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Optimal Sobolev regularity for second order divergence elliptic operators on domains with buried boundary parts

Elmar Schrohe[1], Joachim Rehberg[2]

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[1] Leibniz Universität Hannover
Institute of Analysis
Welfengarten 1
30167 Hannover
Germany
E-Mail: schrohe@math.uni-hannover.de

[2] Weierstrass Institute
Anton-Wilhelm-Amo-Str. 39
10117 Berlin
Germany
E-Mail: joachim.rehberg@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Anton-Wilhelm-Amo-Straße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

We study the regularity of solutions of elliptic second order boundary value problems on a bounded domain Ω in $\mathbb{R}^3$. The coefficients are not necessarily continuous and the boundary conditions may be mixed, i.e. Dirichlet on one part D of the boundary and Neumann on the complementing part. The peculiarity is that D is partly 'buried' in Ω in the sense that the topological interior of $\Omega \cup D$ properly contains Ω. The main result is that the singularity of the solution along the border of the buried contact behaves exactly as the singularity for the solution of a mixed boundary value problem along the border between the Dirichlet and the Neumann boundary part.

1 Introduction

The most disruptive force in semiconductor devices is heat [24], [11]. It leads to the segregation of chemical compounds and eventually to the destruction of the device. In the mathematical theory of semiconductor modeling there exists up to now only one thermodynamically consistent model that includes the electron/hole transport combined with heat transfer [1]. Unfortunately this system lacks parabolicity and therefore defies so far a rigorous mathematical analysis. Our aim in this work is considerably more modest. We study a physical quantity such as the temperature or the electrostatic potential, subject to a corresponding elliptic equation. To fix ideas, consider the *stationary* heat equation or Poisson's equation for the electrostatic potential – here as part of a semiconductor model, see [13].

The crucial feature is that the device contains a so called 'buried contact' B within a much less conducting material: Think of a film of silver that lies inside the device but extends to its boundary. Its thickness is assumed to be negligible compared to the other parameters of the device, so that it can be idealized as an interior *surface* with (possibly non-smooth) boundary. In the stationary heat problem one may think of the film to be heated to a certain temperature from 'outside' and in the semiconductor model that a certain voltage is applied to the 'outer part' of the film.

The question we are addressing here is the following: Given suitable data, which regularity can we expect near B for the solutions to the above equations?

For similar questions in semiconductor modeling, the threshold regularity is known to be $W^{1,3+\epsilon}$ as observed in the pioneering paper [13]. In [6] this was shown to lead to a satisfactory analysis of the van Roosbroeck system even for the case where surface charge densities and avalanche generation are taken into account. Here, in the analysis of an evolution equation, it is necessary to identify the domain of the elliptic operator exactly in order to treat the occurring nonlinearities suitably, see [23].

In mathematical terms, the problem is the following: Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$. We assume that there exists a subset $D \subset \partial\Omega$ such that Ω is a proper subset of the interior $\widehat{\Omega}$ of $\Omega \cup D$, i.e., $\widehat{\Omega} \setminus \Omega \neq \emptyset$.

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Consider an elliptic equation

$$\begin{aligned} -\nabla \cdot \mu \nabla u &= f \text{ in } \Omega \\ u &= 0 \text{ on } D \\ \nu \cdot \mu u &= 0 \text{ on } \partial\Omega \setminus D, \end{aligned}$$

where ν denotes the normal derivative.

We are interested in the regularity of the solution u near points in $\widehat{\Omega} \setminus \Omega$. Our main theorem, resting on results from [18], says that, under moderate assumptions on the geometry, the solution is again $W^{1,3+\epsilon}$ near these points. Our strategy is to localize the problem around the points under consideration and to reduce the localized problem by a C^1-transformation to one for which the resulting geometry fits into a class of model constellations treated in [18] by a symmetrization/antisymmetrization procedure. Interestingly, the symmetrized part of the solution appears exactly as a solution of a *mixed* boundary value problem. Therefore one can, on one hand, expect no better regularity than $W^{1,4-\epsilon}$ in view of Shamir's famous counterexample [25], but obtains, one the other hand, $W^{1,3+\epsilon}$ regularity for right hand sides in $W^{-1,3+\epsilon}$ for (possibly small) $\epsilon > 0$. So, generally speaking, our approach shows that the singularities of the solution at the border of the buried boundary part correspond inevitably to the singularities occurring for a mixed boundary value problem at the border between Dirichlet and Neumann boundary part.

We conjecture that our regularity result also is of use for the investigation of 'rigid inclusions' in mechanics as studied in [21].

2 Preliminaries and general assumptions

In the sequel, $\Omega \subseteq \mathbb{R}^3$ will denote a three dimensional domain, while Λ stands for an open set in $\mathbb{R}^d$. For $\mathrm{x} \in \mathbb{R}^d$, we denote by $B(\mathrm{x}; r)$ the ball of radius r around x. Moreover, $W^{1,q}(\Lambda)$ means the (complex) Sobolev space on Λ. Given a closed subset E of $\partial\Lambda$, we let $W^{1,q}_E(\Lambda)$ be the closure of

$$C^\infty_E(\Lambda) := \{ v|_\Lambda : v \in C^\infty_0(\mathbb{R}^d),\, \operatorname{supp} v \cap E = \emptyset \}$$

in $W^{1,q}(\Lambda)$. As usual, we write $W^{1,q}_0(\Lambda)$ instead of $W^{1,q}_{\partial\Lambda}(\Lambda)$. Finally, $W^{-1,q}_E(\Lambda)$ denotes the (anti)dual to $W^{1,q'}_E(\Lambda)$ with respect to an extension of the L^2 sesquilinear form, where $\frac{1}{q} + \frac{1}{q'} = 1$.

Definition 2.1. A *coefficient function* on an open subset Λ of $\mathbb{R}^d$ is a bounded, measurable function ρ on Λ, taking values in the set of *real, symmetric* $d \times d$ matrices. If ρ additionally satisfies the condition

$$\operatorname*{ess\,inf}_{\mathrm{x} \in \Lambda} \inf_{\|\xi\|_{\mathbb{R}^d} = 1} \rho(\mathrm{x})\xi \cdot \xi > 0, \tag{2.1}$$

then it will be called elliptic.

Given a coefficient function ρ on Λ, we define

$$-\nabla \cdot \rho \nabla : W^{1,2}_E(\Lambda) \to W^{-1,2}_E(\Lambda)$$

by

$$\langle -\nabla \cdot \rho \nabla v, w \rangle = \int_\Lambda \rho \nabla v \cdot \nabla \overline{w} \, d\mathrm{x}, \quad v, w \in W^{1,2}_E(\Lambda),$$

$\langle \cdot, \cdot \rangle$ denoting the sesquilinear pairing between $W^{1,2}_E(\Lambda)$ and $W^{-1,2}_E(\Lambda)$.

Remark 2.2. It is well-known that the part of this operator in $L^2(\Lambda)$ leads to a homogeneous Dirichlet condition (in the sense of traces) on E and a (generalized) homogeneous Neumann condition $\nu \cdot \rho\nabla\psi = 0$ (for ψ in the domain) on $N := \partial\Lambda \setminus E$, see [2, Section 1.2] or [12, Chapter II.2], compare also [3].

In the sequel, we will frequently identify a uniformly continuous function defined on a subset V of $\mathbb{R}^d$ with its (unique) uniformly continuous extension to the closure $\overline{V}$. By $\|\cdot\|_X$ we denote the norm in the Banach space X. Finally, the letter c denotes a generic constant, not always of the same value.

3 Notation, Preliminary Results, Model Constellations

3.1 Notation

We write variables in $\mathbb{R}^3$ in the form $\mathrm{x} = (x_1, x_2, x_3)$, $\mathrm{y} = (y_1, y_2, y_3)$, etc. and denote by $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3$ the unit vectors in the x, y and z direction, respectively. Moreover, we use the following notation:

1 $\iota : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the involutive map given by reflection in the first coordinate $\iota(x_1, x_2, x_3) = (-x_1, x_2, x_3)$.

2 $\mathfrak{H}_j^{\pm} \subset \mathbb{R}^3$, $j = 1, 2, 3$, are the half spaces $\{\mathrm{y} : \pm y_j > 0\}$.

3 $\mathfrak{C}$ stands for the cube $]-1, 1[^3$, and $\mathfrak{Q}$ for its lower half, i.e., $\mathfrak{Q} = \mathfrak{C} \cap \mathfrak{H}_3^-$. Moreover,

$$\mathfrak{C}_{\pm} = \mathfrak{C} \cap \mathfrak{H}_1^{\pm} \text{ and } \mathfrak{Q}_{\pm} = \mathfrak{Q} \cap \mathfrak{H}_1^{\pm}.$$

4 A bijective map $\phi : V \to W$ between two subsets V, W of $\mathbb{R}^d$ is called *bi-Lipschitz*, if there exist positive constants c_1 and c_2 such that

$$c_1|\mathrm{x} - \mathrm{y}| \le |\phi(\mathrm{x}) - \phi(\mathrm{y})| \le c_2|\mathrm{x} - \mathrm{y}|, \ \mathrm{x}, \mathrm{y} \in V. \tag{3.1}$$

It is not hard to see that ϕ then extends (uniquely) to a bi-Lipschitz map $\widehat{\phi} : \overline{V} \to \overline{W}$ between the closures of V and W, satisfying (3.1) with the same constants c_1 and c_2.

Recall that a Lipschitz function on a domain possesses in almost all points a classical (and hence a generalized) derivative, which is in norm not larger than the Lipschitz constant, cf. [10, Sect. 3.1.2].

5 We call a bijective map $\phi : V \to W$ between open sets $V, W \subset \mathbb{R}^3$ a C^1-diffeomorphism, if ϕ and ϕ^{-1} are continuously differentiable with *bounded* derivatives. In this terminology, a C^1-diffeomorphism is, in particular, a bi-Lipschitz mapping between V and W.

6 Following [19, Ch. 2.1], we call a closed set $E \subset \mathbb{R}^3$ a 2-set if there are constants $c_\bullet, c^\bullet$ such that

$$c_\bullet\, r^2 \le \mathcal{H}_2\big(E \cap B(\mathrm{x}, r)\big) \le c^\bullet\, r^2, \quad \mathrm{x} \in E,\ r \in\,]0, 1], \tag{3.2}$$

$\mathcal{H}_2$ being the two-dimensional Hausdorff measure on $\mathbb{R}^3$.

3.2 Localization

We start by recalling Gröger's localization principle, [16, Lemma 2], for elliptic second order operators in the $W^{1,q}$ scale. It shows that local regularity implies global regularity.

Theorem 3.1. *Let $\Lambda \subset \mathbb{R}^d$ be a domain and $D \subset \partial\Lambda$ a closed subset of the boundary. Suppose that ρ is an elliptic coefficient function. Let $U_1, \dots, U_n$ be an open covering of $\overline{\Lambda}$ and define $\Lambda_j = U_j \cap \Lambda$, $N_j = U_j \cap (\partial\Lambda \setminus D) \subset \partial\Lambda_j$, $D_j = \partial\Lambda_j \setminus N_j$.*
Let $q \geq 2$. Suppose that, for every $f_j \in W^{-1,q}_{D_j}(\Lambda_j)$ the solution $u_j \in W^{1,2}_{D_j}(\Lambda_j)$, of $-\nabla \cdot \rho|_{\Lambda_j} \nabla u_j = f_j$, in fact lies in $W^{1,q}_{D_j}(\Lambda_j)$.
Then the solution u of $-\nabla \cdot \rho \nabla u = f \in W^{-1,q}_D(\Lambda)$ belongs to $W^{1,q}_D(\Lambda)$.

The point is here the following: For $\eta_j \in C_0^\infty(U_j)$ and $u \in W^{1,2}_D(\Lambda)$ one has $\eta_j u \in W^{1,2}_{D_j}(\Lambda_j)$. Moreover, $\eta_j u$ fulfills an analogous equation $-\nabla \cdot \rho|_{\Lambda_j} \nabla(\eta_j u) + \eta_j u = f_j \in W^{-1,q}_{D_j}(\Lambda_j)$. So the essential ingredient in the proof of Theorem 3.1 is the knowledge that the functions $\eta_j u$ actually belong to $W^{1,q}_{D_j}(\Omega)$, see Lemma 4.2 below.

It is the aim of this paper to show this for a wide range of geometric constellations by a reduction to a few model situations and the application of bi-Lipschitz transformations (which are in most cases even C^1). In order to distinguish the model situation from the general one, we will denote in the model situation the domain by $\Lambda \subset \mathbb{R}^d$, its Dirichlet boundary by E and the coefficient function by ρ, while, in the general case, we will write $\Omega \subset \mathbb{R}^3$, D and μ for the corresponding items.

In this context, we recall the following transformation theorem:

Proposition 3.2. *Let $\Lambda, \Pi \subseteq \mathbb{R}^d$ be open and bounded with finitely many components and E be a closed subset of $\partial\Lambda$. Assume that $\phi : \Lambda \to \Pi$ is bi-Lipschitz and define $\widehat{\phi}(E) =: F$, $\widehat{\phi}$ being the bi-Lipschitz extension of ϕ to $\overline{\Lambda}$.*

i) ϕ induces a (consistent in $q \in [1, \infty[$) set of linear topological isomorphisms

$$\Psi_q : W^{1,q}_F(\Pi) \to W^{1,q}_E(\Lambda).$$

given by $(\Psi_q f)(\mathrm{x}) = f(\phi(\mathrm{x})) = (f \circ \phi)(\mathrm{x})$.

ii) Let ρ be a coefficient function on Λ. For every $f, g \in W^{1,2}(\Pi)$ the formula

$$\int_\Lambda \rho(\mathrm{x}) \nabla\big(f \circ \phi\big)(\mathrm{x}) \cdot \nabla\big(\overline{g} \circ \phi\big)(\mathrm{x}) \, d\mathrm{x} = \int_\Pi \omega(\mathrm{y}) \nabla f(\mathrm{y}) \cdot \nabla \overline{g}(\mathrm{y}) \, d\mathrm{y}, \tag{3.3}$$

holds with

$$\omega(\mathrm{y}) = (\mathcal{D}\phi)(\phi^{-1}(\mathrm{y}))\rho(\phi^{-1}(\mathrm{y}))\big(\mathcal{D}\phi\big)^T(\phi^{-1}(\mathrm{y})) \frac{1}{\big|\det(\mathcal{D}\phi)(\phi^{-1}\mathrm{y})\big|}, \tag{3.4}$$

where $\mathcal{D}\phi$ denotes the Jacobian of ϕ and $\det(\mathcal{D}\phi)$ the corresponding determinant.

iii) The following formula holds:

$$\big(\Psi_{q'}\big)^* \nabla \cdot \rho \nabla \Psi_q = \nabla \cdot \omega \nabla. \tag{3.5}$$

Finally, if $-\nabla \cdot \rho \nabla : W^{1,q}_E(\Lambda) \to W^{-1,q}_E(\Lambda)$ is a topological isomorphism, then $-\nabla \cdot \omega \nabla : W^{1,q}_F(\Pi) \to W^{-1,q}_F(\Pi)$ also is (and vice versa).

Proof. i) The assertion for $E = \emptyset$ is proved in [22, Section 1.1.7] in case Λ is connected. This carries over to open sets when considering the connected components separately. This shows that Ψ_q maps $W^{1,q}_F(\Pi)$ – as a subspace of $W^{1,q}(\Pi)$ – continuously into $W^{1,q}(\Lambda)$. It remains to prove that $\Psi_q f \in W^{1,q}_E(\Lambda)$ if $f \in W^{1,q}_F(\Pi)$. It suffices to show this for $f \in C^\infty_F(\Pi)$, because $W^{1,q}_E(\Lambda)$ is a *closed* subspace of $W^{1,q}(\Lambda)$. For such f, the function $\Psi_q f = f \circ \phi$ is Lipschitzian, and its support has positive distance to E. By classical results (cf. [10, Theorem 3.1]), the function $\Psi_q f$ has a Lipschitz continuous extension g to $\mathbb{R}^d$. One can easily achieve that the support of g also has positive distance to E. Using a mollifier argument, g is the limit in $W^{1,q}(\mathbb{R}^d)$ for a sequence (g_n) of functions from $C^\infty_0(\mathbb{R}^d)$ whose supports also have positive distance to E. Thus, $g_n|_\Lambda \in C^\infty_E(\Lambda)$ and $(g_n|_\Lambda)$ evidently converges to $g|_\Lambda = \Psi_q f$ in $W^{1,q}(\Lambda)$.

ii) The formulas (3.3) and (3.4) have been derived in [18, Proposition 16] using the the rules for the (weak) differentiation of the compositions $f \circ \phi$ and $g \circ \phi$, respectively, cf. [22, Section 1.1.7].
iii) Formula (3.5) is directly implied by (3.3). The last assertion follows from (i) by duality and (3.5). $\square$

From now on we call ω the *coefficient function obtained from ρ by means of the transformation ϕ*, or, in short, the *transformed coefficient function*.

3.3 The geometric setting

Assumption 3.3. In the sequel we fix a bounded domain $\Omega \subset \mathbb{R}^3$ and a closed part of its boundary, D, which has 3-dimensional Lebesgue measure 0. We let

$$N = \partial\Omega \setminus D.$$

Moreover, we fix an *elliptic* coefficient function μ on Ω, cf. Def. 2.1.

Definition 3.4. $D^{\|} \subset D$ is the set of all points $\mathrm{x} \in D$, for which $\Omega \cup D$ is a neighbourhood of x in $\mathbb{R}^3$; in other words: $D^{\|} = \mathrm{int}(\Omega \cup D) \setminus \Omega$.

Clearly, $D^{\|} \neq \emptyset$ if and only if Ω is a proper subset of $\widehat{\Omega} = \mathrm{int}(\Omega \cup D)$.

Lemma 3.5. *$D^{\|}$ is open in D and every point $\mathrm{x} \in D^{\|}$ has positive distance to N. In particular, $D^{\|} \cap \overline{N} = \emptyset$.*

Proof. For $\mathrm{x} \in D^{\|}$, there exists a ball $B(\mathrm{x}, r)$ which is contained in $\Omega \cup D$. This implies that $B(\mathrm{x}, r) \cap D \subset D^{\|}$; moreover it shows that $\mathrm{dist}(\mathrm{x}, N) \geq r$, since N is disjoint to $\Omega \cup D$. $\square$

It turns out that it makes sense to divide $D^{\|}$ into the following two subclasses:

Definition 3.6. $\mathrm{x} \in D^{\|}$ belongs to $D^{\|}_c$, if $B(\mathrm{x}, r) \cap \Omega$ is connected for $r > 0$ sufficiently small. We let $D^{\|}_d := D^{\|} \setminus D^{\|}_c$.

Before formulating the assumptions on the points from $D^{\|}$, it is our intention to point out already here a significant topological implication for $D^{\|}_d$.

Lemma 3.7. *Let $\Lambda_1, \ldots, \Lambda_m$ be mutually disjoint open sets in $\mathbb{R}^d$ and Λ their union. Then*

$$\partial\Lambda_j \subseteq \partial\Lambda \quad j \in \{1, \ldots, m\}. \tag{3.6}$$

Proof. Suppose this is false. Then there is a point $z \in \partial\Lambda_j$ which is an *inner* point of Λ. Hence, there exists an open ball $B \subset \mathbb{R}^d$ around z, which entirely lies in $\Lambda = \cup_l \Lambda_l$. Since the sets $\Lambda_l \cap B$ are open and disjoint and cover the connected set B, there exists a k such that $B \subseteq \Lambda_k$ and $B \cap \Lambda_l = \emptyset$ for $l \neq k$. But then z is not in $\partial\Lambda_j$ for any j, contradicting the assumption. □

Lemma 3.8. *Suppose $U \subset \Omega \cup D$ for an open set $U \subset \mathbb{R}^d$ and that $\Omega \cap U$ splits up into the components $\Omega_1, \dots, \Omega_m$. Then*

$$\partial\Omega_j \subset D \cup \partial U, \quad j \in \{1, \dots, m\} \tag{3.7}$$

and

$$\partial\Omega_j \cap U \subset D, \quad j \in \{1, \dots, m\}. \tag{3.8}$$

Proof. According to (3.6), one may write (see [4, Section 3.8])

$$\partial\Omega_j \subseteq \partial(\Omega \cap U) \subseteq (\partial\Omega \cap U) \cup (\partial\Omega \cap \partial U) \cup (\Omega \cap \partial U) \subseteq$$

$$\subseteq \big(\partial\Omega \cap (\Omega \cup D)\big) \cup \partial U = D \cup \partial U,$$

because Ω is open, i.e. $\Omega \cap \partial\Omega = \emptyset$.
(3.8) is obtained from (3.7) by intersecting with U and taking into account that U is open, i.e. $\partial U \cap U = \emptyset$.

□

Corollary 3.9. *Adopt the assumptions of Lemma 3.8. For any two functions $f \in W^{1,q}_D(\Omega), \eta \in C^\infty_0(U)$, the function $\eta f|_{\Omega_j}$ belongs to $W^{1,q}_0(\Omega_j)$.*

Proof. It is clear that it suffices to show this, by density, only for functions $f \in C^\infty_D(\Omega)$. (3.7) shows that the support of ηf stays away from $\partial\Omega_j$ because the support of η is by assumption away from ∂U, and that of f is away from D. □

Later on we will discuss the regularity for the solution of the elliptic equation just by considering functions $\eta f|_{\Omega_j}$ with $f \in W^{1,2}_D(\Omega), \eta \in C^\infty_0(U)$. The above considerations make clear that *a priori* these functions do belong to $W^{1,2}_0(\Omega_j)$, i.e. fulfill a pure Dirichlet condition. Exactly this motivates the subsequent assumption on the points in $D^{||}_d$.

Assumption 3.10. i) For $x \in D^{||}_d$ there is an open connected neighborhood $U_x \subset \Omega \cup D$ of x, such that $\Omega \cap U_x$ splits up into finitely many connected components $\Omega_1, \dots, \Omega_m$, each of them being a Lipschitz domain with the property that $x \in \cap_{j=1}^m \overline{\Omega}_j$.

ii) For μ as in Assumption 3.3, let $\mu_j = \mu|_{\Omega_j}$. Then

$$-\nabla \cdot \mu_j \nabla : W^{1,q}_0(\Omega_j) \to W^{-1,q}(\Omega_j), \quad j = 1, \dots, m, \tag{3.9}$$

is a topological isomorphism for some $q > 3$.

Obviously, the isomorphism property (3.9) is a highly implicit condition. We continue by considering several examples where it is known to hold. In fact, these examples will serve as the model problems later on.

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Proposition 3.11. *Let $\Lambda \subset \mathbb{R}^3$ be a bounded Lipschitz graph domain (cf. [15, Definition 1.2.1.1]; equivalently: Λ is a strong Lipschitz domain in the sense of [22, Section 1.1.9]; equivalently: Λ possesses the uniform cone property, cf. [15, Theorem 1.2.2.2]). Suppose the coefficient function ρ is elliptic, uniformly continuous and attains only values in* real, symmetric *matrices. Then there is a $p > 3$ such that*

$$-\nabla \cdot \rho\nabla : W_0^{1,q}(\Lambda) \to W^{-1,q}(\Lambda) \tag{3.10}$$

is a topological isomorphism for all $q \in [2, p[$, cf. [8, Theorem 3.12], compare also [20, Theorem 0.5] for the case of the Dirichlet problem for the Laplacian.

Proposition 3.12. *Let $\Lambda \subset \mathbb{R}^3$ be a bounded Lipschitz domain whose closure is a polyhedron. Let Π be any plane in $\mathbb{R}^3$ which intersects Λ, and assume that the angles between Π and all (parts of) adjacent boundary planes are not larger than π. Suppose, moreover, that the (elliptic) coefficient function ρ is constant on each of the components of $\Lambda \setminus \Pi$.*
Then there is a $p > 3$, such that (3.10) *is a topological isomorphism for all $q \in [2, p[$, cf. [7, Theorem 2.1].*

Corollary 3.13. *Let $\blacksquare \subset \mathbb{R}^2$ be any (open) rectangle and Λ be the cuboid $\blacksquare \times]-1, 0[$. Put $M := \blacksquare \times \{0\}$ and $E := \partial\Lambda \setminus M$. Let the elliptic coefficient function ρ be constant on Λ. Then there is a $p > 3$, such that*

$$-\nabla \cdot \rho\nabla : W_E^{1,q}(\Lambda) \to W_E^{-1,q}(\Lambda) \tag{3.11}$$

is a topological isomorphism for $q \in [2, p[$.

Proof. Reflection across the (x, y)-plane (compare [14, Proposition 4.13]) allows to transform the problem (3.11) to a pure Dirichlet problem which fits into Proposition 3.12. □

Assumption 3.14. For every point $\mathrm{x} \in D_c^{\|}$ there is an open neighborhood $U_\mathrm{x} \subset \Omega \cup D$ for which the following conditions are satisfied:

i) There is a C^1-diffeomorphism ϕ_x from U_x onto the cube $\mathfrak{C}$ (see Subsection 3.1), satisfying

$$\phi_\mathrm{x}(\Omega \cap U_\mathrm{x}) = \mathfrak{C} \setminus \Sigma, \quad \phi_\mathrm{x}(U_\mathrm{x} \cap D^{\|}) = \Sigma, \quad \phi_\mathrm{x}(\mathrm{x}) = 0, \tag{3.12}$$

where Σ is one of the sets $\Sigma_1, \Sigma_2, \Sigma_3$, below:

$$\Sigma_1 := \{\mathrm{y} : y_1 = 0, -1 < y_2 \leq 0, -1 < y_3 < 1\} \tag{3.13}$$

Σ_2 is the *closed* triangle with vertices (3.14)
$(0, 0, 0), (0, -1, -1), (0, -1, 1)$
minus the (open) leg between $(0, -1, -1)$and $(0, -1, 1)$.

$$\Sigma_3 := \big(\mathfrak{C} \cap \{\mathrm{z} : z_1 = 0\}\big) \setminus \mathrm{Int}(\Sigma_2). \tag{3.15}$$

ii) The limit $\lim_{\mathrm{z}\to\mathrm{x},\mathrm{z}\in\Omega} \mu(\mathrm{z}) =: \mu_\mathrm{x}$ exists.

The aim of this paper is to prove elliptic regularity not only around the points from $D^{\|}$, but also around the closure of $D^{\|}$. Therefore, we introduce the following

Definition 3.15. Consider the closure $\overline{D^{\|}}$ of $D^{\|}$ in $\partial\Omega$. In the sequel, we will denote

$$R = \overline{D^{\|}} \setminus D^{\|}.$$

Analogously to $D^{\|}$, the set R is divided into the subsets R_c and R_d, where R_c is the set of points x such that $\Omega \cap B$ is connected for any ball B around x with sufficiently small radius, and

$$R_d := R \setminus R_c.$$

Note that $R \subseteq D$, since D is a *closed* subset of $\partial\Omega$. But, in contrast to the points of $D^{\|}$, it may happen here that D and N *do touch* in points of R. Therefore one must be careful in formulating the analytical conditions on the points of R.

Assumption 3.16. i) For $\mathrm{x} \in R_d$ there is an open, connected neighbourhood U_x of x such that the set $\Omega \cap U_\mathrm{x}$ splits up into at most finitely many connected components $\Omega_1, \ldots, \Omega_m$, each of them being a Lipschitz domain. Moreover, for every pair of indices i, j one has $\mathrm{x} \in \overline{\Omega_j} \cap \overline{\Omega_i} \subseteq D$.

ii) Let $N_j = N \cap \overline{\Omega}_j$ and $D_j := \partial\Omega_j \setminus N_j$. Assume that each D_j is a 2-set, see (3.2). If μ is an elliptic coefficient function on Ω, and μ_j is the restriction to Ω_j, then

$$-\nabla \cdot \mu_j \nabla : W^{1,q}_{D_j}(\Omega_j) \to W^{-1,q}_{D_j}(\Omega_j), \qquad j = 1, \ldots, m,$$

is a topological isomorphism for some $q > 3$.

The following proposition shows model constellations where Assumption 3.16 is fulfilled that will be of interest in the sequel.

Proposition 3.17. *Let* $\blacktriangle \subset \mathbb{R}^2$ *be an open triangle with vertices* P_1, P_2, P_3*. For given real numbers* a, b *define* $\Lambda := \blacktriangle \times]a, b[$.
Let furthermore P *denote the midpoint of the open segment* $\overline{P_1P_2}$*, and let the (open) boundary part* Υ_2 *be*

$$\overline{P_1P_2} \times]a, b[\quad \text{or} \quad \overline{P_1P} \times]a, b[, \tag{3.16}$$

see Figures 1 and 2, and set $E = \partial\Lambda \setminus \Upsilon_2$*. Suppose* $\mathcal{H}$ *to be a plane within* $\mathbb{R}^3$ *that intersects* Λ*, but has a positive distance to the 'ground plate'* $\blacktriangle \times \{a\}$ *and the 'cover plate'* $\blacktriangle \times \{b\}$*. If the elliptic coefficient function* ρ *is constant on both components of* $\Lambda \setminus \mathcal{H}$*, then there is a* $p > 3$ *such that*

$$-\nabla \cdot \rho \nabla : W^{1,q}_E(\Lambda) \to W^{-1,q}_E(\Lambda)$$

is a topological isomorphism for all $q \in [2, p[$.

Proof. The results are proved in [18, Theorem 1 and Theorem 2]. □

Remark 3.18. Recall that the situation described in Proposition 3.17 may be carried over by bi-Lipschitz transformations $\phi : \Lambda \to \Xi$ with $F = \phi(E)$.
Of particular interest are here mappings which are piecewise C^1 since then the subdomains of continuity for the coefficient function are mapped onto subdomains of Ξ where the (transformed) coefficient function again is continuous.

In a next step, we will specify the analytical assumptions on the local geometry of Ω around points in R_c.

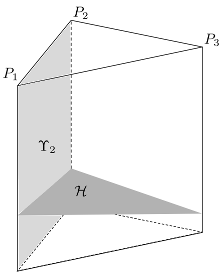

Figure 1: The model set for the first case in (3.16)

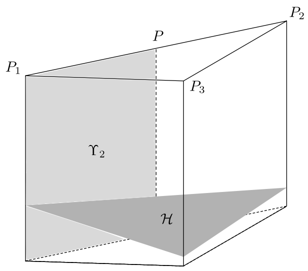

Figure 2: The model set for the second case in (3.16)

Assumption 3.19. For every point $\mathrm{x} \in R_c$, there is an open, connected neighborhood U_x, which satisfies the following conditions:

i) There is a C^1-diffeomorphism $\phi_\mathrm{x} : U_\mathrm{x} \to \mathfrak{C}$ such that

$$\phi_\mathrm{x}(U_\mathrm{x} \cap \Omega) = \mathfrak{Q} \setminus \Sigma_1, \quad \phi_\mathrm{x}(U_\mathrm{x} \cap R^{\|}) = \Sigma_1 \cap \mathfrak{H}_3^-, \quad \phi_\mathrm{x}(\mathrm{x}) = 0, \tag{3.17}$$

with Σ_1 defined in (3.13).

ii) Denoting the transformed (under ϕ_x) coefficient function on $\mathfrak{Q} \setminus \Sigma_1$ by $\underline{\mu}$, we require that both limits $\lim_{\mathrm{z} \in \mathfrak{Q}_\pm, \mathrm{z} \to 0} \underline{\mu}(\mathrm{z}) =: \mu_\pm$ exist and are related by the involution ι from Section 3.1:

$$\mu_- = \iota \mu_+ \iota. \tag{3.18}$$

iii) Unless $U_\mathrm{x} \cap N$ is empty we demand

$$\phi_\mathrm{x}(U_\mathrm{x} \cap N) = (\mathfrak{C} \cap \{\mathrm{z} : z_3 = 0\}) \setminus \Sigma_1,$$

(In this case the whole top surface - minus Σ_1 – is the (local) Neumann part of the boundary.) or

$$\phi_\mathrm{x}(U_\mathrm{x} \cap N) = (\mathfrak{C} \cap \{\mathrm{z} : z_3 = 0\}) \cap \mathfrak{H}_2^+.$$

(In this case half of the top surface is the (local) Neumann boundary part.)

Remark 3.20. Suppose $\mu_- = \iota \mu_+ \iota$ and define the coefficient function $\breve{\mu}$ on $\mathfrak{Q}$ by

$$\breve{\mu}(\mathrm{z}) = \begin{cases} \mu_+, & \text{if } \mathrm{z} \in \mathfrak{Q}_+ \\ \mu_-, & \text{if } \mathrm{z} \in \mathfrak{Q}_- \\ \mathrm{Id}_{\mathbb{R}^3}, & \text{on } \mathfrak{Q} \cap \{\mathrm{z} : z_1 = 0\} \end{cases}.$$

Then $\breve{\mu}$ satisfies the crucial relation

$$\breve{\mu}(\iota(\mathrm{z})) = \iota \breve{\mu}(\mathrm{z}) \iota, \quad \mathrm{z} \in \mathfrak{Q}_+ \cup \mathfrak{Q}_-. \tag{3.19}$$

The essential point is that the coefficient function, resulting from $\breve{\mu}$ by the transformation ι, remains $\breve{\mu}$, cf. Proposition 3.2 above.
Of course, the combination of the mapping properties of ϕ_{x} and condition (3.18) for the transformed matrix is indeed a restriction on the original constellation, see (i) in the concluding remarks.

Proposition 3.21. *Let $\Lambda \subset \mathbb{R}^d$ a bounded domain and $E \subset \partial\Lambda$ a 2-set of positive boundary measure. Suppose that there exists a linear, bounded extension operator $\mathfrak{E} : W^{1,q}_E(\Lambda) \to W^{1,q}_E(\mathbb{R}^d)$. Let ρ be an elliptic coefficient function. Then*

$$-\nabla \cdot \rho\nabla : W^{1,q}_E(\Lambda) \to W^{-1,q}_E(\Lambda) \tag{3.20}$$

is a topological isomorphism for $q = 2$. The set of q's, for which (3.20) *is a topological isomorphism, forms an* open *interval.*

Proof. The case $q = 2$ is implied by Lax-Milgram because the form

$$W^{1,2}_E(\Lambda) \ni u \mapsto \int_\Lambda \rho\nabla u \cdot \nabla u$$

is coercive due to the positive boundary measure of E. The second assertion follows from the interpolation properties of the scales $\{W^{1,q}_E(\Lambda)\}_{q\in]1,\infty[}$, $\{W^{-1,q}_E(\Lambda)\}_{q\in]1,\infty[}$, respectively, see [17, Theorem 5.6]). □

4 The Main Result

We are now in the position to formulate the main result of this paper:

Theorem 4.1. *Let the Assumptions* $3.10, 3.14, 3.16, 3.19$ *be satisfied, and suppose that $u \in W^{1,2}_D(\Omega)$ is the solution of*

$$-\nabla \cdot \mu\nabla u = f \in W^{-1,2}_D(\Omega).$$

Then, for every $\mathrm{x} \in \overline{D^{\|}}$, there is an open neighborhood W_{x} of x in $\mathbb{R}^3$ and a $p = p(\mathrm{x}) > 3$ such that for every $q \in [2, p[$ the following holds: For every $\eta \in C^\infty_0(W_{\mathrm{x}})$, the function ηu belongs to $W^{1,q}_0(\Omega)$, if $\mathrm{x} \in D^{\|}$, and belongs to $W^{1,q}_D(\Omega)$, if $\mathrm{x} \in \overline{D^{\|}} \setminus D^{\|}$, provided that $f \in W^{-1,q}_D(\Omega)$.

4.1 Localization principles

The next lemma provides the possibility of localizing the elliptic problem:

Lemma 4.2. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded Lipschitz domain, E a closed subset of the boundary, and $V \subset \mathbb{R}^d$ bounded and open. Let ρ be an elliptic coefficient function on Λ. Putting $M := \partial\Lambda \setminus E$, define $\Lambda_\bullet := \Lambda \cap V$, $M_\bullet := M \cap V$, $E_\bullet := \partial\Lambda_\bullet \setminus M_\bullet$. Fix an arbitrary Lipschitz function η with support in V. Then, for every $q \in [1, \infty[$, the following holds:*

i) If $v \in W^{1,q}_E(\Lambda)$, then $\eta v|_{\Lambda_\bullet} \in W^{1,q}_{E_\bullet}(\Lambda_\bullet)$. In particular, if $V \cap M = \emptyset$, then $\eta v \in W^{1,q}_0(\Lambda_\bullet)$.

ii) For any $w \in L^1(\Lambda_\bullet)$ denote by $\widetilde{w}$ the extension of w to Λ by 0.
Then the mapping

$$W^{1,q}_{E_\bullet}(\Lambda_\bullet) \ni v \mapsto \widetilde{\eta v}$$

has its image in $W^{1,q}_E(\Lambda)$ and is continuous.

iii) *Suppose that, for $q \in]1,3[$, there is the usual embedding $W^{1,q}_{E_\bullet}(\Lambda_\bullet) \hookrightarrow L^{\frac{3q}{3-q}}(\Lambda_\bullet)$. Let $v \in W^{1,2}_E(\Lambda)$ be the solution of*

$$-\nabla \cdot \rho \nabla v = f \in W^{-1,q}_E(\Lambda) \hookrightarrow W^{-1,2}_E(\Lambda),\ q \in [2,6].$$

Then $v_\bullet := (\eta v)|_{\Lambda_\bullet}$ satisfies an equation

$$-\nabla \cdot \rho_\bullet \nabla v_\bullet = f_\bullet \in W^{-1,q}_{E_\bullet}(\Lambda_\bullet)$$

with $\rho_\bullet = \rho|_{\Lambda_\bullet}$. Moreover, for $q \in [2,6]$, the mappping $W^{-1,q}_E(\Lambda) \ni f \mapsto f_\bullet \in W^{-1,q}_{E_\bullet}(\Lambda_\bullet)$ is linear and continuous.

Proof. See [5, Lemma 5.8 and 5.9]. □

Lemma 4.3. *Let $\Lambda \subset \mathbb{R}^3$ be a bounded domain and E be a closed part of its boundary. Suppose the validity of the embeddings $W^{1,q}_E(\Lambda) \hookrightarrow L^{\frac{3q}{3-q}}$ for $q \in [\frac{6}{5}, 2]$.*
Let Λ be the disjoint union $\Lambda = \Lambda_1 \cup \Lambda_2 \cup (\Lambda \cap M)$, where $\Lambda_{1,2} \subset \Lambda$ are open, and $M \subset \mathbb{R}^3$ is closed and Lebesgue negligible. Assume that a fixed $\mathrm{x} \in M$ is an accumulation point for both Λ_1 and Λ_2, and that $\lim_{\mathrm{y} \in \Lambda_j, \mathrm{y} \to \mathrm{x}} \rho(\mathrm{y}) =: \rho_j$, $j = 1,2$, exists. Define the coefficient function $\hat\rho$ by

$$\hat\rho(\mathrm{y}) = \begin{cases} \rho_j, & \mathrm{y} \in \Lambda_j,\ j = 1,2 \\ \mathrm{Id}_{\mathbb{R}^3}, & \mathrm{y} \in \Lambda \cap M, \end{cases}$$

and suppose that $\nabla \cdot \hat\rho \nabla : W^{1,q}_E(\Lambda) \to W^{-1,q}_E(\Lambda)$ is a topological isomorphism for a $q \in [2,6]$. Then the following holds:

i) *If V is any sufficiently small neighborhood of x in $\mathbb{R}^3$, and ρ is changed to*

$$\rho_V(\mathrm{y}) = \begin{cases} \hat\rho(\mathrm{y}) & \mathrm{y} \in \Lambda \setminus V \\ \rho(\mathrm{y}) & \mathrm{y} \in \Lambda \cap V. \end{cases}$$

then

$$\nabla \cdot \rho_V \nabla : W^{1,q}_E(\Lambda) \to W^{-1,q}_E(\Lambda) \tag{4.1}$$

remains a topological isomorphism for this same q.

ii) *For every sufficiently small neighbourhood V of x and any Lipschitz continuous function η with support in V, the function ηu belongs to $W^{1,q}_E(\Lambda)$, provided that $u \in W^{1,2}_E(\Lambda)$ satisfies $-\nabla \cdot \rho \nabla u = f \in W^{-1,q}_E(\Lambda)$.*

Proof. i) Given $\epsilon > 0$, the assumption on ρ implies that $\|\rho_V - \hat\rho\|_{L^\infty(\Lambda)} < \epsilon$ provided the neighbourhood V is sufficiently small. Taking into account the estimate

$$\|\nabla \cdot (\hat\rho - \rho_V)\nabla\|_{\mathcal{L}(W^{1,q}_E(\Lambda); W^{-1,q}_E(\Lambda))} \le \|\rho_V - \hat\rho\|_{L^\infty(\Lambda)} \tag{4.2}$$

and writing

$$-\nabla \cdot \rho_V \nabla = -\nabla \cdot \hat\rho \nabla + \nabla \cdot (\hat\rho - \rho_V)\nabla,$$

this shows that $-\nabla \cdot \rho_V \nabla$ is a perturbation of $-\nabla \cdot \hat\rho \nabla$, and the difference of both can be made arbitrarily small in norm by taking V small. Hence, the first assertion follows by (4.2) and classical perturbation theory.

ii) Let V be a neighborhood of x, such that (4.1) remains an isomorphism. Further, let W be an open neighborhood of $\overline{\Lambda} \cup \overline{V}$ and η be a Lipschitz continuous function on W with support in $V \subset W$. Assume now that $u \in W^{1,2}_E(\Lambda)$ satisfies $-\nabla \cdot \rho \nabla u = f \in W^{-1,q}_E(\Lambda)$. Applying Lemma 4.2(iii) one sees that the function $\eta|_\Lambda u \in W^{1,2}_E(\Lambda)$ satisfies an equation $-\nabla \cdot \rho \nabla(\eta|_\Lambda u) = f_\bullet \in W^{-1,q}_E(\Lambda)$. Clearly, $\eta|_\Lambda u$ vanishes identically on $\Lambda \setminus V$, and also $f_\bullet$ has no support in $\Lambda \setminus V$. Consequently, the function $\eta|_\Lambda u$ also satisfies the elliptic equation $-\nabla \cdot \rho_V \nabla(\eta|_\Lambda u) = f_\bullet$. Then the isomorphism property (4.1) implies the second assertion. □

Lemma 4.4. *Let $\Lambda_1, \dots, \Lambda_m$ be mutually disjoint bounded open sets in $\mathbb{R}^d$ and $\Lambda := \cup_{j=1}^m \Lambda_j$. Moreover, let $E \subseteq \partial\Lambda$ be closed and $E_j := \partial\Lambda_j \cap E$. Then*

i) E is the union of the sets E_j, $j = 1, \dots, m$.

ii) Suppose that $\overline{\Lambda_j} \cap \overline{\Lambda_i} \subseteq E$ for every pair of indices $i \neq j$. For fixed $j \in \{1, \dots, m\}$ and $\psi \in C^\infty_{E_j}(\Lambda_j)$, let $\Psi \in C^\infty_E(\mathbb{R}^d)$ be any function whose restriction to Λ_j equals ψ. Then $\operatorname{supp}\Psi \cap \overline{\Lambda}_j$ and $\operatorname{supp}\Psi \cap \left(\cup_{k\neq j}\overline{\Lambda_k}\right)$ have a positive distance to each other.

Proof. i) Since $\partial\Lambda \subseteq \cup_j \partial\Lambda_j$, (3.6) shows that $\partial\Lambda = \cup_j \partial\Lambda_j$ and hence the assertion. ii) Suppose that the claim is false. Since both sets, $\operatorname{supp}\Psi \cap \overline{\Lambda}_j$ and $\operatorname{supp}\Psi \cap \left(\cup_{k\neq j}\overline{\Lambda_k}\right)$ are compact, they must then possess a common point, say y. According to the assumption that $\overline{\Lambda_j} \cap \overline{\Lambda_i} \subseteq E$, y then must belong to E, and, due to the definition of E_j, also to this set. But $\operatorname{supp}\Psi$ has an empty intersection with E_j – which is a contradiction. □

Lemma 4.5. *Adopt the notation and assumptions of Lemma 4.4. For fixed $j \in \{1, \dots, m\}$ and $\psi \in C^\infty_{E_j}(\Lambda_j)$ define an extension $\hat{\psi}$ to Λ as follows: Let $\Psi \in C^\infty_{E_j}(\mathbb{R}^d)$ be any function whose restriction to Λ_j equals ψ. Take, according to Lemma 4.4, any function $\eta \in C^\infty_0(\mathbb{R}^d)$ which equals 1 on $\operatorname{supp}\Psi \cap \overline{\Lambda}_j$ and 0 on $\operatorname{supp}\Psi \cap \left(\cup_{k\neq j}\overline{\Lambda_k}\right)$. We then define $\hat{\psi}$ as the restriction of $\eta\Psi$ to Λ. Then:*

i) $\hat{\psi}$ is the extension to Λ by 0 and does neither depend on the function Ψ within the class $C^\infty_E(\mathbb{R}^d)$ nor on η.

ii) $\hat{\psi}$ belongs to $C^\infty_E(\Lambda)$.

iii) $\|\hat{\psi}\|_{W^{1,p}(\Lambda)} = \|\psi\|_{W^{1,p}(\Lambda_j)}$ for $p \in]1, \infty[$. Hence, the mapping

$$C^\infty_{E_j}(\Lambda_j) \ni \psi \mapsto \hat{\psi} \in C^\infty_E(\Lambda)$$

extends by density to an isometric mapping

$$W^{1,p}_{E_j}(\Lambda_j) \ni \psi \mapsto \hat{\psi} \in W^{1,p}_E(\Lambda).$$

Proof. The proof follows from Lemma 4.4. □

Let us, in the terminology of the preceding lemma, associate to every $f \in W^{-1,q}_E(\Lambda)$ elements $f_j \in W^{-1,q}_{E_j}(\Lambda_j)$ by defining

$$\langle f_j, \psi \rangle := \langle f, \hat{\psi} \rangle \quad \text{for} \quad \psi \in W^{1,q'}_{E_j}(\Lambda_j). \tag{4.3}$$

Obviously, for $f \in L^2(\Lambda) \hookrightarrow W^{-1,q}_E(\Lambda)$, $q \in [2, 6]$, f_j is just the restriction of f to Λ_j.

Lemma 4.6. *Adopt the notation and assumptions of Lemma 4.4.*

i) For $u \in W^{1,2}_E(\Lambda)$ the function $v := u|_{\Lambda_j}$ belongs to $W^{1,2}_{E_j}(\Lambda_j)$.

ii) If $u \in W^{1,2}_E(\Lambda)$ satisfies the equation

$$-\nabla \cdot \rho\nabla u = f \in W^{-1,q}_E(\Lambda), \quad q \geq 2$$

then $v := u|_{\Lambda_j}$ satisfies

$$-\nabla \cdot \rho|_{\Lambda_j}\nabla v = f_j \in W^{-1,q}_{E_j}(\Lambda_j).$$

Proof. i) is obvious. ii) The assertion clearly holds for $f \in L^2(\Lambda)$ and extends by density and continuity of all involved operations to all $f \in W^{-1,2}_E(\Lambda)$. □

4.2 Auxiliary results

In this section we are going to establish several results needed later on for the proof of Theorem 4.1. Since we have reason *not* to work with the model constellations established in Proposition 3.17, our first aim is to deduce from these regularity results for slightly modified model problems which are adequate for later purposes.
In order to make the reading easier, we produced several graphics which show the geometry under consideration including the corresponding boundary parts. Note that they only show the left halves of the model constellations (3.12) (with $\Sigma = \Sigma_1$) and (3.17), in order to make the 'buried' part of the Dirichlet boundary visible.

They are to be read as follows: Coordinate axes in $\mathbb{R}^3$ are chosen such that the x-axis points to the right, the y-axis backwards and the z-axis upwards. White surfaces always carry a Dirichlet boundary condition. The black surfaces stand for points from $D^{\|}$, and, consequently, also represents a Dirichlet surface. Also the crosshatched part is Dirichlet – resulting from antisymmetric reflection of the problem (see (4.14) below). The grey part denotes the Neumann part; that is also true for the dotted one. The latter results from symmetric reflection of the original (model) problem, see (4.15) below.

Lemma 4.7. *i) Define $M_- := (]-1,0[\times]0,1[\times\{0\}) \cup (\{0\}\times]0,1[\times]-1,0])$, $M_+ = \iota M_-$, and $E_\pm := \partial\mathfrak{Q}_\pm \setminus M_\pm$. Let $\rho_\pm$ be a* constant *coefficient function on $\mathfrak{Q}_\pm$.*
Then there is a $p > 3$, such that the mapping

$$-\nabla \cdot \rho_\pm\nabla : W^{1,q}_{E_\pm}(\mathfrak{Q}_\pm) \to W^{-1,q}_{E_\pm}(\mathfrak{Q}_\pm) \tag{4.4}$$

is a topological isomorphism for all $q \in [2, p[$.

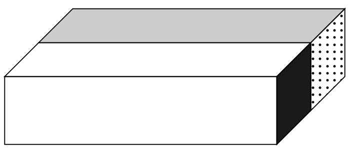

Figure 3: $\mathfrak{Q}_-$ for 4.7.i). The grey and the dotted part form M_- with Neumann b.c., the rest has Dirichlet b.c..

ii) Define $M = \{0\}\times]0,1[\times]-1,1[$*, and* $E_- = \partial\mathfrak{C}_- \setminus M$*. Let* ρ *be a coefficient function on* $\mathfrak{C}_-$ *which is constant on the two subsets* $\mathfrak{Q}_-$*, and* $\mathfrak{C}_- \setminus \mathfrak{Q}_-$*, respectively. Then there is a number* $p > 3$*, such that*

$$-\nabla \cdot \rho\nabla : W^{1,q}_{E_-}(\mathfrak{C}_-) \to W^{-1,q}_{E_-}(\mathfrak{C}_-)$$

is a topological isomorphism for all $q \in [2, p[$.

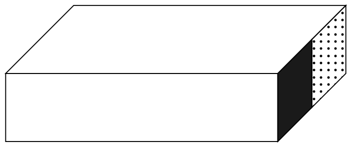

Figure 4: $\mathfrak{Q}_-$ for 4.7.ii). The dotted part is $\{0\} \times]0,1[\times]-1,0[$ with Neumann b.c., the rest has Dirichlet b.c..

iii) ii) *remains true when '*$-$*' is everywhere replaced by '*$+$*'.*

Proof. We prove i), restricting to the 'minus'-case. What we show first is the following:
$\clubsuit$ *There are open, convex sets* $W_0, W_1, \ldots, W_n$ *which form an open covering of* $\overline{\mathfrak{Q}_-}$ *and have the following additional properties:*
a) *each set* $W_j \cap \mathfrak{Q}_-$ *is a Lipschitz domain*
b) *Letting* $\mathfrak{Q}_j := W_j \cap \mathfrak{Q}_-$*,* $M_j := M_- \cap W_j$ *and* $E_j := \partial\mathfrak{Q}_j \setminus M_j$*, the operator*

$$-\nabla \cdot \rho\nabla : W^{1,3}_{E_j}(\mathfrak{Q}_j) \to W^{1,3}_{E_j}(\mathfrak{Q}_j)$$

is a topological isomorphism.

In detail, for every point $\mathrm{x} \in \partial\mathfrak{Q}_- \setminus \{0\}$ one can find a convex set U_x such that $U_\mathrm{x} \cap \mathfrak{Q}_-$ results from a set Λ in Proposition 3.11 or Proposition 3.17 by a Euclidean movement including the corresponding boundary parts. So here Proposition 3.2 applies. Thus, it remains to construct a neighbourhood W_0 of 0 which also satisfies a) and b).
Consider $\mathrm{z}_\bullet := (-1, 0, -\frac{1}{4})$, and take the plane Q which contains $\mathrm{z}_\bullet$ and the y-axis. We define a bi-Lipschitzian transformation $\mathfrak{l} : \mathbb{R}^3 \to \mathbb{R}^3$ as follows: $\mathfrak{l}$ leaves the points which lie on Q or below Q invariant. On the complementary half space $\mathfrak{l}$ acts as the linear mapping $\mathfrak{l}_+$ which is determined as follows: $\mathfrak{l}_+$ acts as the identity on Q and maps the vector $(-1, 0, 0)$ onto $(0, 0, 1)$.
Define $\blacklozenge \subset \mathbb{R}^3$ as the open square with vertices

$$(-1, 0, 0), \quad (0, 1, 0), \quad (1, 0, 0), \quad (0, -1, 0)$$

and $\blacktriangleleft = \blacklozenge \cap \mathfrak{H}_1^-$ as the open left half of this. Further, we put $\mathfrak{P}_\blacklozenge := \blacklozenge\times]-1,1[$ and $\mathfrak{P}_\blacktriangleleft := \blacktriangleleft \times]-1,1[$. Then, for sufficiently small $\alpha > 0$, one has

$$\alpha\mathfrak{P}_\blacklozenge \cap \mathfrak{l}(\mathfrak{Q}_-) = \alpha\mathfrak{P}_\blacktriangleleft, \quad \text{and} \quad \alpha\mathfrak{P}_\blacklozenge \cap \mathfrak{l}(M_-) = \{0\}\times]0,\alpha[\times]-\alpha,\alpha[.$$

Letting $W_0 := \mathfrak{l}^{-1}(\alpha\mathfrak{P}_\blacklozenge)$, one obtains for sufficiently small α

$$\mathfrak{l}(W_0 \cap \mathfrak{Q}_-) = \alpha\mathfrak{P}_\blacktriangleleft, \quad \text{and} \quad \mathfrak{l}(W_0 \cap M_-) = \{0\}\times]0,\alpha[\times]-\alpha,\alpha[.$$

Moreover, it is clear that Q neither intersects the ground plate nor the cover plate of $\alpha\mathfrak{P}_\blacktriangleleft$, and that the transformed coefficient function on $\alpha\mathfrak{P}_\blacktriangleleft$ is constant on both components of $\alpha\mathfrak{P}_\blacktriangleleft \setminus Q$. Thus, one is, concerning $\Lambda := \alpha\mathfrak{P}_\blacktriangleleft$ and $M := \{0\}\times]0,\alpha[\times]-\alpha,\alpha[$ again in the situation of Proposition 3.17, and, hence, an application of Proposition 3.2 shows that W_0 has the required properties. This proves ♣.

Then Proposition 3.1 implies that (4.4) is an isomorphism for $q = 3$. Moreover, $\mathfrak{Q}_-$ is a Lipschitz domain, and E_- evidently is a 2-set, see (3.2). Hence the set of numbers q, for which (4.4) is a topological isomorphism, is an *open interval* by Proposition 3.21 that contains 2 and 3.

ii) is proved along the same lines; this time all points from $\overline{\mathfrak{C}_\pm}$ are even covered by the model sets in Proposition 3.11 and Proposition 3.17. iii) is obtained from ii) by means of Proposition 3.2, there taking ϕ as ι. □

Corollary 4.8. *Assume that $\rho_\pm$ are* constant *coefficient functions on $\mathfrak{Q}_\pm$, respectively. Put*

$$M_- :=]-1,0[\times]0,1[\times \{0\} \tag{4.5}$$

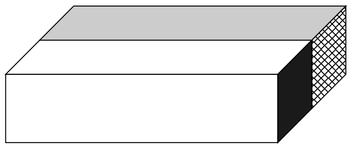

Figure 5: $\mathfrak{Q}_-$ with the grey Neumann surface M_-; the remaining surfaces carry Dirichlet b.c..

or

$$M_- :=]-1,0[\times]-1,1[\times\{0\} \cup \{0\}\times]0,1[\times]-1,0]. \tag{4.6}$$

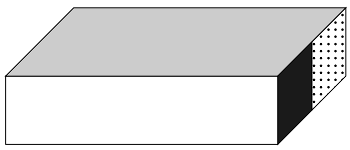

Figure 6: $\mathfrak{Q}_-$ with the Neumann surface M_- of (4.6), consisting of the grey and the dotted part. The remaining surfaces carry Dirichlet b.c..

$M_+ = \iota M_-$ *and* $E_\pm = \partial\mathfrak{Q}_\pm \setminus M_\pm$. *Then the same conclusion as in Lemma 4.7 i) holds.*

Proof. In case of (4.5) the problem is, modulo an affine mapping, the same as in Lemma 4.7 ii) with the roles of the grey and the dotted part exchanged.

In case of (4.6) one reflects (compare [14, Proposition 4.13]) the problem across the $x-y$-plane and again ends up with a problem as in Lemma 4.7 ii). □

Now we will establish the required auxiliary results for points in $D_c^{||}$: In Assumption 3.14 ii) it is only supposed that the corresponding limit exists – whatever this limit is. In the sequel we will modify the C^1 diffeomorphism $\phi_{\mathrm{x}} : U_{\mathrm{x}} \to \mathfrak{C}$ in 3.14 i) in such a manner that the resulting limit commutes with the linear mapping ι.

Lemma 4.9. *Let $\mathfrak{a}$ be a symmetric, positive definite 3×3 matrix. Then there is linear bijection $\mathfrak{b} : \mathbb{R}^3 \to \mathbb{R}^3$, mapping the $y-z$-plane onto itself, such that the matrix $\frac{1}{|\det \mathfrak{b}|}\mathfrak{b}\mathfrak{a}\mathfrak{b}^T$ is the identity.*

Proof. The assumption implies the existence of an orthogonal matrix $\mathfrak{o}$ such that $\mathfrak{o}\mathfrak{a}\mathfrak{o}^T = \mathfrak{diag}(a,b,c)$ with $a,b,c>0$. Next note that for $\mathfrak{s} = \mathfrak{diag}(\sqrt{bc},\sqrt{ac},\sqrt{ab})$

$$(abc)^{-1}\mathfrak{s}\mathfrak{o}\mathfrak{a}\mathfrak{o}^T\mathfrak{s}^T = \mathrm{Id}. \tag{4.7}$$

Moreover, $\det\,\mathfrak{s} = \det\,\mathfrak{a} = abc$. Let H be the image of $\mathrm{span}\{\mathrm{e}_2,\mathrm{e}_3\}$ under $\mathfrak{s}\mathfrak{o}$. Choose an orthonormal basis $\{\mathrm{h}_2,\mathrm{h}_3\}$ of H. Let $\mathfrak{v}$ be an orthogonal map in $\mathbb{R}^3$, taking h_j to e_j, $j=2,3$. $\mathfrak{b} = \mathfrak{v}\mathfrak{s}\mathfrak{o}$ maps $\mathrm{span}\{\mathrm{e}_2,\mathrm{e}_3\}$ onto itself and (4.7) implies

$$\frac{1}{|\det\mathfrak{b}|}\mathfrak{b}\mathfrak{a}\mathfrak{b}^T = (abc)^{-1}\mathfrak{v}\mathfrak{s}\mathfrak{o}\mathfrak{a}\mathfrak{o}^T\mathfrak{s}^T\mathfrak{v}^T = (abc)^{-1}\,\mathfrak{v}\,\mathrm{Id}\,\mathfrak{v}^T = \mathrm{Id},$$

since $\mathfrak{v}^T = \mathfrak{v}^{-1}$. □

Having this at hand, the next lemma allows to reproduce the geometric constellation in Assumption 3.14 in case $\Sigma = \Sigma_1$ *and* the additional property that the limit of the resulting coefficient function towards 0 is a scalar multiple of $\mathrm{Id}_{\mathbb{R}^3}$. Moreover, the cases $\Sigma=\Sigma_2$ and $\Sigma=\Sigma_3$ are reduced in some sense to $\Sigma=\Sigma_1$. Hereby, the resulting coefficient function has limits in $\mathfrak{Q}$ and $\mathfrak{C}\setminus\overline{\mathfrak{Q}}$ for $\mathrm{z}\to 0$ which are of a particularly simple form: They commute with ι.

Lemma 4.10. *i) Under Assumption 3.14, one may find, for* $\mathrm{x}\in D_c^{\|}$*, a neighborhood* $\widetilde{U} = \widetilde{U_{\mathrm{x}}}$*, and a* C^1*-diffeomorphism* $\widetilde{\phi}_{\mathrm{x}}$ *with*

$$\widetilde{\phi}_{\mathrm{x}}(\Omega\cap\widetilde{U}) = \mathfrak{C}\setminus\Sigma_1,\quad \widetilde{\phi}_{\mathrm{x}}(\widetilde{U}\cap D^{\|}) = \widetilde{\phi}_{\mathrm{x}}(\widetilde{U}\cap D_c^{\|}) = \Sigma_1\cap\mathfrak{C},\quad \widetilde{\phi}_{\mathrm{x}}(\mathrm{x}) = 0. \tag{4.8}$$

where Σ_1 *is the set defined in Assumption 3.14.*

ii) Let $\widetilde{\mu}$ *be the coefficient function obtained from* μ *under the transformation* $\widetilde{\phi}_{\mathrm{x}}$ *(cf. Proposition 3.2). In case* $\Sigma=\Sigma_1$ $\widetilde{\mu}$ *has a limit for* $z\to 0$ *in* $\mathfrak{C}\setminus\Sigma_1$ *which equals a scalar multiple of the identity matrix.*
In case of $\Sigma=\Sigma_2$ *or* $\Sigma=\Sigma_3$ *the limits*

$$\lim_{\mathrm{z}\in\mathfrak{Q}\setminus\Sigma_1,\,\mathrm{z}\to 0}\widetilde{\mu}(\mathrm{z}) \quad \textit{and} \quad \lim_{\mathrm{z}\in\mathfrak{C}\setminus\overline{\mathfrak{Q}},\,\mathrm{z}\to 0}\widetilde{\mu}(\mathrm{z})$$

exist and are of the form

$$\begin{pmatrix} \beta & 0 & 0 \\ 0 & a^{\pm}_{2,2} & a^{\pm}_{2,3} \\ 0 & a^{\pm}_{2,3} & a^{\pm}_{3,3}\end{pmatrix},\quad \beta>0. \tag{4.9}$$

Hence they commute with the matrix ι *defined in Section 3.1.*

Proof. Let $\underline{\mu}$ denote the coefficient function which is obtained from μ when transforming under ϕ_{x}, cf. (3.4). Since ϕ_{x} is C^1,

$$\lim_{\mathrm{z}\to 0,\mathrm{z}\in\phi_{\mathrm{x}}(U_{\mathrm{x}}\cap\Omega)}\underline{\mu}(\mathrm{z}) = \lim_{\mathrm{z}\to 0,\mathrm{z}\in\mathfrak{C}\setminus\Sigma}\underline{\mu}(\mathrm{z}) =: \mu_\star \tag{4.10}$$

exists. Denote by Σ one of the sets Σ_1, Σ_2 or Σ_3. By assumption $\phi_{\mathrm{x}}(\Omega\cap U_{\mathrm{x}}) = \mathfrak{C}\setminus\Sigma$. We then apply the transformation $\mathfrak{b}$ constructed in Lemma 4.9 to $\mathfrak{a}=\mu_\star$. We set

$$\widehat{\phi}_{\mathrm{x}} := \mathfrak{b}\phi_{\mathrm{x}}. \tag{4.11}$$

Denoting the resulting coefficient function by $\check{\mu}$, we obtain

$$\lim_{\mathrm{z}\to 0,\mathrm{z}\in\mathfrak{Q}\setminus\Sigma}\check{\mu}(\mathrm{z}) = \mathrm{Id}. \tag{4.12}$$

Now let us focus on the individual cases $\Sigma = \Sigma_1, \Sigma_2, \Sigma_3$. In case $\Sigma = \Sigma_1$ we are nearly finished:

Since the map $\mathfrak{b}$ defined in Lemma 4.9 preserves the $y-z$-plane, we can compose $\widehat{\phi}_{\mathrm{x}}$ in (4.11) with a rotation $\mathfrak{r}$ around the x-axis such that $\mathfrak{r}\mathfrak{b}(\mathrm{e}_3) \in \{\lambda e_3 : \lambda \neq 0\}$ and the set $\{\mathrm{y} : \mathrm{y} = (0, y_2, y_3), y_2 \leq 0\}$ is mapped under $\mathfrak{r}\mathfrak{b}$ onto itself. Evidently, the limit of the coefficient function, resulting under the mapping $\check{\phi}_{\mathrm{x}} := \mathfrak{r}\widehat{\phi}_{\mathrm{x}}$, remains the identity on $\mathbb{R}^3$.

Since 0 is an inner point of the transformed cube $\mathfrak{r}\mathfrak{b}\mathfrak{C}$, there is a number $\alpha \in]0,1[$, such that the cube $\alpha\mathfrak{C}$ is contained in $\mathfrak{r}\mathfrak{b}\mathfrak{C}$. One now shrinks the former neighborhood U_{x} to $\widetilde{U_{\mathrm{x}}} := (\check{\phi}_x)^{-1}\big(\alpha\mathfrak{C}\big)$. The mapping properties of $\mathfrak{r}\mathfrak{b}$ guarantee that the mapping $\widetilde{\phi}_{\mathrm{x}} := \frac{1}{\alpha}\check{\phi}_{\mathrm{x}}$ indeed satisfies (4.8). Moreover, we obtain the asserted form of the limit for the transformed coefficient function.

Let us now treat the cases $\Sigma = \Sigma_2$ and $\Sigma = \Sigma_3$: Evidently, $\mathfrak{b}(\Sigma_2)$ is a triangle $\mathcal{T}$ in the $y-z$ plane with one vertex in $0 \in \mathbb{R}^3$. One then performs a rotation $\mathfrak{r}$ of the $y-z$ plane around the x-axis such that

$$\mathfrak{H}_2^- \supset \mathfrak{r}\mathcal{T} \cap \mathfrak{H}_3^+ \neq \emptyset \neq \mathfrak{r}\mathcal{T} \cap \mathfrak{H}_3^- \subset \mathfrak{H}_2^-.$$

Obviously, $\mathfrak{r}\mathcal{T}$ remains a subset of the $y-z$-plane. It is clear that also after this new transformation the limit of the resulting coefficient function towards 0 remains the identity matrix. Let z_- be the unit vector along the side of $\mathfrak{r}\mathcal{T}$ which is adjacent to $0 \in \mathbb{R}^3$ and situated in $\mathfrak{H}_3^-$ and z_+ the unit vector along the other side of $\mathfrak{r}\mathcal{T}$ which is adjacent to $0 \in \mathbb{R}^3$. In this notation, we define now a bi-Lipschitzian mapping $\omega : \mathbb{R}^3 \to \mathbb{R}^3$ as follows: Within the half space $\overline{\mathfrak{H}_3^-}$ we set ω as the linear mapping which leaves the $x-y$ plane fixed and maps z_- onto $-\mathrm{e}_3$. On the half space $\overline{\mathfrak{H}_3^+}$ we define ω by also leaving the $x-y$ plane fixed and mapping z_+ onto e_3. This transforms $\mathfrak{r}\mathcal{T}$ into a triangle of which one side is the segment between the vertices $(0,0,-1)$ and $(0,0,1)$ and the third vertex lies in the $x-y$ plane. Now one is – locally around $0 \in \mathbb{R}^3$ – in the same situation as in case $\Sigma = \Sigma_1$ as far as the geometry is concerned. Observe that both transformations, $\mathfrak{r}$ and ω in fact only took place in the $y-z$-plane and left the x-direction invariant. Hence, it is clear that then a limit of the derived coefficient function in the half spaces $\mathfrak{H}_3^\pm$ exists and is of the form as postulated in (4.9). Shrinking the neighborhood of x as in the case $\Sigma = \Sigma_1$ and applying the homothety $\mathbb{R}^3 \ni \mathrm{z} \mapsto \frac{1}{\alpha}\mathrm{z}$, one has proved the assertion in case $\Sigma = \Sigma_2$.
If $\Sigma = \Sigma_3$, one proceeds the same way – beginning with the observation that $\mathfrak{b}\big((\mathfrak{C} \cap \{\mathrm{z} : z_1 = 0\}) \setminus \Sigma_3\big)$ also is a triangle in the $y-z$-plane. At the end one applies the mapping $\mathfrak{diag}(\frac{1}{\alpha}, -\frac{1}{\alpha}, \frac{1}{\alpha})$. □

4.3 The proof

Now we present the proof of Theorem 4.1. We start by considering the regularity near points in $D_d^{||}$ and R_d, since this is the easier situation. Treating points in $D_c^{||}$ and R_c requires the analysis of one more model situation, see Lemma 4.11 , below.

4.3.1 Regularity near points in $D_d^{||}$ and R_d

We first localize the problem around any point $\mathrm{x} \in D_d^{||} \cup R_d$, according to Lemma 4.2. Then Assumption 3.10 and Assumption 3.16 allow to apply Lemma 4.6. This permits the *separate* consideration of the equation of the connected components of $\Omega \cap U_{\mathrm{x}}$. Again Assumption 3.10 and Assumption 3.16 assure that the solutions on the separate sets do admit the required $W^{1,q}$-regularity with a $q > 3$. Finally, thanks to Lemma 4.5, then the function on the whole set $U_{\mathrm{x}} \cap \Omega$ belongs to $W_0^{1,q}(U_{\mathrm{x}} \cap \Omega)$ if $\mathrm{x} \in D_d^{||}$ and to $W_D^{1,q}(U_{\mathrm{x}} \cap \Omega)$ if $\mathrm{x} \in R_d$.

4.3.2 Regularity near points in $D_c^{\|}$ and R_c

The strategy which applies to both of the remaining cases $\mathrm{x} \in D_c^{\|}$ and $\mathrm{x} \in R_c$, respectively, is as follows. First – as above – one localizes the problem around the point x under consideration. Here the sets $\widetilde{U}_\mathrm{x}$ from Lemma 4.10 and U_x from Assumption 3.19, respectively, play the role of V in Lemma 4.2. Afterwards one transforms the problem under the mapping $\widetilde{\phi}_\mathrm{x}$ and ϕ_x, respectively, onto the corresponding model sets, thereby preserving the quality of the problems, thanks to Proposition 3.2. Then one 'compares' the resulting problem with one for which the coefficient function is piecewise constant in the spirit of Proposition 4.3.

Let us start by proving a regularity theorem for the corresponding model sets. Here M always denotes the Neumann boundary part.

Lemma 4.11. *Let Λ be one of the domains $\mathfrak{C} \setminus \Sigma_1, \mathfrak{Q} \setminus \Sigma_1$, where Σ_1 is defined in* (3.13)*. In case $\Lambda = \mathfrak{C} \setminus \Sigma_1$ let $M = \emptyset$. In case $\Lambda = \mathfrak{Q} \setminus \Sigma_1$, let either $M = \emptyset$ or*

$$M = \big(] - 1, 1[\times] - 1, 1[\times \{0\}\big) \setminus \Sigma_1, \quad \textit{or} \quad M =] - 1, 1[\times]0, 1[\times \{0\}. \tag{4.13}$$

In any case we set $E := \partial\Lambda \setminus M$.

Let ϱ be an elliptic coefficient function on Λ, which satisfies the invariance property $\varrho(\iota(\cdot)) = \iota\varrho(\cdot)\iota$. Moreover, if $\Lambda = \mathfrak{Q} \setminus \Sigma_1$, then let ρ be constant on $\mathfrak{Q}_+$, and if $\Lambda = \mathfrak{C} \setminus \Sigma_1$, then let ρ be constant on the sets $\mathfrak{C}_+ \cap \mathfrak{H}_3^\pm$.
Then, in any case, there is a $p > 3$ such that $-\nabla \cdot \varrho\nabla : W_E^{1,q}(\Lambda) \to W_E^{-1,q}(\Lambda)$ is a topological isomorphism if $q \in [2, p[$.

Proof. Let us first note that the invariance property $\varrho(\iota(\cdot)) = \iota\varrho(\cdot)\iota$ implies that ρ is also constant on $\mathfrak{Q}_-$, if $\Lambda = \mathfrak{Q} \setminus \Sigma_1$, and ρ is also constant on the sets $\mathfrak{C}_- \cap \mathfrak{H}_3^\pm$, if $\Lambda = \mathfrak{C} \setminus \Sigma_1$.

Define $\Psi : L^2(\Lambda) \to L^2(\Lambda)$ by $(\Psi w)(\mathrm{z}) = w(\iota(\mathrm{z}))$. Since Λ, M and Σ_1 are invariant under ι, Ψ induces topological isomorphisms $\Psi_q : W_E^{1,q}(\Lambda) \to W_E^{1,q}(\Lambda)$ for all $q \in [1, \infty[$, cf. Proposition 3.2. Define furthermmore $\Psi_2^* : W_E^{-1,2}(\Lambda) \to W_E^{-1,2}(\Lambda)$ as the adjoint of $\Psi_2 : W_E^{1,2}(\Lambda) \to W_E^{1,2}(\Lambda)$. Assume that $u \in W_E^{1,2}(\Lambda)$ is a solution of $-\nabla \cdot \varrho\nabla u = f \in W_E^{-1,2}(\Lambda)$. Then one may apply Proposition 3.2 for $\phi = \iota$. The matrix equality $\iota\varrho(\iota(\cdot))\iota = \varrho(\cdot)$ yields $-\nabla \cdot \varrho\nabla\Psi_2 u = \Psi_2^* f$, and this implies that

$$-\nabla \cdot \varrho\nabla(u - \Psi_2 u) = f - \Psi_2^* f, \tag{4.14}$$

and

$$-\nabla \cdot \varrho\nabla(u + \Psi_2 u) = f + \Psi_2^* f. \tag{4.15}$$

First consider (4.14). It is clear that the function $u - \Psi_2 u$ has trace 0 on the set

$$\Xi := \overline{\Lambda} \cap \{\mathrm{y} : y_1 = 0\}. \tag{4.16}$$

Denote the restriction of $u - \Psi_2 u$ to the sets $\Lambda_\pm := \Lambda \cap \mathfrak{H}_1^\pm$ by $v_\pm$. We define

$$E_\pm := (E \cap \mathfrak{H}_1^\pm) \cup (\partial\Lambda_\pm \cap \{\mathrm{z} : z_1 = 0\}).$$

Since the two domains $\Lambda_\pm$ are separated by the Dirichlet boundary part of both sets, we may apply Lemma 4.6 and end up with *separate* equations on both sets. Assume from now on that $q \in [2, \infty[$ and $f \in W_E^{-1,q}(\Lambda)$. This implies $f_\pm \in W_{E_\pm}^{-1,q}(\Lambda_\pm)$, see (4.3).

In case $\Lambda = \mathfrak{C} \setminus \Sigma_1$ one has then $E_\pm = \partial\Lambda_\pm$, see Figure 7.

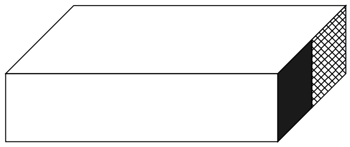

Figure 7: The case $\Lambda = \mathfrak{C} \setminus \Sigma_1$, $M = \emptyset$. The figure shows $\mathfrak{C}_-$ with the surface Σ_1 from $D^{\|}$ in black. The whole boundary is Dirichlet, on the crosshatched part due to antisymmetric reflection. The case $\Lambda = \mathfrak{Q} \setminus \Sigma_1$, $M = \emptyset$ gives an analogous picture for $\mathfrak{Q}_-$.

Hence, one may apply Proposition 3.12. It tells us that $v_\pm \in W_0^{1,q}(\Lambda_\pm)$, if $f \in W^{-1,q}(\Lambda_\pm)$ for all $q \in [2,p[$ for a certain $p > 3$. Since the traces of $v_\pm$ coincide on the common frontier of $\Lambda_\pm$, this even gives $u - \Psi_2 u \in W_0^{1,q}(\Lambda)$ for the same range of q's, cf. Lemma 4.5. If $\Lambda = \mathfrak{Q} \setminus \Sigma_1$ and M is empty, one concludes in the same way.

Consider now the case $\Lambda = \mathfrak{Q} \setminus \Sigma_1$ with M being one of the sets in (4.13). In principle, one can argue here as before – with one difference: The occurring Neumann boundary parts $M_\pm := M \cap \mathfrak{H}_1^\pm$ are nontrivial. Nevertheless, if defining $E_\pm := \partial\Lambda_\pm \setminus M_\pm$ and restricting the problem to each of the two components $\Lambda_\pm$, then the resulting setting fits into Corollary 3.13, if M is the first set in (4.13), and into Corollary 4.8 if M equals the second set in (4.13). This gives $u - \Psi_2 u \in W_{E_\pm}^{1,q}(\Lambda_\pm)$ for $q \in [2,p[$, with $p > 3$. Since both traces on the common interface $\partial\Lambda_+ \cap \partial\Lambda_-$ are zero, $u - \Psi_2 u$ even belongs to $W_E^{1,q}(\Lambda)$ for this same range of q's, thanks to Lemma 4.5.

Next consider (4.15), first assuming $f \in L^2(\Lambda) \hookrightarrow W_E^{-1,q}(\Lambda)$ for $q \in [2,6]$. Then $\Psi_2^* f = \Psi f$, and both $v := u + \Psi_2 u$ and $g := f + \Psi f$ are invariant under Ψ. Let us establish a corresponding equation for $v|_{\Lambda_-}$: Let

$$M_- = (M \cap \overline{\mathfrak{H}_1^-}) \cup (\Lambda \cap \{\mathrm{z} : z_1 = 0\}) \subset \partial\Lambda_-, \; E_- := \partial\Lambda_- \setminus M_-. \tag{4.17}$$

Note that M_- is then *open* in $\partial\Lambda_-$. For $w \in W_{E_-}^{1,r}(\Lambda_-)$ ($r \in]1,\infty[$) we define $\widehat{w}$ on Λ by

$$\widehat{w}(\mathrm{y}) = \begin{cases} w(\mathrm{y}), & \mathrm{y} \in \Lambda_- \\ w(\iota(\mathrm{y})), & \mathrm{y} \in \Lambda_+ \\ \operatorname{tr} w, & \mathrm{y} \in \Lambda \cap \{\mathrm{z} : z_1 = 0\}, \end{cases} \tag{4.18}$$

where tr denotes the corresponding trace. One knows that $\widehat{w} \in W_E^{1,r}(\Lambda)$, if $w \in W_{E_-}^{1,r}(\Lambda_-)$, cf. [9, Proposition 4.4]. Thus, (4.15) in combination with (4.18) yields

$$\begin{aligned} 2\int_{\Lambda_-} \varrho\nabla v \cdot \nabla w \, d\mathrm{y} &= \int_\Lambda \varrho\nabla v \cdot \nabla\widehat{w} \, d\mathrm{y} \\ &= \langle -\nabla\varrho\nabla v, \widehat{w} \rangle = \textstyle\int_\Lambda g\widehat{w}\, dx = 2\int_{\Lambda_-} gw \, d\mathrm{y}. \end{aligned} \tag{4.19}$$

Moreover, Lemma 4.6 tells us that $v \in W_E^{1,2}(\Lambda)$ implies $v|_{\Lambda_-} \in W_{E_-}^{1,2}(\Lambda_-)$. Consequently, (4.19) can be interpreted as the weak formulation of the equation

$$-\nabla \cdot \varrho\nabla v|_{\Lambda_-} = g|_{\Lambda_-}.$$

Concerning the constellations admitted in the assumptions, (4.17) allows for the following possibilities:

- If $M = \emptyset$:

$$\Lambda_- = \mathfrak{C}_-, \quad M_- = \{0\}\times]0,1[\times]-1,1[, \tag{4.20}$$

$$\Lambda_- = \mathfrak{Q}_-, \quad M_- = \{0\}\times]0,1[\times]-1,0[\tag{4.21}$$

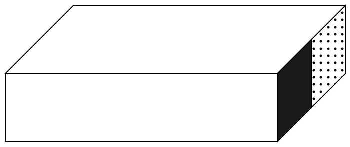

Figure 8: $\mathfrak{Q}_-$ with the dotted Neumann part resulting from symmetric reflection. The rest is Dirichlet boundary, the black part resulting from $D^{\|}$.

- If $M = \big(]-1,1[\times]-1,1[\times\{0\}\big) \setminus \Sigma_1$:

$$\Lambda_- = \mathfrak{Q}_-, \quad M_- =]-1,0[\times]-1,1[\times\{0\} \cup \{0\}\times]0,1[\times]-1,0[, \tag{4.22}$$

- If $M =]-1,1[\times]0,1[\times\{0\}$:

$$\Lambda_- = \mathfrak{Q}_-, \quad M_- = (]-1,0[\times]0,1[\times\{0\}) \cup (\{0\}\times]0,1[\times]-1,0]) \tag{4.23}$$

All these settings are included in our regularity results: (4.20) and (4.21) are covered by Lemma 4.7 ii). (4.22) is treated in Corollary 4.8 and (4.23) is treated in Corollary 4.7 i). Thus, in any case, $v|_{\Lambda_-}$ admits the estimate

$$\|v|_{\Lambda_-}\|_{W^{1,q}_{E_-}(\Lambda_-)} \le c\|g|_{\Lambda_-}\|_{W^{-1,q}_{E_-}(\Lambda_-)} \le c\|g\|_{W^{-1,q}_{E}(\Lambda)}.$$

Since $v \in W^{1,2}_E(\Lambda)$ is invariant under Ψ_2, $\|v\|_{W^{1,q}_E(\Lambda)} = 2\|v|_{\Lambda_-}\|_{W^{1,q}_{E_-}(\Lambda_-)}$, and we obtain the estimate

$$\|v\|_{W^{1,q}_E(\Lambda)} \le c\|g\|_{W^{-1,q}_E(\Lambda)} \tag{4.24}$$

for a suitable constant c and $q \in [2,p[$, if $p > 3$ is sufficiently close to 3. The operator Ψ_2^* transforms $W^{-1,q}_E(\Lambda)$ continuously into itself, and $L^2(\Lambda)$ is dense in $W^{-1,q}_E(\Lambda)$, so (4.24) implies $u + \Psi_2 u \in W^{1,q}_E(\Lambda)$ *for all* $f \in W^{-1,q}_E(\Lambda)$. Together with the discussion of (4.14) this yields $u \in W^{1,q}_E(\Lambda)$, if $f \in W^{-1,q}_E(\Lambda)$. Thus, the assertion is obtained from the open mapping theorem. □

It follows the final step of the proof of Theorem 4.1: Since the cases $\mathrm{x} \in D^{\|}_d$ and $\mathrm{x} \in R_d$ are already established, the remaining ones are $\mathrm{x} \in D^{\|}_c$ and $\mathrm{x} \in R_c$. With a slight change of notation define U_x by:

$$U_\mathrm{x} = \begin{cases} \widetilde{U}_\mathrm{x} \text{ as in Lemma 4.10}, & \mathrm{x} \in D^{\|}_c \\ U_\mathrm{x} \text{ as in Assumption } 3.19, & \mathrm{x} \in R_c. \end{cases} \tag{4.25}$$

We next apply Lemma 4.2 with $V = U_\mathrm{x}$. So, for $\eta \in C^\infty_0(U_\mathrm{x})$, one gets $\eta u \in W^{1,2}_{D_\mathrm{x}}(\Omega_\mathrm{x})$ with $\Omega_\mathrm{x} := \Omega \cap U_\mathrm{x}$ and

$$D_\mathrm{x} = \begin{cases} \partial\Omega_\mathrm{x}, & \mathrm{x} \in D^{\|}_c \\ \partial\Omega_\mathrm{x} \setminus \big(U_\mathrm{x} \cap (\partial\Omega \setminus D)\big), & \mathrm{x} \in R_c. \end{cases}$$

Moreover, Lemma 4.2 tells us that the function $\eta u|_{\Omega \cap U_{\mathrm{x}}} =: u_\bullet$ satisfies

$$-\nabla \cdot \mu_\bullet \nabla u_\bullet = f_\bullet \in W^{-1,q}_{D_{\mathrm{x}}}(\Omega_{\mathrm{x}}), \quad \mu_\bullet := \mu|_{\Omega_{\mathrm{x}}}$$

with $\|f_\bullet\|_{W^{-1,q}_{D_{\mathrm{x}}}(\Omega \cap U_{\mathrm{x}})} \le c\|f\|_{W^{1,q}_D(\Omega)}$, c independent of f.
After passing to this localized problem, we employ now the transformation principle, Proposition 3.2. Then we are almost in the situation of Lemma 4.11: The geometries one has to treat are exactly those. The difference is that the occurring coefficient functions are not constant on the corresponding subsets up to now. But they have limits for $\mathrm{z} \to 0$: For $\mathrm{x} \in D^{\|}_c$ this limit exists, due to Proposition 3.2 and Lemma 4.10 separately in $\Lambda \cap \mathfrak{H}^{\pm}_3$. Modifying the coefficient function by taking it to be the corresponding limit, which is indeed constant on $\Lambda \cap \mathfrak{H}^{\pm}_3$, we can apply Lemma 4.11 in order to obtain the regularity property $W^{1,q}$ for this modified coefficient function with a $q > 3$. Having this at hand, we can finish the proof for the case $\mathrm{x} \in D^{\|}_c$ and $\Lambda = \mathfrak{C} \setminus \Sigma_1$ by applying the 'comparison' result, Lemma 4.3. For $\mathrm{x} \in R_c$ and $\Lambda = \mathfrak{Q} \setminus \Sigma_1$ the argument is the same.

5 Concluding remarks

i) The condition (3.18) in Assumption 3.19 ii) can be perturbed slightly. This means that one can add to a coefficient function, fulfilling this condition, another one which is sufficiently small in the L^∞-norm, and the main result (Theorem 4.1) remains true. This follows by straight forward perturbation arguments, resting on (4.2).
We have tried hard, resting on this argument and gauging, to avoid Assumption 3.19 ii) at all, since it is really a severe restriction concerning the admissable configurations. Unfortunately, all these attempts have failed.

ii) It is not by accident that we had only points from $\overline{D^{\|}}$ under consideration here. If other boundary points are involved, then one can apply the (local) regularity results obtained in [5]. Moreover, Theorem 3.1 then allows to deduce a *global* regularity result within the $W^{1,q}$-scale.

iii) Quite similar results are obtained in two space dimensions, this time under much more general conditions. Namely, if $D \subset \overline{\Omega}$ is a closed 1-set and there is a continuous extension operator $\mathfrak{E} : W^{1,q}_D(\Omega) \to W^{1,q}_D(\mathbb{R}^d)$, then

$$-\nabla \cdot \mu \nabla + 1 : W^{1,q}_D(\Omega) \to: W^{-1,q}_D(\Omega)$$

is a toplogical isomorphism for $q \in]2-\delta, 2+\delta[$ and some $\delta > 0$, see [17].

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