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Pierluigi Colli[1], Gianni Gilardi[2], Andrea Signori[3], Jürgen Sprekels[4]

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[1] Dipartimento di Matematica “F. Casorati”
Università di Pavia
and
Research Associate at the IMATI – C.N.R. Pavia
via Ferrata 5
27100 Pavia
Italy
E-Mail: pierluigi.colli@unipv.it

[2] Dipartimento di Matematica “F. Casorati”
Università di Pavia
via Ferrata 5
27100 Pavia
and
Istituto Lombardo, Accademia di Scienze e Lettere
via Borgonuovo 25
20121 Milano
Italy
E-Mail: gianni.gilardi@unipv.it

[3] Dipartimento di Matematica
Politecnico di Milano
Alexander von Humboldt Research Fellow
via E. Bonardi 9
20133 Milano
Italy
E-Mail: andrea.signori@polimi.it

[4] Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
and
Weierstrass Institute
Anton-Wilhelm-Amo-Str. 39
10117 Berlin
Germany
E-Mail: juergen.sprekels@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

The mathematical analysis of diffuse-interface models for multiphase flows has attracted significant attention due to their ability to capture complex interfacial dynamics, including curvature effects, within a unified, energetically consistent framework. In this work, we study a novel Brinkman–Cahn–Hilliard system, coupling a sixth-order phase-field evolution with a Brinkman-type momentum equation featuring variable shear viscosity. The Cahn–Hilliard equation includes a nonconservative source term accounting for mass exchange, and the velocity equation contains a non divergence-free forcing term. We establish the existence of weak solutions in a divergence-free variational framework, and, in the case of constant mobility and shear viscosity, prove uniqueness and continuous dependence on the forcing. Additionally, we analyze the Darcy limit, providing existence results for the corresponding reduced system.

1 Introduction

Multiphase flow research has been profoundly advanced by diffuse-interface models, which provide a rigorous energetic framework to capture complex interfacial phenomena, including curvature effects. In particular, models based on the Cahn–Hilliard theory and its higher-order variants naturally describe binary mixtures, amphiphilic membranes, and phase separation in soft matter. Such models are especially relevant in biology, where lipid bilayers and cellular membranes undergo curvature-driven transformations, fusion, and budding, processes that are crucial for intracellular organization. When coupled with fluid dynamics, they provide a powerful tool to investigate the interplay between interfacial evolution, transport, and hydrodynamic effects.

Along these lines, we investigate in this paper a novel system of equations of Brinkman–Cahn–Hilliard type. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ. If $T > 0$ is a fixed final time, we set $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$ and

$$Q_t := \Omega \times (0,t) \quad \text{for } t \in (0,T] .$$

Then, the differential system under investigation reads

$$-\operatorname{div} \mathbb{T}(\varphi, \boldsymbol{v}, p) + \lambda(\varphi)\boldsymbol{v} = \mu\nabla\varphi + \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} = 0 \qquad \text{in } Q , \tag{1.1}$$

$$\partial_t\varphi + \boldsymbol{v}\cdot\nabla\varphi - \operatorname{div}(m(\varphi)\nabla\mu) = S(\varphi) \qquad \text{in } Q , \tag{1.2}$$

$$-\varepsilon\Delta w + \tfrac{1}{\varepsilon} f'(\varphi) w + \nu w = \mu \qquad \text{in } Q , \tag{1.3}$$

$$-\varepsilon\Delta\varphi + \tfrac{1}{\varepsilon} f(\varphi) = w \qquad \text{in } Q , \tag{1.4}$$

complemented with the boundary and initial conditions

$$\mathbb{T}(\varphi, \boldsymbol{v}, p)\boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}} w = \partial_{\boldsymbol{n}}\varphi = 0 \qquad \text{on } \Sigma , \tag{1.5}$$

$$\varphi(0) = \varphi_0 \qquad \text{in } \Omega , \tag{1.6}$$

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where $\boldsymbol{n}$ and $\partial_{\boldsymbol{n}}$ denote the outward unit normal vector to Γ and the corresponding outward normal derivative, respectively, and φ_0 is a prescribed function, which serves as the initial datum.

The overall model presents several distinctive features. The velocity field $\boldsymbol{v}$ satisfies a Brinkman-type momentum equation (1.1), where the stress tensor $\mathbb{T}$ involves the symmetrized gradient $D\boldsymbol{v}$ of $\boldsymbol{v}$:

$$\mathbb{T}(\varphi, \boldsymbol{v}, p) = \eta(\varphi) D\boldsymbol{v} - p\mathbb{I}\,,$$

with phase-dependent positive viscosity η, p the pressure, $\mathbb{I} \in \mathbb{R}^{3\times 3}$ the identity matrix, and

$$D\boldsymbol{v} := \frac{1}{2}\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top\right) \tag{1.7}$$

denoting the symmetrized gradient. This formulation interpolates between the Stokes (formally $\lambda \equiv 0$) and Darcy regimes (formally $\eta \equiv 0$) and is particularly suitable for flows in porous or heterogeneous media. Since in this work we also rigorously analyze the Darcy limit of the system, obtained by letting the shear viscosity vanish, it is worth emphasizing that the Darcy regime plays a particularly relevant role in applications to biological and porous media, as it effectively captures the flow through highly permeable tissues and fibrous structures where viscous effects are negligible. The order parameter φ represents the local proportion of one of the two components in the binary material and serves as an order parameter. To simplify the analysis, it is usually normalized in such a way that the pure states correspond to $\varphi = -1$ and $\varphi = 1$, while $\{-1 < \varphi < 1\}$ represents the diffuse interface layer that occurs in an tubular neighborhood of the interface, with thickness parameter $\varepsilon > 0$. Its evolution is governed by a sixth-order variant of the Cahn–Hilliard-type equation with a source term of the form

$$S(\varphi) = -\sigma\varphi + h(\varphi),$$

where h is smooth and bounded, and $\sigma \in \mathbb{R}$ is a real coefficient. This term accounts for mass exchange and explicitly violates the standard mass-conservation property of φ. The chemical potentials μ and w appearing in equations (1.3) and (1.4) correspond to the first variations of the total free energy $\mathcal{E}$ and the Ginzburg–Landau free energy $\mathcal{G}$, respectively:

$$\mu := \frac{\delta\mathcal{E}}{\delta\varphi}, \qquad w := \frac{\delta\mathcal{G}}{\delta\varphi},$$

where

$$\mathcal{E}(\varphi) := \mathcal{F}(\varphi) + \nu\,\mathcal{G}(\varphi) = \frac{1}{2}\int_\Omega (-\varepsilon\Delta\varphi + \tfrac{1}{\varepsilon}f(\varphi))^2 + \nu\int_\Omega \Big(\frac{\varepsilon}{2}\,|\nabla\varphi|^2 + \tfrac{1}{\varepsilon}F(\varphi)\Big)\,. \tag{1.8}$$

Here, F is an everywhere defined double-well potential, f its derivative, and ν a (not necessarily positive) real constant. A prototype is the *standard regular potential*

$$F_{\rm reg}(s) := \frac{1}{4}\,(s^2-1)^2\,, \quad s \in \mathbb{R}\,. \tag{1.9}$$

Finally, $\boldsymbol{u}$ in (1.1) represents a forcing term, which can be interpreted as a control variable in a related control problem (cf. [9, 36]), while the nonlinear term $\mu\nabla\varphi$ corresponds to the Korteweg force, accounting for capillarity effects in the mixture. It is worth mentioning that, due to the regularity of f, the boundary condition $\partial_{\boldsymbol{n}} w = 0$ in (1.5) is equivalent to $\partial_{\boldsymbol{n}}\Delta\varphi = 0$ on Σ. Overall, the system naturally accommodates a wide range of physical regimes depending on the sign and magnitude of ν and combines a sixth-order Cahn–Hilliard type equation with a Brinkman flow through transport and capillarity effects, the nonconservative source term $S(\varphi)$, and the forcing $\boldsymbol{u}$.

Let us now frame the model within the literature. A key application concerns curvature-driven phenomena, notably the evolution of amphiphilic bilayer membranes. The classical Helfrich model [5, 26] describes the bending energy of a membrane Γ_0 as

$$\mathcal{E}_{\text{elastic}} = \frac{k}{2}\int_{\Gamma_0} (H - H_0)^2 \,\mathrm{d}S,$$

where H is the mean curvature, H_0 the spontaneous curvature (often zero), and k the bending rigidity. In a diffuse-interface framework, a phase variable φ distinguishes interior ($\varphi = 1$) and exterior ($\varphi = -1$) regions, and the Helfrich energy can be approximated by a modified Willmore functional

$$\mathcal{E}_\varepsilon(\varphi) = \frac{k}{2\varepsilon}\int_\Omega \Big(-\varepsilon\Delta\varphi + \frac{1}{\varepsilon}(\varphi^2 - 1)\varphi\Big)^2, \tag{1.10}$$

with $\varepsilon > 0$ the interfacial thickness [16, 18], converging to the sharp-interface limit as $\varepsilon \to 0$. Since sharp-interface asymptotics are not considered in this work, we set $\varepsilon = 1$ from now onward for convenience.

It is worth noting that the total energy $\mathcal{E}$ in our model can be viewed as a higher-order extension of the classical Ginzburg–Landau free energy $\mathcal{G}$. When $\nu = 0$, $\mathcal{E}$ reduces to the Willmore functional (1.10), recovering the Canham–Helfrich bending energy of biomembranes. For $\nu > 0$, $\mathcal{E}$ provides a regularized version of $\mathcal{G}$, penalizing high curvature and favoring smoother interfaces. This type of regularization has been used, for example, in [3, 38] to study anisotropy effects in thin film growth and coarsening processes. Conversely, for $\nu < 0$, $\mathcal{E}$ corresponds to the so-called functionalized Cahn–Hilliard energy, originally introduced for amphiphilic mixtures and later applied to polymers and bilayer membranes, see [23]. These different regimes illustrate the versatility of the model, capable of capturing a wide range of interfacial phenomena, from membrane elasticity to pattern formation in soft matter. Without claiming exhaustiveness, we refer to [6, 7, 11, 12, 15, 35, 39] for analytical studies and to [1, 4, 5, 16–19, 29] for numerical investigations. The analysis of sixth-order Cahn–Hilliard equations has attracted considerable attention, with applications ranging from the dynamics of oil-water-surfactant mixtures [32–34], and the phase-field-crystal model [24, 25, 30, 31, 40]. In the context of applications to optimal control, see also [9, 36]. Finally, we mention some works on coupling the Cahn–Hilliard equation with Brinkman-type flow [2, 10, 13, 14, 21, 22, 28].

The main goal of this work is to establish the well-posedness of the initial-boundary value problem (1.1)–(1.6). This is achieved by rewriting the system, especially the velocity equation, in a variational framework where the pressure does not appear explicitly. The structure of the paper is as follows. In Section 2, we introduce the notation, state the precise assumptions, and formulate the problem under consideration. Section 3 is devoted to the proof of the existence theorem. In Section 4, we establish the Darcy limit by letting the shear viscosity tend to zero, thus obtaining an existence result that rigorously recovers the Darcy model, which is formally derived by setting $\eta \equiv 0$ in system (1.1)–(1.6), as a mathematically consistent asymptotic regime. In Section 5, we analyze uniqueness and continuous dependence for the general system by assuming that the coefficients η and m are just positive constants.

2 Notation, assumptions and results

We assume that $\Omega \subset \mathbb{R}^3$ is a bounded and connected open set with smooth boundary Γ. The symbol $|\Omega|$ denotes its Lebesgue measure, and $\partial_{\boldsymbol{n}}$ stands for the outward normal derivative on Γ. For any

Banach space X, the symbols $\|\cdot\|_X$ and X^* indicate the corresponding norm and its dual space, with a few exceptions for the notation of the norm that are listed below. The classical Lebesgue and Sobolev spaces on Ω, corresponding to each $1 \le p \le \infty$ and $k \ge 0$, are denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, respectively. Their associated norms are written as $\|\cdot\|_{L^p(\Omega)} =: \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)}$. Next, we set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{z \in H^2(\Omega) : \partial_{\boldsymbol{n}} z = 0 \text{ on } \Gamma\}, \tag{2.1}$$

$$\boldsymbol{H} := H \times H \times H, \quad \boldsymbol{V} := V \times V \times V \quad \text{and} \quad \boldsymbol{V}_0 := \{\boldsymbol{\zeta} \in \boldsymbol{V} : \operatorname{div} \boldsymbol{\zeta} = 0\}. \tag{2.2}$$

Similarly, we use the boldface characters to denote powers of the Lebesgue and Sobolev spaces, whose elements are vector valued functions. For instance, we employ $\boldsymbol{L}^p(\Omega) := (L^p(\Omega))^3$, for every $1 \le p \le \infty$, and $\boldsymbol{H}^2(\Omega) = (H^2(\Omega))^3$. To simplify notation, the norms in the special cases H and $\boldsymbol{H}$ are both indicated by $\|\cdot\|$, without any subscript. The symbol $\|\cdot\|_\infty$ might denote the norm in each of the spaces $L^\infty(\Omega)$, $L^\infty(Q)$ and $L^\infty(0,T)$, if no confusion can arise. For simplicity, we use the same symbol for the norm in some space and the norm in any power of it, as we have announced for H and $\boldsymbol{H}$. So, for example, the norms in $\boldsymbol{V}$ and $\boldsymbol{V}_0$ are simply denoted by $\|\cdot\|_V$, since $\boldsymbol{V} = V \times V \times V$ and $\boldsymbol{V}_0$ is a subspace of the latter.

It is worth recalling that V is dense in H. Therefore, we can adopt the usual framework of Hilbert triplets obtained by standard identifications. Namely, we have that

$$\langle y, z \rangle = \int_\Omega yz \quad \text{for every } y \in H \text{ and } z \in V, \quad \text{so that} \quad V \hookrightarrow H \hookrightarrow V^*. \tag{2.3}$$

In (2.3), the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality pairing between V^* and V. Besides the space $\boldsymbol{V}_0$ already introduced, we also make use of the space

$$\boldsymbol{H}_0 := \{\boldsymbol{\zeta} \in \boldsymbol{H} : \operatorname{div} \boldsymbol{\zeta} = 0\}, \tag{2.4}$$

where the divergence is understood in the sense of distributions. We notice at once that the embedding

$$\boldsymbol{V}_0 \hookrightarrow \boldsymbol{H}_0 \tag{2.5}$$

is dense (see, e.g., [27, Cor. 2.3]). We additionally remark that all of the embeddings in (2.3) and (2.5) are compact. Next, we recall the symbol $D\boldsymbol{v}$ introduced in (1.7) for the symmetrized gradient of the velocity $\boldsymbol{v}$, whose use will be extended to any vector field $\boldsymbol{\zeta} \in \boldsymbol{V}$. Finally, we recall the standard notation for the scalar product and the norm of matrices, namely

$$A : B := \sum_{i,j=1}^{3} a_{ij} b_{ij} \quad \text{and} \quad |A|^2 := A : A \quad \text{for } A = (a_{ij}),\ B = (b_{ij}) \in \mathbb{R}^{3\times 3}. \tag{2.6}$$

We now list the assumptions regarding the structure of the system under consideration. As for the functions η, λ and m, we assume that

$$\eta : \mathbb{R} \to \mathbb{R} \text{ is Lipschitz continuous,}$$
$$\text{and } \lambda,\ m : \mathbb{R} \to \mathbb{R} \text{ are locally Lipschitz continuous; all are positive, and} \tag{2.7}$$

$$0 < \eta_* \le \eta(s) \le \eta^*, \quad 0 < \lambda_* \le \lambda(s) \le \lambda^* \quad \text{and} \quad 0 < m_* \le m(s) \le m^*,$$
$$\text{for every } s \in \mathbb{R} \text{ and some positive constants } \eta_*, \eta^*, \lambda_*, \lambda^*, m_* \text{ and } m^*. \tag{2.8}$$

Moreover, we postulate that

$$\nu \in \mathbb{R}\,, \quad \sigma \in \mathbb{R}, \quad \text{and} \quad S:\mathbb{R}\to\mathbb{R} \quad \text{is given by } S(\varphi) = -\sigma\varphi + h(\varphi)\,,$$
$$\text{where } h:\mathbb{R}\to\mathbb{R} \text{ is bounded and Lipschitz continuous.} \tag{2.9}$$

Finally, we assume that

$$F:\mathbb{R}\to\mathbb{R} \quad \text{is of class } C^4, \text{ and } f := F' \text{ denotes its derivative;} \tag{2.10}$$

$$\lim_{|s|\to+\infty} \frac{f(s)}{s} = +\infty\,; \tag{2.11}$$

$$f'(s) \ge -C_1\,, \quad |F(s)| \le C_2\left(|sf(s)|+1\right), \quad \text{and} \quad |sf'(s)| \le C_3(|f(s)|+1),$$
$$\text{for every } s\in\mathbb{R} \text{ and some positive constants } C_1, C_2 \text{ and } C_3\,. \tag{2.12}$$

Notice that the last inequality and the continuity of f' imply that

$$|f'(s)| \le C_3'(|f(s)|+1) \quad \text{for every } s\in\mathbb{R}, \tag{2.13}$$

for a suitable constant C_3'.

The conditions (2.10)–(2.13) imposed on the potential can be compared with those in [9, assumptions (2.4)–(2.9)], and are fulfilled, in particular, by the classical regular potential introduced in (1.9), as well as by any lambda-convex potential F exhibiting polynomial growth.

For the data, we assume that

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}), \tag{2.14}$$

$$\varphi_0 \in W\,. \tag{2.15}$$

Our main analytical objective is to establish the well-posedness of the system (1.1)–(1.6). To this end, we first highlight two key properties that allow equivalent formulations of some equations. As outlined in the Introduction, we then recast the velocity equation in a variational framework where the pressure variable p no longer appears explicitly.

To begin with, in place of equation (1.1), we adopt the variational formulation

$$\int_\Omega \eta(\varphi)D\boldsymbol{v} : \nabla\boldsymbol{\zeta} + \int_\Omega \lambda(\varphi)\boldsymbol{v}\cdot\boldsymbol{\zeta} = \int_\Omega \mu\nabla\varphi\cdot\boldsymbol{\zeta} + \int_\Omega \boldsymbol{u}\cdot\boldsymbol{\zeta} \quad \text{for every } \boldsymbol{\zeta}\in\boldsymbol{V}_0\,,$$

where the dot and colon denote the Euclidean inner product and the matrix inner product, respectively. This reformulation is justified by the following observation concerning an equation of type (1.1) for a fixed time and given $\phi \in W$ and $\boldsymbol{f} \in \boldsymbol{H}$. Namely, from [14, Prop. 2.43] (see also [20, Lemma 1.5] and similar results in [21,22]), there exists a unique pair $(\boldsymbol{y},\pi)$ satisfying

$$\boldsymbol{y}\in\boldsymbol{V}_0 \quad \text{and} \quad \pi\in H\,, \tag{2.16}$$

$$-\operatorname{div}\mathbb{T}(\phi,\boldsymbol{y},\pi) + \lambda(\phi)\boldsymbol{y} = \boldsymbol{f} \quad \text{in } \Omega\,, \tag{2.17}$$

$$\mathbb{T}(\phi,\boldsymbol{y},\pi)\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma\,. \tag{2.18}$$

On the other hand, on account of (2.8), it follows from the Lax–Milgram theorem that there exists a unique $\boldsymbol{y}\in\boldsymbol{V}_0$ such that

$$a(\boldsymbol{y},\boldsymbol{\zeta}) := \int_\Omega \big(\eta(\phi)D\boldsymbol{y} : \nabla\boldsymbol{\zeta} + \lambda(\phi)\boldsymbol{y}\cdot\boldsymbol{\zeta}\big) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\zeta} \quad \text{for every } \boldsymbol{\zeta}\in\boldsymbol{V}_0\,. \tag{2.19}$$

Indeed, we have that

$$\int_\Omega \bigl(\eta(\phi) D\boldsymbol{y} : \nabla\boldsymbol{\zeta} + \lambda(\phi)\boldsymbol{y}\cdot\boldsymbol{\zeta}\bigr) \le \max\{\eta^*,\lambda^*\}\, \|\boldsymbol{y}\|_V\, \|\boldsymbol{\zeta}\|_V \quad \text{for every } \boldsymbol{y},\boldsymbol{\zeta}\in \boldsymbol{V}_0\,,$$

that is, the bilinear form $a : \boldsymbol{V}_0 \times \boldsymbol{V}_0 \to \mathbb{R}$ given by the left-hand side of (2.19) is continuous on $\boldsymbol{V}_0 \times \boldsymbol{V}_0$. Moreover, there hold the identity

$$D\boldsymbol{\zeta} : \nabla\boldsymbol{\zeta} = |D\boldsymbol{\zeta}|^2 \quad \text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}\,, \tag{2.20}$$

and the Korn inequality

$$\|\boldsymbol{\zeta}\|_V^2 \le \mathcal{C}_K \int_\Omega (|D\boldsymbol{\zeta}|^2 + |\boldsymbol{\zeta}|^2) \quad \text{for every } \boldsymbol{\zeta}\in\boldsymbol{V}\,, \tag{2.21}$$

with some constant $\mathcal{C}_K$ depending only on Ω. Hence, by setting $\alpha := \min\{\eta_*,\lambda_*\}/\mathcal{C}_K$, it follows that

$$a(\boldsymbol{\zeta},\boldsymbol{\zeta}) = \int_\Omega \bigl(\eta(\phi)|D\boldsymbol{\zeta}|^2 + \lambda(\phi)|\boldsymbol{\zeta}|^2\bigr) \ge \alpha\, \|\boldsymbol{\zeta}\|_V^2 \quad \text{for every } \boldsymbol{\zeta}\in\boldsymbol{V}_0\,. \tag{2.22}$$

Therefore, the bilinear form under consideration is also coercive. Finally, the component $\boldsymbol{y}$ of the solution $(\boldsymbol{y},\pi)$ to (2.16)–(2.18) belongs to $\boldsymbol{V}_0$ and also satisfies (2.19), thanks to the identities

$$\begin{aligned} \int_\Omega \eta(\phi) D\boldsymbol{y} : \nabla\boldsymbol{\zeta} &= \int_\Omega \bigl(\eta(\phi) D\boldsymbol{y} - \pi\mathbb{I}\bigr) : \nabla\boldsymbol{\zeta} \\ &= \int_\Omega \mathbb{T}(\phi,\boldsymbol{y},\pi) : \nabla\boldsymbol{\zeta} = -\int_\Omega \operatorname{div}\mathbb{T}(\phi,\boldsymbol{y},\pi)\cdot\boldsymbol{\zeta}\,, \end{aligned}$$

which, owing to (2.18), hold true for every $\boldsymbol{\zeta}\in\boldsymbol{V}_0$. Thus, it is equivalent to look either for the unique solution $\boldsymbol{y}\in\boldsymbol{V}_0$ to (2.19) or for the first component of the solution $(\boldsymbol{y},\pi)$ to (2.16)–(2.18). Since the component φ of the solution to the problem we want to consider is required to belong to $L^\infty(0,T;W)$ and the right-hand side of (1.1) is expected to belong to $L^2(0,T;\boldsymbol{H})$, we can apply the above correspondence to $\phi = \varphi(t)$ for a.a. $t\in(0,T)$ and replace (1.1) by the forthcoming variational equation (2.28).

We now consider an equivalent formulation of the problem. When appropriate, we eliminate w by substituting the expression provided by (1.4) in (1.3). This leads to the following equation:

$$-\Delta\bigl(-\Delta\varphi + f(\varphi)\bigr) + \bigl(f'(\varphi)+\nu\bigr)\bigl(-\Delta\varphi + f(\varphi)\bigr) = \mu, \tag{2.23}$$

and can recover w and (1.3) by using (1.4) as a definition of w. However, we will write all the equations in their variational forms, for convenience.

Here is the precise formulation of the problem under investigation. We look for a quadruplet $(\boldsymbol{v},\varphi,\mu,w)$ with the regularity

$$\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}_0)\,, \tag{2.24}$$

$$\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;W) \cap L^2(0,T;H^5(\Omega))\,, \tag{2.25}$$

$$\mu \in L^2(0,T;V)\,, \tag{2.26}$$

$$w \in L^\infty(0,T;H) \cap L^2(0,T;H^3(\Omega)\cap W)\,, \tag{2.27}$$

that solves the variational equations

$$\int_\Omega \bigl(\eta(\varphi)D\boldsymbol{v} : \nabla\boldsymbol{\zeta} + \lambda(\varphi)\boldsymbol{v}\cdot\boldsymbol{\zeta}\bigr) = \int_\Omega (\mu\nabla\varphi + \boldsymbol{u})\cdot\boldsymbol{\zeta}$$
$$\text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}_0 \text{ and a.e. in } (0,T)\,, \tag{2.28}$$

$$\langle \partial_t\varphi, z\rangle + \int_\Omega \boldsymbol{v}\cdot\nabla\varphi\, z + \int_\Omega m(\varphi)\nabla\mu\cdot\nabla z = \int_\Omega S(\varphi)z$$
$$\text{for every } z \in V \text{ and a.e. in } (0,T)\,, \tag{2.29}$$

$$\int_\Omega \nabla w\cdot\nabla z + \int_\Omega \bigl(f'(\varphi)+\nu\bigr)wz = \int_\Omega \mu z$$
$$\text{for every } z \in V \text{ and a.e. in } (0,T)\,, \tag{2.30}$$

$$\int_\Omega \nabla\varphi\cdot\nabla z + \int_\Omega f(\varphi)z = \int_\Omega wz$$
$$\text{for every } z \in V \text{ and a.e. in } (0,T)\,, \tag{2.31}$$

as well as the initial condition

$$\varphi(0) = \varphi_0\,. \tag{2.32}$$

Notice that (2.30) and (2.31) contain the homogeneous Neumann boundary condition for w and φ in a generalized sense. However, these equations can be replaced by their strong forms (1.3) and (1.4), thanks to the regularity of w and φ required in (2.27) and (2.25), which also contain the homogeneous Neumann boundary conditions. By taking this into account, we can equivalently replace (2.30)–(2.31) by

$$\int_\Omega \nabla\bigl(-\Delta\varphi + f(\varphi)\bigr)\cdot\nabla z + \int_\Omega \bigl(f'(\varphi)+\nu\bigr)\bigl(-\Delta\varphi + f(\varphi)\bigr)z = \int_\Omega \mu z$$
$$\text{for every } z \in V \text{ and a.e. in } (0,T), \tag{2.33}$$

which is the weak form of (2.23), and keep (2.31), or (1.4), as a definition of w.

Here is our first result.

Theorem 2.1. *Under the assumptions* (2.7)–(2.12) *on the structure and* (2.14)–(2.15) *on the data, there exists at least one quadruplet* $(\boldsymbol{v},\varphi,\mu,w)$ *that fulfills the regularity requirements* (2.24)–(2.27)*, solves Problem* (2.28)–(2.32)*, and satisfies the estimate*

$$\begin{aligned}&\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{V}_0)} + \|\varphi\|_{H^1(0,T;V^*)\cap L^\infty(0,T;W)\cap L^2(0,T;H^5(\Omega))}\\ &\quad + \|\mu\|_{L^2(0,T;V)} + \|w\|_{L^\infty(0,T;H)\cap L^2(0,T;H^3(\Omega)\cap W)} \le K_1\,,\end{aligned} \tag{2.34}$$

with a constant K_1 *that depends only on the structure of the system,* Ω*,* T *and an upper bound for the norms of the data related to* (2.14)–(2.15)*. Moreover, the solution is unique if both* η *and* m *are positive constants.*

We now establish an existence result for a system closely related to (2.28)–(2.32), but governed by the so-called Darcy law, which formally corresponds to setting the shear viscosity $\eta \equiv 0$. In this asymptotic framework, particular care must be devoted to the boundary condition $\mathbb{T}(\varphi,\boldsymbol{v},p)\boldsymbol{n} = \boldsymbol{0}$, which, as is well known (see, e.g., [21] and also [28]), corresponds to the boundary condition $p = 0$ on Σ in the vanishing viscosity limit.

Theorem 2.2. *Suppose that the assumptions* (2.7)–(2.12) *on the structure and* (2.14)–(2.15) *on the data are fulfilled. Furthermore, let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of viscosity functions such that, for each fixed $n \in \mathbb{N}$, η_n is compatible with the assumptions* (2.7) *and* (2.8)*, for two sequences of positive values $\eta_{*,n}$ and ${\eta^*}_{,n}$ satisfying $0 < \eta_{*,n} \le \eta_n(s) \le {\eta^*}_{,n}$ for every $s \in \mathbb{R}$. We further assume that*

$$\|\eta_n\|_{L^\infty(\mathbb{R})} \ \textit{tends to } 0 \ \textit{as } n \to \infty.$$

For any $n \in \mathbb{N}$, let $(\boldsymbol{v}_n, \varphi_n, \mu_n, w_n)$ denote a solution to Problem (2.28)–(2.32) *as found in Theorem 2.1 associated with the function η_n. Then there is a subsequence of the sequence $\{(\boldsymbol{v}_n, \varphi_n, \mu_n, w_n)\}$, which is again labeled by $n \in \mathbb{N}$, that converges to a solution $(\boldsymbol{v}, \varphi, \mu, w)$ of the corresponding Darcy system in the following sense: as $n \to \infty$, it holds that*

$$\boldsymbol{v}_n \to \boldsymbol{v} \quad \textit{weakly in } L^2(0,T;\boldsymbol{H}_0)\,, \tag{2.35}$$

$$\eta_n(\varphi_n) D\boldsymbol{v}_n \to \boldsymbol{0} \quad \textit{strongly in } L^2(0,T;\boldsymbol{H}^3)\,, \tag{2.36}$$

$$\varphi_n \to \varphi \quad \textit{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;W) \cap L^2(0,T;H^5(\Omega))\,, \tag{2.37}$$

$$\mu_n \to \mu \quad \textit{weakly in } L^2(0,T;V)\,, \tag{2.38}$$

$$w_n \to w \quad \textit{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;H^3(\Omega) \cap W)\,, \tag{2.39}$$

where $(\boldsymbol{v}, \varphi, \mu, w)$ solves the variational equations

$$\int_\Omega \lambda(\varphi)\boldsymbol{v}\cdot\boldsymbol{\zeta} = \int_\Omega (\mu\nabla\varphi + \boldsymbol{u})\cdot\boldsymbol{\zeta} \quad \textit{for every } \boldsymbol{\zeta} \in \boldsymbol{H}_0 \textit{ and a.e. in } (0,T)\,, \tag{2.40}$$

$$\langle \partial_t\varphi, z\rangle + \langle \boldsymbol{v}\cdot\nabla\varphi, z\rangle + \int_\Omega m(\varphi)\nabla\mu\cdot\nabla z = \int_\Omega S(\varphi) z \quad \textit{for every } z \in V \textit{ and a.e. in } (0,T)\,, \tag{2.41}$$

as well as (2.30)–(2.31) *and the initial condition* (2.33)*.*

We point out that the equations (2.40) and (2.41) are consistently formulated. Indeed, owing to (2.25) and (2.26), the integral on the right-hand side of (2.40) involves the term

$$\mu\nabla\varphi,$$

which is the product of $\mu \in L^2\big(0,T;L^4(\Omega)\big)$ and $\nabla\varphi \in L^\infty\big(0,T;\boldsymbol{L}^4(\Omega)\big)$. Hence, one has $\mu\nabla\varphi \in L^2(0,T;\boldsymbol{H})$. On the other hand, the second term in (2.41) contains

$$\boldsymbol{v}\cdot\nabla\varphi,$$

where $\boldsymbol{v}$ is just in $L^2(0,T;\boldsymbol{H})$. But, in fact, we have $\boldsymbol{v}\cdot\nabla\varphi \in L^2\big(0,T;L^{4/3}(\Omega)\big)$. Since $L^{4/3}(\Omega)$ is continuously embedded in V^*, we deduce that $\boldsymbol{v}\cdot\nabla\varphi \in L^2(0,T;V^*)$. Let us also remark that, due to the above regularities, it even holds that $\boldsymbol{v}\cdot\nabla\varphi \in L^1(0,T;H)$ so that the corresponding term in (2.41) may be also written as an integral.

We now return to Problem (2.28)–(2.32) and address the issues of uniqueness and continuous dependence of the solution with respect to the forcing term $\boldsymbol{u}$ appearing in the first equation.

Theorem 2.3. *Assume that* (2.7)–(2.12) *are fulfilled and that* η *and* m *are positive constants. Moreover, assume* (2.15) *for the initial datum. Whenever* $\boldsymbol{u}_i$*,* $i = 1, 2$*, belong to* $L^2(0,T;\boldsymbol{H})$ *and* $(\boldsymbol{v}_i, \varphi_i, \mu_i, w_i)$ *are the corresponding solutions, then the estimate*

$$\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{V})} + \|\varphi\|_{C^0([0,T];V)\cap L^2(0,T;H^4(\Omega))} + \|\mu\|_{L^2(0,T;H)} + \|w\|_{L^2(0,T;W)} \le K_2 \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{H})} \tag{2.42}$$

holds true for the differences $(\boldsymbol{v}, \varphi, \mu, w) = (\boldsymbol{v}_1, \varphi_1, \mu_1, w_1) - (\boldsymbol{v}_2, \varphi_2, \mu_2, w_2)$ *and* $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$*, with a constant* K_2 *that depends only on the structure of the system,* Ω*,* T*, the initial datum* φ_0 *and an upper bound for the norms of* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$ *in* $L^2(0,T;\boldsymbol{H})$*.*

The remainder of the section is devoted to the collection of some useful tools. First of all, besides the Hölder and Cauchy–Schwarz inequalities and the Korn inequality (2.21), we widely use Young's inequality

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \in \mathbb{R} \text{ and } \delta > 0. \tag{2.43}$$

We also account for the Sobolev and Poincaré inequalities, as well as for some inequalities associated with the elliptic regularity theory and the compact embeddings between Sobolev spaces (via Ehrling's lemma). More precisely, the following estimates hold:

$$\|z\|_q \le \mathcal{C}_S \|z\|_V \quad \text{for every } z \in V \text{ and } q \in [1,6]. \tag{2.44}$$

$$\|z\|_\infty \le \mathcal{C}_S \|z\|_W \quad \text{for every } z \in W. \tag{2.45}$$

$$\|z\|_V \le \mathcal{C}_P \left(\|\nabla z\| + |\overline{z}|\right) \quad \text{for every } z \in V. \tag{2.46}$$

$$\|z\|_W \le \mathcal{C}_E \left(\|\Delta z\| + \|z\|\right) \quad \text{for every } z \in W. \tag{2.47}$$

$$\|z\|_{H^4(\Omega)} \le \mathcal{C}_E \left(\|\Delta^2 z\| + \|z\|\right) \quad \text{for every } z \in H^4(\Omega) \text{ with } z, \Delta z \in W. \tag{2.48}$$

$$\|z\|_V \le \delta \|\Delta v\| + \mathcal{C}_\delta \|z\| \quad \text{for every } z \in W \text{ and every } \delta > 0. \tag{2.49}$$

$$\|z\|_{H^3(\Omega)} \le \delta \|\Delta^2 z\| + \mathcal{C}_\delta \|z\|_V \quad \text{for every } z \in H^4(\Omega) \text{ with } z, \Delta z \in W \text{ and every } \delta > 0. \tag{2.50}$$

Here, $\overline{z}$ in (2.46) denotes the mean value of z. In general, we set

$$\overline{z} := \frac{1}{|\Omega|} \int_\Omega z \quad \text{for } z \in L^1(\Omega), \tag{2.51}$$

and we use the same notation for time-dependent functions. In the above inequalities (2.44)–(2.48), the constants on the right-hand sides depend only on Ω, while $\mathcal{C}_\delta$ in (2.49)–(2.50) depends on both Ω and δ. Analogous inequalities naturally extend to vector-valued functions as well without the need for repetition. In connection with (2.47) and (2.48), we recall that z belongs to W whenever $z \in V$, $\Delta z \in H$ and the homogeneous Neumann boundary condition is satisfied in the usual weak sense, and that z belongs to $H^4(\Omega)$ whenever both z and Δz belong to W.

We conclude this section by introducing a convention that simplifies some of the subsequent calculations. The lowercase symbol c denotes a generic constant that depends only on Ω, T, the structure of the system, and an upper bound for the norms of the data given in (2.14)–(2.15). In particular, c is independent of the parameter n that we introduce in the next section. Notice that the value of c may change from line to line and even within the same line of computations. Furthermore, whenever some positive constants, e.g., δ or M, occur in a computation, we adopt the corresponding subscript by writing, e.g., c_δ and c_M, instead of a general c, to emphasize that these constants also depend on the parameter under consideration. On the contrary, specific constants referenced explicitly are denoted by different symbols, like in (2.34) and (2.44), where different characters are used.

3 Existence of solutions

This section is devoted to prove the existence part of Theorem 2.1. More precisely, we prove the existence of a quadruplet $(\boldsymbol{v}, \varphi, \mu, w)$ with the regularity specified in (2.24)–(2.27) that solves Problem (2.28)–(2.32) and satisfies the inequality (2.34) with a constant K_1 having the properties stated in the assertion. Our rigorous argument is based on a Faedo–Galerkin scheme associated with the introduction of a viscosity term in the first equation (2.28).

To this end, we introduce the sequence $\{\lambda_j\}_{j\ge1}$ of the eigenvalues and an orthonormal system $\{e_j\}_{j\ge1}$ of corresponding eigenfunctions of the Neumann problem for the Laplace equation, that is,

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \dots \quad \text{and} \quad \lim_{j\to\infty} \lambda_j = +\infty \,, \tag{3.1}$$

$$e_j \in V \quad \text{and} \quad \int_\Omega \nabla e_j \cdot \nabla z = \lambda_j \int_\Omega e_j z \quad \text{for every } z \in V \text{ and } j = 1, 2, \dots, \tag{3.2}$$

$$\int_\Omega e_i e_j = \delta_{ij} \quad \text{for } i, j = 1, 2, \dots, \quad \text{and} \quad \{e_j\}_{j\ge1} \text{ is a complete system in } H, \tag{3.3}$$

where δ_{ij} is the Kronecker symbol. Similarly, we consider the eigenvalue problem

$$\boldsymbol{e}_{0,j} \in \boldsymbol{V}_0 \quad \text{and} \quad \int_\Omega D\boldsymbol{e}_{0,j} : \nabla \boldsymbol{\zeta} = \lambda_{0,j} \int_\Omega \boldsymbol{e}_{0,j} \cdot \boldsymbol{\zeta} \,, \quad \text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}_0. \tag{3.4}$$

Since the bilinear form appearing on the left-hand side of (3.4) is continuous and weakly coercive on $\boldsymbol{V}_0 \times \boldsymbol{V}_0$ (see (2.22) with η and λ replaced by 1), and since $\boldsymbol{V}_0$ is compactly embedded in $\boldsymbol{H}_0$, the general abstract theory can be applied, and the above problem provides the sequence $\{\lambda_{0,j}\}_{j\ge1}$ of the eigenvalues and an orthonormal system $\{\boldsymbol{e}_{0,j}\}_{j\ge1}$ of corresponding eigenfunctions. We thus have

$$0 = \lambda_{0,1} \le \lambda_{0,2} \le \lambda_{0,3} \le \dots \quad \text{and} \quad \lim_{j\to\infty} \lambda_{0,j} = +\infty \,, \tag{3.5}$$

$$\int_\Omega \boldsymbol{e}_{0,i} \cdot \boldsymbol{e}_{0,j} = \delta_{ij} \quad \text{for } i, j = 1, 2, \dots, \quad \text{and} \quad \{\boldsymbol{e}_{0,j}\}_{j\ge1} \text{ is complete in } \boldsymbol{H}_0 \,. \tag{3.6}$$

We then set

$$V_n := \operatorname{span}\{e_1, \dots, e_n\} \quad \text{and} \quad \boldsymbol{V}_{0,n} := \operatorname{span}\{\boldsymbol{e}_{0,1}, \dots, \boldsymbol{e}_{0,n}\}, \quad \text{for } n = 1, 2, \dots, \tag{3.7}$$

and observe that each V_n is included in W and that Δz belongs to V_n for every $z \in V_n$. Moreover, the constant functions belong to V_1 and thus to every V_n. Finally, we stress that the union of the spaces V_n is dense in both V and H. Similarly, the union of the spaces $\boldsymbol{V}_{0,n}$ is dense in both $\boldsymbol{V}_0$ and $\boldsymbol{H}_0$.

Remark 3.1. Let us make some observations regarding the orthogonal projection operator $\mathbb{P}_n : H \to V_n$, where the orthogonality is understood with respect to the standard inner product of H. Let Y

denote any of the spaces H, V, or W. As shown in [9, Rem 3.3] and in [8, Rem. 3.1], we have, for every $z \in Y$, that

$$\|\mathbb{P}_n z\|_Y \le C_\Omega \, \|z\|_Y \quad \text{and} \quad \mathbb{P}_n z \to z \quad \text{strongly in } Y, \tag{3.8}$$

with a constant C_Ω that depends only on Ω. Next, if the operator $\mathbb{P}_n$ is extended to spaces of time-dependent functions (i.e., for $z \in L^2(0,T;Y)$ the function z_n is defined by $z_n(t) := \mathbb{P}_n(z(t))$ for a.a. $t \in (0,T)$), then

$$\|z_n\|_{L^2(0,T;Y)} \le C_\Omega \|z\|_{L^2(0,T;Y)}, \quad \text{and} \quad z_n \to z \quad \text{strongly in } L^2(0,T;Y). \tag{3.9}$$

These observations are useful when we let n tend to infinity in the Faedo–Galerkin scheme introduced below. Unfortunately, it is not clear whether similar inequalities and strong convergence properties hold for the analogous projection operator from $\boldsymbol{H}_0$ onto $\boldsymbol{V}_{0,n}$.

At this point, we are ready to introduce the Faedo–Galerkin scheme mentioned above. It consists in looking for a triplet

$$(\boldsymbol{v}_n, \varphi_n, \mu_n) \in H^1(0,T;\boldsymbol{V}_{0,n}) \times H^1(0,T;V_n) \times L^2(0,T;V_n) \tag{3.10}$$

that solves the variational equations

$$\frac{1}{n}\int_\Omega \partial_t \boldsymbol{v}_n \cdot \boldsymbol{\zeta} + \int_\Omega \big(\eta(\varphi_n) D\boldsymbol{v}_n : \nabla \boldsymbol{\zeta} + \lambda(\varphi_n)\boldsymbol{v}_n \cdot \boldsymbol{\zeta}\big) = \int_\Omega (\mu_n \nabla \varphi_n + \boldsymbol{u}) \cdot \boldsymbol{\zeta}$$
$$\text{for every } \boldsymbol{\zeta} \in \boldsymbol{V}_{0,n} \text{ and a.e. in } (0,T), \tag{3.11}$$

$$\int_\Omega \partial_t \varphi_n \, z + \int_\Omega \boldsymbol{v}_n \cdot \nabla \varphi_n \, z + \int_\Omega m(\varphi_n) \nabla \mu_n \cdot \nabla z = \int_\Omega S(\varphi_n) z$$
$$\text{for every } z \in V_n \text{ and a.e. in } (0,T), \tag{3.12}$$

$$\int_\Omega \big(-\Delta\varphi_n + f(\varphi_n)\big)(-\Delta z) + \int_\Omega \big(f'(\varphi_n) + \nu\big)\big(-\Delta\varphi_n + f(\varphi_n)\big) z = \int_\Omega \mu_n z$$
$$\text{for every } z \in V_n \text{ and a.e. in } (0,T), \tag{3.13}$$

and satisfies the initial conditions

$$\boldsymbol{v}_n(0) = \boldsymbol{0} \quad \text{and} \quad \varphi(0) = \mathbb{P}_n \varphi_0 \, . \tag{3.14}$$

First of all, we have to prove that the above problem is well posed.

Well-posedness of the discrete problem. We show that the problem admits a unique maximal solution. Then, by proving several a priori estimates, we demonstrate that this maximal solution extends globally. This aspect will be revisited later.

We express the unknowns as expansions in terms of the eigenfunctions $\boldsymbol{e}_{0,j}$ and e_j as follows:

$$\boldsymbol{v}_n(t) = \sum_{j=1}^n v_{nj}(t) \boldsymbol{e}_{0,j} \, , \quad \varphi_n(t) = \sum_{j=1}^n \varphi_{nj}(t) e_j \, ,$$
$$\text{and} \quad \mu_n(t) = \sum_{j=1}^n \mu_{nj}(t) e_j \, , \quad \text{for a.a. } t \in (0,T),$$

where the coefficients are required to satisfy

$$v_{nj} \in H^1(0,T), \quad \varphi_{nj} \in H^1(0,T) \quad \text{and} \quad \mu_{nj} \in L^2(0,T)\,.$$

Then, we insert the above expansions in the variational equations (3.11)–(3.14). Since it suffices to choose $\boldsymbol{\zeta} = \boldsymbol{e}_{0,i}$ in (3.11) and $\zeta = e_i$ in (3.12) and (3.13), for $i = 1,2,\dots,n$, we obtain from the orthogonality properties of the eigenfunctions a system of the form

$$\frac{1}{n}\,v'_{ni} = \mathcal{F}_1\big((v_{nj})_{j=1}^n, (\varphi_{nj})_{j=1}^n, (\mu_{nj})_{j=1}^n\big), \tag{3.15}$$

$$\varphi'_{ni} = \mathcal{F}_2\big((v_{nj})_{j=1}^n, (\varphi_{nj})_{j=1}^n, (\mu_{nj})_{j=1}^n\big), \tag{3.16}$$

$$\mu_{ni} = \mathcal{F}_3\big((\varphi_{nj})_{j=1}^n\big), \tag{3.17}$$

all for $i = 1,\dots,n$ and a.e. with respect to time, the functions $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, naturally defined, are locally Lipschitz continuous functions on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, thanks to (2.7), (2.9) and (2.10). By multiplying (3.15) by n and using (3.17) to eliminate every μ_{nj} in (3.15) and (3.16), the problem is reduced to a system of explicit ordinary differential equations in standard form, ruled by locally Lipschitz continuous functions, in the unknowns $(v_{nj})_{j=1}^n$ and $(\varphi_{nj})_{j=1}^n$. On the other hand, (3.14) provide proper initial values. Therefore, the Cauchy problem we obtain has a unique maximal solution $((v_{nj})_{j=1}^n, (\varphi_{nj})_{j=1}^n)$ defined in the interval $[0,T_n)$ for some $T_n \in (0,T]$. Since $(\mu_{nj})_{j=1}^n$ can be recovered from (3.17), we conclude that Problem (3.11)–(3.14) has a unique maximal solution defined in the interval $[0,T_n)$.

Now, we start estimating. As said before, the estimates we establish also guarantee that the solution $(\boldsymbol{v}_n, \varphi_n, \mu_n)$ to Problem (3.11)–(3.14) is global, i.e., that $T_n = T$. For this reason and in order to simplify the notation, we write T instead of T_n from now on. Furthermore, we avoid writing the subscript n in performing calculations and restore the rigorous notation $(\boldsymbol{v}_n, \varphi_n, \mu_n)$ only at the end of each step. Additionally, for any given time-dependent test function z, it is understood that the respective equations are always written at the time t and then tested by $z(t)$ for a.a. $t \in (0,T)$, even though we avoid writing the time, for simplicity.

First a priori estimate. We test equation (3.11) with $\boldsymbol{v}_n$, equation (3.12) with μ_n, and equation (3.13) with $\partial_t\varphi_n + \sigma\varphi_n$. On account of (2.20), we obtain that

$$\frac{1}{2n}\,\frac{d}{dt}\int_\Omega |\boldsymbol{v}|^2 + \int_\Omega \big(\eta(\varphi)|D\boldsymbol{v}|^2 + \lambda(\varphi)|\boldsymbol{v}|^2\big) = \int_\Omega \mu\nabla\varphi\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,,$$

$$\int_\Omega \partial_t\varphi\,\mu + \int_\Omega \boldsymbol{v}\cdot\nabla\varphi\,\mu + \int_\Omega m(\varphi)|\nabla\mu|^2 = \int_\Omega S(\varphi)\mu\,,$$

$$\begin{aligned}
&\int_\Omega \big(-\Delta\varphi + f(\varphi)\big)(-\Delta\partial_t\varphi) + \int_\Omega \big(f'(\varphi)+\nu\big)\big(-\Delta\varphi + f(\varphi)\big)\partial_t\varphi \\
&= -\sigma\int_\Omega \big(-\Delta\varphi + f(\varphi)\big)(-\Delta\varphi) - \sigma\int_\Omega \big(f'(\varphi)+\nu\big)\big(-\Delta\varphi + f(\varphi)\big)\varphi \\
&\quad + \int_\Omega \mu\,(\partial_t\varphi + \sigma\varphi)\,.
\end{aligned}$$

We add these identities to each other. Since some cancellations occur, we have that

$$\frac{1}{2n}\frac{d}{dt}\int_\Omega |\boldsymbol{v}|^2 + \int_\Omega \big(\eta(\varphi)|D\boldsymbol{v}|^2 + \lambda(\varphi)|\boldsymbol{v}|^2\big) + \int_\Omega m(\varphi)|\nabla\mu|^2$$
$$+ \int_\Omega \big(-\Delta\varphi + f(\varphi)\big)(-\Delta\partial_t\varphi) + \int_\Omega \big(f'(\varphi)+\nu\big)\big(-\Delta\varphi + f(\varphi)\big)\partial_t\varphi$$
$$= \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} + \int_\Omega h(\varphi)\mu - \sigma\int_\Omega \big(-\Delta\varphi + f(\varphi)\big)\big(-\Delta\varphi + f'(\varphi)\varphi + \nu\varphi\big).$$

Now, we notice that the time derivative of the energy defined in (1.8) is given by

$$\frac{d}{dt}\,\mathcal{E}(\varphi) = \int_\Omega \big(-\Delta\varphi + f(\varphi)\big)\big(-\Delta\partial_t\varphi + f'(\varphi)\partial_t\varphi\big)$$
$$+ \nu\int_\Omega \big(\nabla\varphi\cdot\nabla\partial_t\varphi + f(\varphi)\partial_t\varphi\big) \tag{3.18}$$

and recall the coercivity inequality (2.22) along with the definition of α (see (2.22)) and the assumptions (2.8). We deduce the inequality

$$\frac{1}{2n}\frac{d}{dt}\,\|\boldsymbol{v}\|^2 + \frac{\alpha}{2}\,\|\boldsymbol{v}\|_V^2 + \frac{1}{2}\int_\Omega \eta(\varphi)|D\boldsymbol{v}|^2 + \frac{\lambda_*}{2}\,\|\boldsymbol{v}\|^2 + m_*\int_\Omega |\nabla\mu|^2 + \frac{d}{dt}\,\mathcal{E}(\varphi)$$
$$\le \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} + \int_\Omega h(\varphi)\mu - \sigma\int_\Omega \big(-\Delta\varphi + f(\varphi)\big)\big(-\Delta\varphi + f'(\varphi)\varphi + \nu\varphi\big), \tag{3.19}$$

written in some complicated form, which is however good for later purposes. At this point, we integrate over $(0,t)$ with respect to time, for any $t\in(0,T]$. However, since a part of the energy is multiplied by ν and this parameter is not required to be positive, it is convenient to rewrite $\mathcal{E}(\varphi)$ in a different form by splitting the first integral into two parts and making explicit calculations in the second one. In the light of (1.8), we have

$$\mathcal{E}(\varphi) = \frac{1}{4}\int_\Omega \big(-\Delta\varphi + f(\varphi)\big)^2 + \frac{1}{4}\int_\Omega \big(|\Delta\varphi|^2 + |f(\varphi)|^2\big)$$
$$+ \frac{1}{2}\int_\Omega f'(\varphi)|\nabla\varphi|^2 + \nu\int_\Omega \Big(\frac{1}{2}\,|\nabla\varphi|^2 + F(\varphi)\Big). \tag{3.20}$$

Therefore, the time integration of (3.19) and some rearrangement yield, for every $t\in(0,T]$,

$$\frac{1}{2n}\,\|\boldsymbol{v}(t)\|^2 + \frac{\alpha}{2}\int_0^t \|\boldsymbol{v}(s)\|_V^2\,ds + \frac{1}{2}\int_{Q_t} \eta(\varphi)|D\boldsymbol{v}|^2 + \frac{\lambda_*}{2}\int_0^t \|\boldsymbol{v}(s)\|^2\,ds$$
$$+ m_*\int_{Q_t} |\nabla\mu|^2 + \frac{1}{4}\int_\Omega \big|\big(-\Delta\varphi + f(\varphi)\big)(t)\big|^2 + \frac{1}{4}\int_\Omega \big(|\Delta\varphi(t)|^2 + |f(\varphi(t))|^2\big)$$
$$\le -\frac{1}{2}\int_\Omega f'(\varphi(t))|\nabla\varphi(t)|^2 - \nu\int_\Omega \Big(\frac{1}{2}\,|\nabla\varphi(t)|^2 + F(\varphi(t))\Big)$$
$$+ \mathcal{E}(\mathbb{P}_n\varphi_0) + \int_{Q_t} \boldsymbol{u}\cdot\boldsymbol{v} + \int_{Q_t} h(\varphi)\mu$$
$$- \sigma\int_{Q_t} \big(-\Delta\varphi + f(\varphi)\big)\big(-\Delta\varphi + f'(\varphi)\varphi + \nu\varphi\big), \tag{3.21}$$

and we have to estimate the terms on the right-hand side. On account of the first assumption in (2.12) and of the compactness inequality (2.49), we have that

$$-\frac{1}{2}\int_\Omega f'(\varphi(t))|\nabla\varphi(t)|^2 - \frac{\nu}{2}\int_\Omega |\nabla\varphi(t)|^2 \le \frac{C_1+|\nu|}{2}\int_\Omega |\nabla\varphi(t)|^2$$
$$\le \frac{1}{8}\int_\Omega |\Delta\varphi(t)|^2 + C'\int_\Omega |\varphi(t)|^2 \,,$$

for a computable constant C'. The next estimate follows a similar approach: from the second assumption in (2.12) and Young's inequality, we obtain that, for a suitable constant C'',

$$-\nu\int_\Omega F(\varphi(t)) \le \frac{1}{8}\int_\Omega |f(\varphi(t)|^2 + C''\int_\Omega |\varphi(t)|^2 + c\,.$$

Now, we observe that the assumptions (2.15) and (2.10) allow us to conclude that

$$\mathcal{E}(\mathbb{P}_n\varphi_0) \le c\,,$$

since $\mathbb{P}_n\varphi_0 \to \varphi_0$ strongly in W by virtue of (3.8), and uniformly, thanks to the compact embedding $W \hookrightarrow C^0(\overline{\Omega})$. Moreover, we immediately obtain from Young's inequality and (2.14) that

$$\int_{Q_t} \boldsymbol{u}\cdot\boldsymbol{v} \le \frac{\lambda_*}{4}\int_0^t \|\boldsymbol{v}(s)\|^2\,ds + c\,.$$

We have to estimate the integral involving S. Recalling the general notation (2.51), we write the term under consideration in the form

$$\int_{Q_t} h(\varphi)(\mu-\overline{\mu}) + \int_{Q_t} h(\varphi)\overline{\mu}$$

and estimate these contributions individually. First, by the Poincaré and Young inequalities, and using the boundedness assumption on h, we find that

$$\int_{Q_t} h(\varphi)(\mu-\overline{\mu}) \le \mathcal{C}_P\int_0^t \|h(\varphi(s))\|\,\|\nabla\mu(s)\|\,ds \le \frac{m_*}{2}\int_{Q_t}|\nabla\mu|^2 + c\,.$$

As for the second term, we recall that the constant functions belong to every V_n and test (3.13) by the constant $1/|\Omega|$ to obtain

$$\overline{\mu} = \frac{1}{|\Omega|}\int_\Omega \big(f'(\varphi)+\nu\big)\big(-\Delta\varphi + f(\varphi)\big) \quad \text{a.e. in } (0,T)\,. \tag{3.22}$$

By also accounting for (2.13) and the above, we infer that

$$\int_{Q_t} h(\varphi)\overline{\mu} \le c\int_{Q_t}|\overline{\mu}| = c\int_0^t |\overline{\mu}(s)|\,ds$$
$$\le c\int_{Q_t} |-\Delta\varphi + f(\varphi)|^2 + c\int_{Q_t}|f(\varphi)|^2 + c\,.$$

Finally, it remains to estimate the last term in (3.21). By exploiting the third assumption in (2.12), we deduce that

$$
\begin{aligned}
& - \sigma \int_{Q_t} \big(-\Delta\varphi + f(\varphi)\big)\big(-\Delta\varphi + f'(\varphi)\varphi + \nu\varphi\big) \\
& \leq |\sigma| \int_{Q_t} \big|-\Delta\varphi + f(\varphi)\big| \big(|\Delta\varphi| + C_3|f(\varphi)| + C_3 + |\nu\varphi|\big) \\
& \leq c \int_{Q_t} |-\Delta\varphi + f(\varphi)|^2 + c \int_{Q_t} |f(\varphi)|^2 + c \int_{Q_t} |\varphi|^2 + c\,.
\end{aligned}
$$

Now we come back to (3.21), collect all these inequalities, and rearrange. We then see that the leading terms of the left-hand side remain, possibly with smaller coefficients, and that just one expression needs some treatment, namely

$$
\frac{1}{8} \int_\Omega |f(\varphi(t))|^2 - (C' + C'') \int_\Omega |\varphi(t)|^2\,.
$$

However, the growth condition (2.11) guarantees the inequality

$$
(C' + C'')s^2 \leq \frac{1}{16}|f(s)|^2 - s^2 + c \qquad \text{for all } s \in \mathbb{R},
$$

and, employing it in this form, we are then in a position to apply Gronwall's lemma. We conclude, in particular, that

$$
\begin{aligned}
& n^{-1/2}\|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{H})} + \alpha^{1/2}\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{V})} + \|\eta(\varphi_n)^{1/2}|D\boldsymbol{v}_n|\|_{L^2(0,T;\boldsymbol{H}^3)} \\
& \quad + \lambda_*^{1/2}\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{H})} + \|\nabla\mu_n\|_{L^2(0,T;\boldsymbol{H})} + \|\varphi_n\|_{L^\infty(0,T;H)} \\
& \quad + \|\Delta\varphi_n\|_{L^\infty(0,T;H)} + \|f(\varphi_n)\|_{L^\infty(0,T;H)} \leq c\,.
\end{aligned}
$$

Then, by using the elliptic regularity estimate (2.47), we finally arrive at

$$
\begin{aligned}
& n^{-1/2}\|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{H})} + \alpha^{1/2}\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{V})} + \|\eta(\varphi_n)^{1/2}|D\boldsymbol{v}_n|\|_{L^2(0,T;\boldsymbol{H}^3)} \\
& \quad + \lambda_*^{1/2}\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{H})} + \|\nabla\mu_n\|_{L^2(0,T;\boldsymbol{H})} + \|\varphi_n\|_{L^\infty(0,T;W)} \leq c\,.
\end{aligned} \tag{3.23}
$$

Let us point out that the constant c in (3.23) depends of course on λ_*, but is independent of α.

Consequences. We recall the computation (3.22) of the mean value of μ and observe that (3.23), (2.45) and our regularity assumption (2.10) on the potential imply that

$$
\|\varphi_n\|_\infty + \|f(\varphi_n)\|_\infty + \|f'(\varphi_n)\|_\infty \leq c\,. \tag{3.24}
$$

Hence, it is straightforward to check that $\overline{\mu}$ is bounded in $L^\infty(0,T)$. By combining this with (3.23) itself and Poincaré's inequality, we conclude that

$$
\|\mu_n\|_{L^2(0,T;V)} \leq c\,. \tag{3.25}
$$

Second a priori estimate. We first observe that (3.23), the estimate (2.44) related to the Sobolev embedding $V \hookrightarrow L^6(\Omega)$, and the Hölder inequality imply that

$$
\begin{aligned}
& \|\boldsymbol{v}_n \cdot \nabla\varphi_n\|_{L^2(0,T;H)} \leq c\,\|\boldsymbol{v}_n \cdot \nabla\varphi_n\|_{L^2(0,T;L^3(\Omega))} \\
& \leq c\,\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{L}^6(\Omega))}\,\|\nabla\varphi_n\|_{L^\infty(0,T;\boldsymbol{L}^6(\Omega))} \leq c\,\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{V})}\,\|\varphi_n\|_{L^\infty(0,T;W)} \leq c\,.
\end{aligned}
$$

At this point, we take any $z \in L^2(0,T;V)$ and test (3.12) by the projection $z_n := \mathbb{P}_n z$. Then, we integrate over $(0,T)$ to obtain that

$$\int_Q \partial_t \varphi \, z_n = - \int_Q \boldsymbol{v} \cdot \nabla \varphi \, z_n - \int_Q m(\varphi) \nabla \mu \cdot \nabla z_n + \int_Q S(\varphi) z_n \, .$$

By virtue of the estimates just obtained, and owing to Remark 3.1, we conclude that

$$\int_Q \partial_t \varphi \, z_n \le c \, \|z_n\|_{L^2(0,T;V)} \le c \, \|z\|_{L^2(0,T;V)} \, .$$

Since $z \in L^2(0,T;V)$ is arbitrary, it is a standard argument to pass to the supremum with respect to z and infer that

$$\|\partial_t \varphi_n\|_{L^2(0,T;V^*)} \le c \, . \tag{3.26}$$

Consequence. As announced at the beginning, we have written T instead of the final value T_n of the interval where the solution $(\boldsymbol{v}_n, \varphi_n, \mu_n)$ to the discrete problem is defined, for simplicity. However, the above estimates imply that this solution is global. Indeed, (3.26) and the boundedness of φ_n in $L^\infty(0,T;W)$ with constants that are independent of T_n ensure that φ_n is bounded in $\overline{\Omega} \times [0,T_n)$, and weakly continuous as a function from $[0,T_n]$ to W. Therefore, if the inequality $T_n < T$ holds, we could initiate a new Cauchy problem by taking, as the initial value at $t = T_n$ the weak limit in W of $\varphi_n(t)$ as t approaches T_n. Similar considerations can be repeated for $\boldsymbol{v}_n$, which is bounded and continuous from $[0,T_n]$ to $\boldsymbol{V}$. Since all this contradicts the definition of maximal solution, we conclude that $T_n = T$. Notably, a posteriori, the above estimates are accurate as presented.

Conclusion. The above estimates and standard weak and weak star compactness results imply that there exists a triplet $(\boldsymbol{v}, \varphi, \mu)$ such that, as $n \to \infty$,

$$\boldsymbol{v}_n \to \boldsymbol{v} \quad \text{weakly in } L^2(0,T;\boldsymbol{V}_0) \, , \tag{3.27}$$

$$\varphi_n \to \varphi \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;W) \, , \tag{3.28}$$

$$\mu_n \to \mu \quad \text{weakly in } L^2(0,T;V) \, , \tag{3.29}$$

at least for a not relabeled subsequence. Moreover, by applying, e.g., [37, Sect. 8, Cor. 4], and invoking the regularity of f, we deduce the strong convergence properties

$$\varphi_n \to \varphi \quad \text{strongly in } C^0([0,T];H^s(\Omega)) \text{ for all } s < 2 \text{ and uniformly in } \overline{Q}, \tag{3.30}$$

$$\begin{aligned} &\eta(\varphi_n) \to \eta(\varphi), \quad \lambda(\varphi_n) \to \lambda(\varphi), \quad m(\varphi_n) \to m(\varphi), \\ &S(\varphi_n) \to S(\varphi), \quad f(\varphi_n) \to f(\varphi), \quad f'(\varphi_n) \to f'(\varphi), \\ &\qquad \text{all uniformly in } \overline{Q}, \text{ i.e., strongly in } C^0(\overline{Q}). \end{aligned} \tag{3.31}$$

We now prove that the quadruplet $(\boldsymbol{v}, \varphi, \mu, w)$, where w is given by (2.31), solves Problem (2.28)–(2.32) and satisfies the estimate (2.34). A part of the latter is immediately established: as a consequence of (3.23), (3.25), (3.26), and of the lower semicontinuity of norms, we have indeed

$$\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{V})} + \|\varphi\|_{H^1(0,T;V^*) \cap L^\infty(0,T;W)} + \|\mu\|_{L^2(0,T;V)} \le c \, , \tag{3.32}$$

where c has the same dependences as the ones required for K_1 in the statement. Before completing the proof of (2.34), we have to show that $(\boldsymbol{v}, \varphi, \mu, w)$ is the sought solution.

We defer the treatment of the first equation, which focuses mainly on $\boldsymbol{v}$.

Now, we take any $z \in L^2(0,T;V)$, test (3.12) by the projection $z_n := \mathbb{P}_n z$, and integrate over $(0,T)$. By accounting for the strong convergence of z_n to z in $L^2(0,T;V)$, ensured by Remark 3.1 (see (3.9) with $Y = V$), we let n tend to infinity and observe, in particular, that $\boldsymbol{v}_n \to \boldsymbol{v}$ weakly in $L^2(0,T;\boldsymbol{H})$ by (3.27); in addition, $(\nabla\varphi_n)z_n \to (\nabla\varphi)z$ strongly in $L^2(0,T;\boldsymbol{H})$ by (3.30) and the continuity of the embedding $H^{s-1}(\Omega) \hookrightarrow L^4(\Omega)$ for $s < 2$ sufficiently large. Moreover, it turns out that $\nabla\mu_n \to \nabla\mu$ weakly in $L^2(0,T;\boldsymbol{H})$ by (3.29) and that $m(\varphi_n)\nabla z_n \to m(\varphi)\nabla z$ strongly in $L^2(0,T;\boldsymbol{H})$ by (3.31). Then, we easily conclude that the time-integrated version of (2.29) is satisfied with the given test function z. Since z is arbitrary in $L^2(0,T;V)$, the time-integrated version of (2.29) is equivalent to (2.29) itself.

A similar procedure applies to equation (3.13), where this time we take $z \in L^2(0,T;W)$. Indeed, we test (3.13) by the projection $z_n := \mathbb{P}_n z$, integrate with respect to time and apply (3.9) with $Y = W$ to ensure the strong convergence of Δz_n to Δz in $L^2(0,T;H)$. To pass to the limit as $n \to \infty$, we note, in particular, that the product $f'(\varphi_n)(-\Delta\varphi_n)$ converges to $f'(\varphi)(-\Delta\varphi)$ weakly star in $L^\infty(0,T;H)$, due to (3.28) and (3.31). Then, letting n tend to infinity in (3.13), we recover the time-integrated version of

$$\int_\Omega \big(-\Delta\varphi + f(\varphi)\big)(-\Delta z) + \int_\Omega \big(f'(\varphi) + \nu\big)\big(-\Delta\varphi + f(\varphi)\big)z = \int_\Omega \mu z$$

for every $z \in W$ and a.e. in $(0,T)$. (3.33)

Now, if we take $w = -\Delta\varphi + f(\varphi)$ as a definition of w, then we infer from (3.32) that

$$\|w\|_{L^\infty(0,T;H)} \le c\,. \tag{3.34}$$

At this point, we notice that w solves the variational equality

$$\int_\Omega w(-\Delta z + z) = \int_\Omega gz \quad \text{for every } z \in W \text{ and a.e. in } (0,T), \tag{3.35}$$

where the source term $g := \mu + (1 - f'(\varphi) - \nu)w$ is uniformly bounded in $L^2(0,T;H)$, by (3.32) and (3.34). Then, we can apply [8, Lemma 4.2] to infer that $w \in L^2(0,T;W)$, w solves $-\Delta w + w = h$ a.e. in Q, and

$$\|w\|_{L^2(0,T;W)} \le c\|h\|_{L^2(0,T;H)} \le c\,. \tag{3.36}$$

Hence, we can integrate by parts in the first term of (3.33) and deduce (2.33), at first for $z \in W$, and then for all $z \in V$, by density. Obviously, (2.33) implies (2.30): if we write the latter in the form

$$\int_\Omega \nabla w \cdot \nabla z = \int_\Omega \big(\mu - (f'(\varphi) + \nu)w\big)z \quad \text{for every } z \in V \text{ and a.e. in } (0,T)\,,$$

then we recognize the weak formulation of the homogeneous Neumann boundary value problem for $-\Delta w = \mu - (f'(\varphi) + \nu)w$, with the forcing term uniformly estimated in $L^2(0,T;V)$, as the reader can easily check using (3.32), (2.10) and (3.36). Therefore, it follows that

$$\|w\|_{L^2(0,T;H^3(\Omega)\cap W)} \le c \tag{3.37}$$

by elliptic regularity. Hence, coming back to the definition of w as $w = -\Delta\varphi + f(\varphi)$ and to the consequent identity (2.31), and since $f(\varphi)$ is uniformly bounded in $L^\infty(0,T;W)$ thanks to (3.32) and (2.10), it is not difficult to deduce that

$$\|\Delta\varphi\|_{L^2(0,T;W)} + \|\varphi\|_{L^2(0,T;H^4(\Omega))} \le c\,, \tag{3.38}$$

still invoking the elliptic regularity theory. Moreover, now we have that $w - f(\varphi) = -\Delta\varphi$ is uniformly bounded in in $L^2(0,T;H^3(\Omega))$ and, consequently, that

$$\|\varphi\|_{L^2(0,T;H^5(\Omega))} \le c\,, \tag{3.39}$$

which concludes the proof of (2.34) for the quadruplet $(\boldsymbol{v},\varphi,\mu,w)$ we have found.

It remains now to verify that $(\boldsymbol{v},\varphi,\mu,w)$ satisfies (2.28) as well. We cannot adopt the argument used for the other equations, since we did not prove properties for the projection operator from $\boldsymbol{H}_0$ onto $\boldsymbol{V}_{0,n}$ that are analogous to those of $\mathbb{P}_n$. Hence, we argue in a different way. We fix some $N \in \mathbb{N}$ and assume that $n \ge N$. Then, any $\boldsymbol{\zeta} \in \boldsymbol{V}_{0,N}$ belongs to $\boldsymbol{V}_{0,n}$ and can be an admissible test function in (3.11). Now, take any piecewise linear function $\boldsymbol{\zeta} : [0,T] \to \boldsymbol{V}_{0,N}$ that vanishes at the endpoints. Then, we can test (3.11) by $\boldsymbol{\zeta}$ and integrate over $(0,T)$. By also integrating by parts in time in the first term, we find that

$$-\frac{1}{n}\int_Q \boldsymbol{v}_n \cdot \partial_t\boldsymbol{\zeta} + \int_Q \bigl(\eta(\varphi_n) D\boldsymbol{v}_n : \nabla\boldsymbol{\zeta} + \lambda(\varphi_n)\boldsymbol{v}_n \cdot \boldsymbol{\zeta}\bigr) = \int_Q (\mu_n \nabla\varphi_n + \boldsymbol{u}) \cdot \boldsymbol{\zeta}\,,$$

and we can let n tend to infinity. By (3.23), we see that the first term tends to zero. On the other hand, the remaining terms tend to those one expects, and just the product $\mu_n\nabla\varphi_n$ needs some further comment. In view of (3.30) and (3.29), φ_n converges to φ strongly, e.g., in $C^0([0,T];W^{1,4}(\Omega))$, whence it follows that $\mu_n\nabla\varphi_n$ converges to $\mu\nabla\varphi$ weakly in $L^2(0,T;\boldsymbol{H})$. Therefore, we obtain that

$$\int_Q \bigl(\eta(\varphi) D\boldsymbol{v} : \nabla\boldsymbol{\zeta} + \lambda(\varphi)\boldsymbol{v} \cdot \boldsymbol{\zeta}\bigr) = \int_Q (\mu \nabla\varphi + \boldsymbol{u}) \cdot \boldsymbol{\zeta}\,, \tag{3.40}$$

where $\boldsymbol{\zeta}$ is as said. Since $N \in \mathbb{N}$ was arbitrary and the union of the spaces $\boldsymbol{V}_{0,N}$ is dense in $\boldsymbol{V}_0$, the set of the piecewise linear functions $\boldsymbol{\zeta}$, which attain their values in this union and vanish at the endpoints of $[0,T]$, is dense in $L^2(0,T;\boldsymbol{V}_0)$. We conclude that (3.40) is valid for every $\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{V}_0)$. But this is equivalent to (2.28), and the proof of the part of Theorem 2.1 we wanted to prove is complete.

Remark 3.2. We emphasize that every solution $(\boldsymbol{v},\varphi,\mu,w)$ provided by Theorem 2.1 enjoys the regularity property (cf. (3.38))

$$\Delta\varphi \in L^2(0,T;W), \tag{3.41}$$

which follows directly from (2.31), together with the regularity properties that $\varphi \in L^\infty(0,T;W)$, $w \in L^2(0,T;W)$, and the smoothness of f (see (2.10)). This observation will play a role in the subsequent analysis.

4 Proof of Theorem 2.2

We start by recalling that, for any $n \in \mathbb{N}$, $(\boldsymbol{v}_n,\varphi_n,\mu_n,w_n)$ represents a solution to Problem (2.28)–(2.32) that corresponds to the function η_n, where

$$0 < \eta_{*,n} \le \eta_n(s) \le \eta^*{}_{,n} \quad \text{for every } s \in \mathbb{R},$$

and the behavior of the sequences is such that

$$\|\eta_n\|_{L^\infty(\mathbb{R})} \ \text{ and, consequently, } \ \eta_{*,n} \text{ tend to } 0 \text{ as } n \to \infty. \tag{4.1}$$

The solution $(\boldsymbol{v}_n, \varphi_n, \mu_n, w_n)$ enjoys the regularity properties (2.24)–(2.27) and satisfies (2.28)–(2.32). In particular, the conditions (2.30) and (2.31) can be equivalently expressed in the following strong form:

$$- \Delta w_n + f'(\varphi_n) w_n + \nu w_n = \mu_n \quad \text{in } Q\,, \tag{4.2}$$

$$- \Delta \varphi_n + f(\varphi_n) = w_n \quad \text{in } Q\,, \tag{4.3}$$

which implies that φ_n and μ_n also satisfy the combination of (4.2)–(4.3). Then, in view of (2.33)), it holds that

$$-\Delta\big(-\Delta\varphi_n + f(\varphi_n)\big) + \big(f'(\varphi_n) + \nu\big)\big(-\Delta\varphi_n + f(\varphi_n)\big) = \mu_n \quad \text{in } Q\,. \tag{4.4}$$

Taking (2.24)–(2.27) into account, together with the regularity assumptions on f and f' in (2.10), a direct inspection of (4.4) shows that all of the terms belong to $L^2(0,T;V)$. Hence, we can reproduce the *First a priori estimate* developed in the previous section directly at the level of the problem, by testing (2.28) with $\boldsymbol{v}_n$, (2.29) with μ_n, and (4.4) with $\partial_t\varphi_n + \sigma\varphi_n \in L^2(0,T;V^*)$.

The subsequent computations follow closely those carried out in the previous section and lead to the estimate (cf. (3.23)–(3.25))

$$\begin{aligned} &\alpha_n^{1/2}\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{V})} + \|\eta_n(\varphi_n)^{1/2}|D\boldsymbol{v}_n|\|_{L^2(0,T;\boldsymbol{H}^3)} + \lambda_*^{1/2}\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{H})} \\ &\quad + \|\mu_n\|_{L^2(0,T;V)} + \|\varphi_n\|_{L^\infty(0,T;W)\cap L^\infty(Q)} \le c\,, \end{aligned} \tag{4.5}$$

where $\alpha_n = \min\{\eta_{*,n}, \lambda_*\}/\mathcal{C}_K \to 0$ as $n \to \infty$ and, importantly, the constant c is uniform with respect to n.

On the other hand, the *Second a priori estimate* of the previous section requires a slight modification. Indeed, due to the Sobolev embeddings $V \hookrightarrow L^4(\Omega)$ and $L^{4/3}(\Omega) \hookrightarrow V^*$, we deduce from Hölder's inequality that

$$\begin{aligned} &\|\boldsymbol{v}_n\cdot\nabla\varphi_n\|_{L^2(0,T;V^*)} \le c\,\|\boldsymbol{v}_n\cdot\nabla\varphi_n\|_{L^2(0,T;L^{4/3}(\Omega))} \\ &\le c\,\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\,\|\nabla\varphi_n\|_{L^\infty(0,T;\boldsymbol{L}^4(\Omega))} \le c\,\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{H})}\,\|\varphi_n\|_{L^\infty(0,T;W)} \le c\,. \end{aligned}$$

Hence, by testing (2.29) with an arbitrary $z \in L^2(0,T;V)$, and using (4.5) to control the other terms, we infer that

$$\begin{aligned} &\int_0^T \langle\partial_t\varphi_n, z\rangle\,dt = \int_Q S(\varphi_n)\,z - \int_Q m(\varphi_n)\nabla\mu_n\cdot\nabla z - \int_0^T \langle\boldsymbol{v}_n\cdot\nabla\varphi_n, z\rangle\,dt \\ &\le c\,\|z\|_{L^2(0,T;V)}\,, \end{aligned}$$

and, consequently, that

$$\|\partial_t\varphi_n\|_{L^2(0,T;V^*)} \le c\,. \tag{4.6}$$

Moreover, the estimate (4.5) allows us to deduce that $w_n = -\Delta\varphi_n + f(\varphi_n)$ satisfies

$$\|w_n\|_{L^\infty(0,T;H)} \le c\,. \tag{4.7}$$

At this point, we can proceed exactly as in the passage from (3.35) to (3.39) to recover the additional estimate

$$\|w_n\|_{L^2(0,T;H^3(\Omega)\cap W)} + \|\varphi_n\|_{L^2(0,T;H^5(\Omega))} \le c\,, \tag{4.8}$$

which remains valid since these steps are independent of the regularity of $\boldsymbol{v}_n$.

Consequently, from standard weak or weak star compactness arguments, we conclude the existence of a subsequence (which is still labeled by $n \in \mathbb{N}$) that has the convergence properties (2.35)–(2.39). In particular, (2.36) follows from (4.5) and (4.1). Comparing with the passage-to-the-limit process carried out in the previous section, we see that (3.30) and all convergence properties in (3.31), except for the first one, still hold and allow us to pass to the limit in the present case.

In particular, (2.40) and (2.41) can be derived by exploiting the strong convergence of $\{\varphi_n\}$ together with the density of $\boldsymbol{V}_0$ in $\boldsymbol{H}_0$. This completes the proof of Theorem 2.2.

5 Uniqueness and continuous dependence

This section is devoted to the conclusion of the proof of Theorem 2.1 and to the proof of Theorem 2.3. Namely, under the assumption that η and m are positive constants, we prove the uniqueness of the solution to Problem (2.28)–(2.32) and the continuous dependence estimate (2.42). Our approach is as follows: first, we prove an estimate in the direction of (2.42) for any pair of functions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and arbitrary corresponding solutions with a constant that also depends on the solutions at hand. Uniqueness is established by focusing on the particular case $\boldsymbol{u}_1 = \boldsymbol{u}_2$. We then combine the previously proved estimate, the uniqueness of the solution, and the stability estimate from the preceding section to conclude the proof of (2.42), with a constant K_2 possessing the properties stated. In what follows, we assume without loss of generality that $\eta \equiv m \equiv 1$.

The basic estimate. For given $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ satisfying (2.14), we pick any pair of corresponding solutions $(\boldsymbol{v}_1, \varphi_1, \mu_1, w_1)$ and $(\boldsymbol{v}_2, \varphi_2, \mu_2, w_2)$, and we set for convenience

$$\boldsymbol{u} := \boldsymbol{u}_1 - \boldsymbol{u}_2\,, \quad \boldsymbol{v} := \boldsymbol{v}_1 - \boldsymbol{v}_2\,, \quad \varphi := \varphi_1 - \varphi_2\,, \quad \mu := \mu_1 - \mu_2 \quad \text{and} \quad w := w_1 - w_2\,.$$

We start proving the estimate mentioned above with a constant that might depend on the norms of the solutions involved. Indeed, this happens for many of the constants termed c (or c_M, etc.) in the proof, which also might depend on the Lipschitz constants of f, f', f'' and ff' encountered during the estimates. However, we observe at once that all of the above norms are related to the regularity requirements (2.24)–(2.27) and that the Lipschitz constants just mentioned depend only on the L^∞ norms of the components φ_i of the solutions we are considering. This remark is important for the conclusion of our proof. We introduce two positive parameters M and δ, whose values are chosen later on. Next, we take the differences of the equations (2.28), (2.29) and (2.33), written for the data

$\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and the corresponding solutions. After some rearrangement, we have, a.e. in $(0,T)$, that

$$\begin{aligned}&\int_\Omega D\boldsymbol{v} : \nabla\boldsymbol{\zeta} + \int_\Omega \big((\lambda(\varphi_1) - \lambda(\varphi_2))\boldsymbol{v}_1 + \lambda(\varphi_2)\boldsymbol{v}\big)\cdot\boldsymbol{\zeta} \\ &= \int_\Omega \big(\mu\nabla\varphi_1 + \mu_2\nabla\varphi + \boldsymbol{u}\big)\cdot\boldsymbol{\zeta}\,, \end{aligned} \tag{5.1}$$

$$\begin{aligned}&\langle\partial_t\varphi, z\rangle + \int_\Omega \big(\boldsymbol{v}\cdot\nabla\varphi_1 + \boldsymbol{v}_2\cdot\nabla\varphi\big)z + \int_\Omega \nabla\mu\cdot\nabla z \\ &= \int_\Omega \big(S(\varphi_1) - S(\varphi_2)\big)z\,, \end{aligned} \tag{5.2}$$

$$\begin{aligned}&\int_\Omega \nabla\big(-\Delta\varphi + f(\varphi_1) - f(\varphi_2)\big)\cdot\nabla z \\ &\quad + \int_\Omega \big[\big(f'(\varphi_1) - f'(\varphi_2)\big)(-\Delta\varphi_1) + f'(\varphi_2)(-\Delta\varphi)\big]z \\ &\quad + \int_\Omega \big(f(\varphi_1)f'(\varphi_1) - f(\varphi_2)f'(\varphi_2)\big)z \\ &\quad + \nu\int_\Omega \big(-\Delta\varphi + f(\varphi_1) - f(\varphi_2)\big)z = \int_\Omega \mu z\,, \end{aligned} \tag{5.3}$$

where $\boldsymbol{\zeta} \in \boldsymbol{V}_0$ is arbitrary in (5.1), $z \in V$ is arbitrary in (5.2), and $z \in V$ is arbitrary in (5.3). First, we test (5.1) with $\boldsymbol{v}$. By recalling the coerciveness inequality (2.22), we obtain

$$\alpha\,\|\boldsymbol{v}\|_V^2 \le \int_\Omega |\lambda(\varphi_1) - \lambda(\varphi_2)|\,|\boldsymbol{v}_1|\,|\boldsymbol{v}| + \int_\Omega \big(|\mu|\,|\nabla\varphi_1| + |\mu_2|\,|\nabla\varphi| + |\boldsymbol{u}|\big)|\boldsymbol{v}|\,. \tag{5.4}$$

Next, we test (5.2) with $M\varphi$ to find that

$$\begin{aligned}&\frac{M}{2}\frac{d}{dt}\int_\Omega |\varphi|^2 + M\int_\Omega \nabla\mu\cdot\nabla\varphi \\ &= -M\int_\Omega \boldsymbol{v}\cdot\nabla\varphi_1\,\varphi \,-\, M\int_\Omega \boldsymbol{v}_2\cdot\nabla\varphi\,\varphi + M\int_\Omega \big(S(\varphi_1) - S(\varphi_2)\big)\varphi\,. \end{aligned} \tag{5.5}$$

We recall the regularity (2.25) of φ_1 and φ_2, and thus of φ. Therefore, $-M\Delta\varphi$ is an admissible test function in (5.2), which leads to

$$\begin{aligned}&\frac{M}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi|^2 + M\int_\Omega \nabla\mu\cdot\nabla(-\Delta\varphi) \\ &= -M\int_\Omega \boldsymbol{v}\cdot\nabla\varphi_1(-\Delta\varphi) \,-\, M\int_\Omega \boldsymbol{v}_2\cdot\nabla\varphi(-\Delta\varphi) \\ &\quad + M\int_\Omega \big(S(\varphi_1) - S(\varphi_2)\big)(-\Delta\varphi)\,. \end{aligned} \tag{5.6}$$

Regarding (5.3), we observe that the first integral can be written in a different form. Indeed, we have

$$\begin{aligned}&\int_\Omega \nabla\big(-\Delta\varphi + f(\varphi_1) - f(\varphi_2)\big)\cdot\nabla z = \int_\Omega w(-\Delta z) \\ &= \int_\Omega (-\Delta w)z = \int_\Omega (-\Delta)\big(-\Delta\varphi + f(\varphi_1) - f(\varphi_2)\big)z\,, \end{aligned}$$

the integration by parts being allowed since $z \in V$ and $w \in L^2(0,T;W)$. It follows that (5.3) can be written as a partial differential equation that holds a.e. in Q and can be multiplied by functions valued in H, in particular, by $-M\Delta\varphi + M\Delta^2\varphi$ and $-M\mu$. By doing this, and integrating over Ω, we obtain

$$\begin{aligned}
&M \int_\Omega |\nabla\Delta\varphi|^2 + M \int_\Omega |\Delta^2\varphi|^2 \\
&= -M \int_\Omega (-\Delta)\big(f(\varphi_1) - f(\varphi_2)\big)\big(-\Delta\varphi + \Delta^2\varphi\big) \\
&\quad - M \int_\Omega \big(f'(\varphi_1) - f'(\varphi_2)\big)(-\Delta\varphi_1)\big(-\Delta\varphi + \Delta^2\varphi\big) \\
&\quad - M \int_\Omega f'(\varphi_2)|\Delta\varphi|^2 - M \int_\Omega f'(\varphi_2)(-\Delta\varphi)\Delta^2\varphi \\
&\quad - M \int_\Omega \big(f(\varphi_1)f'(\varphi_1) - f(\varphi_2)f'(\varphi_2)\big)\big(-\Delta\varphi + \Delta^2\varphi\big) \\
&\quad - M\nu \int_\Omega (-\Delta\varphi)\big(-\Delta\varphi + \Delta^2\varphi\big) - M\nu \int_\Omega \big(f(\varphi_1) - f(\varphi_2)\big)\big(-\Delta\varphi + \Delta^2\varphi\big) \\
&\quad + M \int_\Omega \nabla\mu \cdot \nabla\varphi + M \int_\Omega \nabla\mu \cdot \nabla(-\Delta\varphi)\,, \qquad (5.7)
\end{aligned}$$

as well as

$$\begin{aligned}
&M \int_\Omega |\mu|^2 = M \int_\Omega (\Delta^2\varphi)\mu + M \int_\Omega (-\Delta)\big(f(\varphi_1) - f(\varphi_2)\big)\mu \\
&\quad + M \int_\Omega \big(f'(\varphi_1) - f'(\varphi_2)\big)(-\Delta\varphi_1)\mu + M \int_\Omega f'(\varphi_2)(-\Delta\varphi)\mu \\
&\quad + M \int_\Omega \big(f(\varphi_1)f'(\varphi_1) - f(\varphi_2)f'(\varphi_2)\big)\mu + M\nu \int_\Omega \big(-\Delta\varphi + f(\varphi_1) - f(\varphi_2)\big)\mu\,. \qquad (5.8)
\end{aligned}$$

At this point, we add relations (5.4)–(5.8) to each other and notice that some cancellations occur. The resulting left-hand side is given by

$$\begin{aligned}
&\alpha\, \|\boldsymbol{v}\|_V^2 + \frac{M}{2}\frac{d}{dt} \int_\Omega |\varphi|^2 + \frac{M}{2}\frac{d}{dt} \int_\Omega |\nabla\varphi|^2 \\
&\quad + M \int_\Omega |\nabla\Delta\varphi|^2 + M \int_\Omega |\Delta^2\varphi|^2 + M \int_\Omega |\mu|^2\,,
\end{aligned}$$

and we have to estimate the terms on the resulting right-hand side. In doing this, we repeatedly make use of some of the inequalities (2.44)–(2.50), as well as of the Hölder and Young inequalities (see also Remark 3.2). First, we start with the term that is more delicate in connection with the choice of the values of the parameters. It originates from the right-hand side of (5.4) and can be estimated this way:

$$\begin{aligned}
&\int_\Omega |\mu|\,|\nabla\varphi_1|\,|\boldsymbol{v}| \le \|\mu\|\,\|\nabla\varphi_1\|_4\,\|\boldsymbol{v}\|_4 \le \mathcal{C}_S\|\boldsymbol{v}\|_V\,\|\mu\|\,\|\nabla\varphi_1\|_4 \\
&\le \frac{\alpha}{8}\,\|\boldsymbol{v}\|_V^2 + \frac{2\,\mathcal{C}_S^2\,R^2}{\alpha}\,\|\mu\|^2\,, \qquad (5.9)
\end{aligned}$$

where $\mathcal{C}_S$ is the constant appearing in (2.44) and R is an upper bound for the norm of $\nabla\varphi_1$ in $L^\infty(0,T;\boldsymbol{L}^4(\Omega))$. In this connection, we recall that φ_1 belongs to $L^\infty(0,T;W)$ and that W is continuously embedded in $W^{1,4}(\Omega)$. Next, we have that

$$\int_\Omega |\lambda(\varphi_1) - \lambda(\varphi_2)|\, |\boldsymbol{v}_1|\, |\boldsymbol{v}| \ \le\ c\,\|\varphi\|\,\|\boldsymbol{v}_1\|_4\,\|\boldsymbol{v}\|_4 \ \le\ c\,\|\varphi\|\,\|\boldsymbol{v}_1\|_V\,\|\boldsymbol{v}\|_V$$
$$\le\ \frac{\alpha}{8}\,\|\boldsymbol{v}\|_V^2 + c\,\|\boldsymbol{v}_1\|_V^2\,\|\varphi\|^2\,,$$

and we observe at once that the function $t \mapsto \|\boldsymbol{v}_1(t)\|_V^2$ belongs to $L^1(0,T)$ and is thus suitable for the application of the Gronwall lemma after time integration. Similar remarks will be omitted in the sequel, for brevity. For the same reason, we do not recall the regularity of the solutions specified in (2.24)–(2.27) that we repeatedly owe to. For the next term, we have that

$$\int_\Omega |\mu_2|\,|\nabla\varphi|\,|\boldsymbol{v}| \le \|\boldsymbol{v}\|_4\,\|\mu_2\|_4\,\|\nabla\varphi\| \le \mathcal{C}_S\,\|\boldsymbol{v}\|_V\,\|\mu_2\|_4\,\|\nabla\varphi\| \le \frac{\alpha}{8}\,\|\boldsymbol{v}\|_V^2 + c\,\|\mu_2\|_V^2\,\|\nabla\varphi\|^2\,.$$

The last term on the right-hand side of (5.4) is trivially estimated by

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} \le \frac{\alpha}{8}\,\|\boldsymbol{v}\|_V^2 + c\,\|\boldsymbol{u}\|^2\,.$$

Concerning the right-hand side of (5.5), we have that

$$-\,M\int_\Omega \boldsymbol{v}\cdot\nabla\varphi_1\,\varphi \le M\,\|\boldsymbol{v}\|_4\,\|\nabla\varphi_1\|_4\,\|\varphi\| \le \frac{\alpha}{8}\,\|\boldsymbol{v}\|_V^2 + c_M\,\|\varphi\|^2\,,$$
$$-\ M\int_\Omega \boldsymbol{v}_2\cdot\nabla\varphi\,\varphi \le M\,\|\boldsymbol{v}_2\|_4\,\|\nabla\varphi\|\,\|\varphi\|_4 \le c_M\,\|\boldsymbol{v}_2\|_V\,\|\varphi\|_V^2\,,$$
$$M\int_\Omega \big(S(\varphi_1) - S(\varphi_2)\big)\varphi \le c_M\,\|\varphi\|^2\,.$$

Next, we address the right-hand side of (5.6), taking into account the estimates (2.44) and (2.50). We deduce that

$$-\,M\int_\Omega \boldsymbol{v}\cdot\nabla\varphi_1(-\Delta\varphi) \le M\|\boldsymbol{v}\|_4\,\|\nabla\varphi_1\|_4\,\|\Delta\varphi\|$$
$$\le \frac{\alpha}{8}\,\|\boldsymbol{v}\|_V^2 + \frac{2\mathcal{C}_S^2 M^2}{\alpha}\,\|\nabla\varphi_1\|_4^2\,\|\Delta\varphi\|^2$$
$$\le \frac{\alpha}{8}\,\|\boldsymbol{v}\|_V^2 + c_M\,\|\Delta\varphi\|^2 \le \frac{\alpha}{8}\,\|\boldsymbol{v}\|_V^2 + \delta\,\|\Delta^2\varphi\|^2 + c_{\delta,M}\,\|\varphi\|_V^2$$

and

$$-\ M\int_\Omega \boldsymbol{v}_2\cdot\nabla\varphi(-\Delta\varphi) \le M\,\|\boldsymbol{v}_2\|_4\,\|\nabla\varphi\|\,\|\Delta\varphi\|_4 \le c_M\,\|\boldsymbol{v}_2\|_V\,\|\nabla\varphi\|\,\|\varphi\|_{H^3(\Omega)}$$
$$\le c_M\,\|\boldsymbol{v}_2\|_V^2\,\|\nabla\varphi\|^2 + \delta\,\|\Delta^2\varphi\|^2 + c_{\delta,M}\,\|\varphi\|_V^2\,.$$

Similarly, we have that

$$M\int_\Omega \big(S(\varphi_1) - S(\varphi_2)\big)(-\Delta\varphi) \le c\,M\|\varphi\|\,\|\Delta\varphi\| \le \delta\,\|\Delta^2\varphi\|^2 + c_{\delta,M}\,\|\varphi\|_V^2\,.$$

We now proceed to estimate the terms on the right-hand sides of (5.7) and (5.8). Regarding the first of the latter terms, although straightforward to handle, it is important to note that

$$M\int_\Omega (\Delta^2\varphi)\mu \le \frac{M}{2}\int_\Omega |\Delta^2\varphi|^2 + \frac{M}{2}\int_\Omega |\mu|^2\,.$$

Next, we consider the first integral on the right-hand side of (5.7) and the second one on the right-hand side of (5.8). Since they can be treated in the same way, we provide an estimate involving a generic function $z \in L^2(0,T;H)$. We have

$$
\begin{aligned}
& - M \int_\Omega (-\Delta)\bigl(f(\varphi_1) - f(\varphi_2)\bigr) z \\
& = M \int_\Omega \bigl(f''(\varphi_1)|\nabla\varphi_1|^2 - f''(\varphi_2)|\nabla\varphi_2|^2 + f'(\varphi_1)\Delta\varphi_1 - f'(\varphi_2)\Delta\varphi_2\bigr) z \\
& = M \int_\Omega \bigl[\bigl(f''(\varphi_1) - f''(\varphi_2)\bigr)|\nabla\varphi_1|^2 + f''(\varphi_2)\nabla(\varphi_1+\varphi_2)\cdot\nabla\varphi\bigr] z \\
& \quad + M \int_\Omega \bigl[\bigl(f'(\varphi_1) - f'(\varphi_2)\bigr)\Delta\varphi_1 + f'(\varphi_2)\Delta\varphi\bigr] z \\
& \le Mc\,\|\varphi\|_6 \|\nabla\varphi_1\|_6^2\,\|z\| + Mc\,\|\nabla(\varphi_1+\varphi_2)\|_4\,\|\nabla\varphi\|_4\,\|z\| \\
& \quad + Mc\,\|\varphi\|_4\,\|\Delta\varphi_1\|_4\,\|z\| + Mc\,\|\Delta\varphi\|\,\|z\| \\
& \le \delta\,\|z\|^2 + c_{\delta,M}\,\|\varphi_1\|_{H^2(\Omega)}^2\,\|\varphi\|_V^2 + c_{\delta,M}\,\|\varphi_1+\varphi_2\|_{H^2(\Omega)}^2\,\|\varphi\|_{H^2(\Omega)}^2 \\
& \quad + c_{\delta,M}\,\|\varphi_1\|_{H^3(\Omega)}^2\,\|\varphi\|_V^2 + c_{\delta,M}\,\|\Delta\varphi\|^2 \\
& \le \delta\,\|z\|^2 + c_{\delta,M}\,\|\varphi_1\|_{H^3(\Omega)}^2\,\|\varphi\|_V^2 + c_{\delta,M}\,\|\varphi\|_{H^2(\Omega)}^2 \\
& \le \delta\,\|z\|^2 + c_{\delta,M}\,\|\varphi_1\|_{H^3(\Omega)}^2\,\|\varphi\|_V^2 + \delta\,\|\Delta^2\varphi\|^2 + c_{\delta,M}\,\|\varphi\|_V^2 \,.
\end{aligned}
$$

By choosing $z = -\Delta\varphi + \Delta^2\varphi - \mu$, we derive an inequality for the sum of the two terms we have to estimate. Namely, we obtain that

$$
\begin{aligned}
& - M \int_\Omega (-\Delta)\bigl(f(\varphi_1) - f(\varphi_2)\bigr)\bigl(-\Delta\varphi + \Delta^2\varphi\bigr) + M \int_\Omega (-\Delta)\bigl(f(\varphi_1) - f(\varphi_2)\bigr)\mu \\
& \le \delta\,\|-\Delta\varphi + \Delta^2\varphi - \mu\|^2 + c_{\delta,M}\,\|\varphi_1\|_{H^3(\Omega)}^2\,\|\varphi\|_V^2 + \delta\,\|\Delta^2\varphi\|^2 + c_{\delta,M}\,\|\varphi\|_V^2 \,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \delta\,\|-\Delta\varphi + \Delta^2\varphi - \mu\|^2 \le 3\delta\,\|\Delta\varphi\|^2 + 3\delta\,\|\Delta^2\varphi\|^2 + 3\delta\,\|\mu\|^2 \\
& \le 4\delta\,\|\Delta^2\varphi\|^2 + c_\delta\,\|\varphi\|_V^2 + 3\delta\,\|\mu\|^2 \,.
\end{aligned}
$$

The other integrals on the right-hand side of (5.7) that remain after cancellation can be treated by using the same arguments as before. Namely, if I is any of them, we have that

$$
I \le \delta\,\|\Delta^2\varphi\|^2 + \psi_1\,\|\varphi\|_V^2 \,,
$$

with some (possibly constant) function $\psi_1 \in L^1(0,T)$ depending on M and δ. Similarly, if J is any of the other terms on the right-hand side of (5.8), we have that

$$
J \le \delta\,\|\mu\|^2 + \delta\,\|\Delta^2\varphi\|^2 + \psi_2\,\|\varphi\|_V^2 \,,
$$

where ψ_2 has analogous properties. At this stage, we can collect all the inequalities we have established and combine them with the sum of the relations (5.4)–(5.8). After straightforward rearrangements, we arrive at the inequality

$$\begin{aligned}&\frac{2\alpha}{8}\,\|\boldsymbol{v}\|_{\boldsymbol{V}}^2+\frac{M}{2}\frac{d}{dt}\,\|\varphi\|_V^2+M\,\|\nabla\Delta\varphi\|^2\\&\quad+\Big(\frac{M}{2}-n_1\delta\Big)\|\Delta^2\varphi\|^2+\Big(\frac{M}{2}-\frac{2\,\mathcal{C}_S^2\,R^2}{\alpha}-n_2\delta\Big)\|\mu\|^2\\&\le\psi\,\|\varphi\|_V^2+c\,\|\boldsymbol{u}\|^2\,,\end{aligned}\tag{5.10}$$

for some positive integers n_1 and n_2 and some function ψ belonging to $L^1(0,T)$. We notice that ψ depends only on the structure of the original system, upper bounds for some of the norms of the solutions at hand related to (2.24)–(2.27), M and δ. Therefore, we can first choose, e.g., $M = 8\,\mathcal{C}_S^2\,R^2/\alpha$, and then δ small enough in order that all the coefficients on the left-hand side of (5.10) remain positive. At this point, we can integrate with respect to time and apply Gronwall's lemma. Combining the resulting inequality with elliptic regularity, and invoking (2.48) and Remark 3.2, we finally conclude that

$$\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{V})}+\|\varphi\|_{C^0([0,T];V)\cap L^2(0,T;H^4(\Omega))}+\|\mu\|_{L^2(0,T;H)}\le c\,\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{H})}\,.\tag{5.11}$$

From this, and (2.31) written for both solutions, we deduce the similar estimate

$$\|w\|_{L^2(0,T;W)}\le c\,\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{H})}\,.\tag{5.12}$$

Conclusions. Although the constants termed c in (5.11) and (5.12) depend on the particular solutions we have considered, these inequalities prove that the solution to Problem (2.28)–(2.32) is unique. Indeed, it suffices to take $\boldsymbol{u}_1=\boldsymbol{u}_2$ and to recall that the above estimates hold for arbitrary solutions. This concludes the proof of Theorem 2.1.

For the proof of Theorem 2.3, i.e., of the estimate (2.42), we observe that the uniqueness result just established implies that the solutions involved in (5.11)–(5.12) must coincide with the ones we have constructed in the previous section. Therefore, they satisfy the estimate (2.34). Since the constants that occur in the proof of (5.11)–(5.12), in particular those of (5.11)–(5.12), depend on the solutions only through upper bounds for some of the norms related to (2.24)–(2.27), we deduce that they have the same properties as those required for K_2 in the statement.

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