L^\infty(0,T;H)$. Since the initial datum σ_0 belongs to $L^\infty(\Omega)$ (cf. (3.1)), maximal parabolic regularity (see, e.g., [37, Chapter **III**, Theorem 7.1, p. 181]) yields

$$\|\sigma_\varepsilon\|_{L^\infty(Q)} \le C. \tag{3.4}$$

SECOND ESTIMATE. We test (2.11) by $-\Delta\mu_t$ and integrate over $(0,t)$. Integrating by parts in space, and using the assumptions **(A3)**, **(A4)**, (3.1), (3.2) along with the estimates (2.30), (2.31), we obtain from the Hölder and Young inequalities that

$$\begin{aligned}
&\frac{\alpha}{2}\|\nabla\partial_t\mu(t)\|^2 + \frac{1}{2}\|\Delta\mu(t)\|^2 \\
&= \frac{\alpha}{2}\|\mu_0'\|_V^2 + \frac{1}{2}\|\Delta\mu_0\|^2 - \int_{Q_t} \nabla\partial_t\varphi\cdot\nabla\partial_t\mu \\
&\quad + \int_{Q_t} \big[P'(\varphi)(\sigma + \chi(1-\varphi) + \mu) - u_1\mathbb{h}'(\varphi)\big]\nabla\varphi\cdot\nabla\partial_t\mu \\
&\quad + \int_{Q_t} P(\varphi)\nabla(\sigma - \chi\varphi + \mu)\cdot\nabla\partial_t\mu - \int_{Q_t} \mathbb{h}(\varphi)\nabla u_1\cdot\nabla\partial_t\mu \\
&\le C + C\int_{Q_t} |\nabla\partial_t\varphi|^2 \\
&\quad + C\int_0^t \|\big(|\sigma| + 1 + |\varphi| + |\mu| + |u_1|\big)(s)\|_3^2 \|\nabla\varphi(s)\|_6^2\,ds \\
&\quad + C\int_{Q_t} |\nabla(\sigma - \chi\varphi + \mu)|^2 + C\int_{Q_t} |\nabla u_1|^2 + \int_0^t \|\nabla\partial_t\mu\|^2\,.
\end{aligned}$$

Note that all terms on the right-hand side except the last are already bounded due to (2.30), (2.31), (3.2) and Sobolev embeddings. Then, applying now a standard Gronwall lemma, we derive the regularity estimate

$$\|\mu_\varepsilon\|_{H^1(0,T;V)\cap L^\infty(0,T;W)} \le C\,. \tag{3.5}$$

THIRD ESTIMATE. In view of (3.5), it turns out that μ is bounded in $L^\infty(0,T;W)$, hence in $L^\infty(Q)$. The same can be concluded for $F_2'(\varphi)$, due to (2.8) and **(A2)**. Then, let us rewrite the equation (2.22) as

$$\tau\partial_t\varphi - \Delta\varphi + F'_{1,\varepsilon}(\varphi) = h \quad \text{a.e. in } Q, \\ \text{with } h = \mu + \chi\,\sigma - F_2'(\varphi) \text{ bounded in } L^\infty(Q). \tag{3.6}$$

To prove the third property in (3.3), it is enough to derive a uniform $L^\infty(Q)$-bound for $F'_{1,\varepsilon}(\varphi)$. Let us outline the argument by proceeding formally and pointing out that just a truncation of the test functions would be needed for a rigorous proof. We take any $p > 2$ and test (3.6) by $|F'_{1,\varepsilon}(\varphi)|^{p-2}F'_{1,\varepsilon}(\varphi)$, a function of φ which is increasing and attains the value 0 at 0 (cf. (2.20)). Then, we integrate from 0 to $t \in (0,T]$, obtaining

$$\begin{aligned} &\tau\int_\Omega\Big(\int_0^{\varphi(t)} |F'_{1,\varepsilon}(s)|^{p-2}F'_{1,\varepsilon}(s)ds\Big) \\ &+ (p-1)\iint_{Q_t}|F'_{1,\varepsilon}(\varphi)|^{p-2}F''_{1,\varepsilon}(\varphi)|\nabla\varphi|^2 + \iint_{Q_t}|F'_{1,\varepsilon}(\varphi)|^p \\ &= \tau\int_\Omega\Big(\int_0^{\varphi_0}|F'_{1,\varepsilon}(s)|^{p-2}F'_{1,\varepsilon}(s)ds\Big) + \iint_{Q_t} h|F'_{1,\varepsilon}(\varphi)|^{p-2}F'_{1,\varepsilon}(\varphi)\,. \end{aligned} \tag{3.7}$$

Note that the first term on the left-hand side is nonnegative since $F'_{1,\varepsilon}$ is monotone increasing with $F'_{1,\varepsilon}(0) = 0$; moreover, the second term on the left-hand side is nonnegative since the derivative $F''_{1,\varepsilon}$ is nonnegative everywhere. About the right-hand side we may recall (2.21) and observe that

$$\tau\int_\Omega\Big(\int_0^{\varphi_0}|F'_{1,\varepsilon}(s)|^{p-2}F'_{1,\varepsilon}(s)ds\Big) \le \tau\|F'_{1,\varepsilon}(\varphi_0)\|_\infty^{p-1}\|\varphi_0\|_\infty|\Omega| \le C\|(\partial F_1)^\circ(\varphi_0)\|_\infty^{p-1}.$$

On the other hand, thanks to $p' = p/(p-1)$ and the Young inequality, we have that

$$\begin{aligned} &\iint_{Q_t} h|F'_{1,\varepsilon}(\varphi)|^{p-2}F'_{1,\varepsilon}(\varphi) \le \|h\|_{L^p(Q_t)}\,\|\,|F'_{1,\varepsilon}(\varphi)|^{p-1}\|_{L^{p'}(Q_t)} \\ &\quad = \|h\|_{L^p(Q_t)}\,\|F'_{1,\varepsilon}(\varphi)\|_{L^p(Q_t)}^{p/p'} \le \frac{1}{p}\,\|h\|_{L^p(Q_t)}^p + \frac{1}{p'}\,\|F'_{1,\varepsilon}(\varphi)\|_{L^p(Q_t)}^p\,. \end{aligned}$$

By rearranging from (3.7), and taking $t = T$, we infer that

$$\begin{aligned} \|F'_{1,\varepsilon}(\varphi)\|_{L^p(Q)} &\le \Big(p\,C\|(\partial F_1)^\circ(\varphi_0)\|_\infty^{p-1} + \|h\|_{L^p(Q)}^p\Big)^{1/p} \\ &\le \big(p\,C\|(\partial F_1)^\circ(\varphi_0)\|_\infty^{p-1}\big)^{1/p} + \|h\|_{L^p(Q)}. \end{aligned}$$

Then, passing to the limit as $p \to +\infty$ in the above chain of inequalities, we conclude that

$$\|F'_{1,\varepsilon}(\varphi)\|_{L^\infty(Q)} \le \|f_1'(\varphi_0)\|_\infty + \|h\|_{L^\infty(Q)}.$$

Hence, as $\xi_\varepsilon = F'_{1,\varepsilon}(\varphi_\varepsilon)$ (now using the notation with dependence on ε), we finally arrive at

$$\|\xi_\varepsilon\|_{L^\infty(Q)} \le C. \tag{3.8}$$

We now collect the estimates (3.4), (3.5), and (3.8), and recall the limiting procedure as $\varepsilon \searrow 0$, already carried out in the proof of Theorem 2.2. Since the uniform estimates are preserved by lower semicontinuity, the proof is thus concluded. □

Remark 3.2. Let us point out that the regularity $\xi \in L^\infty(Q)$ established in (3.3) implies, in particular, the so-called *separation property* in the case of the logarithmic potential (1.7). Indeed, since

$$\xi = \ln \frac{1+\varphi}{1-\varphi} \in L^\infty(Q),$$

there exist two real numbers r_* and r^*, depending on the structure of the system, such that

$$-1 < r_* \leq \varphi(x,t) \leq r^* < 1 \quad \text{for every } (x,t) \in \overline{Q}. \tag{3.9}$$

Hence, the boundedness of ξ prevents the phase variable φ from approaching the singular values ± 1, ensuring that φ remains within physically meaningful bounds throughout the evolution.

4 The case when P is constant

In this section, we restrict ourselves to the case where the proliferation function P reduces to a positive constant and investigate the asymptotic behavior of the problem as $\alpha \searrow 0$. Accordingly, we strengthen assumption **(A3)** by imposing

$$P \text{ is a positive constant.} \tag{4.1}$$

Besides, we allow the initial data for μ, $\partial_t \mu$, and σ to depend on α, while we keep φ_0, the initial value of φ, fixed. This choice is made for simplicity, in view of the restrictions on φ_0 stated in (2.15).

Thus, for $0 < \alpha \leq 1$, we consider families of data $\{\mu_{0,\alpha}, \mu'_{0,\alpha}, \sigma_{0,\alpha}\}$ such that

$$\mu_{0,\alpha} \text{ is uniformly bounded in } V, \tag{4.2}$$

$$\mu'_{0,\alpha} \text{ is uniformly bounded in } H, \tag{4.3}$$

$$\begin{aligned} &\sigma_{0,\alpha} \text{ is uniformly bounded in } V \\ &\qquad \text{and strongly converges to } \sigma_0 \text{ in } H \text{ as } \alpha \searrow 0. \end{aligned} \tag{4.4}$$

Of course, it follows from (4.4) that $\sigma_0 \in V$ (cf. (2.15)) and $\sigma_{0,\alpha} \to \sigma_0$ weakly in V. We can state the following convergence result.

Theorem 4.1. *Assume that* **(A1)–(A4)***,* (4.1) *hold, and let the initial data satisfy* (2.15) *and* (4.2)–(4.4)*. Moreover, let*

$$(u_1, u_2) \in L^2(Q) \times L^2(Q) \quad \textit{with} \quad u_1 \in H^1(0,T;H) \cap L^2(0,T;L^3(\Omega)), \tag{4.5}$$

so that also (2.17) *is satisfied. For all $\alpha \in (0,1]$, let the quadruple $(\mu_\alpha, \varphi_\alpha, \xi_\alpha, \sigma_\alpha)$, with*

$$\mu_\alpha \in W^{2,\infty}(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \tag{4.6}$$

$$\varphi_\alpha \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \cap C^0(\overline{Q}), \tag{4.7}$$

$$\xi_\alpha \in L^\infty(0,T;H), \tag{4.8}$$

$$\sigma_\alpha \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{4.9}$$

be the solution to the initial value problem

$$\langle \alpha\partial_{tt}\mu_\alpha, v\rangle + \int_\Omega \partial_t\varphi_\alpha\, v + \int_\Omega \nabla\mu_\alpha \cdot \nabla v$$
$$= \int_\Omega P\,(\sigma_\alpha + \chi(1-\varphi_\alpha) - \mu_\alpha)v - \int_\Omega \mathbb{h}(\varphi_\alpha)u_1 v$$
$$\text{for every } v \in V \text{ and a.e. in } (0,T), \tag{4.10}$$

$$\tau\partial_t\varphi_\alpha - \Delta\varphi_\alpha + \xi_\alpha + F_2'(\varphi_\alpha) = \mu_\alpha + \chi\,\sigma_\alpha, \quad \xi_\alpha \in \partial F_1(\varphi_\alpha), \text{ a.e. in } Q, \tag{4.11}$$

$$\partial_t\sigma_\alpha - \Delta\sigma_\alpha = -\chi\Delta\varphi_\alpha - P\,(\sigma_\alpha + \chi(1-\varphi_\alpha) - \mu_\alpha) + u_2 \quad \text{a.e. in } Q, \tag{4.12}$$

$$\mu_\alpha(0) = \mu_{0,\alpha}\,, \quad (\partial_t\mu_\alpha)(0) = \mu_{0,\alpha}'\,, \quad \varphi_\alpha(0) = \varphi_0, \quad \sigma(0) = \sigma_{0,\alpha} \quad \text{a.e. in } \Omega. \tag{4.13}$$

Then there exists a quadruple $(\mu, \varphi, \xi, \sigma)$ such that, for some subsequence α_k tending to 0, there holds

$$\mu_{\alpha_k} \to \mu \quad \text{weakly star in } L^\infty(0,T;V), \tag{4.14}$$
$$\alpha_k\mu_{\alpha_k} \to 0 \quad \text{weakly star in } W^{2,\infty}(0,T;V^*) \text{ and strongly in } W^{1,\infty}(0,T;H), \tag{4.15}$$
$$\varphi_{\alpha_k} \to \varphi \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)$$
$$\text{and strongly in } C^0([0,T];V) \cap C^0(\overline{Q}), \tag{4.16}$$
$$\xi_{\alpha_k} \to \xi \quad \text{weakly star in } L^\infty(0,T;H), \tag{4.17}$$
$$\sigma_{\alpha_k} \to \sigma \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W). \tag{4.18}$$

Moreover, $(\mu, \varphi, \xi, \sigma)$ is a solution to the viscous Cahn–Hilliard system

$$\int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega P\,(\sigma + \chi(1-\varphi) - \mu)v - \int_\Omega \mathbb{h}(\varphi)u_1 v$$
$$\text{for every } v \in V \text{ and a.e. in } (0,T), \tag{4.19}$$

$$\tau\partial_t\varphi - \Delta\varphi + \xi + F_2'(\varphi) = \mu + \chi\,\sigma, \quad \xi \in \partial F_1(\varphi), \text{ a.e. in } Q, \tag{4.20}$$
$$\partial_t\sigma - \Delta\sigma = -\chi\Delta\varphi - P\,(\sigma + \chi(1-\varphi) - \mu) + u_2 \quad \text{a.e. in } Q, \tag{4.21}$$
$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \text{a.e. in } \Omega. \tag{4.22}$$

Proof. We go back to the proof of Theorem 2.2 and consider the First Estimate. We focus, in particular, on the equality (2.23). Now, we change the treatment of the term coming from the right-hand side of (4.10). Please, let us use the notation without any index when doing the computation. Then, by integrating by parts in time, in place of (2.24) we obtain

$$\int_{Q_t} \bigl[P\,(\sigma + (1-\chi)\varphi - \mu) - \mathbb{h}(\varphi)u_1\bigr]\partial_t\mu$$
$$= -\frac{P}{2}\int_\Omega |\mu(t)|^2 + \frac{P}{2}\int_\Omega |\mu_{0,\alpha}|^2 + \int_\Omega [P\,(\sigma + (1-\chi)\varphi) - \mathbb{h}(\varphi)u_1](t)\,\mu(t)$$
$$- \int_\Omega \bigl[P\,(\sigma_{0,\alpha} + (1-\chi)\varphi_0) - \mathbb{h}(\varphi_0)u_1(0)\bigr]\mu_{0,\alpha}$$
$$- \int_{Q_t} [P\,(\partial_t\sigma - \chi\,\partial_t\varphi) - \mathbb{h}'(\varphi)\partial_t\varphi\, u_1 - \mathbb{h}(\varphi)\partial_t u_1]\mu,$$

whence, from **(A4)**, (4.1) and Hölder's inequality, it follows that

$$\begin{aligned}
&\int_{Q_t} \big[P\,(\sigma + (1-\chi)\varphi - \mu) - \mathbb{h}(\varphi)u_1\big]\partial_t\mu \\
&\leq -\frac{P}{2}\int_\Omega |\mu(t)|^2 + C + \frac{P}{4}\int_\Omega |\mu(t)|^2 + C\int_\Omega \big(|\sigma(t)|^2 + |\varphi(t)|^2\big) \\
&\quad + C\|u_1\|^2_{L^\infty(0,T;H)} + C\big(\|\sigma_{0,\alpha}\| + \|\varphi_0\| + \|u_1\|_{L^\infty(0,T;H)}\big)\|\mu_{0,\alpha}\| \\
&\quad + \frac{1}{16}\int_{Q_t} |\partial_t\sigma|^2 + C\int_{Q_t} \big(|\partial_t\varphi|^2 + |\mu|^2\big) \\
&\quad + C\int_0^t \|\partial_t\varphi(s)\|\,\|u_1(s)\|_3\,\|\mu(s)\|_6 ds + C\|\partial_t u_1\|^2_{L^2(0,T;H)} + C\int_{Q_t} |\mu|^2 .
\end{aligned}$$

Therefore, in view of (4.2)–(4.5), and using Sobolev's embeddings and Young's inequality, we have that

$$\begin{aligned}
&\int_{Q_t} \big[P\,(\sigma + (1-\chi)\varphi - \mu) - \mathbb{h}(\varphi)u_1\big]\partial_t\mu \\
&\leq -\frac{P}{4}\int_\Omega |\mu(t)|^2 + C + C\left(\|\sigma_{0\alpha}\|^2 + \int_{Q_t} \sigma\partial_t\sigma\right) + C\left(\|\varphi_0\|^2 + \int_{Q_t} \varphi\partial_t\varphi\right) \\
&\quad + \frac{1}{16}\int_{Q_t} |\partial_t\sigma|^2 + C\int_{Q_t} \big(|\partial_t\varphi|^2 + |\mu|^2\big) + C\int_0^t \|u_1(s)\|_3^2\,\|\mu(s)\|_V^2 ds,
\end{aligned}$$

and, consequently,

$$\begin{aligned}
&\int_{Q_t} \big[P\,(\sigma + (1-\chi)\varphi - \mu) - \mathbb{h}(\varphi)u_1\big]\partial_t\mu \\
&\leq -\frac{P}{4}\int_\Omega |\mu(t)|^2 + C + \frac{2}{16}\int_{Q_t} |\partial_t\sigma|^2 \\
&\quad + C\int_{Q_t} \big(|\sigma|^2 + |\varphi|^2 + |\partial_t\varphi|^2 + |\mu|^2\big) + C\int_0^t \|u_1(s)\|_3^2\,\|\mu(s)\|_V^2 ds . \qquad (4.23)
\end{aligned}$$

By virtue of (4.23), the inequality (2.26) now becomes

$$\begin{aligned}
&\frac{\alpha}{2}\|\partial_t\mu(t)\|^2 + \frac{P}{4}\int_\Omega |\mu(t)|^2 + \frac{1}{2}\int_\Omega |\nabla\mu(t)|^2 + \frac{\tau}{2}\|\partial_t\varphi(t)\|^2 + \int_{Q_t} |\nabla\partial_t\varphi|^2 \\
&\leq\, C + C\int_{Q_t} \big(|\mu|^2 + |\varphi|^2 + |\sigma|^2 + |\partial_t\varphi|^2\big) \\
&\quad + \frac{3}{16}\int_{Q_t} |\partial_t\sigma|^2 + C\int_0^t \|u_1(s)\|_3^2\,\|\mu(s)\|_V^2 ds , \qquad (4.24)
\end{aligned}$$

so that in this context the subsequent inequality (2.29) reads

$$\begin{aligned}
&\frac{\alpha}{2}\|\partial_t\mu(t)\|^2 + \min\left\{\frac{P}{4},\frac{1}{2}\right\}\|\mu(t)\|_V^2 + \frac{\tau}{2}\|\partial_t\varphi(t)\|^2 + \frac{\chi^2}{2}\|\varphi(t)\|_V^2 \\
&\quad + 4\tau\chi^2\int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2}\int_{Q_t} |\nabla\partial_t\varphi|^2 + \frac{1}{4}\int_{Q_t} |\partial_t\sigma|^2 + \frac{1}{4}\|\sigma(t)\|_V^2 \\
&\leq\, C + C\int_{Q_t} \big(|\mu|^2 + |\varphi|^2 + |\sigma|^2 + |\partial_t\varphi|^2\big) + C\int_0^t \|u_1(s)\|_3^2\,\|\mu(s)\|_V^2 ds , \qquad (4.25)
\end{aligned}$$

where all of the constants C in the above sequence of estimates are independent of α. Note that the function $s \mapsto \|u_1(s)\|_3^2$ belongs to $L^1(0,T)$ (see (4.5)), whence the application of the Gronwall lemma to (4.25) leads us to

$$\alpha^{1/2}\|\mu\|_{W^{1,\infty}(0,T;H)} + \|\mu\|_{L^\infty(0,T;V)} + \|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \\ + \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq C\,, \tag{4.26}$$

which replaces (2.30). Next, a closer inspection of the proof of Theorem 2.2, Second Estimate, reveals that the estimates (2.31)–(2.33) still hold, with constants independent of α, and that (2.34) can be replaced by

$$\alpha\|\mu\|_{W^{2,\infty}(0,T;V^*)} \leq C\,, \tag{4.27}$$

on account of the fact that (cf. (4.5)) the right-hand side of (4.10) is now bounded in $L^\infty(0,T;H)$. In conclusion, the boundedness properties (2.31)–(2.33), (4.26), (4.27) are also valid for the solution $(\mu_\alpha, \varphi_\alpha, \xi_\alpha, \sigma_\alpha)$ of (4.10)–(4.13). This solution is unique due to Theorem 2.2, since (4.5) implies (2.17).

Then, by a standard weak star compactness argument, we deduce the existence of a subsequence $\alpha_k \searrow 0$ and a quadruple $(\mu, \varphi, \xi, \sigma)$ such that (4.14)–(4.18) hold. At this point, we can perform the limit procedure as in the passage to the limit as $\varepsilon \searrow 0$ in the proof of Theorem 2.2. We just point out that, in order to obtain the inclusion in (4.20), we should instead use [3, Lemma 2.3, p. 38], since the subdifferential structure explicitly appears in (4.11) as well. This concludes the proof. $\square$

Remark 4.2. Note that Theorem 4.1 implicitly guarantees the existence of solutions to system (4.19)–(4.22) for all potentials F satisfying **(A2)**, and therefore for every convex and lower semicontinuous function $F_1 : \mathbb{R} \to [0, +\infty]$ with $F_1(0) = 0$. The obtained solution is in fact a strong solution. Indeed, using (4.14)–(4.18), treating (4.19) appropriately, and invoking elliptic regularity theory, we see that (4.19) can be rewritten as

$$\partial_t\varphi - \Delta\mu = P\,(\sigma + \chi(1-\varphi) - \mu) - h(\varphi)u_1 \qquad \text{a.e. in } Q, \tag{4.28}$$

supplemented with the boundary condition

$$\partial_{\mathbf{n}}\mu = 0 \qquad \text{a.e. on } \Sigma, \tag{4.29}$$

which yields the additional regularity $\mu \in L^\infty(0,T;W)$. Moreover, we remark that if $\sigma_0 \in V \cap L^\infty(\Omega)$ and $u_2 \in L^\infty(0,T;H)$, then, by arguing as in the First Estimate in the proof of Theorem 3.1, we obtain the regularity $\sigma \in L^\infty(Q)$.

At this stage, if $(\partial F_1)^\circ(\varphi_0) \in L^\infty(\Omega)$, we may also repeat the Third Estimate from the proof of Theorem 3.1, since the right-hand side h in (3.6) belongs to $L^\infty(Q)$. Hence, we deduce that $\xi \in L^\infty(Q)$, which in particular yields a separation property (cf. Remark 3.2) in the case of the logarithmic potential (1.7).

We now derive an error estimate for the differences $\varphi_\alpha - \varphi$ and $\sigma_\alpha - \sigma$, measured in suitable norms and quantified in terms of the parameter α.

Theorem 4.3. *Under the same assumptions as in Theorem 4.1, we let $(\mu_\alpha, \varphi_\alpha, \xi_\alpha, \sigma_\alpha)$ denote the solution to* (4.10)–(4.13)*, for $\alpha \in (0,1]$, and $(\mu, \varphi, \xi, \sigma)$ be the solution to* (4.19)–(4.22) *found by the asymptotic limit in* (4.14)–(4.18)*. Moreover, in addition to* (4.4) *we assume as well that*

$$\|\sigma_{0,\alpha} - \sigma_0\| \leq C_\sigma\, \alpha^{1/4} \quad \textit{for every } \alpha \in (0,1], \tag{4.30}$$

for some constant $C_\sigma > 0$. Then there is a constant $K_2 > 0$, which depends on the structure of the system but is independent of α, such that

$$\alpha^{1/2}\|\mu_a\|_{L^\infty(0,T;H)} + \|1 * (\mu_\alpha - \mu)\|_{L^\infty(0,T;V)} + \|\varphi_\alpha - \varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\sigma_\alpha - \sigma\|_{L^2(0,T;H)} + \|1 * (\sigma_\alpha - \sigma)\|_{L^\infty(0,T;V)} \le K_2\,\alpha^{1/4} \quad \textit{for every } \alpha \in (0,1]. \tag{4.31}$$

Proof. We proceed partly as in the proof of Theorem 2.2, specifically for the steps concerning Uniqueness and Continuous Dependence. Let us introduce the auxiliary elements $\rho_\alpha := \sigma_\alpha - \chi\varphi_\alpha,\ \rho := \sigma - \chi\varphi$ and set

$$\overline{\mu} := \mu_\alpha - \mu, \quad \overline{\varphi} := \varphi_\alpha - \varphi, \quad \overline{\xi} := \xi_\alpha - \xi,$$
$$\overline{\sigma} := \sigma_\alpha - \sigma, \quad \overline{\rho} := \rho_\alpha - \rho \quad \text{and} \quad \overline{\sigma}_0 := \sigma_{0,\alpha} - \sigma_0.$$

The plan is to subtract (4.19)–(4.21) from the corresponding relations (4.10)–(4.12). We begin by integrating in time the difference between (4.10) and (4.19). Using (4.13), we then deduce that

$$\int_\Omega \alpha\,\partial_t\mu_\alpha\,v + \int_\Omega \nabla(1*\overline{\mu})\cdot\nabla v + \int_\Omega P(1*\overline{\mu})v = \int_\Omega \alpha\,\mu_{0,\alpha}'\,v - \int_\Omega \overline{\varphi}\,v + \int_\Omega P(1*\overline{\rho})v - \int_\Omega [1*((\mathbb{h}(\varphi_\alpha) - \mathbb{h}(\varphi))u_1)]v \quad \text{for every } v \in V \text{ and a.e. in } (0,T). \tag{4.32}$$

From (4.11) and (4.20) it follows that

$$\tau\partial_t\overline{\varphi} - \Delta\overline{\varphi} + \overline{\xi} + F_2'(\varphi_\alpha) - F_2'(\varphi) = \overline{\mu} + \chi\,\overline{\rho} + \chi^2\,\overline{\varphi},$$
$$\xi_\alpha \in \partial F_1(\varphi_\alpha),\ \xi \in \partial F_1(\varphi), \ \text{a.e. in } Q. \tag{4.33}$$

We then perform another time integration on the difference of (4.12) and (4.21), using (4.13) together with (4.22). After adding $-\chi\,\overline{\varphi}$ to both sides, it is straightforward to arrive at

$$\overline{\rho} - \Delta(1*\overline{\rho}) + P(1*\overline{\rho}) = \overline{\sigma}_0 + P(1*\overline{\mu}) - \chi\,\overline{\varphi} \quad \text{a.e. in } Q. \tag{4.34}$$

We also note that (cf. (2.3))

$$(1*\overline{\mu})(0) = 0, \quad \overline{\varphi}(0) = 0, \quad (1*\overline{\rho})(0) = 0, \quad \text{a.e. in } \Omega. \tag{4.35}$$

Now we take $v = \overline{\mu}$ in (4.32) and test the equality in (4.33) by $\overline{\varphi}$. Summing the two relations, observing a cancellation, and integrating with respect to time, we obtain the inequality below. Since the product $\overline{\xi}\,\overline{\varphi}$ is nonnegative – by the inclusions in (4.33) and the monotonicity of ∂F_1 – we easily derive

$$\begin{aligned}
&\frac{\alpha}{2}\|\mu_\alpha(t)\|^2 + \frac{K}{2}\|(1*\overline{\mu})(t)\|_V^2 + \frac{\tau}{2}\|\overline{\varphi}(t)\|^2 + \int_{Q_t}|\nabla\overline{\varphi}|^2 \\
&\le \frac{\alpha}{2}\|\mu_{0,\alpha}\|^2 + \int_{Q_t}\alpha\,\partial_t\mu_\alpha\,\mu + \int_\Omega \alpha\,\mu_{0,\alpha}'(1*\overline{\mu})(t) \\
&\quad + \int_{Q_t} P(1*\overline{\rho})\,\overline{\mu} - \int_{Q_t}[1*((\mathbb{h}(\varphi_\alpha) - \mathbb{h}(\varphi))u_1)]\,\overline{\mu} \\
&\quad - \int_{Q_t}\big(F_2'(\varphi_\alpha) - F_2'(\varphi) - \chi^2\,\overline{\varphi}\big)\overline{\varphi} + \int_{Q_t}\chi\,\overline{\rho}\,\overline{\varphi}
\end{aligned} \tag{4.36}$$

for all $t \in (0,T]$, where $K := \min\{P,1\}$. Now, recalling the boundedness properties (4.2) and (4.3), the estimate (4.26) for $\alpha^{1/2}\|\partial_t\mu_\alpha\|_{L^\infty(0,T;H)}$, as well as the regularity $\mu \in L^\infty(0,T;H)$, we deduce from Young's inequality

$$\begin{aligned}
&\frac{\alpha}{2}\|\mu_{0,\alpha}\|^2 + \int_{Q_t} \alpha\,\partial_t\mu_\alpha\,\mu + \int_\Omega \alpha\,\mu'_{0,\alpha}(1*\overline{\mu})(t)\\
&\le C\alpha + C\alpha^{1/2} + \frac{K}{16}\int_\Omega |(1*\overline{\mu})(t)|^2 + C\alpha^2\|\mu'_{0,\alpha}\|^2\\
&\le C\alpha^{1/2} + \frac{K}{16}\int_\Omega |(1*\overline{\mu})(t)|^2. \qquad (4.37)
\end{aligned}$$

We integrate by parts in time the next two terms on the right side of (4.36). With the aid of the Lipschitz continuity of $\mathbb{h}$ (see (2.43)), the estimates (2.4) and (2.5) with $p=6$, and Hölder and Young's inequalities we obtain

$$\begin{aligned}
&\int_{Q_t} P(1*\overline{\rho})\,\overline{\mu} - \int_{Q_t}[1*((\mathbb{h}(\varphi_\alpha)-\mathbb{h}(\varphi))u_1)]\,\overline{\mu}\\
&= \int_\Omega P(1*\overline{\rho})(t)(1*\overline{\mu})(t) - \int_\Omega [1*((\mathbb{h}(\varphi_\alpha)-\mathbb{h}(\varphi))u_1)](t)(1*\overline{\mu})(t)\\
&\quad - \int_{Q_t} P\,\overline{\rho}\,(1*\overline{\mu}) + \int_{Q_t}(\mathbb{h}(\varphi_\alpha)-\mathbb{h}(\varphi))u_1(1*\overline{\mu})\\
&\le \frac{K}{16}\int_\Omega |(1*\overline{\mu})(t)|^2 + \frac{4P^2}{K}\int_\Omega |(1*\overline{\rho})(t)|^2 + L\|(1*\overline{\mu})(t)\|_6\int_0^t \|\overline{\varphi}(s)\|\|u_1(s)\|_3\,ds\\
&\quad + \frac{1}{4}\int_{Q_t}|\overline{\rho}|^2 + P^2\int_0^t \|(1*\overline{\mu})(s)\|^2\,ds + L\int_0^t \|\overline{\varphi}(s)\|\|u_1(s)\|_3\|(1*\overline{\mu})(s)\|_6\,ds.
\end{aligned}$$

Hence, thanks to the Sobolev embeddings and again to the Hölder and Young inequalities, we infer that

$$\begin{aligned}
&\int_{Q_t} P(1*\overline{\rho})\overline{\mu} - \int_{Q_t}[1*((\mathbb{h}(\varphi_\alpha)-\mathbb{h}(\varphi))u_1)]\,\overline{\mu}\\
&\le \frac{K}{8}\|(1*\overline{\mu})(t)\|_V^2 + \frac{4P^2}{K}\int_\Omega |(1*\overline{\rho})(t)|^2\,ds + \frac{1}{4}\int_{Q_t}|\overline{\rho}|^2\\
&\quad + C\int_0^t \|(1*\overline{\mu})(s)\|_V^2\,ds + C\int_0^t \|u_1(s)\|_3^2\,\|\overline{\varphi}(s)\|^2\,ds. \qquad (4.38)
\end{aligned}$$

As for the last two terms of (4.36), by virtue of the Lipschitz continuity of F_2' and the Young inequality, we have that

$$-\int_{Q_t}\big(F_2'(\varphi_\alpha) - F_2'(\varphi) - \chi^2\,\overline{\varphi}\big)\overline{\varphi} + \int_{Q_t}\chi\,\overline{\rho}\,\overline{\varphi} \le \frac{1}{4}\int_{Q_t}|\overline{\rho}|^2 + C\int_{Q_t}|\overline{\varphi}|^2. \qquad (4.39)$$

At this point, collecting the estimates (4.37)–(4.39), it follows from (4.36) that

$$\begin{aligned}
&\frac{\alpha}{2}\|\mu_\alpha(t)\|^2 + \frac{K}{4}\big\|(1*\overline{\mu})(t)\big\|_V^2 + \frac{\tau}{2}\|\overline{\varphi}(t)\|^2 + \int_{Q_t}|\nabla\overline{\varphi}|^2\\
&\le\ C\alpha^{1/2} + \frac{4P^2}{K}\int_\Omega |(1*\overline{\rho})(t)|^2\,ds + \frac{1}{2}\int_{Q_t}|\overline{\rho}|^2\\
&\quad + C\int_0^t \big\|(1*\overline{\mu})(s)\big\|_V^2\,ds + C\int_0^t \big(1+\|u_1(s)\|_3^2\big)\,\|\overline{\varphi}(s)\|^2\,ds \qquad (4.40)
\end{aligned}$$

for all $t \in (0,T]$.

Next, we test (4.34) by $M\overline{\rho}$, where the constant $M > 0$ will be specified below. Integrating with respect to time and applying Young's inequality, we obtain

$$M \int_{Q_t} |\overline{\rho}|^2 + \frac{MK}{2} \big\|(1*\overline{\rho})(t)\big\|_V^2 \leq \frac{M}{2} \int_{Q_t} |\overline{\sigma}_0 + P(1*\overline{\mu}) - \chi\,\overline{\varphi}|^2 + \frac{M}{2} \int_{Q_t} |\overline{\rho}|^2,$$

so that the assumption (4.30) allows us to deduce that

$$\frac{M}{2} \int_{Q_t} |\overline{\rho}|^2 + \frac{MK}{2} \big\|(1*\overline{\rho})(t)\big\|_V^2 \leq MC\alpha^{1/2} + MC \int_{Q_t} \big(|1*\overline{\mu}|^2 + |\overline{\varphi}|^2\big), \tag{4.41}$$

for all $t \in (0,T]$.

Before adding (4.40) and (4.41), we choose M such that

$$\frac{M}{2} > \frac{1}{2} \quad \text{and} \quad \frac{MK}{2} > \frac{4P^2}{K}, \qquad \text{i.e.,} \qquad M > \max\left\{1,\, \frac{8P^2}{K^2}\right\}.$$

With this choice, we may sum (4.40) and (4.41) and then apply the Gronwall lemma, using the fact that $s \mapsto \|u_1(s)\|_3^2$ belongs to $L^1(0,T)$ (cf. (4.5)). Therefore, we finally obtain the estimate

$$\begin{aligned} &\alpha^{1/2}\, \|\mu_\alpha\|_{L^\infty(0,T;H)} + \|1*\overline{\mu}\|_{L^\infty(0,T;V)} + \|\overline{\varphi}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\qquad + \|\overline{\rho}\|_{L^2(0,T;H)} + \|1*\overline{\rho}\|_{L^\infty(0,T;V)} \leq C\,\alpha^{1/4}, \end{aligned} \tag{4.42}$$

from which (4.31) follows immediately, recalling that $\overline{\rho} = \rho_\alpha - \rho = \sigma_\alpha - \sigma - \chi(\varphi_\alpha - \varphi)$. □

Remark 4.4. We point out that applying a similar procedure to a pair of arbitrary solutions to problems (4.19)–(4.22) would allow us to prove the uniqueness of the quadruple $(\mu, \varphi, \xi, \sigma)$, with

$$\mu \in L^\infty(0,T;V), \qquad \xi \in L^\infty(0,T;H), \tag{4.43}$$

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \cap C^0(\overline{Q}), \tag{4.44}$$

$$\sigma \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{4.45}$$

solving (4.19)–(4.22). Indeed, obtaining an estimate analogous to (4.42) would directly yield uniqueness for $1*\mu$, φ, and ρ (and consequently for σ), while the uniqueness of ξ then follows from a comparison in (4.20). Moreover, the uniqueness property implies that the convergences (4.14)–(4.18) stated in Theorem 4.1 hold not only along a subsequence $\alpha_k \to 0$, but for the entire family as $\alpha \to 0$.

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