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Alexander Mielke[1], André Schlichting[2], Artur Stephan[3]

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[1] Weierstraß-Institut für Angewandte Analysis und Stochastik
Anton-Wilhelm-Amo-Str. 39
10117 Berlin
Germany
E-Mail: alexander.mielke@wias-berlin.de

[2] Institut für Angewandte Analysis
Universität Ulm
Helmholtzstraße 18
89081 Ulm
Germany
E-mail: andre.schlichting@uni-ulm.de

[3] Technische Universität Wien
Institut für Analysis und Scientific Computing
Wiedener Hauptstraße 8–10
1040 Wien
Austria
E-mail: artur.stephan@tuwien.ac.at

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Anton-Wilhelm-Amo-Straße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

We provide a derivation of the fourth-order DLSS equation based on an interpretation as a chemical reaction network. We consider the rate equation on the discretized circle for a process in which pairs of particles occupying the same site simultaneously jump to the two neighboring sites; the reverse process involves pairs of particles at adjacent sites simultaneously jumping back to the site located between them. Depending on the rates, in the vanishing-mesh-size limit we obtain either the classical DLSS equation or a variant with nonlinear mobility of power type. Via EDP convergence, we identify the limiting gradient structure to be driven by entropy with respect to a generalization of diffusive transport with nonlinear mobility. Interestingly, the DLSS equation with power-type mobility shares qualitative similarities with the fast diffusion and porous medium equation, since we find traveling wave solutions with algebraic tails or compactly supported polynomials, respectively.

1 Introduction

In this paper, we provide a microscopic derivation of a generalization of the Derrida-Lebowitz-Speer-Spohn (DLSS) with nonlinear mobility given by

$$\partial_t \rho = -\partial_{xx}(\rho^\alpha \partial_{xx} \log \rho) \tag{1.1}$$

on the torus $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. We consider mobility exponents $\alpha > 0$. The case with linear mobility $\alpha = 1$ was derived in [DL*91a, DL*91b] as the law of fluctuations of the toom interface model, but also occurs in the quantum-drift diffusion [DMR05]. The generalization with mobility $\alpha \neq 1$ is so far not studied in the literature. We will derive (1.1) as the macroscopic limit of rate equations for a binary reaction network. In this situation $\alpha = 2$ is a natural choice, but our approach provides a derivation of (1.1) for all $\alpha > 0$.

The work focuses on the derivation of (1.1) from a microscopic model and provides a first analytic framework identifying possible weak solutions and the thermodynamic consistent a priori bound for (1.1). However, we expect that (1.1) has a rich dynamical behaviour mimicking that of the fast diffusion ($\alpha < 1$) and porous medium equation ($\alpha > 1$). The details are left for further studies and we provide at the end of the introduction some first indication in this direction. In detail, we provide traveling wave solutions on $\mathbb{R}$ with algebraic tails ($\alpha < 1$) or compact support ($\alpha > 1$) as well as numerical tests of source-type solutions mimicking the according behaviour of the tails.

Our derivation of (1.1) is motivated by a physical interpretation of the recently proposed numerical scheme [MR*25] in the case of $\alpha = 1$ as a chemical network of $N \in \mathbb{N}$ binary

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reactions of the type

$$X_{k-1} + X_{k+1} \rightleftharpoons 2X_k\,, \qquad \text{for } k \in [N] := \{1, \dots, N\}\,. \tag{1.2}$$

We consider the vector of concentrations $c = (c_k)_{k=1,\dots,,N} \in \mathcal{P}^N$, that is a discrete probability measure on $[N]$, where each entry provides the concentration of the species X_k in (1.2) (with the convention $c_0 = c_N$). They evolve according to the chemical reaction-rate equation by

$$\dot c_k = N^2\big(J_{\alpha,k-1}[c] - 2J_{\alpha,k}[c] + J_{\alpha,k+1}[c]\big) \tag{1.3a}$$

$$\text{with} \quad J_{\alpha,k}[c] = \sigma_\alpha(c_{k-1}, c_k, c_{k+1}) N^2 \Big(c_k^2 - c_{k-1}c_{k+1}\Big)\,, \tag{1.3b}$$

where the activity σ_α will be specified below. A possible choice for σ_α is motivated from the context of chemical reactions by the mass action kinetics and is given as $\sigma_2 \equiv 1$. In general, σ_α will be chosen to be homogeneous of degree $\alpha - 2$, which will turn (1.3) into a discrete approximation of (1.1) (see Assumption 2.1 below). Since the system (1.3) originates from a rate equation of the chemical network (1.2), it has a good global existence theory and also positivity properties. Crucially, the solution of the chemical reaction-rate equation (1.3) can be characterized variationally as a generalized gradient flow [LM*17, PeS23, Mie23a] in continuity equation form of the entropy

$$E_N(c) := \frac{1}{N}\sum_{k=1}^{N}(c_k \log c_k - c_k + 1)\,. \tag{1.4}$$

The role of the continuity equation is taken by the first equation in (1.3a) and a general curve $(c, J) : [0, T] \to \mathcal{P}^N \times \mathbb{R}^N$ solving (1.3a) is denoted by $(c, J) \in \mathrm{CE}_N$. The specific choice of the flux J_α in (1.3b) is encoded by an energy-dissipation balance (EDB), which includes the time-integrated dissipation functional $D_{\alpha,N}(c, J)$ (see (2.12) below), and is given by the full energy-dissipation functional

$$L_{\alpha,N}(c, J) := E_{\alpha,N}(c(T)) - E_{\alpha,N}(c(0)) + D_{\alpha,N}(c, J)\,. \tag{1.5}$$

For positive and smooth curves $(c, J) \in \mathrm{CE}_N$ with $c : [0, T] \to \mathcal{P}^N_{>0}$, the construction in Section 2.4 ensures by the Young-Fenchel inequality that $L_{\alpha,N}(c, J) \geq 0$. In our first result, we identify the classic solutions to (1.3) having E_N as Lyapunov function as global minimizers of the energy-dissipation functional, that is $L_{\alpha,N}(c, J) = 0$ and are called *energy-dissipation balance (EDB) solutions*. This will be the starting point for the convergence analysis of the system.

Result A (Well-posedness and variational characterization of (1.3))**.** *Assume the acitivity σ_α satisfies Assumption 2.1. For all $c^0 \in \mathcal{P}^N$, the system* (1.3) *has a global differentiable solution $c : [0, \infty) \to \mathcal{P}^N$ such that $c(t) \in \mathcal{P}^N_{>0}$ is strictly positive for all $t > 0$. The constructed solution is an EDB solution satisfying $L_{\alpha,N}(c, J) = 0$.*

The Result A is proven in Proposition 2.5 and Proposition 2.6 in Section 2.

We turn next to the limit $N \to \infty$ and note that the total scaling N^4 in (1.3) is chosen to obtain a macroscopic limit, if the concentration vector c^N is embedded into the space of continuous densities on the torus $\mathbb{T}$. In this scaling, we arrive at the fourth-order equation (1.1). For the heuristic argument, we observe that the outer Laplacian in (1.1) is already explicit as formal limit of the discrete second-order continuity equation in (1.3a). The diffusive flux term $\rho^2\partial_{xx}\log\rho$ is generated by the *geometric* Laplacian in (1.3b) by a formal expansion

$$\begin{aligned} N^2\Big(c_k^2 - c_{k-1}c_{k+1}\Big) &= -N^2c_k^2[\exp(\log c_{k-1} + \log c_{k+1} - 2\log c_k) - 1] \\ &\approx -\rho(x)^2\partial_{xx}\log\rho + O(N^{-2}). \end{aligned} \tag{1.6}$$

The rigorous proof of the limit is based on the energy-dissipation principle, which is a thermodynamic formulation for the discrete (1.3) as well as continuous (1.1) gradient structure. In fact, our results shows that the limiting gradient structure for (1.1) is driven by the continuous entropy given by

$$\mathcal{E}(\rho) = \int_{\mathbb{T}}(\rho(x)\log\rho(x) - \rho(x) + 1)\,\mathrm{d}x. \tag{1.7}$$

We formulate the limit (1.1) in terms of a continuous second-order continuity-type equation and a constitutive relation for the flux, that is

$$\partial_t\rho = \Delta j_\alpha \qquad \text{and} \qquad j_\alpha = -\rho^\alpha\partial_{xx}\log\rho\,. \tag{1.8}$$

Again, as in the discrete case, a general solution pair $(\rho, j) : [0,T] \to \mathrm{L}^1(\mathbb{T}) \times \mathrm{W}^{2,1}(\mathbb{T})$ to the second-order continuity equation $\partial_t\rho = \Delta j$ is denoted by $(\rho, j) \in \mathrm{CE}$, understood in a suitable weak sense. The specific flux $j_\alpha = -\rho^\alpha\partial_{xx}\log\rho$ for the solution in (1.8) is encoded through a suitable total dissipation functional $\mathcal{D}_\alpha$, see (3.6). These ingredients provide the limiting gradient structure by defining the full energy-dissipation functional for a curve $(\rho, j) \in \mathrm{CE}$ by

$$\mathcal{L}_\alpha(\rho, j) := \mathcal{E}(\rho(T)) - \mathcal{E}(\rho(0)) + \mathcal{D}_\alpha(\rho, j). \tag{1.9}$$

The total dissipation $\mathcal{D}_\alpha$ consists of the time integral of a primal dissipation $\mathcal{R}_\alpha(\rho, j)$ and the slope $\mathcal{S}_\alpha(\rho)$. The primal dissipation is given for $j \in \mathrm{L}^1(\mathbb{T})$ and $j\,\mathrm{d}x \ll \rho\,\mathrm{d}x$ by

$$\mathcal{R}_\alpha(\rho, j) := \frac{1}{2}\int_{\mathbb{T}}\frac{j^2}{\rho^\alpha}\,\mathrm{d}x, \qquad \text{for } j \in \mathrm{L}^1(\mathbb{T}) \text{ such that } j\,\mathrm{d}x \ll \rho\,\mathrm{d}x\,. \tag{1.10}$$

The slope term $\mathcal{S}_\alpha$ is formally defined by inserting the driving force $-\Delta\mathrm{D}\mathcal{E}(\rho) = -\Delta\log\rho$ into the dual dissipation potential (3.1), obtained as the Legendre-Fenchel dual of $\mathcal{R}_\alpha$ from (1.10), that is $\mathcal{S}_{\alpha,+}(\rho) = \mathcal{R}_\alpha^*(\rho, -\Delta\mathrm{D}\mathcal{E}(\rho))$. Due to the presense of the logarithm and Laplacian, this is only justified for positive and smooth concentrations, where we indeed get

$$\mathcal{S}_{\alpha,+}(\rho) := \frac{1}{2}\int_{\mathbb{T}}\rho^\alpha\,(\Delta\log\rho)^2\,\mathrm{d}x. \tag{1.11}$$

The main ingredient of the analysis is its relaxed lower-semicontinuous envelope $\mathcal{S}_\alpha$ defined on a suitable family of non-negative Sobolev functions (see Lemma 3.1).

For this reason, the non-negativity of $\mathcal{L}_\alpha$ is only ensured for sufficient regular curves $(\rho, j) \in \mathrm{CE}$. Such regularity is a priori not known for the obtained solutions to (1.1). Hence, we distinguish two notion of solutions, those satisfying the *energy-dissipation balance* (EDB) $\mathcal{L}_\alpha(\rho, j) = 0$ and the weaker form satisfying the *energy-dissipation inequality* (EDI) $\mathcal{L}_\alpha(\rho, j) \le 0$.

As we will see, the convergence result in Section 3 constructs a curve ρ that is a priori only an EDI solution.

Result B (EDP convergence)**.** *Assume the acitivity σ_α satisfies Assumption 2.1.*
Let $(c^N, J^N) \in \mathrm{CE}_N$ be such that $\sup_N L_{\alpha,N}(c^N, J^N) < \infty$ and $\sup_N E_N(c^N(0)) < \infty$. Then

there exist suitable embeddings $(\iota_N c^N, \mathcal{I}_N J^N) \in \mathrm{CE}$ *such that the sequence* $(\iota_N c^N, \mathcal{I}_N J^N)$ *converges to* $(\rho, j) \in \mathrm{CE}$ *and it holds*

$$\forall\, t > 0 : \ \liminf_{N\to\infty} E_N(c^N(t)) \geq \mathcal{E}(\rho(t)) \qquad \text{and} \qquad \liminf_{N\to\infty} D_{\alpha,N}(c^N, J^N) \geq \mathcal{D}_\alpha(\rho, j).$$

If, in addition the curves c^N *are EDB solutions to* (1.3)*, that is* $L_{\alpha,N}(c^N, J^N) = 0$*, and have well-prepared initial data, i.e.* $\iota_N c^N(0) \to \rho(0)$ *and* $E_N(c^N(0)) \to \mathcal{E}(\rho(0))$*, then the limit curve* (ρ, j) *is an EDI solution to* (1.1)*, that is* $\mathcal{L}_\alpha(\rho, j) \leq 0$*.*

The full statement is contained in Theorem 3.6 and proven in Section 3.

Under suitable assumptions, we can show that the so-obtained EDI solution is indeed an EDB solution by showing that the energy-dissipation functional $\mathcal{L}_\alpha$ is non-negative on its domain. For doing so, we show a *chain-rule* in Proposition 4.2, which allows us to conclude that any EDI solution (ρ, j) is already an EDB solution, and at the same time identifies a suitable weak solution to (1.1).

Result C (EDB and weak solutions)**.** *Consider* $\alpha > 0$ *and an EDI solution* (ρ, J) *satisfying one of the following conditions:*

$$\alpha = 1; \tag{1.12a}$$
$$\alpha \in \,]0, 2] \ \ \text{and} \ \ \rho \in \mathrm{L}^\infty([0,T]\times\mathbb{T}); \tag{1.12b}$$
$$\alpha > 0, \quad \rho \in \mathrm{L}^\infty([0,T]\times\mathbb{T}), \ \text{and} \ \exists\, \delta > 0 : \ \rho(t,x) \geq \delta \ \text{a.e.} \tag{1.12c}$$

Then, (ρ, j) *is an EDB solution. Moreover, these EDB solutions are also weak solutions of* $\partial_t\rho + \frac{1}{\alpha}\Delta\Big(\Delta\rho^\alpha - 4|\nabla\rho^{\alpha/2}|^2\Big) = 0$*, namely for all* $\psi \in \mathrm{C}^2_\mathrm{c}([0,T)\times\mathbb{T})$ *it holds*

$$\int_\mathbb{T} \rho(0)\,\psi(0)\,\mathrm{d}x + \int_0^T\!\!\int_\mathbb{T} \rho\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\mathbb{T} \frac{1}{\alpha}\Big(\Delta\rho^\alpha - 4|\nabla\rho^{\alpha/2}|^2\Big)\ \Delta\psi\,\mathrm{d}x\,\mathrm{d}t\,,$$

where, setting $p_\alpha = \max\Big\{(4{+}\alpha)/(2{+}\alpha), 4/3\Big\}$*, the flux terms satisfies*

$$j = -\frac{1}{\alpha}\Big(\Delta\rho^\alpha - 4|\nabla\rho^{\alpha/2}|^2\Big) = -\frac{2}{\alpha}\rho^{\alpha/2}\Big(\Delta\rho^{\alpha/2} - 4|\nabla\rho^{\alpha/4}|^2\Big) = -\rho^{\alpha/2}V \in \mathrm{L}^{p_\alpha}([0,T]\times\mathbb{T}).$$

If $\rho \in \mathrm{L}^\infty([0,T]\times\mathbb{T})$*, then we always have* $j \in \mathrm{L}^2([0,T]\times\mathbb{T})$*.*

The well-posedness of the term on the right-hand side is ensured by finiteness of the relaxed slope $\mathcal{S}_\alpha$ for variational solutions and a posteriori for the identified weak solutions demanding to satisfy the energy dissipation balance, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\rho) = -\frac{4}{\alpha^2}\int_\mathbb{T}\Big(\Delta\rho^{\alpha/2} - 4|\nabla\rho^{\alpha/4}|^2\Big)^2\,\mathrm{d}x \qquad \text{for a.e. } t \in [0,T]\,.$$

Relation to the literature

We note that our Results A–C lead to several generalizations of the work [MR*25], even in the case $\alpha = 1$. We provide a complete variational EDP convergence and identification of the same weak solution constructed as in [MR*25]. Moreover, we allow for a greater range of semi-discretizations based on the choice of the activity function σ_α, where our Assumption 2.1

covers the specific choice from [MR*25], however at the drawback that our scheme in general preserves only the entropy as Lyapunov function, whereas (1.1) for $\alpha = 1$ has a rich family of further Lyapunov functions. At this point, we refer to [MR*25, Sec. 1] for an extensive discussion of the origin, the structural properties and various numerical schemes for the DLSS equation ($\alpha = 1$) as well as to [MRS25] for the discussion of the induced distance from the primal dissipation (1.10).

In this work, we pioneer the derivation of (1.1) for the full range $\alpha > 0$ via EDP convergence and provide the existence of variational solution satisfying an energy-dissipation inequality or a conditional energy-dissipation balance. Both concepts are a generalization of curves of maximal slope introduced by De Giorgi [DGMT80] in the form developed in [AGS05, RMS08, Mie23a] relying on the energy dissipation balance based on a primal and a dual dissipation potential. The notion of EDP convergence was introduced (informally and without name) in [LM*17] and then conceptually studied in [DFM19, MMP21, PeS23] and [Mie23a, Sec. 5.4]. It is a refinement of the Sandier-Serfaty approach to Γ-convergence for gradient flows [SaS04, Ser11] and allows to study general multiscale limits like homogenization [Mie16], layer-to-membrane limits [LM*17, FrL21, Mie23b], fast-slow reaction systems [FrL21, MPS21, Ste21, Mie23b], or discrete-to-continuum limits [DiL15, HST24, EHS25, HPS24, HMS25] as in the present paper.

The derivation of thermodynamically consistent continuum models from stochastic or discrete dynamics is a recent undertaking and we comment on some recent literature. In the context of linear regular Markov jump processes the work [PR*22] provides an extensive framework. The justification of the macroscopic, exponential kinetic relation (Marcellin-De Donder kinetic) from stochastic jump processes was derived via large-deviation theory in [MPR14, MP*17] which leads to the so-called *cosh-gradient structure* used for our discrete model as well. In the context of discrete coagulation-fragmentation equations, which show a similar quadratic structure as (1.3), the $\cosh$-gradient structure was also identified in [HLS25].

The obtained gradient structure for our continuum model (1.1) can be seen as a second order generalization of the recent novel gradient structure obtained for the porous medium equation in [GeH25, FeG23], that is

$$\partial_t \rho = \tfrac{1}{\alpha} \partial_{xx}(\rho^\alpha) = \partial_x(\rho^\alpha \partial_x \log \rho) \,. \tag{1.13}$$

The authors derive (1.13) as the contiuum limit of a suitable rescaled zero-range process in the *thermodynamic scaling* limit of infinite many particles and large volume such that the density stays order one. In our situation, the ODE system (1.3) can be already seen as the infinite particle limit of a stochastic model, which in the context of chemical reaction corresponds to the chemical master equation [Van07, MaM20]. The considered limit $N \to \infty$ of the manuscript corresponds to the infinite volume limit in that language, see [GeH25, Figure 1].

The authors in [GeH25] identify the driving energy as the entropy $\mathcal{E}$ as in our case (1.7), see [GeH25, Chapter 11]. Their variational formulation is obtained in terms of curves $(\rho, g) : [0, T] \to \mathcal{P} \times \mathrm{L}^2$ solving the skeleton equation

$$\partial_t \rho = \tfrac{1}{\alpha} \partial_{xx} \rho^\alpha + \partial_x(\rho^{\alpha/2} g) \,,$$

with the control $g \in \mathrm{L}^2([0, T]; \mathrm{L}^2)$ and a large-deviation rate functional given by $\mathcal{J}_\alpha(\rho, g) = \frac{1}{2}\|g\|^2_{\mathrm{L}^2 \mathrm{L}^2}$, where $\mathcal{J}_\alpha \equiv 0$ characterizes suitable weak solutions to (1.13). Their result [GeH25, Thm. 4] implies that the functional $\mathcal{J}_\alpha$ has indeed the structure of an energy-dissipation functional similar to $\mathcal{L}_\alpha$ in (1.9). Indeed, along suitable curves $(\rho(t), j(t))_{t \in [0,T]}$ solving the (first-order) continuity equation $\partial\rho + \partial_x j = 0$ in the weak sense, it holds

$$\mathcal{J}_\alpha(\rho, g) = \mathcal{E}(\rho(T)) - \mathcal{E}(\rho(0)) + \int_0^T \Big(\mathcal{R}_\alpha(\rho, j) + \mathcal{S}_\alpha^{\mathrm{PME}}(\rho)\Big)\,\mathrm{d}t\,,$$

where $\mathcal{R}_\alpha$ is as in (1.10) and $\mathcal{S}_\alpha^{\mathrm{PME}} = \frac{2}{\alpha^2}\int|\nabla\rho^{\alpha/2}|^2\,\mathrm{d}x$. Again as in our case $\mathcal{S}_\alpha^{\mathrm{PME}}$ can be seen as the lower semicontinuous relaxation of the functional $\int\rho^\alpha|\nabla\log\rho|^2\,\mathrm{d}x$. In this sense, the DLSS equation with mobility $\alpha > 0$ in (1.1) is a generalization of the porous medium equation with respect to a second-order thermodynamic (degenerate) metric induced by $\mathcal{R}_\alpha$ from (1.10) together with the second-order continuity equation given in terms of the first equation in (1.8).

The classical DLSS equation ($\alpha = 1$) has, besides the gradient structure based on diffusive transport and driving functional given by the entropy, another gradient structure, which is based on classic Otto-Wasserstein tensors driven by the Fisher information [GST09]. We are not aware of any formal rewriting of (1.1) for $\alpha \neq 1$ in terms of a Otto-Wasserstein gradient flow, even with some mobility and suitable generalization of the Fisher-Information as considered for gradient-flow formulations of the thin-film equation [Ott98, GiO01, MMS09].

Traveling fronts and similarity profiles

We close the introduction with and illustration of some of the expected novel features of (1.1) due to the nonlinear mobility by investigating traveling front solutions and doing some numerical experiments.

First, we provide on the real line $\mathbb{R}$ solutions to the equation

$$\partial_t\rho = -\partial_{xx}\Big(\rho^\alpha\partial_{xx}\log\rho\Big) = -\partial_{xx}\Big(\rho^{\alpha-2}\Big(\rho\partial_{xx}\rho - (\partial_x\rho)^2\Big)\Big), \qquad t > 0,\ x \in \mathbb{R},$$

We observe that for $\alpha > 1$ there are explicit solutions that have a moving support. We consider the ansatz

$$\rho(t, x) = \kappa\Big(ct - x\Big)^\delta \text{ for } x < ct \quad\text{and}\quad \rho(t, x) = 0 \text{ for } x > ct,$$

with $\kappa > 0$. We obtain an explicit solution for $x < ct$ if we choose

$$\delta = \frac{3}{\alpha-1} \quad\text{and}\quad c = \kappa^{\alpha-1}\,\frac{3(\alpha+2)}{(\alpha-1)^2} > 0.$$

For $\alpha \in\,]1, 7/4[$ the solution is a classical solution lying in $\mathrm{C}^4(\mathbb{R})$. For $\alpha \geq 7/4$ the solution is still a weak solution if properly defined. Note that $\rho^{\alpha-1}\partial_{xx}\rho$ and $\rho^{\alpha-2}(\partial_x\rho)^2$ both behave like $(ct - x)^\sigma$ with $\sigma = \alpha\delta - 2 = (\alpha + 2)/(\alpha - 1) > 1$.

Moreover, we are able to look for self-similar solutions of having the dynamical scaling form $\rho(t, x) = t^{-\gamma}\Phi(x/t^\gamma)$ for a profile function $\Phi : \mathbb{R} \to \mathbb{R}$. As the right-hand side is homogeneous of degree α in ρ we find $\gamma = \frac{1}{3+\alpha}$. Using the similarity variable $y = xt^{-\gamma}$, we find the ODE for the profile function

$$-\gamma\Phi(y) - \gamma y\Phi'(y) = -\Big(\Phi^{\alpha-2}\Big(\Phi\Phi'' - (\Phi')^2\Big)\Big)''.$$

The left-hand side is a derivative, so we can integrate ones and are left with

$$\frac{1}{3+\alpha}y\,\Phi = \Big(\Phi^{\alpha-2}\Big(\Phi\Phi'' - (\Phi')^2\Big)\Big)'.$$

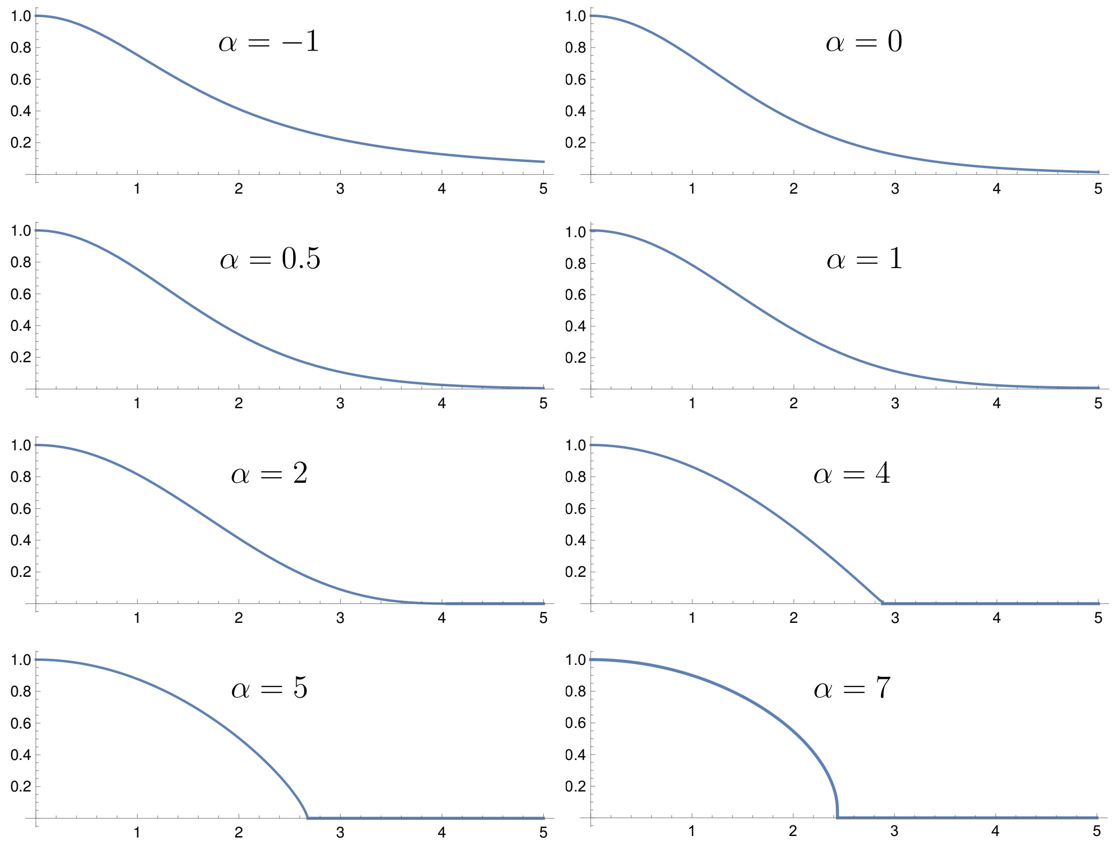

Figure 1: Numerically obtained similarity profiles Φ_α for $\alpha \in \{-1, 0, 0.5, 1, 2, 3, 4, 5\}$ normalized by $\Phi_\alpha(0) = 1$. For $\alpha < 1$ the solutions have algebraic decay like $|y|^{-3/(1-\alpha)}$; for $\alpha = 1$ we have $\Phi_1(y) = \mathrm{e}^{-y^2/4}$, for $\alpha > 1$ the solutions have compact support and behave like $(y_\alpha - y)^{3/(\alpha-1)}$ to the left of the right boundary of the support $[-y_\alpha, y_\alpha]$.

For the DLSS case with $\alpha = 1$ there is the explicit solution $\Phi(y) = \mathrm{e}^{-y^2/4}$, see [BLS94].

In general, we expect symmetric solutions (i.e. $\Phi(-y) = \Phi(y)$) and thus can produce solutions by a shooting method starting with $\Phi(0) = 1$, $\Phi'(0) = 0$, and $\Phi''(0) = b$, where b needs to be varied to find a sufficiently smooth, non-negative solution in $L^1(\mathbb{R})$. Figure 1 displays the corresponding solutions Φ_α for $\alpha \in \{-1, \ldots, 5\}$. Clearly, for $\alpha > 1$ the behavior close to the moving boundary of the support is given by the traveling fronts as constructed above.

For comparison we also provide numerical experiments for the equation (1.3), which provides already a spatial discrete approximation to (1.1) once the activity σ_α is specified. We use the specific choice

$$\sigma_\alpha(c_{k-1}, c_k, c_{k+1}) = \frac{4}{\alpha^2}\left(\frac{c_k^{\frac{\alpha}{2}} - \sqrt{c_{k-1}c_{k+1}}^{\frac{\alpha}{2}}}{c_k - \sqrt{c_{k-1}c_{k+1}}}\right)^2,$$

which satisfies Assumption 2.1. We use an implicit Euler scheme for the time integration, which is solved using a Newton method. The resulting scheme is implemented in the Julia language [BE*17] and the obtained solutions are depicted in Figure 2.

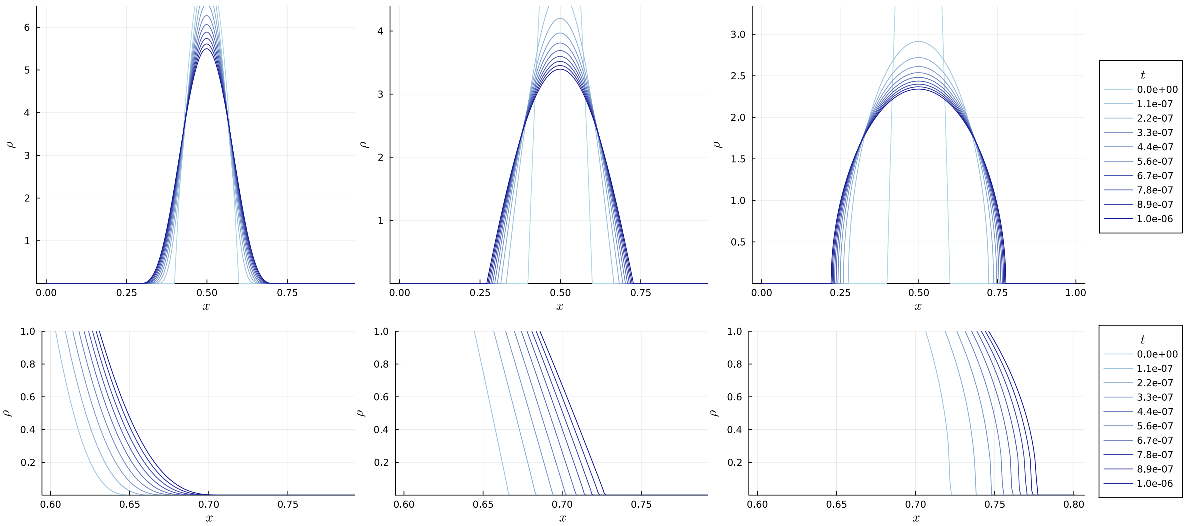

Figure 2: Numerically obtained solutions to (1.3) for $\alpha \in \{2, 4, 7\}$ (from left to right). Starting from a discrete bump function $c_k^0 = \max\{0, 1 - ((N/2 - k)/(\ell N))^2\}$ with $\ell = 0.1$ and $N = 2^{10}$. First line show the overall evolution and propagation of fronts, whereas the second line are zooms towards the tip of the support. We emphasize that in accordance to our result about positivity of solutions to the discrete system, the obtained numerical are also positive.

2 The discrete model

2.1 Discrete spaces and operators

Let $N \in \mathbb{N}$ be fixed in this section. On $\mathrm{L}^1(\mathbb{T}_N) \simeq \mathbb{R}^N$ we consider the canonical scalar product

$$\langle v, \psi \rangle_N = \frac{1}{N} \sum_{k=1}^{N} v_k \psi_k. \tag{2.1}$$

We write $f_+, f_- : [N] \to \mathbb{R}$ for the left/right translates of $f : [N] \to \mathbb{R}$ given by $(f_\pm)_k = f_{k\pm1}$ for all $k \in [N]$. We also have the forward and backward differential operators as well as discrete Laplace operator for such $f : [N] \to \mathbb{R}$ defined by

$$\partial_+^N f = N(f_+ - f)\,, \quad \partial_-^N f = N(f - f_-) \qquad \text{and} \quad \Delta^N f = N^2(f_- - 2f + f_+)\,.$$

We also note, that $\Delta^N = \partial_+^N \partial_-^N = \partial_-^N \partial_+^N$. The operator Δ^N is a symmetric linear operator with respect to the product (2.1), that is the integration by parts formulas hold

$$\langle \Delta^N \varphi, \psi \rangle_N = \langle \partial_+^N \partial_-^N \varphi, \psi \rangle_N = -\langle \partial_-^N \varphi, \partial_-^N \psi \rangle_N = \langle \varphi, \partial_+^N \partial_-^N \psi \rangle_N = \langle \varphi, \Delta^N \psi \rangle_N\,. \tag{2.2}$$

Next, we introduce the set of probability densities on the discrete torus by

$$\mathcal{P}^N = \left\{ c \in \mathrm{L}^1(\mathbb{T}_N) \;:\; \forall\, k \in [N] :\; c_k \geq 0, \;\; N^{-1} \sum_{k=1}^{N} c_k = 1 \right\} \subset \mathrm{L}^1(\mathbb{T}_N)\,,$$

as well as the subset of positive probability densities by

$$\mathcal{P}^N_{>0} = \left\{ c \in \mathcal{P}^N : c_k > 0 \;\forall\, k \in [N] \right\}.$$

For later reference, we define the variational derivative of a functional $E_N : \mathcal{P}^N \to \mathbb{R}$ at $c^* \in \mathcal{P}^N$ as the dual function $E'_N(c^*)$ with respect to the product (2.1); more explictly

$$(E'_N(c^*))_k = \left.\frac{\partial E_N(c)}{\partial c_k}\right|_{c=c^*} . \tag{2.3}$$

2.2 Well-posedness of the ODE system

We specify the assumptions on the activity function σ_α that describes the jump rates. As it turns out, we have some flexibility for the choice, which ensures both well-posedness and the convergence to the limit system (1.1). We recall that σ_α is homogeneous of degree $\alpha-2$, hence the function $\bar{\jmath}_\alpha$ defined below will be homogeneous of degree $\alpha > 0$.

Assumption 2.1. *For* $\alpha > 0$ *the* activity $\sigma_\alpha = \sigma_\alpha(c_{k-1}, c_k, c_{k+1}) =: \overline{\sigma}_\alpha(c_k, \sqrt{c_{k-1}c_{k+1}})$ *satisfies:*

- *(A1) The function* $\bar{\jmath}$ *defined via* $]0,\infty[^2 \ni (x,y) \mapsto \bar{\sigma}_\alpha(x,y)(x^2-y^2) =: \bar{\jmath}_\alpha(x,y)$ *has a continuous extension to* $[0,\infty[^2$*.*
- *(A2) The extended function* $\bar{\jmath}_\alpha : [0,\infty[^2 \to \mathbb{R}$ *satisfies* $\bar{\jmath}_\alpha(x,0) > 0$ *for all* $x > 0$*.*
- *(A3) For all* $x, y > 0$ *the following bounds hold:*

$$\max\{x^\alpha, y^\alpha\} \geq \overline{\sigma}_\alpha(x,y)\; xy \quad \textit{and} \quad \overline{\sigma}_\alpha(x,y) \;\geq\; \left(\frac{(x^\alpha - y^\alpha)}{\alpha(x-y)}\right)^2 \min\{1/x^\alpha, 1/y^\alpha\}. \tag{2.4}$$

It will be beneficial to rewrite the bound (2.4) in terms of the Stolarsky mean, which is a generalization of the logarithmic mean, to make use of many of its properties, see [Sto75].

Definition 2.2 (Stolarsky mean)**.** *For* $p \in \mathbb{R}$*, the* Stolarsky mean $\mathrm{s}_p : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is given by*

$$\mathrm{s}_p(x,y) = \left(\frac{x^p - y^p}{p\,(x-y)}\right)^{1/(p-1)},$$

with the cases $p = 0, 1$ *and* $\mathrm{s}_p(x,x) = x$ *understood as a limit.*

Hence, we can equivalently rewrite the lower bound (2.4) in terms of the Stolarsky mean as

$$\overline{\sigma}_\alpha(x,y) \quad \geq \quad \mathrm{s}_\alpha(x,y)^{2\alpha-2} \min\{1/x^\alpha, 1/y^\alpha\}.$$

In particular, we observe that the activity σ_α is thus $(\alpha-2)$-homogeneous and non-negative.

We also define the so-called *mobility function* by

$$m_\alpha(c_{k-1}, c_k, c_{k+1}) = \overline{m}_\alpha(c_k, \sqrt{c_{k-1}c_{k+1}}) := \overline{\sigma}_\alpha(c_k, \sqrt{c_{k-1}c_{k+1}}) \cdot (c_k\sqrt{c_{k-1}c_{k+1}}), \tag{2.5}$$

we observe that it satisfies thanks to Assumption (2.4) the inequality

$$\forall\, x, y \in [0,\infty[: \quad \max\{x^\alpha, y^\alpha\} \;\geq\; \overline{m}_\alpha(x,y) \;\geq\; \mathrm{s}_\alpha(x,y)^{2\alpha-2}\, xy\, \min\{1/x^\alpha, 1/y^\alpha\}. \tag{2.6}$$

Remark 2.3 (Discussion on the assumptions and examples). *The continuity in Assumption 2.1 is used in the proof of existence of the ODE system* (1.3)*. The upper bound on $\overline{\sigma}_\alpha$ provides that the mobility is of order α and is crucial for compactness to pass to the limit. The lower bound is needed for the proof of a liminf-estimate for the slope in the gradient-flow formulation.*

Some important cases are $\alpha = 1$ and $\alpha = 2$. For $\alpha = 1$, the classical DLSS equation, we observe that $\overline{\sigma}_\alpha$ has to satisfy

$$\max\{1/x, 1/y\} \quad \geq \quad \overline{\sigma}_\alpha(x,y) \quad \geq \quad \min\{1/x, 1/y\}.$$

*In particular, the scheme introduced in [MR*25] with the activitiy function $\frac{2}{x+y}$ is included in our analysis.*

For $\alpha = 2$, we have the inequality

$$\forall\, x, y \in [0,\infty[: \quad \frac{\max\{x^2, y^2\}}{xy} \quad \geq \quad \overline{\sigma}_2(x,y) \quad \geq \quad \frac{(x+y)^2}{4\max\{x^2,y^2\}},$$

where we have used that $\mathrm{s}_2(x,y) = \frac{x+y}{2}$ is the arithmetic mean. In particular, $\overline{\sigma}_2 = 1$, corresponding to the mass-action law for chemical reactions, is admissible.

The next lemma shows, that there is indeed for all $\alpha > 0$ an activity function σ_α that satisfies Assumptions 2.1.

Lemma 2.4 (Existence of activity σ_α).

1 *For all $\alpha \geq 0$, $x, y \geq 0$ we have the inequality* $\mathrm{s}_\alpha(x,y)^{\alpha-1}\sqrt{xy} \leq \frac{x^\alpha + y^\alpha}{2} \leq \max\{x^\alpha, y^\alpha\}$.

2 *The choices $\overline{\sigma}_\alpha(x,y) = \mathrm{s}_\alpha(x,y)^{2\alpha-2}\frac{2}{x^\alpha+y^\alpha}$ satisfy Assumptions 2.1.*

Proof. We prove the inequality of the first part by distinguishing two cases $\alpha \leq 1$ and $\alpha \geq 1$.

First, let $\alpha \in]0,1]$. Because the Stolarsky means s_p are monotone decreasing in their parameter $p \in \mathbb{R}$ and s_{-1} is the geometric mean [Sto75], we have that

$$\mathrm{s}_\alpha(x,y) \geq \mathrm{s}_{-1}(x,y) = \sqrt{xy} \quad \Rightarrow \quad \mathrm{s}_\alpha(x,y)^{\alpha-1} \leq \sqrt{xy}^{\alpha-1}.$$

Multiplying with $\sqrt{xy}$, we get with the AM-GM inequality the claim.

Now, let $\alpha \geq 1$. Observe that the generalized Stolarsky mean[1] $\mathrm{s}_{r,2r}$ [Sto75, (10)] is just the power mean $\left(\frac{x^r+y^r}{2}\right)^{1/r}$. Again, by using the monotonicity in both parameters of the generalized Stolarsky mean [Sto75, Theorem p. 89] and using that $\alpha \geq 1$, we have

$$\mathrm{s}_\alpha(x,y) = \mathrm{s}_{\alpha,1}(x,y) \leq \mathrm{s}_{\alpha,2\alpha}(x,y) \quad \Rightarrow \quad \mathrm{s}_\alpha(x,y)^{\alpha-1} \leq \mathrm{s}_{\alpha,2\alpha}(x,y)^{\alpha-1} = \Big(\frac{x^\alpha+y^\alpha}{2}\Big)^{\frac{\alpha-1}{\alpha}}.$$

Multiplying with $\sqrt{xy}$ and using that the power means are monotone, i.e. $\sqrt{xy} \leq \left(\frac{x^\alpha+y^\alpha}{2}\right)^{1/\alpha}$ for $\alpha \geq 1$ we get

$$\mathrm{s}_\alpha(x,y)^{\alpha-1}\sqrt{xy} \leq \Big(\frac{x^\alpha+y^\alpha}{2}\Big)^{\frac{\alpha-1}{\alpha}}\Big(\frac{x^\alpha+y^\alpha}{2}\Big)^{\frac{1}{\alpha}} = \frac{x^\alpha+y^\alpha}{2}.$$

[1]defined for $\alpha, \beta \in \mathbb{R}$ as $\mathrm{s}_{\alpha,\beta}(x,y) = \left(\frac{\beta(x-y)^\alpha}{\alpha(x-y)^\beta}\right)^{\frac{1}{\alpha-\beta}}$ with the undefined cases understood as limits.

Finally, we observe that by continuity the inequality also holds for $\alpha = 0$.

By the first part it follows immediately, that $\overline{\sigma}_\alpha$ satisfies (2.4). Hence, it suffices to show the properties of $\overline{\jmath}_\alpha$, where we now have $\overline{\jmath}_\alpha(x,y) = \mathrm{s}_\alpha(x,y)^{2\alpha-2}\frac{2}{x^\alpha+y^\alpha}(x^2-y^2)$ for $x,y>0$. The only singular limits are for $x = y$, $x = 0$, or $y = 0$, where the latter can be excluded by symmetry. For $y = 0$ and $x > 0$, we easily compute the continuous extension $\overline{\jmath}_\alpha(x,0) = \frac{2}{\alpha^2}x^\alpha$. For $x = y$, we use the monotonicity of the Stolarsky mean, which provides $\min\{x,y\} \le \mathrm{s}_\alpha(x,y) \le \max\{x,y\}$. Hence, for both cases $\alpha \ge 1$ and $\alpha < 1$, we compute the extension $\overline{\jmath}_\alpha(x,x) = 0$. This shows, that $\overline{\jmath}_\alpha$ can be extended continuously on the whole $[0,\infty)^2$. The definiteness in Assumption 2 is also clear. □

Our first result regards the well-posedness of (1.3) under Assumption 2.1.

Proposition 2.5 (Well-posedness of (1.3))**.** *For $\alpha > 0$, let σ_α satisfy Assumption 2.1. Then for all $c^0 \in \mathcal{P}^N$ exists a differentiable global solution $c : [0,\infty) \to \mathcal{P}^N$ solving* (1.3) *such that $c(t) \in \mathcal{P}^N_{>0}$ for all $t > 0$. For those solutions the discrete entropy* (1.4) *is a Lyapunov function.*

Proof. *Step 1. Local existence of positive solutions*: We start with a positive initial datum $c^0 \in \mathcal{P}^N_{>0}$. By Assumption 2.1, the diffusive flux

$$J_{\alpha,k}[c] = \sigma_\alpha(c_{k-1},c_k,c_{k+1})(c_k^2 - c_{k-1}c_{k+1}) = \bar{\sigma}_\alpha(c_k,\sqrt{c_{k-1}c_{k+1}})(c_k^2 - c_{k-1}c_{k+1})$$

in (1.3) is continuous for $c \in \mathcal{P}^N_{>0}$ and hence we obtain for some $\tau > 0$ a local solution $c : [0,\tau) \to \mathcal{P}^N_{>0}$ by the Peano existence theorem. We note that any such solution is mass conserving due to the presence of the discrete Laplacian in (1.3). Hence, the constructed solution is also bounded by N.

Step 2. Preservation of positivity: We now consider a local maximal positive solution on $[0,\tau)$. Since, it is bounded, we find a sequence $t_n \to \tau$ such that the limit $\lim_{n\to\infty} c(t_n) =: c(\tau) \in \mathcal{P}^N$ exists. Suppose that $c(\tau) \in \mathcal{P}^N \setminus \mathcal{P}^N_{>0}$, then we find some $k \in [N]$ such that

$$c_k(\tau) = 0, \quad \dot{c}_k(\tau) \le 0 \quad \text{and} \quad c_{k+1}(\tau) > 0$$

where we used the mass conservation and the continuity of the fluxes from Assumption 2.1 (A1). From the form of the flux and Assumption 2.1 (A2), we conclude

$$J_{\alpha,k}[c] \le 0, \quad J_{\alpha,k-1}[c] \ge 0 \quad \text{and} \quad J_{\alpha,k+1}[c] > 0\,.$$

Hence, from the discrete Laplacian in (1.3), we conclude

$$\dot{c}_k = N^2\Big(J_{\alpha,k-1}[c] - 2J_{\alpha,k}[c] + J_{\alpha,k+1}[c]\Big) > 0$$

which provides a contradiction to $\dot{c}_k \le 0$. Hence, we get $c(\tau) \in \mathcal{P}^N_{>0}$ and the constructed solutions exists globally in time on $[0,\infty)$.

Step 3. Extension to $\mathcal{P}^N$ and generation of positivity: For $c^0 \in \mathcal{P}^N$, we consider a sequence $c^{0,n} \in \mathcal{P}^N_{>0}$ such that $c^{0,n} \to c^0$. Consider the according solutions $c^n : [0,\infty) \to \mathcal{P}^N_{>0}$. Again by the continuity of the diffusive flux from Assumption 2.1 (A1), we find a pointwise limit $c(t) := \lim_{n\to\infty} c^n(t)$ starting from c^0, which is differentiable satisfying (1.3). Suppose $c(t) \notin \mathcal{P}^N_{>0}$ for $t > 0$, then continuity implies that there exists $k \in [N]$ such that $c_k(0) = 0 = \dot{c}_k(0)$ and without loss of generality $c_{k+1}(0) > 0$, which by the same argument as in Step 2 provides a contradiction.

Step 4. Lyapunov functional: By the positivity and differentiability of the constructed solutions, we can calculate for all $t > 0$ using the integration by parts (2.2) and exponential rewriting of the diffusive flux (1.6)

$$\frac{\mathrm{d}}{\mathrm{d}t} E_N(c(t)) = \frac{1}{N} \sum_{k\in[N]} \log c_k(t) \Delta^N J_{\alpha,k}(c(t))$$
$$= -\frac{1}{N} \sum_{k\in[N]} \Delta^N \log c_k(t)\ \sigma_\alpha(c_{k-1}(t), c_k(t), c_{k+1}(t))\ c_k(t)^2\ N^2 \Big[\exp\Big(N^{-2}\Delta^N \log c_k(t)\Big) - 1\Big] \le 0\,,$$

where the last inequality follows from the fact that $\mathbb{R} \ni x \mapsto x(\exp(x) - 1) \ge 0$. Since $E_N : \mathcal{P}^N \to [0, \infty)$ is continuous, we conclude that $t \mapsto E_N(c(t))$ is decreasing on $[0, \infty)$. □

2.3 Discrete gradient structure in continuity equation format

We now describe the gradient structure for (1.3) on $\mathcal{P}^N$ in more detail. For this we use gradient systems in continuity equation format as in [PeS23, HPS24, HMS25]. Note that here the situation simplifies because the spaces for the concentrations and for the fluxes have the same dimension, because we are in one dimension. The first ingredient is the abstract gradient map which is given by the discrete Laplace operator Δ^N. Secondly, the discrete energy is given in terms of the entropy (1.4). By using the notation (2.3), we have the identity

$$(E_N'(c))_{k\in[N]} = \Big(\log c_k\Big)_{k\in[N]}.$$

The final element is the (dual) dissipation potential $R^*_{\alpha,N}$. For fixed $\alpha > 0$, let the activity $\sigma_\alpha : [0, \infty[^3 \to [0, \infty[$ be given and recall the mobility m_α from (2.5):

$$m_\alpha(c_{k-1}, c_k, c_{k+1}) := \sigma_\alpha(c_k, c_{k-1}, c_{k+1}) \cdot (c_k \sqrt{c_{k-1}c_{k+1}})\,,$$

we observe that provided σ_α is homogeneous of degree $\alpha - 2$, then m_α is homogeneous of degree α, justifying its name. Now, we define for $c \in \mathcal{P}^N$ and any $\xi : [N] \to \mathbb{R}$ the dual dissipation potential by

$$R^*_{\alpha,N}(c, \xi) = \frac{1}{N} \sum_{k=1}^{N} N^4 m_\alpha(c_{k-1}, c_k, c_{k+1}) \mathsf{C}^*\Big(N^{-2}\xi_k\Big),$$

where we have $\mathsf{C}^*(r) = 4\,(\cosh(r/2) - 1)$. Here, the superscript $*$ refers to the Legendre duality. Since, C^* is convex, we get its convex primal function in terms of

$$\mathsf{C}(s) = \inf_{r\in\mathbb{R}} (rs - \mathsf{C}^*(r))\,.$$

Instead of providing the explicit form, it is more conventient, to recall the basic identity

$$\mathsf{C}^{*,\prime}(r) = 2\sinh(r/2) \qquad \text{and} \qquad \mathsf{C}'(s) = (\mathsf{C}^{*,\prime})^{-1}(s) = 2\,\mathrm{arsinh}(s/2)\,,$$

that is C is the primitive of $2\,\mathrm{arsinh}(s/2)$ with $\mathsf{C}(0) = 0$. Besides C, we define its perspective function $\mathbb{C} : \mathbb{R} \times [0, \infty]$ defined by

$$\mathbb{C}(s|w) = \begin{cases} w\mathsf{C}\Big(\frac{s}{w}\Big), & w > 0\,; \\ 0, & s = 0, w = 0\,; \\ +\infty, & s \ne 0, w = 0\,. \end{cases} \tag{2.7}$$

The function C and its perspective function $\mathfrak{C}$ satisfy the following crucial estimate

$$\forall\, s \in \mathbb{R}, w \in [0,\infty[\quad \forall\, q > 1: \quad \mathsf{C}(s) \le \frac{q}{q-1}\mathfrak{C}(s|w) + \frac{4w^q}{q-1} \tag{2.8}$$

and the monotonicity property

$$\forall\, s \in \mathbb{R}, \forall\, w \in [0,\infty[: \quad]0,\infty[\ni \lambda \mapsto \mathfrak{C}(\lambda s|\lambda^2 w) \quad \text{is increasing,} \tag{2.9}$$

For the proofs of (2.8) and (2.9), we refer to [HMS25].

Fixing a time horizon $T > 0$, and an initial datum $c_0 \in \mathcal{P}^N_{>0}$ the gradient system with components $(\mathcal{P}^N, \Delta^N, E_N, R^*_{\alpha,N})$ induces on $[0,T]$ the *gradient-flow* equation

$$\begin{aligned} \dot c &= \Delta^N J^N, && \text{on }]0,T] && (2.10a)\\ J^N &= \partial_\xi R^*_{\alpha,N}(c, -\Delta^N E'_N(c)), && \text{on }]0,T] && (2.10b)\\ c(0) &= c_0. \end{aligned}$$

The first equation (2.10a) is the second-order continuity equation involving the discrete Laplacian. Note that for all curves $c \in \mathsf{AC}([0,T];\mathcal{P}^N)$, thanks to the conservation of mass, $\dot c$ has mean-zero, that is $\sum_{k\in[N]} \dot c_k = 0$. Since Δ^N is invertible on mean-zero densities and hence for a.e. $t \in [0,T]$ exists a curve of fluxes $J^N(t) : [N] \to \mathbb{R}$ such that $\Delta^N J^N = \dot c$. In the following, the set of curves (c^N, J^N) satisfying (2.10a) is denoted by CE_N.

To make the equation (2.10b) more explicit, we observe that $\partial_{\xi_k} R^*_{\alpha,N} = N^2 m_\alpha \mathsf{C}^{*\prime}(N^{-2}\xi_k)$. Inserting $\xi_k = -(\Delta^N E'_N(c))_k$, using the logarithmic rule $\mathsf{C}^{*\prime}(\log c_{k-1} - 2\log c_k + \log c_{k+1}) = \frac{c_k^2 - c_{k-1}c_{k+1}}{c_k\sqrt{c_{k-1}c_{k+1}}}$ and the explicit choice of the mobility m_α, we obtain

$$J^N_k = N^2 \sigma_\alpha(c_{k-1}, c_k, c_{k+1})(c_k^2 - c_{k-1}c_{k+1}),$$

which provides exactly the equation (1.3).

2.4 Variational characterization via the energy dissipation principle

One important tool between the gradient system and the evolution equation is the so-called *energy-dissipation principle (EDP)*, which provides a variational formulation for the system (2.10) in which we pass to the limit $N \to \infty$. For this, we define by Legendre transform the primal dissipation potential $R_{\alpha,N}$, which for $c \in \mathcal{P}^N$ and $J : [n] \to \mathbb{R}$ is given by

$$R_{\alpha,N}(c,J) = \frac{1}{N}\sum_{k=1}^N N^2 \mathfrak{C}\Big(J_k | N^2 m_\alpha(c_{k-1}, c_k, c_{k+1})\Big) = \frac{1}{N}\sum_{k=1}^N \mathfrak{C}\Big(N^2 J_k | N^4 m_\alpha(c_{k-1}, c_k, c_{k+1})\Big),$$

where the so-called *perspective function* $\mathfrak{C}$ is defined in (2.7). In addition, we define the *discrete relaxed slope* for $c \in \mathcal{P}^N_{\ge 0}$ by

$$S_{\alpha,N}(c) := R^*_{\alpha,N}(c, -\Delta^N E'_N(c)) = \frac{1}{N}\sum_{k=1}^N 2N^4 \sigma_\alpha(c_{k-1}, c_k, c_{k+1}) \left(c_k - \sqrt{c_{k-1}c_{k+1}}\right)^2 , \tag{2.11}$$

where we use for $a, b > 0$ the identity

$$\sqrt{ab}\, \mathsf{C}^*(\log a - \log b) = 2\Big(\sqrt{a} - \sqrt{b}\Big)^2 .$$

Equation (2.11) and the rewriting in terms of $\overline{\jmath}_\alpha$ with Assumption 2.1 thus shows, that the definition of $S_{\alpha,N} : \mathcal{P}^N_{>0} \to \mathbb{R}$ can be continuously extended to $\mathcal{P}^N_{\geq 0}$, which is what we use from now on.

The primal dissipation functional and the slope-term give rise to the total dissipation functional defined for curve of a pair of concentration $c \in \mathsf{AC}([0,T],\mathcal{P}^N)$ and flux $J^N : [0,T] \to \mathrm{L}^1(\mathbb{T}_N)$ by

$$D_{\alpha,N}(c,J) = \begin{cases} \int_0^T \Big(R_{\alpha,N}(c,J) + S_{\alpha,N}(c)\Big)\,\mathsf{d}t & \text{for } (c,J) \in \mathrm{CE}_N\,; \\ +\infty & \text{for } (c,J) \notin \mathrm{CE}_N\,. \end{cases} \tag{2.12}$$

In total, we defined now all ingredients for the full energy-dissipation functional $L_{\alpha,N}$ in (1.5) and are now in the position to prove the second part of Result A.

Proposition 2.6. *For $\alpha > 0$, let σ_α satisfy Assumptions 2.1. The solution $c : \mathsf{AC}([0,T],\mathcal{P}^N)$ of equation* (1.3) *minimizes the full energy-dissipation functional, i.e. $L_{\alpha,N}(c,J) = 0$.*

Proof. By Proposition 2.5, we now that $[0,T] \ni t \mapsto c(t)$ is a classical solution of (1.3) and we have $c(t)_k > 0$ for all $k \in [N]$ and $t > 0$. In particular, we have $E'_N(c)_k = \log c_k$. The energy is differentiable, and we obtain on any subinterval $[s,t] \subset [0,T]$ that

$$E_N(c(t)) - E_N(c(s)) = \int_s^t \frac{\mathsf{d}}{\mathsf{d}r} E_N(c(r))\,\mathsf{d}r = \int_s^t \frac{1}{N} \sum_{k\in[N]} \Delta^N \log c_k(t) J_{\alpha,k}(c(t))\,\mathsf{d}t,$$

where we have used that Δ^N is self-adjoint. By (2.10b) we have $J_{\alpha,N} = \partial_\xi R^*_{\alpha,N}(c, -\Delta \log c)$, which provides by Young-Fenchel duality, that

$$E_N(c(t)) - E_N(c(s)) = -\int_s^t \tfrac{1}{N} \sum_{k\in[N]} (-\Delta^N \log c_k) J_{\alpha,k}(c)\,\mathsf{d}t = -\int_s^t S_{\alpha,N}(c) + R_{\alpha,N}(c,J)\,\mathsf{d}t.$$

Hence, $L_{\alpha,N}(c,J) = 0$. □

Remark 2.7. *Although not needed in the main existence result, we briefly remark that also the converse statement of Proposition 2.6 holds true, i.e., curves $(c,J) \in \mathrm{CE}_N$, $c \in \mathsf{AC}([0,T],\mathcal{P}^N)$ with $L_{\alpha,N}(c,J) = 0$ are solutions of the evolution equation* (1.3) *satisfying an entropy dissipation balance for E_N. For this, one uses that $\overline{\sigma}_\alpha(x,y) = 0$ iff $x = y = 0$ (by the lower bound) and Prop. 6.1 and equation (6.3) from [HMS25], which provides the identification of the flux J^N.*

3 EDP convergence to continuous system

3.1 Continuous spaces and functionals

The continuous space is $\mathbb{T}$, identified with $[0,1]$ and periodic boundary conditions. We consider the state space $\mathrm{L}^1(\mathbb{T})_{\geq 0}$ with the usual dual pairing by $\langle v, \xi\rangle = \int_\mathbb{T} v(x)\xi(x)\,\mathsf{d}x$. We will also use the Bochner space on the time interval $[0,T]$, denoted by $\mathrm{L}^p([0,T],\mathrm{L}^q(\mathbb{T})) =: \mathrm{L}^p\,\mathrm{L}^q$, and similarly $\mathrm{L}^p([0,T],\mathrm{W}^{1,q}(\mathbb{T})) =: \mathrm{L}^p\,\mathrm{W}^{1,q}$. The differential operators are denoted as usual by

$$\partial_x : \mathrm{W}^{1,1}(\mathbb{T}) \to \mathrm{L}^1(\mathbb{T}), \qquad \Delta : \mathrm{W}^{2,1}(\mathbb{T}) \to \mathrm{L}^1(\mathbb{T}).$$

For the gradient system in continuity equation format $(X, \Delta, \mathcal{E}, \mathcal{R})$ with $X := \mathrm{L}^1(\mathbb{T})_{\geq 0}$, we consider the Laplacian Δ as the abstract gradient map. Moreover, we define the (quadratic) dual dissipation potential as

$$\mathcal{R}^*_\alpha(\rho, \eta) = \frac{1}{2} \int_{\mathbb{T}} \rho(x)^\alpha \eta(x)^2 \, \mathrm{d}x. \tag{3.1}$$

For a fixed time horizon $T > 0$ the gradient system $(X, \Delta, \mathcal{E}, \mathcal{R}^*_\alpha)$ induces on $[0, T]$ a *gradient-flow equation*

$$\dot\rho = \Delta j \tag{3.2a}$$
$$j = \partial_\eta \mathcal{R}^*_\alpha(\rho, -\Delta \mathrm{D}\mathcal{E}(\rho)) \tag{3.2b}$$
$$\rho(0) = \rho_0.$$

The first equation (3.2a) is the second-order continuity equation involving the Laplacian, which is understood in the weak-sense, i.e.

$$\forall\, \phi \in \mathrm{C}^2(\mathbb{T}): \qquad \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} \phi\, \rho \, \mathrm{d}x = \int_{\mathbb{T}} \Delta\phi\, j \, \mathrm{d}x. \tag{3.3}$$

Note, that restricting the domain of the Laplacian to functions with average zero, it becomes invertible and selfadjoint i.e.

$$\Delta : H^2_{\#}(\mathbb{T}) := H^2(\mathbb{T}) \cap \Big\{ f \text{ periodic } : \int_{\mathbb{T}} f = 0 \Big\} \to \mathrm{L}^2(\mathbb{T}).$$

Similarly, we use $\mathrm{L}^p_{\#}(\mathbb{T})$, etc. for other function spaces with average zero. We note that Δ^{-1} : $\mathrm{L}^p_{\#}(\mathbb{T}) \to \mathrm{W}^{2,p}_{\#}(\mathbb{T})$ is bounded for all $p \in [1, \infty]$. In particular, for a curve $\rho \in \mathrm{W}^{1,1}(0, T; \mathbb{T})$ it follows that there exists for a.e. $t \in [0, T]$ a flux $j(t) \in \mathrm{W}^{2,1}(\mathbb{T})$ such that $\dot\rho = \Delta j$. The set of curves (ρ, j) is denoted by CE.

For the second equation (3.2b), we consider positive concentrations ρ and observe $\mathrm{D}\mathcal{E}(\rho) = \log \rho$. Using $\partial_\eta \mathcal{R}^*_\alpha(\rho, \eta) = \rho^\alpha \eta$, we derive formally the flux $j = -\rho^\alpha \Delta(\log \rho)$ which provides exactly the equation (1.1). We highlight the two cases $\alpha = 1$, corresponding to the usual DLSS equation $\dot\rho = -\Delta\Big(\Delta\rho + \frac{(\partial_x \rho)^2}{\rho}\Big)$, and $\alpha = 2$, which originates on the discrete level from the mass-action law in chemical kinetics and the according space-continuous equation is of the form $\dot\rho = -\Delta\Big(\rho\Delta\rho - (\partial_x\rho)^2\Big)$.

3.2 Variational characterization of DLSS$_\alpha$

Analogously to the discrete setting, we now derive the energy-dissipation balance. The primal dissipation potential $\mathcal{R}_\alpha$ is again quadratic and, by duality to the dual (3.1), given by (1.10).

The next lemma shows that the tentative definition of the slope given by the functional $\mathcal{S}_{\alpha,+}$ in (1.11) has a relaxed lower semicontinuous envelope for non-negative Sobolev functions, which we call the *relaxed slope* $\mathcal{S}_\alpha$ and it is the main ingredient in our analysis.

Lemma 3.1 (Relaxed slope)**.** *for all smooth positive ρ it holds that $\mathcal{S}_{\alpha,+}(\rho) = \mathcal{S}_\alpha(\rho)$. Hereby, the relaxed slope $\mathcal{S}_\alpha$ is equivalently defined for all non-negative density $\rho : \mathbb{T} \to [0, \infty[$ with*

$\rho^{\alpha/2} \in \mathrm{W}^{2,2}(\mathbb{T})$ *and* $\rho^{\alpha/4} \in \mathrm{W}^{1,4}(\mathbb{T})$ *by*

$$\mathcal{S}_\alpha(\rho) := \frac{2}{\alpha^2} \int_{\mathbb{T}} \Big(\Delta\rho^{\alpha/2} - 4(\partial_x \rho^{\alpha/4})^2 \Big)^2 \mathrm{d}x \tag{3.4a}$$

$$= \frac{2}{\alpha^2} \int_{\mathbb{T}} \Big(\big(\Delta\rho^{\alpha/2}\big)^2 + \frac{16}{3}\big(\partial_x \rho^{\alpha/4}\big)^4 \Big) \,\mathrm{d}x \tag{3.4b}$$

$$= \frac{2}{\alpha^2} \int_{\mathbb{T}} \frac{\Big(\rho^{\alpha/2}\Delta\rho^{\alpha/2} - \big(\partial_x \rho^{\alpha/2}\big)^2 \Big)^2}{\rho^\alpha} \,\mathrm{d}x \tag{3.4c}$$

$$= \frac{1}{2} \int_{\mathbb{T}} \Big(\frac{(\Delta\rho)^2}{\rho^{2-\alpha}} + \frac{2\alpha-3}{3} \frac{(\partial_x\rho)^4}{\rho^{4-\alpha}} \Big) \,\mathrm{d}x \tag{3.4d}$$

Remark 3.2. *The several different forms of the slope will have their different advantages in various situations:*

(3.4a) *shows that the integrand of* $\mathcal{S}_\alpha$ *equals* $\frac{1}{2}\Sigma^2$ *with* $\Sigma = -\frac{2}{\alpha}\Big(\Delta\rho^{\alpha/2} - 4(\partial_x\rho^{\alpha/4})^2\Big)$ *which plays a prominent role in the proof of the chain-rule (Proposition 4.2);*

(3.4b) *shows that* $\mathcal{S}_\alpha$ *controls the* L^2 *norms of* $\Delta\rho^{\alpha/2}$ *and* $(\partial_x\rho^{\alpha/4})^2$ *justifying the assumption of the Lemma on* ρ*;*

(3.4c) *characterizes the integrand of the slope* $\mathcal{S}_\alpha$ *by a jointly convex function* u^2/v*, which is used in the proof of the liminf estimate;*

(3.4d) *shows that* $\mathcal{S}_\alpha$ *is convex for the range* $\alpha \in [3/2, 2]$.

The proof of (3.4) relies on the classical chain rule for Sobolev functions and the integration-by-parts rule

$$0 = \int_{\mathbb{T}} \partial_x\Big(u^\gamma\,(\partial_x u)^3\Big)\,\mathrm{d}x = \gamma \int_{\mathbb{T}} u^{\gamma-1}(\partial_x u)^4\,\mathrm{d}x + 3\int_{\mathbb{T}} u^\gamma\,(\partial_x u)^2 \Delta u\,\mathrm{d}x, \tag{3.5}$$

which holds for smooth and positive u and all $\gamma \in \mathbb{R}$.

Proof. For a positive smooth function $\rho : \mathbb{T} \to]0,\infty[$, a direct computations shows

$$\rho^\alpha\,(\Delta\log\rho)^2 = \Big(\rho^{\alpha/2-1}\Delta\rho - \rho^{\alpha/2-2}\,(\partial_x\rho)^2\Big)^2.$$

Using that $\rho^{\alpha/2-1}\Delta\rho = \frac{2}{\alpha}\Delta\rho^{\alpha/2} + (1-\alpha/2)\rho^{\alpha/2-2}\,(\partial_x\rho)^2$ and $\partial_x\rho^{\alpha/4} = \frac{\alpha}{4}\rho^{\alpha/4-1}\partial_x\rho$, we obtain $\rho^\alpha\,(\Delta\log\rho)^2 = \Big(\frac{2}{\alpha}\Big(\Delta\rho^{\alpha/2} - (2\partial_x\rho^{\alpha/4})^2\Big)\Big)^2$, which proves the first formula (3.4a).

Moreover, we have

$$\Big(\frac{2}{\alpha}\Big(\Delta\rho^{\alpha/2} - 4(\partial_x\rho^{\alpha/4})^2\Big)\Big)^2 = \frac{4}{\alpha^2}\Big((\Delta\rho^{\alpha/2})^2 - 8\Delta\rho^{\alpha/2}(\partial_x\rho^{\alpha/4})^2 + 16(\partial_x\rho^{\alpha/4})^4\Big).$$

The integral of the middle term in the last expression can be simplified by using (3.5) with $\gamma = 1$ and $u = \rho^{\alpha/4}$ and exploiting the identity $\Delta(u^2) = 2u\Delta u + 2(\partial_x u)^2$, namely

$$\int_{\mathbb{T}}\Big(\partial_x\rho^{\alpha/4}\Big)^2\Delta\rho^{\alpha/2}\,\mathrm{d}x = \int_{\mathbb{T}}(\partial_x u)^2\Delta u^2\,\mathrm{d}x = 2\int_{\mathbb{T}}\Big(u(\partial_x u)^2\Delta u + (\partial_x u)^4\Big)\,\mathrm{d}x = \frac{4}{3}\int_{\mathbb{T}}\Big(\partial_x\rho^{\alpha/4}\Big)^4\,\mathrm{d}x.$$

With this, (3.4b) is established.

Using $\frac{(\partial_x \rho^{\alpha/2})^2}{\rho^{\alpha/2}} = 4(\partial_x \rho^{\alpha/4})^2$, we obtain (3.4c) from (3.4a). To obtain the last form (3.4d), we re-express the integrand in (3.4b) by powers of ρ, $\partial_x\rho$ and $\nabla\rho$. From $(\Delta\rho^{\alpha/2})^2$ we obtain a mixed term $\rho^{\alpha-3}(\partial_x\rho)^2\Delta\rho$ which is integrated by parts via (3.5) with $\gamma = \alpha-3$ and $u = \rho$. After some cancellations we obtain the desired form (3.4d).

Thus, (3.4a)–(3.4d) hold for positive and smooth ρ. Since the right-hand sides can be continuously extended to all non-negative ρ with $\rho^{\alpha/2} \in \mathrm{W}^{2,2}(\mathbb{T})$ and $\rho^{\alpha/4} \in \mathrm{W}^{1,4}(\mathbb{T})$, the claim follows. □

Remark 3.3 (Critical spaces)**.** *Equation* (3.4b) *shows that the slope is given as a sum of two terms involving a second and first order differential operator, respectively. However, the term* $(\partial_x\rho^{\alpha/4})^4$ *cannot be treated as a lower order term w.r.t.* $(\Delta\rho^{\alpha/2})^2$ *in the proof of the convergence result in Section 3. Indeed, one can show that the mapping* $v \mapsto w = \sqrt{v}$ *is bounded and weakly continuous from* $\mathrm{H}^2_{\geq 0}(\mathbb{T}) := \left\{ v \in \mathrm{H}^2(\mathbb{T}) \,\middle|\, v \geq 0 \text{ a.e.} \right\}$ *into* $\mathrm{W}^{1,4}(\mathbb{T})$*, but not compact.*
The boundedness follows by observing that from $\Delta v = \Delta w^2 = 2w\Delta w + 2(\nabla w)^2$ *combined in* (3.5) *with* $\gamma = 1$ *and* $u = v$*, we obtain*

$$\int_{\mathbb{T}} (\nabla w)^4 \,\mathrm{d}x = \frac{3}{4}\int_{\mathbb{T}} (\nabla w)^2 \Delta v \,\mathrm{d}x \quad \textit{implying} \quad \int_{\mathbb{T}} (\nabla w)^4 \,\mathrm{d}x \leq \frac{9}{16}\int_{\mathbb{T}} (\Delta v)^2 \,\mathrm{d}x\,.$$

From this, we easily see that $v \in \mathrm{H}^2_{\geq 0}(\mathbb{T})$ *implies* $w = \sqrt{v} \in \mathrm{W}^{1,4}(\mathbb{T})$ *with* $\|w\|^4_{\mathrm{W}^{1,4}} \leq \|v\|^2_{\mathrm{H}^2}$.
For the non-compactness, we consider a sequence $v_m \rightharpoonup v$ *in* $\mathrm{H}^2_{\geq 0}(\mathbb{T})$ *and define* $w_m = \sqrt{v_m}$ *and* $w = \sqrt{v}$. *Using* $|\sqrt{a}-\sqrt{b}| \leq \sqrt{|a-b|}$ *we find* $\|w_m - w\|^4_{\mathrm{L}^4} \leq \|v_m - v\|^2_{\mathrm{L}^2} \to 0$ *and conclude* $w_m \rightharpoonup w$ *in* $\mathrm{W}^{1,4}(\mathbb{T})$. *Hence, to show that the mapping is not compact, we construct a weakly converging sequence* $(u_m)_m$ *in* $\mathrm{H}^2(\mathbb{T})$ *such that* $w_m = \sqrt{v_m}$ *does not converge strongly. For* $m \in \mathbb{N}$*, we set*

$$v_m(x) = \frac{1}{m^{1/2}}|\sin(2\pi x)|^{3/2+2/m} \quad \textit{and} \quad w_m(x) = \sqrt{v_m(x)} = \frac{1}{m^{1/4}}|\sin(2\pi x)|^{3/4+1/m}$$

We easily obtain $\|w_m\|^4_{\mathrm{L}^4} = \|u_m\|^2_{\mathrm{L}^2} \leq C/m$. *Moreover, an explicit calculation gives*

$$\|\nabla w_m\|^4_{\mathrm{L}^4} \to C_1 > 0 \quad \textit{and} \quad \|\Delta v_m\|^2_{\mathrm{L}^2} \to C_2 > 0.$$

Thus, $w_m \rightharpoonup 0$ *in* $\mathrm{W}^{1,4}(\mathbb{T})$ *and* $v_m \rightharpoonup 0$ *in* $\mathrm{H}^2(\mathbb{T})$*, but strong convergence fails in both cases.*

With the relaxed slope and the primal dissipation potential we now define the total dissipation functional for a curve of a pair of density ρ and flux j by

$$\mathcal{D}_\alpha(\rho, j) := \begin{cases} \int_0^T \Big(\mathcal{R}_\alpha(\rho, j) + \mathcal{S}_\alpha(\rho)\Big)\,\mathrm{d}t & \text{for } (\rho, j) \in \mathrm{CE}, \\ +\infty & \text{for } (\rho, j) \notin \mathrm{CE}, \end{cases} \tag{3.6}$$

where CE denotes solutions the second-order continuity equation (3.3). In this way, we have introduced all ingredients to define energy dissipation functional (1.9) to formulate our notions of solutions. For this, we distinguish curves that satisfy the *energy-dissipation inequality* (EDI) and the *energy-dissipation balance* (EDB).

Definition 3.4. *A curve* $\rho : [0, T] \to \mathrm{L}^1(\mathbb{T})$ *is an EDI solution of* (1.1) *if* $\sup_{t\in[0,T]} \mathcal{E}(\rho(t)) < \infty$ *and if there exists a curve of fluxes* $j : [0, T] \to \mathrm{L}^1(\mathbb{T})$ *such that* $(\rho, j) \in \mathrm{CE}$ *and* $\mathcal{L}_\alpha(\rho, j) \leq 0$. *If, in addition,* (ρ, j) *satisfies for all* $[s, t] \subset [0, T]$ *the balance equation*

$$\mathcal{E}(\rho(t)) - \mathcal{E}(\rho(s)) + \int_s^t \Big(\mathcal{R}_\alpha(\rho(r), j(r)) + \mathcal{S}_\alpha(\rho(r))\Big)\,\mathrm{d}r = 0,$$

then $\rho : [0, T] \to \mathrm{L}^1(\mathbb{T})$ *is called EDB solution.*

3.3 Embeddings

The convergence result of the discrete energy-dissipation functional $L_{\alpha,N}$ towards its continuous analog $\mathcal{L}_\alpha$ will be formulated with the help of suitable embeddings of the discrete quantities into the continuum.

Embedding of densities

A discrete concentration vector $c \in \mathrm{L}^1(\mathbb{T}_N)$ is turned into a density on $\mathbb{T} = [0,1]_{0\sim 1}$ by a piecewise constant interpolation $\iota : \mathrm{L}^1(\mathbb{T}_N) \to \mathrm{L}^1(\mathbb{T})$ defined by

$$\Big(\iota_N c^N\Big)(x) := \rho^N(x) := \sum_{k=0}^{N-1} c_k^N \mathbb{1}_{[k/N,(k+1)/N[}(x)\,. \tag{3.7}$$

The dual embedding is an integral average given by

$$(\iota_N^* \phi)_k = N \int_{k/N}^{(k+1)/N} \phi(x)\,\mathsf{d}x\,.$$

Indeed, we have the identity $\langle \iota_N c^N, \phi\rangle = \frac{1}{N}\sum_{k=1}^N c_k^N \; N\int_{k/N}^{(k+1)/N} \phi\,\mathsf{d}x = \langle c^N, \iota_N^*\phi\rangle_N$. In particular, we have for the embedded density from (3.7) the identity

$$\int_{\mathbb{T}} \rho^N \,\mathsf{d}x = \sum_{k=1}^N \int_{k/N}^{(k+1)/N} c_k^N \,\mathsf{d}x = \frac{1}{N}\sum_k c_k^N\,.$$

For the discrete spaces we use for $p \geq 1$ the following norms

$$\|c^N\|_{\mathrm{L}^p_N} := \left(\frac{1}{N}\sum_{k=1}^N |c_k^N|^p\right)^{1/p}$$

and observe that the embedding $\iota_N : (\mathrm{L}^1(\mathbb{T}_N), \|\cdot\|_{\mathrm{L}^p_N}) \to (\mathrm{L}^1(\mathbb{T}), \|\cdot\|_{\mathrm{L}^p(\mathbb{T})})$ is norm-preserving

$$\|\iota_N c\|^p_{\mathrm{L}^p(\mathbb{T})} = \int_{\mathbb{T}} |\iota_N c(x)|^p \,\mathsf{d}x = \frac{1}{N}\sum_{k=1}^N |c_k|^p = \|c\|^p_{\mathrm{L}^p_N}\,.$$

Embedding of fluxes

Using the discrete Laplacian Δ^N and the usual Laplacian Δ, we define for the fluxes the following embedding operator

$$\mathcal{I}_N J^N(x) := \Big(\Delta^{-1}\iota_N \Delta^N J^N\Big)(x). \tag{3.8}$$

A simple calculation shows that this embedding is consistent with both continuity equations, the discrete (2.10a) and the continuous (3.2a). Indeed, we have the equivalent formulations

$$\Big(\iota_N c^N, \mathcal{I}_N J^N\Big) \in \mathrm{CE} \quad \Leftrightarrow \quad \Big(c^N, J^N\Big) \in \mathrm{CE}_N, \tag{3.9}$$

because a simple computation shows for all $\phi \in \mathcal{D}(\mathbb{T})$ and $\xi = \iota_N^* \phi \in \mathcal{D}(\mathbb{T}_N)$ that

$$\Big(\iota_N c^N, \mathcal{I}_N J^N\Big) \in \mathrm{CE} \quad \Leftrightarrow \quad \forall\, \phi \in \mathcal{D}(\mathbb{T}):\ \langle \iota_N \dot{c}^N, \phi\rangle = \langle \mathcal{I}_N J^N, \Delta\phi\rangle = \langle \iota_N \Delta^N J^N, \phi\rangle,$$
$$\text{and } \Big(c^N, J^N\Big) \in \mathrm{CE}_N \quad \Leftrightarrow \quad \forall\, \xi \in \mathcal{D}(\mathbb{T}_N):\ \langle \dot{c}^N, \xi\rangle_N = \langle \Delta^N J^N, \xi\rangle_N.$$

Moreover, we introduce the discrete analogue of the classical (homogeneous) Sobolev spaces by the norms

$$\|\phi\|^p_{\mathrm{W}^{2,p}_N} := \frac{1}{N}\sum_k \Big|(\Delta^N \phi)_k\Big|^p = \|\Delta^N \phi\|^p_{\mathrm{L}^p_N},$$

In particular, we have that

$$\Delta^N : \mathrm{W}^{2,p}_N \to \mathrm{L}^p_N \qquad \text{is an isomorphism.}$$

Since, $\iota_N : \mathrm{L}^p_N \to \mathrm{L}^p$ is bounded, and $\Delta^{-1} : \mathrm{L}^p \to \mathrm{W}^{2,p}$ is bounded, we conclude for the flux embedding in (3.8) that $\mathcal{I}_N = \Delta^{-1}\iota_N \Delta^N : \mathrm{W}^{2,p}_N \to \mathrm{W}^{2,p}$ is also bounded.

Lemma 3.5 (Flux embedding)**.** *for all discrete flux $J^N \in \mathrm{L}^1_N$ it holds $\|\mathcal{I}_N J^N\|_{\mathrm{L}^1(\mathbb{T})} \le 2\|J^N\|_{\mathrm{L}^1_N}$. If the sequence $(J^N)_{N\in\mathbb{N}}$ is uniformly bounded, i.e. $\sup_N \|J^N\|_{\mathrm{L}^1_N} < \infty$, then there exists a subsequence such that both embedded fluxes $\iota_N J^N \in \mathrm{L}^1$ and $\mathcal{I}_N J^N \in \mathrm{L}^1$ converges as finite measures in $\mathcal{M}(\mathbb{T})$ and their limits coincide.*

Proof. We first derive the estimate for the fluxes. Let us fix $N \in \mathbb{N}$. First, take $\phi \in \mathrm{W}^{2,\infty}(\mathbb{T}) \subset C^{1,\gamma}(\mathbb{T})$ for all $\gamma \in [0,1[$. By the mean-value theorem, for all $x \in [0,1]$ there are $x_-, x_+ \in [0, \frac{1}{N}]$ such that

$$\phi(x - N^{-1}) - \phi(x) = N^{-1}\phi'(x - x_-), \quad \phi(x) - \phi(x + N^{-1}) = N^{-1}\phi'(x + x_+).$$

Hence, we have that

$$\phi(x-N^{-1}) - 2\phi(x) + \phi(x+N^{-1}) = \frac{1}{N}\Big(\phi'(x-x_-) - \phi'(x+x_+)\Big) = -\frac{1}{N}\int_{-x_-}^{x_+} \phi''(x+t)\,\mathrm{d}t,$$

where we have used that ϕ' is absolutely continuous. In particular, we have for all $x \in [0,1]$ that

$$\Big|\phi(x - N^{-1}) - 2\phi(x) + \phi(x + N^{-1})\Big| \le \frac{2}{N^2}\|\phi''\|_\infty.$$

Hence, we get that for all k that

$$\Big|\Big(\Delta^N \iota_N^* \phi\Big)_k\Big| = N\bigg|\int_{k/N}^{(k+1)/N} N^2\Big(\phi(x - N^{-1}) - 2\phi(x) + \phi(x + N^{-1})\Big)\mathrm{d}x\bigg| \le 2\|\phi''\|_\infty,$$

which implies that

$$\langle \mathcal{I}_N J^N, \Delta\phi\rangle = \langle \iota_N \Delta^N J^N, \phi\rangle = \langle J^N, \Delta^N \iota_N^*\phi\rangle_N = \frac{1}{N}\sum_k J^N_k\Big(\Delta^N \iota_N^*\phi\Big)_k \le 2\|\phi''\|_\infty \frac{1}{N}\sum_k |J^N_k|.$$

Using that $\Delta^{-1} : \mathrm{L}^\infty(\mathbb{T}) \to \mathrm{W}^{2,\infty}(\mathbb{T})$ is an isomorphism, we hence obtain

$$\|\mathcal{I}_N J^N\|_{\mathrm{L}^1(\mathbb{T})} = \sup_{f\in \mathrm{L}^\infty(\mathbb{T}):\|f\|_{\mathrm{L}^\infty}\le 1} \langle \mathcal{I}_N J^N, f\rangle = \sup_{\phi\in \mathrm{W}^{2,\infty}(\mathbb{T}):\|\phi\|_{\mathrm{W}^{2,\infty}}\le 1} \langle \mathcal{I}_N J^N, \Delta\phi\rangle \le 2\|J^N\|_{\mathrm{L}^1_N}.$$

This shows the desired estimate.

Assume now, that $\sup_N \|J^N\|_{\mathrm{L}^1_N} < \infty$. Then, the induced measures $\iota_N J^N$ and $\mathcal{I}_N J^N$ are bounded in the space of finite measures. Hence, by the Theorem of Banach-Alaoglu, there exists a subsequence that converge in $\mathcal{M}(\mathbb{T})$. To identify the limit, let $\xi \in C^\infty(\mathbb{T})$ be fixed. Similar to the expansion in the beginning of the proof, we can do a third orde rTaylor expansion around $x \in [0,1]$ to find $x_- \in [x - \frac{1}{N}, x]$ and $x_+ \in [x, x + \frac{1}{N}]$ such that

$$\xi\Big(x - \tfrac{1}{N}\Big) - 2\xi(x) + \xi\Big(x + \tfrac{1}{N}\Big) = \tfrac{1}{N^2}\xi''(x) + \tfrac{1}{6N^3}\Big(\xi'''(x_+) - \xi'''(x_-)\Big),$$

which implies that

$$(\iota_N^* \Delta\xi)_k - (\Delta^N \iota_N^* \xi)_k = N \int_{k/N}^{(k+1)/N} \frac{1}{6N}(\xi'''(x_-) - \xi'''(x_+))\,\mathrm{d}x.$$

In particular, we get the following commutator estimate

$$\Big|(\iota_N^* \Delta\xi)_k - \Big(\Delta^N \iota_N^* \xi\Big)_k\Big| \le \frac{1}{3N}\|\xi'''\|_{\mathrm{L}^\infty}, \tag{3.10}$$

which allows us to identitfy the limit, because of

$$|\langle \iota_N J^N, \Delta\xi\rangle - \langle \mathcal{I}_N J^N, \Delta\xi\rangle| = |\langle J^N, \iota_N^*\Delta\xi - \Delta^N \iota_N^*\xi\rangle_N| \le \frac{1}{N}\sum_{k=1}^N |J_k^N| \frac{1}{3N}\|\xi'''\|_{\mathrm{L}^\infty} \to 0\,. \;\square$$

3.4 EDP convergence

We are now in the position to state the detailed version of Result B.

Theorem 3.6. *Assume the acitivity σ_α satisfies Assumption 2.1.*
Let $(c^N, J^N) \in \mathrm{CE}_N$ be solutions of the discrete continuity equation with $\sup_N L_{\alpha,N}(c^N, J^N) < \infty$. Let $\rho^N := \iota_N c^N, j^N := \mathcal{I}_N J^N$ be their embeddings. Then there exists an absolutely continuous curve of densities $\rho \in \mathrm{W}^{1,1}([0,T], \mathrm{L}^1(\mathbb{T}))$ and fluxes $j \in \mathrm{L}^1([0,T], \mathrm{L}^1(\mathbb{T}))$ such that $(\rho, j) \in \mathrm{CE}$, $\rho^N \rightharpoonup \rho \in \mathrm{W}^{1,1}([0,T], \mathrm{L}^1(\mathbb{T}))$ and $j^N \overset{*}{\rightharpoonup} j$ in $\mathcal{M}([0,T] \times \mathbb{T})$ up to a subsequence. Moreover, it holds

$$\forall\, t > 0 : \liminf_{N\to\infty} E_N(c^N(t)) \ge \mathcal{E}(\rho(t)) \qquad \text{and} \qquad \liminf_{N\to\infty} D_{\alpha,N}(c^N, J^N) \ge \mathcal{D}_\alpha(\rho, j).$$

If, in addition the curves c^N are EDB solutions to (1.3) and have well-prepared initial data, that is $\iota_N c^N(0) \to \rho(0)$ and $E_N(c^N(0)) \to \mathcal{E}(\rho(0))$, then the limit curve (ρ, j) is an EDI solution to (1.1).

In the rest of the section, we prove Theorem 3.6 by deriving first strong compactness for the embedded curves ρ^N and showing then the liminf estimates.

Proof of EDP convergence: compactness

The aim of the section is to derive enough compactness for the family of solutions the discrete gradient-flow equation in order to prove Theorem 3.6 and so Result B. Let $\alpha > 0$ and an activity σ_α be fixed such that Assumption 2.1 is satisfied. Throughout this section, we fix a time horizon $T > 0$.

Let $[0,T] \ni t \mapsto c^N(t) \in \mathrm{L}^1_N(\mathbb{T})$ and $[0,T] \ni t \mapsto J^N(t)$ be two trajectories with $(c^N, J^N) \in \mathrm{CE}_N$ such that

$$\sup_{N\in\mathbb{N}} D_{\alpha,N}(c^N, J^N) \le C_{\mathrm{diss}}, \quad \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} E_N(c^N(t)) \le C_{\mathrm{en}}. \tag{3.11}$$

In particular, we have from the definition of $D_{\alpha,N}$ in (2.12) the two priori bounds

$$\int_0^T \frac{1}{N}\sum_{k=1}^N N^2 \mathbb{C}\left(J_k^N | N^2 m_\alpha(c_{k-1}, c_k, c_{k+1})\right) \mathrm{d}t \le C_{\mathrm{diss}} ;$$

$$\int_0^T \frac{1}{N}\sum_{k=1}^N N^4 \sigma_\alpha(c_{k-1}, c_k, c_{k+1}) \left(c_k - \sqrt{c_{k-1}c_{k+1}}\right)^2 \mathrm{d}t \le C_{\mathrm{diss}} .$$

Using the embedding operators from Section 3.3, we define pointwise in time the embedded sequences

$$\left(\rho^N(t) := \iota_N c^N(t)\right)_{N\in\mathbb{N}}, \quad \left(j^N(t) := \mathcal{I}_N J^N(t)\right)_{N\in\mathbb{N}} .$$

To derive strong relative compactness of $(\rho^N)_{N\in\mathbb{N}}$ in $\mathrm{L}^1(0,T;\mathbb{T})$ we rely on an Aubin-Lions argument. For this, we first derive spatial regularity.

Lemma 3.7 (Discrete spatial regularity)**.** *For all $c \in \mathrm{L}^1(\mathbb{T}_N)$ it holds*

$$S_{\alpha,N}(c) \ge \frac{1}{24\alpha^2}\,\frac{1}{N}\sum_{k=1}^N \left(\left(\Delta^N c^{\alpha/2}\right)_k^2 + \left(\partial_-^N c^{\alpha/4}\right)_k^4 + \left(\partial_+^N c^{\alpha/4}\right)_k^4\right).$$

Proof. The assumption on $\overline{\sigma}_\alpha$ from (2.4) allows us to obtain a lower bound on the slope. Indeed, we have

$$S_{\alpha,N}(c) \ge \frac{1}{N}\sum_{k=1}^N \frac{N^4}{\alpha^2}\frac{(c_k^\alpha - \sqrt{c_{k-1}c_{k+1}}^\alpha)^2}{\max\{c_k^\alpha, \sqrt{c_{k-1}c_{k+1}}^\alpha\}} \ge \frac{1}{N}\sum_{k=1}^N \frac{N^4}{\alpha^2}\left(c_k^{\alpha/2} - \sqrt{c_{k-1}c_{k+1}}^{\alpha/2}\right)^2 ,$$

where in the second inequality, we have used that $\frac{\left(c_k^{\alpha/2} + \sqrt{c_{k-1}^{\alpha/2} c_{k+1}^{\alpha/2}}\right)^2}{\max\{c_k^\alpha, \sqrt{c_{k-1}^\alpha c_{k+1}^\alpha}\}} \ge 1$. Introducing $w_k := c_k^{\alpha/2}$, we need to show that

$$\sum_{k=1}^N N^4 \left(w_k - \sqrt{w_{k-1}w_{k+1}}\right)^2 \ge \frac{1}{24}\left(\sum_{k=1}^N (\Delta^N w)_k^2 + (\left(\partial_-^N c^{\alpha/4}\right)_k^4 + \left(\partial_+^N c^{\alpha/4}\right)_k^4\right).$$

We use discrete partial integration to observe that

$$\sum_{k=1}^N \Big[(\sqrt{w_{k+1}} - \sqrt{w_k})^3(\sqrt{w_{k+1}} + \sqrt{w_k}) - (\sqrt{w_k} - \sqrt{w_{k-1}})^3(\sqrt{w_k} + \sqrt{w_{k-1}})\Big] = 0.$$

Adding this to the left-hand side, writing $u_k = \sqrt{w_k}$, it hence suffices to show that for all $k \in [N]$ we have

$$\begin{aligned}(u_k^2 - u_{k+1}u_{k-1})^2 + \frac{1}{4}\Big\{(u_{k+1} - u_k)^3(u_{k+1} + u_k) - (u_k - u_{k-1})^3(u_{k-1} + u_k)\Big\}& \\ \ge \frac{1}{24}\Big((u_{k-1}^2 - 2u_k^2 + u_{k+1}^2)^2 + (u_{k-1} - u_k)^4 + (u_{k+1} - u_k)^4\Big)&.\end{aligned}$$

First, we use that both sides of the inequality are 1-homogeneous and so we may set $u_k = 1$. Moreover, introducing the parametrization $u_{k-1} = x + \sqrt{t}$ and $u_{k+1} = x - \sqrt{t}$ with $x \geq 0$ and $t \in [0, x^2]$, putting all terms on one side, and rearranging, a straight-forward computation shows that it suffices to show that

$$\forall\, x \geq 0 \;\forall\, t \in [0, x^2] : \quad S(x,t) := 15t^2 + 2t(x{-}11)(x{-}1) + (x{-}1)^2(3{+}22x{+}15x^2) \geq 0.$$

We observe $S(x,0) \geq 0$ for all $x \geq 0$. For fixed $x > 0$ we minimize over $t \in [0, x^2]$ and obtain $S(x,t) \geq S(x,0)$ for $x \in [0,1] \cup [11,\infty)$, as the prefactor of the term for t^1 is non-negative for these x.

For $x \in (1, 11)$ we use that $t \mapsto S(x,t)$ is quadratic, hence its minimum is given at $t_x = \frac{1}{15}(11{-}x)(x{-}1) \in [0, x^2]$. Inserting the critical value into S we find

$$S(x, t_x) = \frac{1}{45}(x-1)^2(56x^2 + 88x - 19) \geq 0,$$

for $x \geq 1$. Hence, we conclude that $S(x,t) \geq 0$, which proves the inequality. □

The discrete spatial regularity has important consequences for the embedded sequence $(\rho^N)_{N\in\mathbb{N}}$ as it provides a uniform spatial L^∞-bound and spatial regularity. We summarize static properties next, which are later also needed for time-integrated bounds.

Corollary 3.8. *Let* $(c^N)_{N\in\mathbb{N}}$ *satisfy* $\sup_{N\in\mathbb{N}}\Big\{S_{\alpha,N}(c^N) + \|c^N\|_{\mathrm{L}^\alpha_N}\Big\} < \infty$. *Then:*

(A) The sequence $\rho^N = \iota_N c^N$ *is uniformly bounded in* $\mathrm{L}^\infty(\mathbb{T})$*;*

(B) The sequence $\iota_N \partial^N_\pm (c^N)^{\alpha/2}$ *is compact in* $\mathrm{L}^2(\mathbb{T})$*;*

(C) We have $\|(\rho^N)^{\alpha/4}\|_{\mathrm{BV}(\mathbb{T})} \leq \|\partial^N_-(c^N)^{\alpha/4}\|_{\mathrm{L}^4_N}$.

Proof. Let $u^N := (c^N)^{\alpha/4}$ and $w^N := (c^N)^{\alpha/2}$. The bound on $c^N \in \mathrm{L}^\alpha_N$ implies the bounds $w^N \in \mathrm{L}^2_N$ and $u_N \in \mathrm{L}^4_N$. From the assumptions in combination with Lemma 3.7, we conclude that

$$\sup_{N\in\mathbb{N}} \|u^N\|_{\mathrm{W}^{1,4}_N} = \sup_{N\in\mathbb{N}}\Big\{\|\iota_N \partial_\pm u^N\|_{\mathrm{L}^4} + \|\iota_N u^N\|_{\mathrm{L}^4}\Big\} < \infty,$$

which implies that all sequences $\iota_N u^N, \iota_N w^N, \rho^N$ are uniformly bounded in $\mathrm{L}^\infty(\mathbb{T})$ by $\mathrm{W}^{1,4}_N \subset \mathrm{L}^\infty_N$, the discrete analog of the continuous embedding $\mathrm{W}^{1,4}(\mathbb{T}) \subset \mathrm{L}^\infty(\mathbb{T})$. This proves (A).

The bound on the slope also provides the uniform bound $\iota_N \Delta^N w^N \in \mathrm{L}^2$. Since $\partial^N_\pm w^N$ has mean-average zero, we conclude by a discrete PoincarÃ© inequality the uniform bound $\iota_N \partial^N_\pm w^N \in \mathrm{L}^2$. Compactness of that sequence in L^2 now follows from a discrete Rellich theorem. Indeed, by the Arzela-Ascoli theorem it suffices show that $\omega^N = \iota_N w^N$ is HÃ¶lder-$1/2$ continuous if $\sup_N \|\partial^N_- w^N\|_{\mathrm{L}^2_N} \leq C$. To show this, we let x, y such that $x \in [k/N, k{+}1/N[$, $y \in [l/N, l+1/N[$ and estimate

$$\Big|\omega^N(x) - \omega^N(y)\Big| \leq \Big|\iota_N w^N(x) - \iota_N w^N(y)\Big| = |w_k - w_l| = \sum_{j=k}^{l-1} |w_j - w_{j+1}|$$

By the Cauchy-Schwartz inequality and introducing factors of $\sqrt{N}$, we conclude

$$\begin{aligned}\Big|\omega^N(x) - \omega^N(y)\Big| &\leq \left(\sum_{j=k}^{l-1} N|w^N_j - w^N_{j+1}|^2\right)^{1/2} \left(\sum_{j=k}^{l-1} \frac{1}{N}\right)^{1/2} \\ &\leq \|\partial_- w\|_{\mathrm{L}^2_N} \left(\frac{l-1-k}{N}\right)^{1/2} \leq \|\partial_- w^N\|_{\mathrm{L}^2_N} |x-y|^{1/2},\end{aligned}$$

which proves (B).

For (C), we recall that the BV-norm is defined in duality with functions $\phi \in C^1(\mathbb{T})$ with $\|\phi\|_{\mathrm{L}^\infty(\mathbb{T})} \leq 1$. We observe that for all $h \in]0, 1/N[$ we have

$$\int_{\mathbb{T}} \iota_N u(x) \frac{\phi(x+h) - \phi(x)}{h} \, \mathrm{d}x \leq \frac{1}{h} \|\phi\|_{\mathrm{L}^\infty(\mathbb{T})} \int_{\mathbb{T}} |\iota_N u(x-h) - \iota_N u(x)| \, \mathrm{d}x. \tag{3.12}$$

Since

$$\begin{aligned} |\iota_N u(x-h) - \iota_N u(x)| &= \left| \sum_{k=1}^N u_k^N \mathbb{1}_{[k/N, k+1/N[}(x-h) - u_k^N \mathbb{1}_{[k/n, k+1/N[}(x) \right| \\ &\leq \sum_{k=1}^N |u_k^N - u_{k-1}^N| \mathbb{1}_{[k/N, k+h[}(x) \,, \end{aligned}$$

we get from (3.12), that

$$\begin{aligned} \int_{\mathbb{T}} \iota_N u^N(x) \frac{\phi(x+h) - \phi(x)}{h} \, \mathrm{d}x &\leq \frac{1}{h} \|\phi\|_{\mathrm{L}^\infty(\mathbb{T})} \int_{\mathbb{T}} \sum_{k=1}^N |u_k^N - u_{k-1}^N| \mathbb{1}_{[k/N, k/N+h[} \, \mathrm{d}x \\ \leq \|\phi\|_{\mathrm{L}^\infty(\mathbb{T})} \sum_{k=1}^N |u_k^N - u_{k-1}^N| &= \|\phi\|_{\mathrm{L}^\infty(\mathbb{T})} \|\partial_-^N u^N\|_{\mathrm{L}^1_N} \leq \|\phi\|_{\mathrm{L}^\infty(\mathbb{T})} \|\partial_-^N u^N\|_{\mathrm{L}^4_N}. \end{aligned}$$

Taking the limit $h \to 0$, we conclude by dominated convergence that

$$\int_{\mathbb{T}} \iota_N u^N(x) \phi'(x) \, \mathrm{d}x \, \mathrm{d}t \leq \|\phi\|_{\mathrm{L}^\infty(\mathbb{T})} \|\partial_-^N u^N\|_{\mathrm{L}^4_N}.$$

Taking the supremum over $\phi \in C^1(\mathbb{T})$ with $\|\phi\|_{\mathrm{L}^\infty(\mathbb{T})} \leq 1$ and noting that $(\rho^N)^{\alpha/4} = (\iota_N c^N)^{\alpha/4} = \iota_N (c^N)^{\alpha/4} = \iota_N u_N$, we conclude that

$$\|(\rho^N)^{\alpha/4}\|_{\mathrm{BV}(\mathbb{T})} = \|\iota_N u^N\|_{\mathrm{BV}(\mathbb{T})} \leq \|\partial_-^N u^N\|_{\mathrm{L}^4_N}.$$ □

Lemma 3.9 (Improved integrability)**.** *Let the family of trajectories* $c^N \in \mathrm{L}^1(0, T; \mathrm{L}^1(\mathbb{T}))$ *satisfy the a priori bounds* (3.11)*. Then, there exists an exponent* $q_{\mathrm{crit}} := \alpha + 1$ *such that*

$$\sup_{N \in \mathbb{N}} \|\rho^N\|_{\mathrm{L}^{q_{\mathrm{crit}}}([0,T] \times \mathbb{T})} < \infty.$$

Proof. Step 1. We set $u^N := (c^N)^{\alpha/4}$ and $\upsilon^N := \iota_N u^N$. We first show, that υ^N is uniformly bounded in $\mathrm{L}^4 \, \mathrm{L}^\infty$. To arrive at the claim, we observe the static estimate

$$\|u^N\|_{\mathrm{L}^\infty_N} \leq \|u^N\|_{\mathrm{L}^1_N} + \|\partial_\pm^N u^N\|_{\mathrm{L}^1_N}, \tag{3.13}$$

which is the discrete analog of the continuous embedding $\mathrm{W}^{1,1} \subset \mathrm{L}^\infty$. Now, let $\gamma > 0$ sufficiently small. Using HÃ¶lder's inequality with exponents $q, q^* \geq 1$, i.e. $1/q + 1/q^* = 1$, we have

$$\|\upsilon^N\|_{\mathrm{L}^1} = \int_{\mathbb{T}} (\upsilon^N)^{1-\gamma} (\upsilon^N)^\gamma \, \mathrm{d}x \leq \left(\int_{\mathbb{T}} (\upsilon^N)^{(1-\gamma) q^*} \, \mathrm{d}x \right)^{1/q^*} \left(\int_{\mathbb{T}} (\upsilon^N)^{\gamma q} \, \mathrm{d}x \right)^{1/q}.$$

Choosing, $\gamma q = 4/\alpha$, which is possible for $\gamma < 4/\alpha$, the second term is just $\left(\int_{\mathbb{T}} (\upsilon^N)^{\gamma q} \, \mathrm{d}x \right)^{1/q} = \|\rho^N\|_{\mathrm{L}^1}^{\alpha\gamma/4}$. For this q we have $q^* = 4/(4-\alpha\gamma)$, and hence we conclude

$$\|\upsilon^N\|_{\mathrm{L}^1} \leq \|\upsilon^N\|_{\mathrm{L}^p}^{1-\gamma} \cdot \|\rho^N\|_{\mathrm{L}^1}^{\alpha\gamma/4}, \quad \text{with} \quad p > \max\Big\{ \frac{4-4\gamma}{4-\alpha\gamma}, 1 \Big\}.$$

Note that for all $\alpha > 0$, there exists $\gamma \in]0, 4/\alpha[$ such that such a $p > 1$ exists. The last estimate implies with Young's inequality that

$$\|\upsilon^N\|_{\mathrm{L}^1} \leq \|\upsilon^N\|_{\mathrm{L}^\infty}^{1-\gamma} \cdot \|\rho^N\|_{\mathrm{L}^1}^{\alpha\gamma/4} \leq (1-\gamma)\|\upsilon^N\|_{\mathrm{L}^\infty} + \gamma \|\rho^N\|_{\mathrm{L}^1}^{\alpha/4}.$$

Inserting this into (3.13), we hence obtain

$$\|\upsilon^N\|_{\mathrm{L}^\infty_N} \leq \|\upsilon^N\|_{\mathrm{L}^1_N} + \|\partial_\pm^N u^N\|_{\mathrm{L}^1_N} \leq (1-\gamma)\|\upsilon^N\|_{\mathrm{L}^\infty} + \gamma \|\rho^N\|_{\mathrm{L}^1}^{\alpha/4} + \|\partial_\pm^N u^N\|_{\mathrm{L}^1_N}.$$

Hence, we conclude that $\|\upsilon^N\|_{\mathrm{L}^\infty_N} \leq \|\rho^N\|_{\mathrm{L}^1}^{\alpha/4} + \frac{1}{\gamma}\|\partial_\pm^N u^N\|_{\mathrm{L}^1_N} \leq \|\rho^N\|_{\mathrm{L}^1}^{\alpha/4} + \frac{1}{\gamma}\|\partial_\pm^N u^N\|_{\mathrm{L}^4_N}$, where in the last inequality we have used Jensen's inequality. Since ρ^N is uniformly bounded in $\mathrm{L}^\infty \mathrm{L}^1$ by the bound on the energy (3.11), and, in addition, the discrete spatial regularity in Lemma 3.7 which provides $\|\partial_\pm^N u^N\|_{\mathrm{L}^4 \mathrm{L}^4_N} \leq C_{\mathrm{diss}}$, we conclude that υ^N is uniformly bounded in $\mathrm{L}^4 \mathrm{L}^\infty$.

Step 2. To prove the statement, we now recall the classical interpolation result for Lebesgue spaces, i.e. for $\mathrm{L}^{q_1}(\mathbb{T}), \mathrm{L}^{q_2}(\mathbb{T})$ with $q_1, q_2 \geq 1$ and $p_1, p_2 \geq 1$ and $\theta \in [0,1]$ we have

$$\Big[\mathrm{L}^{p_1}([0,T], \mathrm{L}^{q_1}(\mathbb{T})), \mathrm{L}^{p_2}([0,T], \mathrm{L}^{q_2}(\mathbb{T}))\Big]_\theta \simeq \mathrm{L}^{p_\theta}([0,T], [\mathrm{L}^{q_1}(\mathbb{T}), \mathrm{L}^{q_2}(\mathbb{T})]_\theta) \simeq \mathrm{L}^{p_\theta}([0,T], \mathrm{L}^{q_\theta}(\mathbb{T})),$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$. Let first $\alpha \in]0, 4]$. By the time-uniform bound on ρ^N, we conclude the uniform bound $\upsilon^N \in \mathrm{L}^\infty \mathrm{L}^{4/\alpha}$. Hence, we conclude that $u^N \in \mathrm{L}^\infty \mathrm{L}^{4/\alpha} \cap \mathrm{L}^4 \mathrm{L}^\infty \subset \mathrm{L}^{4+4/\alpha}([0,T], \mathbb{T})$, which implies that $\rho^N = (u^N)^{4/\alpha} \in \mathrm{L}^{1+\alpha}([0,T] \times \mathbb{T})$ is uniformly bounded. For $\alpha \geq 4$, we observe that $\upsilon^N \in \mathrm{L}^4 \mathrm{L}^\infty$ is equivalent to $\rho^N = (\upsilon^N)^{4/\alpha} \in \mathrm{L}^\alpha \mathrm{L}^\infty$. Hence, in addition with the uniform energy bound, we have $\rho^N \in \mathrm{L}^\infty \mathrm{L}^1 \cap \mathrm{L}^\alpha \mathrm{L}^\infty \subset \mathrm{L}^{\alpha+1}([0,T] \times \mathbb{T})$. This proves the claim. □

The time regularity of the curve will be implied by suitable integrability of the fluxes. Hence, we first show compactness for the fluxes by uniform integrability. Here the improved regularity from Lemma 3.9 and the estimate (2.8) is crucial. The proof is analogously to [HMS25].

Lemma 3.10 (Uniform integrability of fluxes)**.** *For the admissible curves (c^N, J^N), there is a constant $C_{\mathrm{flux}} > 0$, such that for all $N \in \mathbb{N}$ we have*

$$\frac{1}{N} \sum_{k=1}^N \int_0^T \mathsf{C}(J_k^N) \, \mathrm{d}t \leq C_{\mathrm{flux}}.$$

Moreover, there exists a curve of fluxes $j \in \mathrm{L}^1(0,T;\mathbb{T})$ with $\int_0^T \int_{\mathbb{T}} \mathsf{C}(j) \, \mathrm{d}x \, \mathrm{d}t \leq C_{\mathrm{flux}}$ and $\iota_N J^N \rightharpoonup j$ weakly in $\mathrm{L}^1(0,T;\mathbb{T})$ (up to a subsequence). In addition, $\mathcal{I}_N J^N = j^N$ converges in the space $\mathcal{M}([0,T] \times \mathbb{T})$ (up to a subsubsequence) also to $j \in \mathrm{L}^1([0,T] \times \mathbb{T})$.

Proof. By the estimate (2.8), we have

$$\mathsf{C}(J^N) \leq \frac{q}{q-1}\mathfrak{C}\Big(J^N\Big|m_\alpha(c_{k-1},c_k,c_{k+1})\Big) + \frac{4m_\alpha^q}{q-1}.$$

Using Lemma 1, we estimate

$$m_\alpha = \sigma_\sigma(c_{k-1}^N,c_k^N,c_{k+1}^N)\Big(c_k^N\sqrt{c_{k-1}^Nc_{k+1}^N}\Big) \leq \max\Big\{c_k^\alpha,\sqrt{c_{k-1}c_{k+1}}^\alpha\Big\} \leq c_{k-1}^\alpha + c_k^\alpha + c_{k+1}^\alpha.$$

Choosing $q = p$ from Lemma 3.9 together with a discrete HÃ¶lder inequality, we get that $m_\alpha^q \in \mathrm{L}^1(0,T;\mathrm{L}^1_N)$. Moreover, we have by monotonicity (2.9)

$$\mathfrak{C}(J^N|m_\alpha(c_{k-1},c_k,c_{k+1}) \leq \mathfrak{C}(N^2J_k^N|N^4m_\alpha).$$

Together this implies that

$$\frac{1}{N}\sum_{k=1}^N\int_0^T \mathsf{C}(J^N)\,\mathsf{d}t \leq \frac{1}{N}\sum_{k=1}^N\int_0^T \mathfrak{C}(N^2J_k^N|N^4m_\alpha)\,\mathsf{d}t + \|m_\alpha^q\|_{\mathrm{L}^1\mathrm{L}^1_N} \leq C_{\mathrm{diss}} + C\|c^N\|_{\mathrm{L}^{\alpha q}\mathrm{L}^{\alpha q}} < \infty\,,$$

and proves the first claim.

For the embedded fluxes we esimate

$$\int_0^T\int_{\mathbb{T}} \mathsf{C}(\iota_NJ^N)\,\mathsf{d}t \leq \frac{1}{N}\sum_{k=1}^N\int_0^T \mathsf{C}(J^N)\,\mathsf{d}t < \infty.$$

Hence, by the criterion of de la VallÃ©e Poussin, we conclude that there exists $j \in \mathrm{L}^1(0,T;\mathrm{L}^1(\mathbb{T}))$ with $\iota_NJ^N \rightharpoonup j$ in $\mathrm{L}^1([0,T]\times\mathbb{T})$ for a subsequence. By Lemma 3.5, we also conclude that there is a subsubsequence such that the induced measures $\mathcal{I}_NJ^N = j^N$ converge in $\mathcal{M}([0,T],\mathbb{T})$ to the same limit $j \in \mathrm{L}^1([0,T],\mathbb{T})$ finishing the proof. □

Lemma 3.11 (Time regularity). *Let (c^N,J^N) be admissible curves. Then, we have that $\rho^N \in \mathrm{BV}([0,T],\big(\mathrm{W}^{2,\infty}(\mathbb{T})\big)^*)$ uniformly in $N \in \mathbb{N}$, i.e. we have a uniform bound on*

$$\|\rho^N\|_{\mathrm{TV}} = \sup\left\{\sum_{\ell=1}^L \|\rho^N(t_\ell)-\rho^N(t_{\ell-1})\|_* : \quad 0 = t_0 \leq t_1 \leq \cdots \leq t_\ell \leq \ldots\cdots \leq t_L = T\right\},$$

where the dual norm for functions in $f \in \mathrm{L}^1(\mathbb{T})$ with respect to $\mathrm{W}^{2,\infty}$ is defined by

$$\|f\|_* := \sup_{\phi\in\mathrm{W}^{2,\infty}}\left\{\int_{\mathbb{T}} f\,\phi\;\mathsf{d}x : \|\phi\|_{\mathrm{L}^\infty(\mathbb{T})} + \|\phi'\|_{\mathrm{L}^\infty(\mathbb{T})} + \|\phi''\|_{\mathrm{L}^\infty(\mathbb{T})} \leq 1\right\}.$$

Proof. Fixing two time values, say $t_1,t_2 \in [0,T]$, and a test function $\phi \in \mathrm{W}^{2,\infty}(\mathbb{T})$, we observe by the use of $(c^N,J^N) \in \mathrm{CE}_N$ and Lemma 3.5 that

$$\begin{aligned}\Big\langle\phi,\rho^N(t_2)-\rho^N(t_1)\Big\rangle &= \Big\langle\phi,\iota_N(c^N(t_2)-c^N(t_1))\Big\rangle = \Big\langle\iota_N^*\phi,c^N(t_2)-c^N(t_1)\Big\rangle = \int_{t_1}^{t_2}\Big\langle\iota_N^*\phi,\dot{c}^N(t)\Big\rangle\,\mathsf{d}t.\\ &\overset{\mathrm{CE}_N}{=}\int_{t_1}^{t_2}\Big\langle\iota_N^*\phi,\Delta^NJ^N(t)\Big\rangle\,\mathsf{d}t = \int_{t_1}^{t_2}\Big\langle\phi,\Delta\mathcal{I}_NJ^N(t)\Big\rangle\,\mathsf{d}t\\ &\leq \int_{t_1}^{t_2}\|\Delta\phi\|_{\mathrm{L}^\infty}\times\|\mathcal{I}_NJ^N(t)\|_1\,\mathsf{d}t \leq 2\int_{t_1}^{t_2}\|\iota_NJ^N(t)\|_1\,\mathsf{d}t = 2|\iota_NJ^N|([t_1,t_2]\times\mathbb{T}).\end{aligned}$$

In particular, this implies that

$$\forall\, t_1, t_2 \in [0,T]: \quad \|\rho^N(t_2) - \rho^N(t_1)\|_* \le 2|\iota_N J^N|([t_1,t_2]\times\mathbb{T}). \tag{3.14}$$

Summing up, using the bound on the embedded fluxes from Lemma 3.5 and the boundedness of $J^N \in \mathrm{L}^1([0,T],\mathrm{L}^1_N)$ from Lemma 3.10, we arrive at the uniform TV-bound

$$\sum_{\ell=1}^{L} \|\rho^N(t_\ell) - \rho^N(t_{\ell-1})\|_* \le 2\sum_\ell \int_{t_{\ell-1}}^{t_\ell} \|J^N(t)\|_1 \,\mathrm{d}t = 2|J^N|([0,T]\times\mathbb{T}_N) < \infty\,. \qquad \square$$

Proposition 3.12 (Strong compactness)**.** *There exists $\rho \in \mathrm{L}^{\alpha+1}([0,T],\mathrm{L}^1(\mathbb{T}))$ such that the strong convergence $\rho^N \to \rho$ in $\mathrm{L}^1([0,T],\mathrm{L}^1(\mathbb{T}))$ holds up to a subsequence.*

Proof. The proof relies on the Aubin-Lions type result from [RoS03] (more precisely, Thm. 2 in addition with Prop. 1.10). As usual it combines spatial regularity, which we will deduce from Lemma 3.7 together with temporal regularity deduced from Lemma 3.11. By Lemma 3.9, we already now that ρ^N is uniformly bounded in $\mathrm{L}^{\alpha+1}([0,T]\times\mathbb{T})$. Hence, it suffices to show that $\rho^N \to \rho$ in L^1. We perform the proof in three steps.

Step 1. On $\mathrm{L}^1(\mathbb{T})$, we define the functional

$$\mathcal{F}(\rho) := \mathcal{E}(\rho) + \|\rho^{\alpha/4}\|_{\mathrm{L}^1(\mathbb{T})} + \|\rho^{\alpha/4}\|^4_{\mathrm{BV}(\mathbb{T})}.$$

From the bound on the energy, the bound on the dissipation functional with Lemma 3.7 and the improved integrability from Lemma 3.9, we conclude that the sequence ρ^N satisfies

$$\int_0^T \mathcal{E}(\rho^N(t))\,\mathrm{d}t < \infty, \quad \int_0^T \|(\rho^N)^{\alpha/4}\|_{\mathrm{L}^1(\mathbb{T})}\,\mathrm{d}t < \infty, \quad \text{and} \quad \int_0^T \|\partial^N_\pm (c^N)^{\alpha/4}\|^4_{\mathrm{L}^4_N(\mathbb{T})}\,\mathrm{d}t < \infty.$$

By Corollarly 3.8, the last term provides a bound on $\int_0^T \|(\rho^N)^{\alpha/4}\|^4_{\mathrm{BV}(\mathbb{T})}\,\mathrm{d}t$, which shows that the sequence ρ^N is tight w.r.t. $\mathcal{F}$, i.e. $\sup_{N\in\mathbb{N}} \int_0^T \mathcal{F}(\rho^N(t))\,\mathrm{d}t < \infty$.

Step 2. We show that $\mathcal{F} : \mathrm{L}^1(\mathbb{T}) \to [0,\infty]$ is a normal coercive integrand (in the sense of [RoS03]). In particular, we have to show

$$\forall\, c > 0 : \{\rho \in \mathrm{L}^1(\mathbb{T}) : \mathcal{F}(\rho) \le c\} \quad \text{is compact in } \mathrm{L}^1(\mathbb{T}).$$

To see this we recall the classical Helly's selection criteria, which is stated as following. For a given sequence $f_n : \mathbb{T} \to \mathbb{R}$ with $\sup_{n\in\mathbb{N}}\Big(\|f_n\|_{\mathrm{L}^1(\mathbb{T})} + \|f_n\|_{\mathrm{BV}(\mathbb{T})}\Big) < \infty$ there exists a subsequence f_{n_k} and a function $f \in \mathrm{BV}(\mathbb{T})$, such that $f_{n_k} \to f$ pointwise almost everywhere, $\|f_{n_k} - f\|_{\mathrm{L}^1(\mathbb{T})} \to 0$, and $\|f\|_{\mathrm{BV}(\mathbb{T})} \le \liminf_{k\to\infty} \|f_{n_k}\|_{\mathrm{BV}(\mathbb{T})}$.

Now, to prove that $\mathcal{F}$ has compact sublevels, we take $c > 0$ and any sequence ρ^N in the sublevelset of $\mathcal{F}$. Defining pointwise $\upsilon^N := (\rho^N)^{\alpha/4}$ (possible because $\rho^N \ge 0$), we conclude that $\sup_{N\in\mathbb{N}}\Big(\|\upsilon^N\|_{\mathrm{L}^1(\mathbb{T})} + \|\upsilon^N\|_{\mathrm{BV}(\mathbb{T})}\Big) < \infty$, where we used that $x \mapsto x^4$ is monotone. Hence, by Helly's selection criteria we conclude that it exists $\upsilon \in \mathrm{L}^1(\mathbb{T})$ such that (up to subsequence) we have the convergence $\upsilon_N \to \upsilon$ in L^1. In particular, we have that $\upsilon^N \to \upsilon$ in measure. The power is continuous and hence, $\rho^N = (\upsilon^N)^{4/\alpha} \to \upsilon^{4/\alpha}$ in measure[2]. Because the energy $\mathcal{E}$

[2] We recall that on a set with finite measure we have that a sequence f_n converges to f in measure if and only if every subsequence has in turn a subsequence that converges to f almost everywhere. In particular, applying a continuous function does not change convergence in measure.

is superlinear, we know that the sequence ρ^N is uniformly integrable by the de la VallÃ©e-Poussin theorem. Convergence in measure together with uniform integrability implies the strong convergence $\rho^N \to \rho$ in $\mathrm{L}^1(\mathbb{T})$. In particular, we also observe that $\mathcal{F}$ is lower semicontinuous.

Step 3. Finally we show the weak integral equicontinuity. We have to show that

$$\lim_{h\to 0} \sup_{N\in\mathbb{N}} \int_0^{T-h} \|\rho^N(t+h) - \rho^N(t)\|_* \,\mathsf{d}t = 0.$$

By the bound (3.14), it is sufficient to show

$$\forall\, \varepsilon > 0 \exists\, h > 0 : \quad \sup_{N\in\mathbb{N}} \int_0^{T-h} |\iota_N J^N|([t,t+h]\times\mathbb{T})\,\mathsf{d}t < T\varepsilon\,.$$

Indeed, because $\iota_N J^N$ is uniformly integrable, we find for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $h \in]0,\delta[$ (note: $|\mathbb{T}| = 1$) we have $\sup_N |\iota_N J^N|([t,t+h]\times\mathbb{T}) < \varepsilon$. Hence, $\sup_{N\in\mathbb{N}} \int_0^{T-h} |\iota_N J^N|([t,t+h]\times\mathbb{T})\,\mathsf{d}t < T\varepsilon$.

Applying [RoS03, Thm. 2], we conclude that the sequence ρ^N is relatively compact in $\mathcal{M}(0,T;\mathrm{L}^1(\mathbb{T}))$. Since $\rho^N \in \mathcal{M}(0,T;\mathrm{L}^1(\mathbb{T}))$ is uniformly integrable in $\mathrm{L}^1([0,T]\times\mathbb{T})$ by the bound on the energy, we conclude by [RoS03, Prop. 1.10] that ρ^N is relatively compact in $\mathrm{L}^1(0,T;\mathrm{L}^1(\mathbb{T}))$, which finishes the proof. □

Proof of EDP convergence: liminf estimates

In this part of the section, we make use of the just derived compactness to obtain $\liminf$ estimates of the entropy and dissipation potentials. For doing so, we consier a sequence $(c^N, J^N) \in \mathrm{CE}_N$ such that

$$\sup_N L_{\alpha,N}(c^N, J^n) < \infty \qquad \text{and} \qquad \sup_N E_N(c^N(0)) < \infty\,. \tag{3.15}$$

In particular, the sequence (c^N, J^N) satisfies the a priori bounds (3.11).

We start with the proof of the $\liminf$-estimate for the energy in Result B and Theorem 3.6.

Lemma 3.13 (Liminf of energy)**.** *Let $(c^N, J^N) \in \mathrm{CE}_N$ satisfy (3.15). Let ρ be the limit from Proposition 3.12. Then, we have for all $t \geq 0$ the following liminf estimate*

$$\liminf_{N\to\infty} E_N(c^N(t)) \geq \mathcal{E}(\rho(t)).$$

Proof. Since (c^N, J^N) satisfy the a priori estimates (3.11), we conclude from Proposition 3.12, that $\iota_N c^N = \rho^N \to \rho$ in $\mathrm{L}^1([0,T],\mathrm{L}^1(\mathbb{T}))$. Moreover, we observe that

$$\begin{aligned}
\mathcal{E}(\iota_N c^N(t)) &= \int_{\mathbb{T}} \Big(\rho^N(t)\log\rho^N(t) - \rho^N(t) + 1\Big)\,\mathsf{d}x \\
&= \sum_{k=1}^N \int_{k/N}^{(k+1)/N} \Big(c_k^N(t)\log(c_k^N(t)) - c_k(t)^N + 1\Big)\,\mathsf{d}x \\
&= \frac{1}{N}\sum_{k=1}^N \Big(c_k^N(t)\log(c_k^N(t)) - c_k^N(t) + 1\Big) = E_N(c^N(t)).
\end{aligned}$$

Hence, the liminf estimate follows by the lower semicontinuity of the entropy $\mathcal{E}$. □

Proposition 3.14 (Liminf for primal dissipation potential)**.** *Let $(c^N, J^N) \in \mathrm{CE}_N$ satisfy (3.15). Let ρ be the limit from Proposition 3.12 and $j \in \mathrm{L}^1([0,T],\mathbb{T})$ be the weak-* limit from Lemma 3.10. Then, the primal dissipation potentials satisfy the* $\liminf$ *estimate*

$$\liminf_{N\to\infty} \int_0^T R_{\alpha,N}(c^N, J^N)\,\mathrm{d}t \ge \int_0^T \mathcal{R}_\alpha(\rho, j)\,\mathrm{d}t\,.$$

Proof. For all $N \in \mathbb{N}$, we have $(c^N, J^N) \in \mathrm{CE}_N$, and hence thanks to the embeddings (3.9) also $(\rho^N, j^N) \in \mathrm{CE}$, which holds in the sense of distributions. Since $\rho^N \to \rho$ in $\mathrm{L}^1\,\mathrm{L}^1$ and $j^N \overset{*}{\rightharpoonup} j$ in $\mathcal{M}([0,T]\times\mathbb{T})$, we conclude that $(\rho, j) \in \mathrm{CE}$. Hence, we only need to show

$$\liminf_{N\to\infty} \int_0^T \frac{1}{N}\sum_{k=1}^N \mathsf{C}\left(N^2 J_k^N \middle| N^4 m_\alpha(c_{k-1}, c_k, c_{k+1})\right)\mathrm{d}t \ge \frac{1}{2}\int_0^T \int_{\mathbb{T}} \frac{j^2}{\rho^\alpha}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.16}$$

We are going to show that by duality. To do so, we recall from the a priori estimates (3.11) and the improved integrability from Lemma 3.9, that $\rho^N \to \rho \in \mathrm{L}^1([0,T]\times\mathbb{T})$ and that ρ^N is uniformly bounded in $\mathrm{L}^{\alpha+1}([0,T]\times\mathbb{T})$. In the first step, we show that for all $\xi \in C^\infty([0,T]\times\mathbb{T})$ we have that

$$\limsup_{N\to\infty} \int_0^T R^*_{\alpha,N}(c^N, \Delta^N \iota_N^*\xi)\,\mathrm{d}t \le \int_0^T \mathcal{R}^*_\alpha(\rho, \Delta\xi)\,\mathrm{d}t,$$

where we recall

$$R^*_{\alpha,N}(c, \Delta^N\iota_N^*\xi) = \frac{1}{N}\sum_{k=1}^N N^4 m_\alpha(c_{k-1}, c_k, c_{k+1})\,\mathsf{C}^*\Big(\tfrac{1}{N^2}\big(\Delta^N\iota_N^*\xi\big)_k\Big)\,,$$
$$\mathcal{R}^*_\alpha(\rho, \Delta\xi) = \frac{1}{2}\int_{\mathbb{T}} \rho^\alpha |\Delta\xi|^2\,\mathrm{d}x.$$

Using the symmetry of C^*, the monotonicity of $]0,\infty] \ni r \mapsto \mathsf{C}^*(r)$ and the commutator estimate (3.10), we compute

$$\begin{aligned} R^*_{\alpha,N}(c^N, \Delta^N\iota_N^*\xi) &\le \frac{1}{N}\sum_{k=1}^N N^4 m_\alpha(c^N_{k-1}, c^N_k, c^N_{k+1})\mathsf{C}^*\Big(\tfrac{1}{N^2}\left(\Delta^N\iota_N^*\xi\right)_k\Big) \\ &\le \frac{1}{N}\sum_{k=1}^N N^4 m_\alpha(c^N_{k-1}, c^N_k, c^N_{k+1})\mathsf{C}^*\Big(\tfrac{1}{N^2}\iota_N^*(|\Delta\xi|)_k + \frac{1}{3N^3}\|\xi'''\|_{\mathrm{L}^\infty}\Big)\,. \end{aligned}$$

Using the elementary inequality $N^4\mathsf{C}^*(r/N^2) \le \frac{r^2}{2}\cosh(r/N^2)$, we get

$$R^*_{\alpha,N}(c^N, \Delta^N\iota_N^*\xi) \le \omega_N \frac{1}{N}\sum_{k=1}^N m_\alpha(c^N_{k-1}, c^N_k, c^N_{k+1})\frac{\left(\iota_N^*(|\Delta\xi|)_k + \frac{1}{3N}\|\xi'''\|_\infty\right)^2}{2},$$

where we have introduced the positive factor

$$\omega_N := \max_k \cosh\Big(\frac{1}{N^2}\iota_N^*(|\Delta\xi|)_k + \frac{1}{3N^3}\|\xi'''\|_{\mathrm{L}^\infty}\Big),$$

which converges to one as $N \to \infty$ since $\xi \in C^\infty(\mathbb{T})$. Moreover, using (2.6) we have

$$m_\alpha(c_{k-1}, c_k, c_{k+1}) = \sigma_\alpha(c_{k-1}, c_k, c_{k+1})\,(c_k\sqrt{c_{k-1}c_{k+1}}) \le \max\{c_k^\alpha, \sqrt{c_{k-1}c_{k+1}}^{\,\alpha}\} =: A^N_{\alpha,k}(c),$$

which provides the estimate

$$R^*_{\alpha,N}(c^N, \Delta^N \iota^*_N \xi) \le \omega_N \frac{1}{2}\frac{1}{N}\sum_{k=1}^N A_{\alpha,k}(c)\left\{\iota^*_N(|\Delta\xi|)^2_k + \frac{1}{3N}\|\xi'''\|_{\mathrm{L}^\infty}\iota^*_N(|\Delta\xi|)_k + \frac{1}{9N^2}\|\xi'''\|^2_{\mathrm{L}^\infty}\right\}$$
$$= I^N_1 + I^N_2 + I^N_3,$$

with

$$I^N_1 := \omega_N \frac{1}{2}\frac{1}{N}\sum_{k=1}^N A_{\alpha,k}(c)\iota^*_N(|\Delta\xi|)^2_k,$$
$$I^N_2 := \omega_N \frac{1}{2}\frac{1}{N}\sum_{k=1}^N A_{\alpha,k}(c)\frac{1}{3N}\|\xi'''\|_{\mathrm{L}^\infty}\iota^*_N(|\Delta\xi|)_k,$$
$$I^N_3 := \omega_N \frac{1}{2}\frac{1}{N}\sum_{k=1}^N A_{\alpha,k}(c)\frac{1}{9N^2}\|\xi'''\|^2_{\mathrm{L}^\infty}.$$

Since, $\xi \in C^\infty([0,T]\times\mathbb{T})$ and $\iota_N c^N$ is uniformly bounded in $\mathrm{L}^{\max\{\alpha,1\}}([0,T]\times\mathbb{T})$, we easily see that $\int_0^T I^N_2 \,\mathrm{d}t, \int_0^T I^N_3 \,\mathrm{d}t \to 0$ as $N\to\infty$ by dominated convergence

For I^N_1, we use Jensen's inequality and the strong convergence $(\iota_N c^N)^\alpha \rightharpoonup \rho^\alpha$ from 3.12, to conclude

$$\limsup_{N\to\infty}\int_0^T R^*_{\alpha,N}(c^N, \Delta^N\iota^*_N)\,\mathrm{d}t = \limsup_{N\to\infty}\int_0^T I^N_1\,\mathrm{d}t \le \frac{1}{2}\int_0^T\int_{\mathbb{T}}\rho^\alpha|\Delta\xi|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \mathcal{R}^*_\alpha(\rho,\Delta\xi).$$

With that estimate, we can now show the desired liminf-estimate (3.16). Exploiting the duality of $\mathsf{C}-\mathsf{C}^*$, we have $\langle J^N, \Delta^N\iota^*_N\xi\rangle_N \le R^*_N(c^N,\Delta^N\iota^*_N\xi) + R_N(c^N, J^N)$. Hence, we conclude for $\xi \in C^2([0,T]\times\mathbb{T}^d)$ that

$$\int_0^T \langle j,\Delta\xi\rangle - \mathcal{R}^*_\alpha(\rho,\Delta\xi)\,\mathrm{d}t \le \liminf_{N\to\infty}\int_0^T\langle j^N,\Delta\xi\rangle\,\mathrm{d}t - \limsup_{N\to\infty}\int_0^T R^*_{\alpha,N}(c^N,\iota^*_N\Delta^N\xi)\,\mathrm{d}t$$
$$\le \liminf_{N\to\infty}\int_0^T\langle J^N,\Delta^N\iota^*_N\xi\rangle - R^*_{\alpha,N}(c^N,\Delta^N\iota^*_N\xi)\,\mathrm{d}t$$
$$\le \liminf_{N\to\infty}\int_0^T R_{\alpha,N}(c^N,J^N)\,\mathrm{d}t.$$

Now, the desired liminf-estimate follows by taking the supremum over test functions for the quadratic functional $\xi \mapsto \int_0^T(\langle j,\Delta\xi\rangle - \mathcal{R}^*(\rho,\xi))\,\mathrm{d}t$ □

Proposition 3.15 (Liminf estimate for the slope)**.** *If $\iota_N c^N \to \rho$ in $\mathrm{L}^{\max\{\alpha,1\}}(\mathbb{T})$, then*

$$\liminf_{N\to\infty} S_{\alpha,N}(c^N) \;\ge\; \mathcal{S}_\alpha(\rho). \tag{3.17}$$

Moreover, if $(c^N, J^N)\in \mathrm{CE}_N$ satisfies (3.15) *and ρ is the limit from Proposition 3.12, then, it holds*

$$\liminf_{N\to\infty}\int_0^T S_{\alpha,N}(c^N)\,\mathrm{d}t \ge \int_0^T \mathcal{S}_\alpha(\rho)\,\mathrm{d}t. \tag{3.18}$$

Proof. We first show that (3.17) implies (3.18). For this we use $S_{\alpha,N}(c)\ge 0$ and apply Fatou's lemma:

$$\liminf_{N\to\infty}\int_0^T S_{\alpha,N}(c^N(t))\,\mathrm{d}t \overset{\text{Fatou}}{\ge} \int_0^T \liminf_{N\to\infty} S_{\alpha,N}(c^N(t))\,\mathrm{d}t \overset{(3.17)}{\ge} \int_0^T \mathcal{S}_\alpha(\rho(t))\,\mathrm{d}t,$$

where for the last estimate we used that Proposition 3.12 implies $\iota_N c^N(t) = \rho^N(t) \to \rho(t)$ for a.e. $t \in [0,T]$ in $\mathrm{L}^{\max\{\alpha,1\}}(\mathbb{T})$.

To establish the static liminf estimate (3.17), we use the lower bound from Assumption 2.1:

$$
\begin{aligned}
S_{\alpha,N}(c) &= \frac{1}{N}\sum_{k=1}^{N} 2N^4\,\sigma_\alpha(c_{k-1},c_k,c_{k+1})\Big(c_k - \sqrt{c_{k-1}c_{k+1}}\Big)^2 \\
&\geq \frac{1}{N}\sum_{k=1}^{N} N^4 \frac{2}{\alpha^2}\frac{\Big(c_k^\alpha - (c_{k-1}c_{k+1})^{\alpha/2}\Big)^2}{\max\{c_k^\alpha,(c_{k-1}c_{k+1})^{\alpha/2}\}} = \frac{2}{\alpha^2}\int_{\mathbb{T}} \frac{\Big(N^2\iota_N(c_k^\alpha - (c_{k-1}c_{k+1})^{\alpha/2})\Big)^2}{\iota_N\Big(\max\{c_k^\alpha,(c_{k-1}c_{k+1})^{\alpha/2}\}\Big)}\,\mathrm{d}x\,,
\end{aligned}
$$

where the last equality holds because the functions in the fraction are piecewise constant.

Introducing $w_k := c_k^{\alpha/2}$, we can now rewrite

$$
\begin{aligned}
N^2\Big(\sqrt{c_{k-1}^\alpha c_{k+1}^\alpha} - c_k^\alpha\Big) &= N^2(w_{k-1}w_{k+1} - w_k^2) \\
&= N^2\Big(w_k(w_{k-1} - 2w_k + w_{k+1}) - (w_k - w_{k-1})(w_{k+1} - w_k)\Big) \\
&= w_k\left(\Delta^N w\right)_k - \left(\partial_-^N w\right)_k\left(\partial_+^N w\right)_k .
\end{aligned}
$$

We now replace c by c^N and w by w^N, respectively, and assume $\iota_N c^N \to \rho$ in $\mathrm{L}^{\max\{\alpha,1\}}(\mathbb{T})$ and $\liminf_{N\to\infty} S_{\alpha,N}(c^N) =: \beta$. In the case $\beta = \infty$ there is nothing to be shown. We thus consider the case $\beta < \infty$ and further assume (after extracting a subsequence, not relabeled) that $S_{\alpha,N}(c^N) \to \beta$. With this, we are going to show that

$$
\iota_N\Big(\max\{c_k^\alpha,(c_{k-1}c_{k+1})^{\alpha/2}\}\Big) \;\to\; \rho^\alpha \qquad \text{in } \mathrm{L}^1(\mathbb{T}) \tag{3.19a}
$$

$$
\iota_N\Big(w_k^N\left(\Delta^N w^N\right)_k - \left(\partial_-^N w^N\right)_k\left(\partial_+^N w^N\right)_k\Big) \;\rightharpoonup\; \omega\Delta\omega - (\partial_x\omega)^2 \quad \text{in } \mathrm{L}^1(\mathbb{T}). \tag{3.19b}
$$

Since $(\iota_N c^N)_N$ converges to ρ in $\mathrm{L}^{\max\{\alpha,1\}}(\mathbb{T})$ the convergence of (3.19a) follows by dominated convergence.

To show (3.19b), we treat both terms separately. By Lemma 3.7, we have that $(\iota_N\Delta^N w^N)_N$ is uniformly bounded in $\mathrm{L}^2(\mathbb{T})$, and hence it converges weakly in $\mathrm{L}^2(\mathbb{T})$ to a function that can be identified by $\Delta\omega$ (because the differential operator is outside). Moreover, since $\iota_N c^N$ converges to ρ in $\mathrm{L}^{\max\{\alpha,1\}}(\mathbb{T})$ have the strong convergence $\iota_N w^N \to \omega$ in $\mathrm{L}^2(\mathbb{T})$. Hence, for the product of a weakly and a strongly converging sequence, we obtain the weak convergence $\iota_N(w^N\Delta^N w^N) = (\iota_N w^N)(\iota_N\Delta^N w^N) \rightharpoonup \omega\Delta\omega$.

Corollary 3.8 implies that the sequence $(\iota_N\partial_\pm^N w^N)_{N\in\mathbb{N}}$ is compact in $\mathrm{L}^2(\mathbb{T})$. Hence, they converge strongly and the limit is $\partial_x\omega$ (because the differential operator is outside). So also their product converges strongly in $\mathrm{L}^1(\mathbb{T})$. Hence, the convergence (3.19b) is established.

To show the liminf estimate, we exploit the joint convexity of the function $(u,v) \mapsto \frac{u^2}{v}$ and the convergences (3.19) and conclude that

$$
\liminf_{N\to\infty}\int_0^T S_{\alpha,N}(c^N)\,\mathrm{d}t \geq \frac{2}{\alpha^2}\int_0^T\int_{\mathbb{T}} \frac{\Big(\omega\Delta\omega - (\partial_x\omega)^2\Big)^2}{\rho^\alpha}\,\mathrm{d}x\,\mathrm{d}t = \mathcal{S}_\alpha(\rho),
$$

by the rewriting of the slope (3.4c). This proves the static liminf estimate (3.17). □

4 Chain rule and weak solutions

Our approach to the chain rule is stimulated by the usage of the Hilbert space $\mathrm{L}^2\mathrm{L}^2 = \mathrm{L}^2([0,T]\times\mathbb{T})$ as in [FeG23, GeH25], but we use a much more direct (and probably less general) approach by introducing the *modified flux* V and and the *modified slope* Σ given by

$$V = \rho^{-\alpha/2} j \quad \text{and} \quad \Sigma = -\frac{2}{\alpha}\Big(\Delta\Big(\rho^{\alpha/2}\Big) - 4\Big|\nabla(\rho^{\alpha/4})\Big|^2\Big). \tag{4.1}$$

They are chosen such that the dissipation functional $\mathcal{D}_\alpha$ in (2.12) takes thanks to (3.4a) the form

$$\mathcal{D}_\alpha(\rho,J) = \int_0^T \int_\mathbb{T} \Big[\frac{j^2}{2\rho^\alpha} + \frac{\rho^\alpha}{2}(\Delta\log\rho)^2\Big]\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\mathbb{T} \Big[\frac{1}{2}|V|^2 + \frac{1}{2}|\Sigma|^2\Big]\,\mathrm{d}x\,\mathrm{d}t.$$

The advantage is that $\mathcal{D}_\alpha(\rho,j) < \infty$ gives a clear control of V and Σ in $\mathrm{L}^2\mathrm{L}^2$.

The desired chain rule then takes the form

$$\mathcal{E}(\rho(s)) - \mathcal{E}(\rho(r)) = -\int_r^s \int_\mathbb{T} \Sigma\; V\,\mathrm{d}x\,\mathrm{d}t,$$

see Proposition 4.2 below. However, to achieve this goal for general ρ with $\mathcal{D}_\alpha(\rho,j) < \infty$, we exploit the relation first for smooth approximations $\rho_{\varepsilon,\delta}$, $\Sigma_{\varepsilon,\delta}$, and $V_{\varepsilon,\delta}$ and need to control the passages to the limit for $\varepsilon \to 0$ and $\delta \to 0$.

The following result shows that using mollifications ρ_ε and V_ε allows a control of V_ε and Σ_ε for suitable α. In other cases, we can only proceed by assuming additionally boundedness of ρ, namely $\rho \in \mathrm{L}^\infty([0,T]\times\mathbb{T})$. Choosing a non-negative mollifier $\phi \in \mathrm{H}^2(\mathbb{R})$ with $\mathrm{supp}(\phi) \subset [-1,1]$, $\int_\mathbb{R} \phi\,\mathrm{d}y = 1$ and setting $\phi_\varepsilon(y) = \varepsilon^{-1}\phi(y/\varepsilon)$, we define the smoothed density and flux

$$\rho_\varepsilon(t,x) = \Big(\rho(t,\cdot) * \phi_\varepsilon\Big)(x) \quad \text{and} \quad j_\varepsilon(t,x) = \Big(j(t,\cdot) * \phi_\varepsilon\Big)(x). \tag{4.2}$$

The following result is classical for convex functionals, but our proof for $\alpha \in (1,3/2)$ is slightly more general, because we only know convexity of the slope $\rho \mapsto \mathcal{S}_\alpha(\rho)$ for $\alpha \in [3/2,2]$, see (3.4d).

Lemma 4.1 (Bounds for convolutions)**.** *Assume that* (ρ,j) *satisfy* $\mathcal{D}_\alpha(\rho,j) < \infty$ *and that* $\sup_{t\in[0,T]} \mathcal{E}(\rho(t)) < \infty$. *Define* $(\rho_\varepsilon, j_\varepsilon)$ *via* (4.2).

(A) If $\alpha \in [0,1]$, *then* $V_\varepsilon = \rho_\varepsilon^{-\alpha/2} j_\varepsilon \to V = \rho^{-\alpha/2} j$ *in* $\mathrm{L}^2\mathrm{L}^2$.

(B) If $\alpha \in [1,2]$, *then* $\Sigma_\varepsilon = -\frac{2}{\alpha}\Big(\Delta\rho_\varepsilon^{\alpha/2} - 4|\nabla\rho_\varepsilon^{\alpha/4}|^2\Big) \to \Sigma$ *in* $\mathrm{L}^2\mathrm{L}^2$.

Proof. For part (A) we use $\alpha \in [0,1]$ giving the convexity of the primal dissipation potential $(\rho,J) \mapsto \mathcal{R}(\rho,j) = \frac{1}{2}\iint \rho^{-\alpha} j^2\,\mathrm{d}x\,\mathrm{d}t$. By Jensen's inequality for convolutions we have $\mathcal{R}(\rho_\varepsilon, j_\varepsilon) = \mathcal{R}(\rho,j) \leq \mathcal{D}_\alpha(\rho,j) < \infty$ and conclude boundedness of $V_\varepsilon = \rho_\varepsilon^{-\alpha/2} j_\varepsilon$, namely $\|V_\varepsilon\|_{\mathrm{L}^2\mathrm{L}^2} \leq \|V\|_{\mathrm{L}^2\mathrm{L}^2} < \infty$. Hence, along a subsequence (not relabeled) we have $V_\varepsilon \rightharpoonup W$ in $\mathrm{L}^2\mathrm{L}^2$, and $\|W\|_{\mathrm{L}^2\mathrm{L}^2} \leq \liminf_{\varepsilon\to 0} \|V_\varepsilon\|_{\mathrm{L}^2\mathrm{L}^2} \leq \|V\|_{\mathrm{L}^2\mathrm{L}^2}$. To identify the limit, we recall by Proposition 3.9 that $\rho_\varepsilon^{\alpha/2} \in \mathrm{L}^r\,\mathrm{L}^r$ with $r > 2$, and thus obtain that $j_\varepsilon = \rho_\varepsilon^{\alpha/2} V_\varepsilon$ is bounded in $\mathrm{L}^q\mathrm{L}^q$ with $q > 1$. Passing to the weak limit in the last relation we obtain $j = \rho^{\alpha/2} W$, which implies $W = V$. Hence, part (A) is established.

For part (B) we define the convex functional

$$\mathfrak{S}(\rho) := \int_0^T \int_{\mathbb{T}} \Big(\frac{(\Delta\rho)^2}{\rho^{2-\alpha}} + \frac{|\nabla\rho|^4}{\rho^{4-\alpha}} \Big) \, \mathrm{d}x \, \mathrm{d}t.$$

The convexity for $\alpha \in [1,2]$ follows from the convexity of $\mathbb{R}\times]0,\infty[\ni (a,b) \mapsto a^\beta/b^\gamma$ for $\gamma \geq 0$ and $\beta \geq 1+\gamma$. Moreover, the slope representations (3.4d) implies $\mathfrak{S}(\rho) \leq c_\alpha \int_0^T \mathcal{S}_\alpha(\rho) \, \mathrm{d}t \leq c_\alpha \mathcal{D}_\alpha(\rho, j)$.

Arguing as for part (A) with Jensen's inequality for convolutions we find

$\nabla\rho_\varepsilon^{\alpha/4} = \frac{4}{\alpha}\rho_\varepsilon^{\alpha/4-1}\nabla\rho_\varepsilon \to \frac{4}{\alpha}\rho^{\alpha/4-1}\nabla\rho = \nabla\rho^{\alpha/4}$ in $\mathrm{L}^4\mathrm{L}^4$ and
$\Delta\rho_\varepsilon^{\alpha/2} = \frac{2}{\alpha}\rho_\varepsilon^{\alpha/2-1}\Delta\rho_\varepsilon + \frac{4-2\alpha}{\alpha^2}\rho_\varepsilon^{\alpha/2-2}(\nabla\rho_\varepsilon)^2 \to \frac{2}{\alpha}\rho^{\alpha/2-1}\Delta\rho + \frac{4-2\alpha}{\alpha^2}\rho^{\alpha/2-2}(\nabla\rho)^2 = \Delta\rho^{\alpha/2}$ in $\mathrm{L}^2\mathrm{L}^2$.

Inserting this into the definition of Σ_ε we arrive at $\Sigma_\varepsilon \to \Sigma$, which is part (B). □

We are now ready to establish our chain rule, where the function ρ with $\mathcal{D}_\alpha(\rho, j) < \infty$ has to satisfy additional bounds depending on α, see (1.12). Only for $\alpha = 1$ we obtain the full result without further conditions, in all other case we need boundedness of ρ or even strict positivity. The proof relies on approximation via smooth and positive functions.

Proposition 4.2 (Chain rule). *For $\alpha > 0$ consider a pair $(\rho, j) \in \mathrm{CE}$, i.e. $\partial_t\rho = \Delta j$ in the sense of distribution, satisfying the bound $\mathcal{D}_\alpha(\rho, j) < \infty$ and $\sup_{t\in[0,T]} \mathcal{E}(\rho(t)) < \infty$. Moreover, assume one of the following three additional conditions:*

$$\alpha = 1; \tag{1.12a}$$
$$\alpha \in]0,2] \;\; \textit{and} \;\; \rho \in \mathrm{L}^\infty([0,T]\times\mathbb{T}); \tag{1.12b}$$
$$\alpha > 0, \;\; \rho \in \mathrm{L}^\infty([0,T]\times\mathbb{T}), \;\; \textit{and} \;\; \exists\, \delta > 0 : \;\; \rho(t,x) \geq \delta \;\; \textit{a.e.} \tag{1.12c}$$

Then, for all subintervals $[r,s] \subset [0,T]$ we have the identity

$$\mathcal{E}(\rho(s)) - \mathcal{E}(\rho(r)) = -\int_r^s \int_{\mathbb{T}} \Sigma \; V \, \mathrm{d}x \, \mathrm{d}t. \tag{4.3}$$

where the modified flux V and and the modified slope Σ are defined in (4.1).

Proof. *<u>Step 1. Smooth and positive case</u>*: We first consider the case that $\rho \in \mathrm{W}^{1,2}([0,T]; \mathrm{L}^2(\mathbb{T})) \cap \mathrm{L}^2\mathrm{H}^2$ with $\rho(t,x) \geq \delta > 0$. In this case, we can first apply the classical chain rule for convex functionals in $\mathrm{L}^2(\mathbb{T})$, see e.g. [Bré73, Lem. 3.3], and then integrate by parts to obtain

$$\begin{aligned} \mathcal{E}(\rho(s)) - \mathcal{E}(\rho(r)) &= \int_r^s \int_{\mathbb{T}} \log\rho \, \partial_t\rho \, \mathrm{d}x \, \mathrm{d}t = \int_r^s \int_{\mathbb{T}} \log\rho \, (\Delta j) \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_r^s \int_{\mathbb{T}} \Delta\big(\log\rho\big) \, j \, \mathrm{d}x \, \mathrm{d}t = \int_r^s \int_{\mathbb{T}} \Big(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{\rho^2} \Big) \, j \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_r^s \int_{\mathbb{T}} \rho^{\alpha/2} \Big(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{\rho^2} \Big) \, \frac{j}{\rho^{\alpha/2}} \, \mathrm{d}x \, \mathrm{d}t = -\int_r^s \int_{\mathbb{T}} \Sigma \, V \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.4}$$

This establishes the desired identity for the smooth case.

<u>Step 2. Smoothing of positive ρ</u>: We start now with a general pair (ρ, j) satisfying $\mathcal{D}_\alpha(\rho, j) < \infty$ and $\rho(t,x) \in [\delta, M]$ with $0 < \delta < M = \|\rho\|_{\mathrm{L}^\infty} < \infty$.

Using bound $|V|^2 = \rho^{-\alpha}|j|^2 \in \mathrm{L}^1(0,T;\mathrm{L}^1(\mathbb{T}))$ and $\rho \leq M$, we obtain $j \in \mathrm{L}^2\mathrm{L}^2$. Using the mollification from (4.2) the pair $(\rho_\varepsilon, j_\varepsilon)$ still satisfies the linear continuity equation $\partial_t\rho + \Delta j = 0$ and the upper and lower bound $\rho_\varepsilon \in [\delta, M]$. For $\varepsilon > 0$ we have $\rho_\varepsilon \in \mathrm{L}^2\mathrm{H}^2$. Moreover, $V_\varepsilon \in \mathrm{L}^2\mathrm{L}^2$ yields $\rho_\varepsilon \in \mathrm{W}^{1,2}\mathrm{L}^2$. Hence, Step 1 can be applied, i.e. (4.4) holds:

$$\mathcal{E}(\rho_\varepsilon(s)) - \mathcal{E}(\rho_\varepsilon(r)) = -\int_r^s \int_{\mathbb{T}} \Sigma_\varepsilon V_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \tag{4.5}$$
$$\text{with } \Sigma_\varepsilon = -\frac{2}{\alpha}\Big(\Delta(\rho_\varepsilon)^{\alpha/2} - 4|\nabla(\rho_\varepsilon^{\alpha/4})|^2\Big) \text{ and } V_\varepsilon = \rho_\varepsilon^{-\alpha/2} j_\varepsilon.$$

Step 3. Limit $\varepsilon \to 0$: We keep $\delta > 0$ fixed and consider the limit $\varepsilon \to 0$ in (4.5). As $\mathcal{E}$ is convex we have $\mathcal{E}(\rho_\varepsilon(t)) \leq \mathcal{E}(\rho(t))$ by Jensen's inequality, and using the lower semicontinuity $\liminf_{\varepsilon\to 0} \mathcal{E}(\rho_\varepsilon(t)) \geq \mathcal{E}(\rho(t)$ we conclude $\mathcal{E}(\rho_\varepsilon(t)) \to \mathcal{E}(\rho(t))$ as $\varepsilon \to 0$, for all $t \in [0,T]$.

Moreover, for $\varepsilon \to 0$ we obtain $V_\varepsilon \to V$ strongly in $\mathrm{L}^2\mathrm{L}^2$ by using the pointwise convergence $\rho_\varepsilon(t,x)^{-\alpha/2} \to \rho(t,x)^{-\alpha/2} \in \big[M^{-\alpha/2}, \delta^{-\alpha/2}\big]$ a.e. in $[0,T]\times\mathbb{T}$ and the strong L^2 convergence $j_\varepsilon \to j$.

For the slope we use $\Sigma \in \mathrm{L}^2\mathrm{L}^2$ which implies $\nabla\rho^{\alpha/4} \in \mathrm{L}^4\mathrm{L}^4$ and $\Delta\rho^{\alpha/2} \in \mathrm{L}^2\mathrm{L}^2$. For $\alpha > 4$ we exploit $\rho \geq \delta > 0$ and for $\alpha \in]0,4[$ we use $\rho \leq M$ to conclude $\nabla\rho \in \mathrm{L}^4\mathrm{L}^4$. Similarly, we find $\Delta\rho \in \mathrm{L}^2\mathrm{L}^2$ by using $\rho \in [\delta, M]$. Hence, we have $\nabla\rho_\varepsilon \to \nabla\rho$ in $\mathrm{L}^4\mathrm{L}^4$ and $\Delta\rho_\varepsilon \to \Delta\rho$ in $\mathrm{L}^2\mathrm{L}^2$ and find

$$\nabla\rho_\varepsilon^{\alpha/4} = \rho_\varepsilon^{\alpha/4-1}\nabla\rho_\varepsilon \to \rho^{\alpha/4-1}\nabla\rho = \nabla\rho^{\alpha/4} \quad \text{strongly in } \mathrm{L}^4\mathrm{L}^4.$$

Similarly, we obtain $\Delta\rho_\varepsilon^{\alpha/2} \to \Delta\rho$ strongly in $\mathrm{L}^2\mathrm{L}^2$ and hence, $\Sigma_\varepsilon \to \Sigma$ strongly in $\mathrm{L}^2\mathrm{L}^2$.

Now the limit $\varepsilon \to 0$ in (4.5) yields the chain rule (4.3) under the conditions in (1.12c).

Step 4. The case $\alpha \leq 2$: In this case, we have to show that the lower bound $\rho \geq \delta > 0$ is not needed. We do this by considering $\rho_\delta = \delta + \rho$ and taking the limit $\delta \to 0$. For $\delta > 0$ we can apply Step 3 and easily see that $\mathcal{E}(\rho_\delta(t)) \to \mathcal{E}(\rho(t))$ as $\delta \to 0$, for all $t \in [0,T]$. Moreover, $V_\delta = \rho_\delta^{-\alpha/2} j$ satisfies $|V_\delta| \leq |V|$ such that $V_\delta \to V$ strongly in $\mathrm{L}^2\mathrm{L}^2$ by dominated convergence. This part works for all $\alpha > 0$.

For the convergence $\Sigma_\delta \to \Sigma$, we observe the explicit representation

$$-\Sigma_\delta = (\rho+\delta)^{\alpha/2}\Big(\frac{\Delta\rho}{\rho+\delta} - \frac{|\nabla\rho|^2}{(\rho+\delta)^2}\Big) = \Big(\frac{\rho}{\rho+\delta}\Big)^{1-\alpha/2}\Big(-\Sigma + \frac{16\,\delta}{\alpha^2(\rho+\delta)}\big|\nabla\rho^{\alpha/4}|^2\Big). \tag{4.6}$$

Thus, using $\alpha \in]0,2]$ we find $|\Sigma_\delta| \leq |\Sigma| + C_\alpha|\nabla\rho^{\alpha/4}|^2$, which is an integrable pointwise majorant for $|\Sigma_\delta|^2$. Moreover, we easily see $\Sigma_\delta(t,x) \to \Sigma(t,x)$ a.e. for $\delta \to 0$. Thus, $\Sigma_\delta \to \Sigma$ strongly in $\mathrm{L}^2\mathrm{L}^2$. Hence, the final chain rule (4.3) for ρ follows from that for $\rho+\delta$, under the conditions (1.12b).

Step 5. The case $\alpha = 1$: We now exploit the result of Lemma 4.1, which shows $V_\varepsilon \to V$ and $\Sigma_\varepsilon \to \Sigma$ in $\mathrm{L}^2\mathrm{L}^2$ without any upper or lower bound. Again starting from the smooth and positive case (4.5) we can pass to the limit to obtain the chain rule (4.3) for $\alpha = 1$ and general (ρ, j) with $\mathcal{D}_\alpha(\rho, j) < \infty$. □

We note that the restriction $\alpha \leq 2$ in Step 4 of the above proof cannot be removed easily. For $\alpha > 2$ we can choose γ with $3/\alpha < \gamma < 3/2$ and consider ρ with $\rho(x) = |x|^\gamma$ for $|x| \leq 1/4$ and smooth otherwise. Then, $\Sigma \in \mathrm{L}^2\mathrm{L}^2$ with $\Sigma(x) = -\gamma|x|^{\alpha\gamma/2-2}$ for $|x| \leq 1/4$. However, Σ_δ

in (4.6) satisfies $\Sigma_\delta(x) \approx -(\delta+|x|^\gamma)^{\alpha/2-1}\gamma|x|^{\gamma-1}$, and hence does not lie in $\mathrm{L}^2\mathrm{L}^2$. It remains open to show the chain rule (under L^∞ bounds) for $\alpha > 2$ when no positivity bound is assumed.

We can now prove our main result on EDB and weak solutions (Result C), which again uses the additional conditions (1.12) on ρ if $\alpha \neq 1$.

Proof of Result C. We first show the EDB, i.e. for all r, s with $0 \le r < s \le T$ we have

$$\mathcal{E}(\rho(s)) + \int_r^s\!\!\int_{\mathbb{T}} \Big(\frac{j^2}{2\rho^\alpha} + \frac{\Sigma^2}{2}\Big)\,\mathrm{d}x\,\mathrm{d}t = \mathcal{E}(\rho(r)). \tag{4.7}$$

Applying Proposition 4.2 for EDI solutions (ρ, j) satisfying $\mathcal{D}_\alpha(\rho, j) < \infty$, we obtain with the energy-dissipation inequality and the chain rule (4.3) that

$$0 \ge \mathcal{E}(\rho(T)) - \mathcal{E}(\rho(0)) + \mathfrak{D}(\rho) = \int_0^T\!\!\int_{\mathbb{T}} \Big(-\Sigma\, V + \frac{1}{2}|V|^2 + \frac{1}{2}|\Sigma|^2\Big)\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_{\mathbb{T}} \frac{1}{2}\Big|V - \Sigma\Big|^2 \mathrm{d}x\,\mathrm{d}t.$$

Thus, we conclude $V = \Sigma = \frac{1}{2}V^2 + \frac{1}{2}\Sigma^2$. Thus, (4.7) follows from the chain rule (4.3) and the relation $V = \rho^{-\alpha/2} j$.

To show that (ρ, j) is a weak solution with a well-defined flux $j = \rho^{\alpha/2} V$ we use higher integrability of ρ (cf. Lemma 3.9 for the discrete case). Using $\int_0^T \mathcal{S}_\alpha(\rho)\,\mathrm{d}t \le \mathcal{D}_\alpha(\rho, j) < \infty$ and $\mathcal{E}(\rho(t)) \le \mathcal{E}(\rho(0)) < \infty$ we obtain that EDB solutions ρ satisfy $\rho \in \mathrm{L}^{q_*}\mathrm{L}^{q_*}$ with $q_* = \max\{4+\alpha, 2\alpha\}$. For this, we consider first $\alpha \in]0, 4]$ and use $u = \rho^{\alpha/4} \in \mathrm{L}^\infty\mathrm{L}^{4/\alpha} \cap \mathrm{L}^4\mathrm{W}^{1,4}$. With interpolation and a version of the Gagliardo-Nirenberg estimate, see e.g. [HMS25, Append. C], we obtain $u \in \mathrm{L}^{4+16/\alpha}\mathrm{L}^{4+16/\alpha}$, and $\rho = u^{4/\alpha} \in \mathrm{L}^{4+\alpha}\mathrm{L}^{4+\alpha}$ follows. For $\alpha \ge 4$ we have $u = \rho^{\alpha/4} \in \mathrm{L}^\infty\mathrm{L}^1 \cap \mathrm{L}^4\mathrm{W}^{1,4} \subset \mathrm{L}^8\mathrm{L}^8$ and conclude $\rho = u^{4/\alpha} \in \mathrm{L}^{2\alpha}\mathrm{L}^{2\alpha}$.

Moreover, we know $\rho^{\alpha/2} \in \mathrm{L}^2\mathrm{H}^2$ and $\rho^{\alpha/4} \in \mathrm{L}^4\mathrm{W}^{1,4}$. Hence, for a.a. $t \in [0, T]$ we can apply the product rule in Sobolev spaces:

$$\begin{aligned} &\Delta(\rho^\alpha) = \Delta\Big(\rho^{\alpha/2}\,\rho^{\alpha/2}\Big) = 2\rho^{\alpha/2}\Delta(\rho^{\alpha/2}) + 2\Big|\nabla(\rho^{\alpha/2})\Big|^2, \\ &\nabla(\rho^{\alpha/2}) = \nabla\Big(\rho^{\alpha/4}\,\rho^{\alpha/4}\Big) = 2\rho^{\alpha/4}\,\nabla(\rho^{\alpha/4}) \quad \Longrightarrow \quad \Big|\nabla\rho^{\alpha/2}\Big|^2 = 4\rho^{\alpha/2}\Big|\nabla\rho^{\alpha/4}\Big|^2 \end{aligned}$$

With this we find the identity

$$j = \rho^{\alpha/2}V = \rho^{\alpha/2}\Sigma = -\rho^{\alpha/2}\,\frac{2}{\alpha}\Big(\Delta(\rho^{\alpha/2}) - 4\Big|\nabla(\rho^{\alpha/4})\Big|^2\Big) = -\frac{1}{\alpha}\Delta(\rho^\alpha) - \frac{4}{\alpha}\Big|\nabla(\rho^{\alpha/2})\Big|^2.$$

From $\rho^{\alpha/2} \in \mathrm{L}^{2q_*/\alpha}$ and $\Delta(\rho^{\alpha/2}), |\nabla(\rho^{\alpha/4})|^2 \in \mathrm{L}^2\mathrm{L}^2$ we conclude $j \in \mathrm{L}^{p_\alpha}\mathrm{L}^{p_\alpha}$ with exponent $p_\alpha = 2q_*/(q_*+\alpha) = \max\Big\{(4+\alpha)/(2+\alpha), 4/3\Big\}$.

Of course, if we additionally know that ρ is bounded, then $j \in \mathrm{L}^2\mathrm{L}^2$.

Combining this with the weak form of the continuity equation finishes the proof. □

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