Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

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submitted: September 4, 2025

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No. 3216
Berlin 2025

2020 *Mathematics Subject Classification.* 49J45, 74B20, 74K20.

Key words and phrases. Reissner–Mindlin plate model, dimension reduction, finite elasticity.

The first author was supported by Einstein foundation (EGR-2022-731). This work benefitted from discussions with K. Buryachenko, A. Glitzky and M. Liero in the context of project AA2-21 within MATH+ which is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany´s Excellence Strategy – The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID 390685689).

Edited by
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Abstract

We discuss how the Reissner–Mindlin plate model can be derived from three-dimensional finite elasticity in terms of Γ-convergence. The presence of transverse shear effects in the Reissner–Mindlin model requires to scale different components of the three-dimensional elastic strain differently. A main technical tool is then the combination of rigidity estimates for the deformation and suitably averaged versions.

1 Introduction

The rigorous justification of simplified models for the elastic behaviour of thin bodies has a long standing history. In this paper, we follow the approach to derive static lower-dimensional models from three-dimensional elasticity by means of Γ-convergence. The latter is a concept of convergence that -if complemented by appropriate compactness properties- ensures convergence of (almost) minimizers (see e.g. [7, 9]). The idea to use Γ-convergence in the context of dimension reduction in elasticity dates back to [1, 13, 19, 20, 32], and since then, a large body of literature has been devoted to a refined analysis in various settings (for recent overviews see e.g. [22, 27, 33]), including for instance nonlinear bending models for shells as well as Kirchhoff-Love's or von Kármán models for plates (see e.g. [2, 4, 5, 13, 14, 17, 21, 30, 39] and the references therein) and models for beams or rods (see e.g. [10, 11, 18, 25, 26, 29, 37, 38] and the references therein).

In the present work we address the question, whether it is possible to derive the Reissner–Mindlin plate model ([23, 36]) from nonlinear elasticity. Before discussing the related literature let us briefly explain the setting and the challenges. We denote by $S \subseteq \mathbb{R}^2$ the midsurface of a thin plate $S \times (-h/2, h/2)$, and consider the energy functionals for linearized and nonlinear Reissner–Mindlin models, namely

$$J(\varphi, v) = \sum_{i=1}^{2} a_i \int_S (\varphi_i + \partial_i v)^2 dx_1 dx_2 + \sum_{i,j=1}^{2} b_{ij} \int_S (\partial_i \varphi_j + \partial_j \varphi_i)^2 dx_1 dx_2 \tag{1}$$

and

$$I(u, \varphi, v) = \sum_{i=1}^{2} a_i \int_S (\varphi_i + \partial_i v)^2 dx_1 dx_2 + \sum_{i,j=1}^{2} b_{ij} \int_S (\partial_i \varphi_j + \partial_j \varphi_i)^2 dx_1 dx_2 + \sum_{i,j=1}^{2} c_{ij} \int_S (\partial_i u_j + \partial_j u_i + \frac{1}{2} \partial_i v_j \partial_j v_i)^2 dx_1 dx_2, \tag{2}$$

where $v(x_1, x_2)$ is the (renormalized) vertical displacement and $u_i(x_1, x_2)$ are the (renormalized) in-plane displacements of the midsurface and $\varphi_i(x_1, x_2)$ are the angles of deflection of the orthogonal

DOI 10.20347/WIAS.PREPRINT.3216

cross-sections of the plate. On $S \times (-h/2, h/2)$, this corresponds to a deformation of the form (we use the notation $x' := (x_1, x_2)$)

$$y^h(x', x_3) := \begin{pmatrix} x' \\ x_3 \end{pmatrix} + \begin{pmatrix} h^\alpha u(x') \\ h^\beta v(x') \end{pmatrix} + x_3 \begin{pmatrix} h^\gamma \varphi(x') \\ 0 \end{pmatrix}. \tag{3}$$

The corresponding strain can be seen to be of the form

$$\begin{aligned} &(\nabla y^h)^T(\nabla y^h) \\ = \; & Id + \begin{pmatrix} 2h^\alpha(\nabla' u)_{\mathsf{sym}} + 2x_3 h^\gamma (\nabla'\varphi)_{\mathsf{sym}} + h^{2\beta}\nabla' v \otimes \nabla' v & h^\gamma\varphi + h^\beta \nabla' v \\ (h^\gamma\varphi + h^\beta\nabla' v)^T & 0 \end{pmatrix} + \text{ h.o.t,} \end{aligned}$$

where h.o.t. stands for higher order terms. Hence, for an energy density W that has the typical behavior $W(F) \sim |F^TF - \mathsf{Id}|^2$ close to $SO(3)$ we find

$$\begin{aligned} &\int_{S\times(-h/2,h/2)} W(\nabla y^h)\, dx \\ \sim & h \int_S 2\left|h^\gamma\varphi + h^\beta\nabla' v\right|^2 + \frac{h^{2\gamma+2}}{3}\left|(\nabla'\varphi)_{\mathsf{sym}}\right|^2 + 4\left|h^\alpha(\nabla' u)_{\mathsf{sym}} + \frac{1}{2}h^{2\beta}\nabla' v \otimes \nabla v\right|^2 \, dx_1\, dx_2. \end{aligned} \tag{4}$$

Comparing with (1) and (2) we notice the following:

- it must hold $\gamma = \beta$ and $\alpha = 2\beta$;
- a renormalization of (4) by $h^{2\gamma+3}$ will allow to obtain the term $\int_S |(\nabla'\varphi)_{\mathsf{sym}}|^2 \, d\mathcal{L}^2$ in the limit $h \to 0$ but necessarily forces $\varphi = \nabla' v$, c.f., for example, [14];
- in order to derive all terms in (2) or (1) from our ansatz the elastic energy density has to weight different entries of the elastic strain with different powers in h.

Taking into account the above considerations, we will derive (1) and (2) from an elastic energy that weights different entries of the elastic strain differently. In our setting (see Section 1.1 for the precise definitions) the derivation of (2) will be performed for $\beta = \gamma > 1$ whereas (1) will be derived for $\gamma = \beta = 1$ and $\alpha = 2\beta = 2\gamma + 2 = 4$.

Let us briefly discuss some related literature. Falach, Paroni, Podio-Guidugli and Tomasetti [12], [34, 35] used an anisotropic (transversely isotropic) linear three-dimensional energy containing second-gradient terms. In fact, the idea was to consider for a given three-dimensional problem (the so-called "real problem") an approximation whose variational limit coincides with the variational limit of the "real problem". Another idea is to consider anisotropic elasticity and to scale the elastic constants in different ways, which allows to avoid the inconsistency with the fact that for the Mindlin-Timoshenko-type models the shear modulus cannot be "too large". However, the parent models in [12, 34] consider linearized strains and the Saint Venant–Kirchhoff model for the strain energy, which does not cover the case of nonlinear Lagrangian strains and important models such as neo-Hookean, Mooney-Rivlin, Ogden, Fung etc. (see [31] and references therein). Another approach starting from isotropic, linear elasticity with microrotations is presented in [28]. For further discussion of the challenges and related literature we refer to the above references.

The rest of the paper is organized as follows: in Subsection 1.1 we set the main notation, state the parent problem and formulate the main assumptions on the energy density and external forces, we also present an example of a nonlinear energy functional satisfying these assumptions. In Subsection 1.2 we formulate and discuss our main result. In Section 2 rigidity estimates for the displacement gradient to the case of anisotropic elasticity are adjusted. In Section 3 we prove main results on Γ-convergence of the three-dimensional problem to the Reissner–Mindlin energy functional in case of the energy scaling of power greater than 4. Finally, Section 4 contains a Γ-convergence result in case of the scaling of power 4.

1.1 Notation and setting.

Throughout the text, we denote by C generic constants that may change from expression to expression.
Let $S \subseteq \mathbb{R}^2$ be an open, bounded Lipschitz domain, and consider for $h > 0$ the (thin) domain

$$\Omega_h = S \times \left(-\frac{h}{2}, \frac{h}{2}\right).$$

We will always assume that $h \in (0, 1)$. For a deformation $w \in W^{1,2}(\Omega_h; \mathbb{R}^3)$, we consider the elastic energy

$$\mathcal{E}_1^h(w) := \int_{\Omega_h} W_1(\nabla' w'(z)^T \nabla' w'(z) + \nabla' w_3(z) \otimes \nabla' w_3(z))dz + h^2 \int_{\Omega_h} W_2(\nabla w(z))dz, \tag{5}$$

where we use the notation $x' = (x_1, x_2)$, $w' = (w_1, w_2)$ and $\nabla' w = w_{,1} \otimes e_1 + w_{,2} \otimes e_2$. Here and in the following, $w_{,i}$ denotes the i-th partial derivative $\partial_i w$ for $i = 1, 2, 3$, and similarly for second order partial derivatives. For $M = (m_{ij})_{i,j=1\dots3} \in \mathbb{R}^{3\times3}$, we similarly set $M' := (m_{ij})_{i,j=1,2} \in \mathbb{R}^{2\times2}$. Now, consider the fixed domain $\Omega = S \times I$, where $I := (-\frac{1}{2}, \frac{1}{2})$ and introduce the change of variables $z(x) = (x_1, x_2, \frac{x_3}{h})$ and the rescaled deformation $y : \Omega \to \mathbb{R}^3$, $y(x) = w(z(x))$. It then holds

$$\nabla w = \left(\nabla' y, \frac{1}{h} y_{,3}\right) =: \nabla_h y$$

and

$$\frac{1}{h}\mathcal{E}^h(y) := \int_{\Omega} W_1(\nabla' y'(x)^T \nabla' y'(x) + \nabla' y_3(x) \otimes \nabla' y_3(x))dx + h^2 \int_{\Omega} W_2(\nabla_h y(x))dx. \tag{6}$$

Slightly abusing notation, we identify functions $f : S \to \mathbb{R}^n$ with their trivial extensions $f : \Omega \to \mathbb{R}^n$ given by $f(x', x_3) := f(x')$. We sometimes write $u(x')$ or $u(x_3)$ instead of u to point out on which components the function depends.

Free energy densities. We will always impose (without further mentioning) that the following assumptions on the free energy densities $W_i : \mathbb{R}^{(i+1)\times(i+1)} \to [0, \infty]$, $i = 1, 2$ hold.

Assumption 1. *There exist constants $c_0, c_1, c_2 > 0$ such that*

(A1) $W_2(QF) = W_2(F)$ *for all* $Q \in SO(3)$ *and all* $F \in \mathbb{R}^{3\times3}$*,*

(A2) $W_1(Id) = \min W_1 = 0 = \min W_2 = W_2(Q)$ *for all* $Q \in SO(3)$*,*

(A3) $W_2(F) \geq c_0 \operatorname{dist}^2(F, SO(3))$ *for all* $F \in \mathbb{R}^{3\times3}$*,*

(A4) $W_1(F'^T F' + (f_{31}, f_{32}) \otimes (f_{31}, f_{32})) + c_1|(f_{31}, f_{32})|^4 \geq c_2 \mathrm{dist}^2(F', SO(2))$ *for all* $F = (f_{ij}) \in \mathbb{R}^{3\times3}$*, and*

(A5) W_2 *is twice continuously differentiable in a neighborhood of* $SO(3)$*, and* W_1 *is twice differentiable in a neighbourhood of Id.*

Let us briefly discuss an example of free energy densities W_1 and W_2 satisfying Assumption 1.

Example 1. *We consider a special case of an orthotropic neo-Hookian strain energy (see e.g [3, 6, 40]). Precisely, for* $\beta > 0$ *we set* $W_1 : \mathbb{R}^{2\times2} \to [0, \infty]$,

$$W_1(C') := \begin{cases} \beta\left[\mathrm{tr}\,(C'^2) - 2\,\mathrm{tr}\,(C') + 2\right] & \text{if } C' \in \mathcal{A} \\ +\infty & \text{otherwise,} \end{cases} \tag{7}$$

where

$$\mathcal{A} := \left\{C' \in \mathbb{R}^{2\times2} :\ \exists F \in \mathbb{R}^{3\times3} \text{ s.t. } \det F' > 0 \text{ and } C' = F'^T F' + (f_{31}, f_{32}) \otimes (f_{31}, f_{32})\right\}.$$

In addition, for $\lambda, \mu > 0$*, we set* $W_2 : \mathbb{R}^{3\times3} \to [0, \infty]$,

$$W_2(F) := \begin{cases} \mu(\mathrm{tr}(F^T F) - 3) - \mu \ln \det(F^T F) + \lambda(\det(F^T F) - 1)^2 & \text{if } \det F > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

We note that the assumption $\det F > 0$ *is related to non-interpenetrability of matter, and the assumption* $\det F' > 0$ *is typically satisfied in the case of infinitesimal planar strain.*
Clearly, the functions W_1 *and* W_2 *satisfy assumptions (A1) and (A5). To see that* W_1 *satisfies (A2) note that all* $C' \in \mathcal{A}$ *are symmetric, and hence, denoting the eigenvalues of* C' *by* ν_1 *and* ν_2*, we have*

$$W_1(C') = \beta\left[\nu_1^2 + \nu_2^2 - 2(\nu_1 + \nu_2) + 2\right] = \beta\left[(\nu_1 - 1)^2 + (\nu_2 - 1)^2\right] \geq 0$$

with equality if and only if $\nu_1 = \nu_2 = 1$.
To see the other properties, we note that the matrix $F^T F$ *is symmetric and positive definite and hence has three positive real eigenvalues that we denote by* λ_1^2, λ_2^2 *and* λ_3^2 *with* $\lambda_i > 0$ *for* $i = 1, \ldots, 3$*. Then*

$$\begin{aligned} W_2(F^T F) &\geq \mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - \mu(\ln \lambda_1^2 + \ln \lambda_2^2 + \ln \lambda_3^2) \geq \mu \sum_{i=1}^3 (\lambda_i^2 - \ln \lambda_i^2 - 1) \\ &\geq \mu \sum_{i=1}^3 (\lambda_i - 1)^2 = \mu|\sqrt{F^T F} - I|^2 = \mu\,\mathrm{dist}^2(F, SO(3)), \end{aligned}$$

where we used that $\ln \lambda_i^2 = 2 \ln \lambda_i \leq 2(\lambda_i - 1)$*. This shows (A3) and the first part of (A2). Next, we consider* W_1*. For* $F \in \mathbb{R}^{3\times3}$*, we have*

$$\begin{aligned} & W_1\left(F'^T F' + (f_{31}, f_{32}) \otimes (f_{31}, f_{32})\right) \\ \geq\ & \beta\left[\left((f_{11}^2 + f_{21}^2 + f_{31}^2)^2 + 2(f_{11}f_{12} + f_{21}f_{22} + f_{31}f_{32})^2 + (f_{12}^2 + f_{22}^2 + f_{32}^2)^2\right)\right. \\ & \left. -2\left(f_{11}^2 + f_{21}^2 + f_{31}^2 + f_{12}^2 + f_{22}^2 + f_{32}^2\right) + 2\right] \\ \geq\ & \beta\left[(f_{11}^2 + f_{21}^2 - 1)^2 + 2(f_{11}f_{12} + f_{21}f_{22})^2 + (f_{12}^2 + f_{22}^2 - 1)^2\right. \\ & \left. + f_{31}^4 + f_{32}^4 + 2f_{31}^2 f_{32}^2 + 2(f_{11}^2 + f_{21}^2 - 1)f_{31}^2 + 2(f_{12}^2 + f_{22}^2 - 1)f_{32}^2\right] \\ \geq\ & c_2|F'^T F' - I|^2 - c_1|(f_{31}, f_{32})|^4. \end{aligned} \tag{8}$$

Let us briefly explain how the previous example arises in modelling (see e.g [3, 6, 40]). Orthotropic materials are characterized by symmetry relations with respect to three orthogonal planes. The corresponding preferred directions are chosen as the intersections of these planes and are given by unit vectors a, b and c. If the vectors a, b and c are oriented along the coordinate axes, we can introduce the matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for structural tensors of the material, corresponding to the first two directions. With the notation $F = \nabla y$, $F' = \nabla' y'$, $\mathbf{C} = F^T F$, the orthotropic part has the form

$$W_1^{(M_1,M_2)}(\mathbf{C}) = \begin{cases} \beta \left[\sum_{i=1}^{2} (\operatorname{tr} \mathbf{C}^2 M_i - 2 \operatorname{tr} \mathbf{C} M_i + 1) \right] & \text{if} \quad \det F' > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Since for $C' = F'^T F' + (f_{31}, f_{32}) \otimes (f_{31}, f_{32})$, there holds

$$\operatorname{tr} C'^2 = \operatorname{tr} C^2 (M_1 + M_2), \qquad \operatorname{tr} C' = \operatorname{tr} C(M_1 + M_2),$$

we find that $W_1^{(M_1,M_2)}$ defined in this way coincides with (7).

Regularized functionals and constraints. We impose Dirichlet boundary conditions on admissible deformations, and following [34], we consider for technical reasons regularized versions of (6). Precisely, we set $I^h : W^{1,2}(\Omega; \mathbb{R}^3) \to [0, \infty]$,

$$I^h(y) := \begin{cases} \int_\Omega W_1 \left(\nabla' y'(x)^T \nabla' y'(x) + \nabla' y_3(x) \otimes \nabla' y_3(x) \right) dx \\ + c_1 \int_\Omega |\nabla' y_3(x)|^4 dx + h^2 \int_\Omega W_2(\nabla_h y(x)) dx \\ + \frac{1}{h^\epsilon} \int_\Omega |y'{,}_{33}(x)|^2 dx \qquad \text{if } y(x) = (x', hx_3) \text{ on } \partial S \times (0,1), \\ +\infty \qquad \text{otherwise} \end{cases} \tag{9}$$

with some $\epsilon > 0$. Here, the first expression is understood in the sense that it is $+\infty$ if $\nabla' y_3 \notin L^4$ or $y'_{,33} \notin L^2$. We note that we impose the specific boundary condition on all of ∂S in (9) only for the ease of notation. It can be easily relaxed to hold only on part of the boundary. We will point out during the proof explicitly which parts require the boundary condition and which ones also hold without it.

Note that the last term in (9) coincides with the second-gradient term in the linear three-dimensional functionals in [34].

Forces. To include forces $f^h \in L^2(S; \mathbb{R}^3)$, we follow [14] and assume that the total force and the total moment applied to the reference configuration is zero, i.e.

$$\int_\Omega f^h dx = 0, \qquad \int_\Omega x \wedge f^h dx = 0. \tag{10}$$

We suppose that there is some $\alpha \in \mathbb{R}$ such that

$$\frac{1}{h^\alpha} f^h \rightharpoonup f \quad \text{in} \quad L^2(S; \mathbb{R}^3) \quad \text{with} \quad f_1 = f_2 = 0, \tag{11}$$

and denote the functional with forces by $J^h : W^{1,2}(\Omega;\mathbb{R}^3) \to (-\infty,\infty]$,

$$J^h(y) = I^h(y) - \int_\Omega f^h(x')\cdot y\,dx. \tag{12}$$

1.2 Main results and discussion.

Our main result is the Γ-limit of the sequence of functionals (9) as $h \to 0$. It turns out that the Γ-limit has the form of the linearized Reissner–Mindlin energy (2), and coincides with the Γ-limit of (6). Precisely, we have the following result.

Theorem 1.1. *(i)* Compactness and lower bound.
Suppose that $\sigma \geq 4$. Let $(y^h)_{h>0} \subseteq W^{1,2}(\Omega;\mathbb{R}^3)$ be such that

$$\limsup_{h\to 0} \frac{1}{h^\sigma} I^h(y^h) < \infty. \tag{13}$$

Then there exist $Q_h : S \to SO(2)$, $u \in W^{1,2}(S;\mathbb{R}^2)$, $v \in W^{1,2}(S)$, and $\varphi \in W^{1,2}(S;\mathbb{R}^2)$ identified with the constant in x_3-direction function $\varphi \in W^{1,2}(\Omega;\mathbb{R}^2)$ such that

$$u^h \rightharpoonup u \quad \text{in } W^{1,2}(S;\mathbb{R}^2), \quad \text{where} \quad u^h(x') := \frac{1}{h^{\sigma/2}} \int_I \left(\begin{pmatrix} y_1^h(x',x_3) \\ y_2^h(x',x_3) \end{pmatrix} - x' \right) dx_3, \tag{14}$$

$$v^h \rightharpoonup v \quad \text{in } W^{1,2}(S), \quad \text{where} \quad v^h(x') := \frac{1}{h^{\sigma/2-1}} \int_I y_3^h(x',x_3)\,dx_3, \tag{15}$$

$$\varphi^h \rightharpoonup \varphi \quad \text{in } L^2(\Omega;\mathbb{R}^2), \quad \text{where} \quad \varphi^h(x',x_3) := \frac{1}{h^{\sigma/2}} \left(Q_h^T \partial_3 y'(x',x_3)\right), \tag{16}$$

$$\frac{1}{h^{\sigma/2}} Q_h^T \left(\nabla'(y^h)' - \int_I \nabla'(y^h)' dx_3 \right) \rightharpoonup x_3 \nabla'\varphi \text{ in } L^2(\Omega;\mathbb{R}^2). \tag{17}$$

Moreover, if $\sigma > 4$ set

$$\mathcal{Q}_3^i(A) := \frac{\partial^2 W_i(A)}{\partial A^2}(Id)(A,A), \quad i=1,2, \quad \text{and} \quad \mathcal{Q}_2^2(\tilde{G}) := \min_{c\in\mathbb{R}} \mathcal{Q}_3^2\left(\tilde{G} + c e_3 \otimes e_3\right).$$

Then we have

$$\begin{aligned} &\liminf_{h\to 0} \frac{1}{h^\sigma} I^h(y^h) \\ \geq &\frac{1}{2}\int_S \mathcal{Q}_2^2(\tilde{G})\,dx' + \frac{1}{2}\int_S \mathcal{Q}_3^1(2\,\mathrm{sym}\,\nabla' u)\,dx' + \frac{1}{6}\int_S \mathcal{Q}_3^1(\mathrm{sym}\,\nabla'\varphi)\,dx' \end{aligned} \tag{18}$$

with

$$\tilde{G}(x') := \begin{pmatrix} 0 & 0 & \varphi_1(x') + \partial_1 v(x') \\ 0 & 0 & \varphi_2(x') + \partial_2 v(x') \\ \varphi_1(x') + \partial_1 v(x') & \varphi_2(x') + \partial_2 v(x') & 0 \end{pmatrix}. \tag{19}$$

(ii) Optimality of lower bound in the case $\sigma > 4.$
If $u \in W^{1,2}(S;\mathbb{R}^2)$*,* $v \in W^{1,2}(S)$*, and* $\varphi \in W^{1,2}(S;\mathbb{R}^2)$*, then there exists a recovery sequence* $(\hat{y}^h)_{h>0}$ *such that* (14)–(17) *hold and*

$$\liminf_{h\to 0} \frac{1}{h^\sigma} I^h(\hat{y}^h) = \frac{1}{2}\int_S \mathcal{Q}_2^2(\tilde{G})dx' + \frac{1}{2}\int_S \mathcal{Q}_3^1(2\,\mathrm{sym}\,\nabla' u)dx' + \frac{1}{6}\int_S \mathcal{Q}_3^1(\mathrm{sym}\,\nabla'\varphi(x'))dx'. \tag{20}$$

As a corollary of Theorem 1.1 one can infer convergence of minimizers to minimizers of a linear Reissner–Mindlin energy. Precisely, we have the following result.

Theorem 1.2. *Suppose that* $\sigma > 4$*. Consider external forces* $(f^h)_{h>0}$ *such that* (10) *and* (11) *hold with* $\sigma = 2\alpha - 2$*. Let* $(y^h)_{h>0}$ *be a* σ*-minimizing sequence for* $(J^h)_{h>0}$*, i.e.,*

$$\limsup_{h\to 0} \frac{1}{h^\sigma}\left(J^h(y^h) - \inf J^h\right) = 0.$$

Then there exist $Q_h : S \to SO(2)$*,* $u \in W^{1,2}(S;\mathbb{R}^2)$*,* $v \in W^{1,2}(S)$*, and* $\varphi \in W^{1,2}(S;\mathbb{R}^2)$ *such that*

$$\nabla_h y^h \to Id \quad \text{in} \quad L^2(\Omega;\mathbb{R}^3),$$

(14) *holds with* $u = 0$*, and* (15)–(16) *hold up to a subsequence. Moreover,* $0 \geq \inf J^h \geq -Ch^\sigma$ *and the limit function* (v,φ) *minimizes the functional*

$$J(v,\varphi) = \frac{1}{2}\int_S \mathcal{Q}_2^2(\tilde{G})\,dx' + \frac{1}{6}\int_S \mathcal{Q}_3^1(\mathrm{sym}\,\nabla'\varphi(x'))\,dx' - \int_S f_3(x')v(x')\,dx'$$

with $\tilde{G}$ *defined in* (19)*. Finally,* $\lim_{h\to 0}\frac{1}{h^\sigma}J^h(y^h) = \min J.$

Let us briefly discuss the main difficulties in our analysis. Starting point is the seminal work in [14] and [15]on dimension reduction in nonlinear elasticity. However, the scaling of the energy functional considered there yields a higher rigidity than we expect in the Reissner–Mindlin model. In particular, one obtains relations between the angles of rotation of the cross-sections of the plate and the transversal displacement of the mid surface $(\phi_1,\phi_2) = -\nabla v$. The Reissner–Mindlin model on the other hand takes into account transverse shear effects, which contradict this relation.
To overcome this, we rescale parts of the energy differently. More precisely, if we denote the strain matrix by $S = (s_{ij})_{i,j=1,2,3}$, we rescale the part depending on the upper part $S' = (s_{ij})_{i,j=1,2}$ by h^σ while the rest of the energy is rescaled by $h^{\sigma-2}$. While this rescaling allows us to avoid unwanted rigidity, it also leads to a lack of compactness. In case $\sigma > 4$, one can overcome this difficulty by adding regularizing terms (see (9)), which vanish in the limit. Here we use Dirichlet-type boundary conditions, which allow us to use Korn-type inequalities (see e.g. Proposition 1 [14]).
However, for the case $\sigma = 4$, we still experience the lack of compactness to perform the limit transition in the nonlinear term $\nabla' y_3(x) \otimes \nabla' y_3(x)$ in (9). The way to overcome this difficulty by adding a second gradient term is discussed in Section 4 but the limit functional differs from the classical Reissner–Mindlin energy.
The main steps in the proof are the following:

(a) *Scaled rigidity estimates in thin domain.* In a thin domain $\Omega_h = S \times (-h/2, h/2)$ we use different rigidity estimates. We derive approximations of the scaled deformation gradient $\nabla_h y$ by a rotation $R_h(x')$ in $SO(3)$ depending on x', the averaged with respect to the third component 2D gradient $\int_I \nabla' y' dx_3$ by a rotation $Q_h(x')$ in $SO(2)$ depending on x' (see Theorem 2.2), and $\nabla' y'$ by a rotation $T_h(x)$ in $SO(2)$ depending on x (see Theorem 2.7). In the first case the approximation rate is $h^{\sigma-2}$ and in the second and the third cases h^σ. We also construct a rotation $L_h \in SO(3)$ associated with Q_h

$$L_h = \begin{pmatrix} Q_h & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) *Scaling of deformations.* Next we obtain the rates of convergence of rotations to the identity in L^2 and in any L^p for $2 < p < \infty$ and normalize and scale in-plane and out-of-plane deformations according to these estimates. We note that the scaling of the out-of-plane components does not coincide with that introduced in [14].

(c) *Convergence of scaled deformations.* We prove weak convergence of scaled and averaged with respect to the third component deformations. Using the boundary conditions we show that the deformations converge to the same functions even without normalization.

(d) *Identification of the limiting strain.* We then estimate in L^2 three approximate strain components and find that their weak limits in L^2 up to a subsequence have the following structures:

$$G_h := \frac{L_h^T \nabla_h y^h - Id}{h^{\frac{\sigma}{2}-1}} \rightharpoonup \begin{pmatrix} 0 & 0 & \varphi_1(x') \\ 0 & 0 & \varphi_2(x') \\ \partial_1 v(x') & \partial_2 v(x') & G_{33}(x). \end{pmatrix}$$

$$\operatorname{sym} F_h := \operatorname{sym} \frac{Q_h^T \int_I \nabla'(y^h)' dx_3 - Id}{h^{\frac{\sigma}{2}}} \rightharpoonup \operatorname{sym} \nabla' u(x')$$

$$K_h := \frac{Q_h^T(\nabla'(y^h)' - \int_I \nabla'(y^h)' dx_3)}{h^{\frac{\sigma}{2}}} \rightharpoonup x_3 \nabla' \varphi(x').$$

(e) Γ*-convergence.* Using the Taylor's expansion and this relations we obtain the lower bound in Theorem 1.1. For the proof of the optimality of the lower bound we construct a recovering sequence (136).

2 Geometric rigidity

To prove the results on the approximation of the deformation gradient by rotations we make use of the celebrated Friesecke-James-Müller rigidity theorem:

Theorem 2.1 ([13]). *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. There exists a constant $C(U)$ such that for each $v \in W^{1,2}(U, \mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ satisfying*

$$\|\nabla v - R\|_{L^2(U)} \leq C(U) \|\operatorname{dist}(\nabla v, SO(n)\|_{L^2(U)}.$$

Remark 1. *The constant $C(U)$ in Theorem 2.1 can be chosen uniformly for a family of domains which are Bilipschitz equivalent with controlled Lipschitz constant (see also [8, Section 5]). The constant $C(U)$ is invariant under dilations.*

Building on this estimate, we show the following result.

Theorem 2.2. *Let* $S \subseteq \mathbb{R}^2$ *be a Lipschitz domain and* $\Omega := S \times (-\frac{1}{2}, \frac{1}{2})$. *Let* $y \in W^{1,2}(\Omega, \mathbb{R}^3)$ *and set*

$$E_1 := \int_\Omega \operatorname{dist}^2(\nabla_h y, SO(3))dx \le Ch^2, \text{ and } E_2 := \int_\Omega \operatorname{dist}^2(\nabla' y', SO(2))dx \le Ch^2. \quad (21)$$

Then there exist maps $R_h : S \to SO(3)$, $\tilde{R}_h : S \to \mathbb{R}^{3\times 3}$, $Q_h : S \to SO(2)$, *and* $\tilde{Q}_h : S \to \mathbb{R}^{2\times 2}$ *such that* $|\tilde{R}_h| \le C$, $|\tilde{Q}_h| \le C$ *and* $\tilde{R}_h \in W^{1,2}(S, \mathbb{R}^{3\times 3})$, $\tilde{Q}_h \in W^{1,2}(S, \mathbb{R}^{2\times 2})$. *Moreover,*

$$\|\nabla_h y(x', x_3) - R_h(x')\|^2_{L^2(\Omega)} \le CE_1, \quad \|\tilde{R}_h(x') - R_h(x')\|^2_{L^2(S)} \le CE_1, \quad (22)$$

$$\|\nabla' \tilde{R}_h(x')\|^2_{L^2(S)} \le \frac{C}{h^2}E_1, \quad \|\tilde{R}_h(x') - R_h(x')\|^2_{L^\infty(S)} \le \frac{C}{h^2}E_1, \quad (23)$$

$$\|\int_I \nabla' y'(x)dx_3 - Q_h(x')\|^2_{L^2(S)} \le CE_2, \ \|\tilde{Q}_h(x') - Q_h(x')\|^2_{L^2(S)} \le CE_2, \quad (24)$$

$$\|\nabla' \tilde{Q}_h(x')\|^2_{L^2(S)} \le \frac{C}{h^2}E_2, \quad \text{and} \quad \|\tilde{Q}_h(x') - Q_h(x')\|^2_{L^\infty(S)} \le \frac{C}{h^2}E_2. \quad (25)$$

Proof. Estimates (22) and (23) can be derived as in [14, Theorem 6], so we omit the proof here.

Let U be an open subset in $\mathbb{R}^2$ and let $K \subset U$ be compact and such that $\operatorname{dist}_\infty(K, \partial U) > 3h$, where $\operatorname{dist}_\infty$ is the distance with respect to the norm $\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$. For each point $x' \in K$ we consider the square

$$S_{x',h} := x' + (0,h)^2$$

with lower left corner x'. Let $\psi \in C_c^\infty((0,1)^2)$ be a standard mollifier, i.e. $\psi \ge 0$ and $\int_{\mathbb{R}^n} \psi = 1$, and set $\psi_h(\cdot) = h^{-2}\psi(\cdot/h)$. We set

$$F(x', x_3) := \nabla' y'(x', x_3) \quad \text{and} \quad \bar{F}(x') := \int_I F(x', x_3)dx_3 \quad (26)$$

and consider the map

$$\hat{Q}_h(x') := (\psi_h * \bar{F})(x') = \int_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right) \int_I F(z', z_3)dz_3 dz'.$$

Applying Theorem 2.1 to $S_{x',h}$ we obtain that for any fixed $z_3 \in I$ there exists a rotation $Q_{x',z_3,h}$ such that

$$\int_{S_{x',h}} |F(z', z_3) - Q_{x',z_3,h}|^2 dz' \le C \int_{S_{x',h}} \operatorname{dist}^2(F(z', z_3), SO(2))dz'. \quad (27)$$

Hence, by Hölder's inequality and Fubini, we also obtain

$$\int_{S_{x',h}} \left| \int_I F(z', z_3)dz_3 - \int_I Q_{x',z_3,h}dz_3 \right|^2 dz'$$
$$\le \int_I \int_{S_{x',h}} |F(z', z_3) - Q_{x',z_3,h}|^2 dz' dz_3 \le C \int_{S_{x',h}\times I} \operatorname{dist}^2(F(z), SO(2))dz. \quad (28)$$

Consequently, using Hölder's inequality and (28) we have

$$\begin{aligned} & \left|\hat{Q}_h(x') - \int_I Q_{x',z_3,h} dz_3\right|^2 \\ \leq & \left|\int_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right)\int_I F(z',z_3)dz_3dz' - \int_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right)\int_I Q_{x',z_3,h}dz_3dz'\right|^2 \\ \leq & \frac{C}{h^2}\int_{S_{x',h}}\left|\int_I F(z',z_3)dz_3 - \int_I Q_{x',z_3,h}dz_3\right|^2 dz' \\ \leq & \frac{C}{h^2}\int_{S_{x',h}\times I} \operatorname{dist}^2(F(z),SO(2))dz. \end{aligned} \tag{29}$$

Since $\int \nabla\psi_h = 0$, for any point $\tilde{x}' \in S_{x',h}$ we have from Hölder's inequality and (28)

$$\begin{aligned} & \left|\nabla\hat{Q}_h(\tilde{x}')\right|^2 \\ = \quad & h^{-6}\left|\int_{S_{\tilde{x}',h}} \nabla\psi\left(\frac{\tilde{x}'-z'}{h}\right)\int_I F(z',z_3)dz_3dz' - \int_{S_{\tilde{x}',h}} \nabla\psi\left(\frac{\tilde{x}'-z'}{h}\right)\int_I Q_{x',z_3,h}dz_3dz'\right|^2 \\ \leq \quad & \frac{C}{h^4}\int_{S_{x',2h}}\left|\int_I F(z',z_3)dz_3 - \int_I Q_{x',z_3,h}dz_3\right|^2 dz' \\ \leq \quad & \frac{C}{h^4}\int_{S_{x',2h}\times I} \operatorname{dist}^2(F(z),SO(2))dz, \end{aligned} \tag{30}$$

and integrating (30) over $S_{x',h}$ we find

$$\int_{S_{x',h}} \left|\nabla\hat{Q}_h(z')\right|^2 dz' \leq \frac{C}{h^2}\int_{S_{x',2h}\times I} \operatorname{dist}^2(F(z),SO(2))dz. \tag{31}$$

For any point $\tilde{x}' \in S_{x',h}$ we have again using (28)

$$\begin{aligned} & \left|\hat{Q}_h(x') - \hat{Q}_h(\tilde{x}')\right|^2 \\ \leq \quad & \left|\int_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right)\int_I F(z',z_3)dz_3dz' - \int_{S_{\tilde{x}',h}} h^{-2}\psi\left(\frac{\tilde{x}'-z'}{h}\right)\int_I F(z',z_3)dz_3dz'\right|^2 \\ \leq \quad & C\left|\int_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right)\int_I F(z',z_3)dz_3dz' - \int_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right)\int_I Q_{x',z_3,h}dz_3dz'\right|^2 \end{aligned}$$

$$
\begin{aligned}
&+ \quad C\left|\int\limits_{S_{\tilde{x}',h}} h^{-2}\psi\left(\frac{\tilde{x}'-z'}{h}\right)\int\limits_I F(z',z_3)dz_3dz' - \int\limits_{S_{\tilde{x}',h}} h^{-2}\psi\left(\frac{\tilde{x}'-z'}{h}\right)\int\limits_I Q_{x',z_3,h}dz_3dz'\right|^2 \\
&\le \quad \frac{C}{h^2}\int\limits_{S_{x',2h}}\left|\int\limits_I F(z',z_3)dz_3 - \int\limits_I Q_{x',z_3,h}dz_3\right|^2 dz' \\
&\le \quad \frac{C}{h^2}\int\limits_{S_{x',2h}\times I} \operatorname{dist}^2(F(z),SO(2))dz. \qquad (32)
\end{aligned}
$$

Combining (28), (29), and (32) we obtain

$$
\begin{aligned}
&\int\limits_{S_{x',h}}\left|\hat{Q}_h(z') - \int\limits_I F(z',z_3)dz_3\right|^2 dz' \\
&\le C\Bigg(\int\limits_{S_{x',h}}\left|\hat{Q}_h(z')-\hat{Q}_h(x')\right|^2 dz' + \int\limits_{S_{x',h}}\left|\hat{Q}_h(x') - \int\limits_I Q_{x',z_3,h}dz_3\right|^2 dz' \\
&\quad + \int\limits_{S_{x',h}}\left|\int\limits_I Q_{x',z_3,h}dz_3 - \int\limits_I F(z',z_3)dz_3\right|^2 dz'\Bigg) \\
&\le C\int\limits_{S_{x',h}\times I}\operatorname{dist}^2(F(z),SO(2))dz. \qquad (33)
\end{aligned}
$$

Finally, setting $g(\zeta) := \operatorname{dist}(\hat{Q}_h(x'+h\zeta),SO(2))$ and taking into account (30), (33) we arrive at

$$
\begin{aligned}
&\int_{(0,1)^2}|g|^2d\zeta + \sup_{(0,1)^2}|\nabla' g|^2 \\
= &\int\limits_{S_{x',h}}\operatorname{dist}^2(\hat{Q}_h(z'),SO(2))dz' + \sup_{S_{x',h}}|\nabla'\operatorname{dist}(\hat{Q}_h(z'),SO(2))|^2 \\
\le\ & C\Bigg(\int\limits_{S_{x',h}}|\hat{Q}_h(z') - \int\limits_I F(z',z_3)dz_3|^2dz' + \int\limits_{S_{x',h}}\operatorname{dist}^2(\int\limits_I F(z',z_3)dz_3,SO(2))dz' \\
&+\sup_{S_{x',h}}|\nabla'\hat{Q}_h(z')|^2\Bigg) \le \frac{C}{h^2}\int\limits_{S_{x',2h}\times I}\operatorname{dist}^2(F(z),SO(2))dz.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\operatorname{dist}^2(\hat{Q}_h(\tilde{x}'),SO(2)) \le \sup_{(0,1)^2}|g|^2 \le C(\int_{(0,1)^2}|g|^2d\zeta + \sup_{(0,1)^2}|\nabla' g|^2) \\
\le \frac{C}{h^2}\int\limits_{S_{x',2h}\times I}\operatorname{dist}^2(F(z),SO(2))dz, \qquad (34)
\end{aligned}
$$

and integrating (34) we obtain

$$\int_{S_{x',h}} \operatorname{dist}^2(\hat{Q}_h(\tilde{x}'), SO(2))d\tilde{x}' \leq \frac{C}{h^2} \int_{S_{x',h}} \int_{S_{x',2h}\times I} \operatorname{dist}^2(F(z), SO(2))dzd\tilde{x}'$$
$$\leq C \int_{S_{x',h}\times I} \operatorname{dist}^2(F(z), SO(2))dz. \quad (35)$$

Now we consider a lattice of squares of size h in $\mathbb{R}^2$ and sum (31), (33) over all squares which intersect K. This yields

$$\int_K \left(|\hat{Q}_h(x') - \int_I F(x', x_3)dx_3|^2 + h^2|\nabla'\hat{Q}_h(x')|^2 \right) dx' \leq C \int_{U\times I} \operatorname{dist}^2(F(x), SO(2))dx. \quad (36)$$

Following [14] we consider S to be locally the epigraph of a Lipschitz function, then arguing as in [14] and applying the above estimates we get

$$\int_{K\cap S} \left(|\hat{Q}_h(x') - \int_I F(x', x_3)dx_3|^2 + h^2|\nabla'\hat{Q}_h(x')|^2 \right) dx' \quad (37)$$
$$\leq C \int_{U\times I} \operatorname{dist}^2(F(x), SO(2))dx.$$

Since S is Lipschitz, its closure $\bar{S}$ can be covered by a finite number of open cubes $U_0, ..., U_l$, where $\bar{U}_0 \subset S$. Denote $\hat{Q}_i$ the maps constructed in previous steps for U_i and consider a partition of unity corresponding to the cover $\{U_i\}$

$$\eta_i \in C_0^\infty(U_i), \qquad \eta_i \geq 0, \qquad \sum_{i=0}^{l} \eta_i(x') = 1, \, \forall x' \in S.$$

If we set

$$\tilde{Q}_h = \sum_{i=0}^{l} \eta_i \hat{Q}_i \quad (38)$$

then

$$\tilde{Q}_h - \int_I F dx_3 = \sum_{i=0}^{l} \eta_i(\hat{Q}_i - \int_I F dx_3) \text{ and } \nabla'\tilde{Q}_h = \sum_{i=0}^{l} \eta_i \nabla'\hat{Q}_i + \sum_{i=0}^{l} \nabla'\eta_i(\hat{Q}_i - \int_I F dx_3).$$

Consequently, applying (36) and (37) with $K_i = \operatorname{supp} \eta_i$ we obtain

$$\int_S \left(|\tilde{Q}_h(x') - \int_I F(x', x_3)dx_3|^2 + h^2|\nabla'\tilde{Q}_h(x')|^2 \right) dx' \leq CE_2. \quad (39)$$

Note that this implies in particular that $\|\nabla'\tilde{Q}_h(x')\|^2_{L^2(S)} \leq \frac{C}{h^2}E_2$, which is the first part of assertion of (24). From (34) we obtain

$$\sup_S \operatorname{dist}^2(\tilde{Q}_h(x'), SO(2)) \leq \frac{C}{h^2} \sup_{x'\in S} \int_{(B_{x',c_0h}\cap S)\times I} \operatorname{dist}^2(\nabla' y(z), SO(2))dz \leq \frac{C}{h^2}E_2 \leq C \quad (40)$$

where c_0 depends only on S. Note that this estimate implies in particular

$$\|\tilde{Q}_h\|_{L^\infty(S)} \leq C. \tag{41}$$

It follows from (35) that

$$\int_S \operatorname{dist}^2(\tilde{Q}_h(\tilde{x}'), SO(2)) d\tilde{x}' \leq CE_2. \tag{42}$$

Now the rotation Q_h can be obtained as projection of $\tilde{Q}_h$ onto $SO(2)$. More precisely, since $SO(2)$ is a smooth manifold, there exists a tubular neighbourhood $\mathcal{U}$ of $SO(2)$ such that the projection $\pi : \mathcal{U} \to SO(2)$ is smooth. In particular, there exists some $\delta > 0$ such that for all $M \notin \mathcal{U}$, there holds $\operatorname{dist}(M, SO(2)) > \delta$. Let

$$\tilde{\pi} : \mathbb{R}^{2\times 2} \to SO(2), \qquad \tilde{\pi}(M) := \begin{cases} \pi(M) & M \in \mathcal{U}, \\ Id & M \notin \mathcal{U}, \end{cases} \tag{43}$$

and set $Q_h := \tilde{\pi} \circ Q_h$. If $\tilde{Q}_h(x') \in \mathcal{U}$ then $|Q_h(x') - \tilde{Q}_h(x')| = \operatorname{dist}(\tilde{Q}_h, SO(2))$. If $\tilde{Q}_h(x') \notin \mathcal{U}$ then $\operatorname{dist}(\tilde{Q}_h(x'), SO(2)) > \delta$, and with (41) we obtain $|Q_h(x') - \tilde{Q}_h(x')| \leq C\operatorname{dist}(\tilde{Q}_h, SO(2))$. Consequently, with (40) we obtain

$$\|Q_h(x') - \tilde{Q}_h(x')\|^2_{L^\infty(S)} = \sup_S |Q_h(x') - \tilde{Q}_h(x')|^2 \leq C \sup_S \operatorname{dist}^2(\tilde{Q}_h, SO(2)) \leq \frac{C}{h^2} E_2, \tag{44}$$

which shows the second assertion of (25). Similarly, with (42) we deduce

$$\int_S |Q_h(x') - \tilde{Q}_h(x')|^2 dx' \leq C \int_S \operatorname{dist}^2(\tilde{Q}_h, SO(2)) dx' \leq CE_2, \tag{45}$$

which is the second assertion of (24). Finally, combining (39) and (45) we obtain the first assertion of (24). This concludes the proof of Theorem 2.2. □

Remark 2. *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be neighborhoods of $SO(3)$ and $SO(2)$ such that the respective projections onto $SO(3)$ and $SO(2)$ are smooth. If we assume that $E_i \leq Ch^{2+\varepsilon}$ for some $\varepsilon > 0$ then (for h small enough) we obtain from* (23) *and* (25) *that $\tilde{R}_h(x') \in \mathcal{U}_1$ and $\tilde{Q}(x') \in \mathcal{U}_2$. It follows that the the maps $R_h : S \to SO(3)$ and $Q_h : S \to SO(2)$ which are obtained by projections onto $SO(3)$ and $SO(2)$, respectively, are actually also in $W^{1,2}$, and it holds*

$$\|\nabla' R_h(x')\|^2_{L^2(S)} \leq \|\nabla' \tilde{R}_h(x')\|^2_{L^2(S)} \leq \frac{C}{h^2} E_1, \tag{46}$$

$$\|\nabla' Q_h(x')\|^2_{L^2(S)} \leq \|\nabla' \tilde{Q}_h(x')\|^2_{L^2(S)} \leq \frac{C}{h^2} E_2. \tag{47}$$

Thus in this case the estimates from Theorem 2.2 (more precisely the first assertions of (23) *and* (25)*) hold for R_h and Q_h directly.*

Corollary 1. *Under the assumptions of Theorem 2.2 there exists a constant rotation $P_h' \in SO(2)$ such that*

$$\| \int_I \nabla' y'(x) dx_3 - P_h' \|^2_{L^2(S)} \leq \frac{C}{h^2} E_2, \qquad \textit{and} \tag{48}$$

$$\|Q_h(x') - P_h'\|^2_{L^p(S)} \leq \frac{C}{h^2} E_2, \quad 1 \leq p < \infty. \tag{49}$$

Proof. We use the notation from Theorem 2.2 and from (43) and define

$$P_h' = \tilde{\pi}\left(\frac{1}{|S|}\int_S \tilde{Q}_h(x')dx'\right).$$

It follows from Poincaré's inequality that

$$\begin{aligned}
&\|\tilde{Q}_h(x') - P_h'\|^2_{L^2(S)}\\
\leq &C\left(\left\|\tilde{\pi}(\tilde{Q}_h(x')) - \tilde{\pi}\left(\frac{1}{|S|}\int_S \tilde{Q}_h(x')dx'\right)\right\|^2_{L^2(S)} + \left\|\tilde{Q}_h(x') - \tilde{\pi}(\tilde{Q}_h(x'))\right\|^2_{L^2(S)}\right)\\
\leq &C\left(\|\nabla'\tilde{Q}_h(x')\|^2_{L^2(S)} + \int_S \operatorname{dist}^2(\tilde{Q}_h(x'), SO(2))dx'\right) \leq C\frac{E_2}{h^2}.
\end{aligned}$$

Hence by the Sobolev embedding, we have for $1 \leq p < \infty$

$$\begin{aligned}
\|\tilde{Q}_h(x') - P_h'\|^2_{L^p(S)} \leq &C\|\tilde{Q}_h(x') - P_h'\|^2_{W^{1,2}(S)} \leq C\left(\|\tilde{Q}_h(x') - P_h'\|^2_{L^2(S)} + \|\nabla'\tilde{Q}_h(x')\|^2_{L^2(S)}\right)\\
\leq &\frac{C}{h^2}E_2,
\end{aligned}$$

where in the last step we used the first estimate in (25). Then (48) and (49) follow with (24), (25). □

Now we define R_h' as 2×2 submatrix of the rotation matrix R_h, and the rotation $P_h \in SO(3)$ via

$$R_h = \begin{pmatrix} R_h' & & 0\\ & & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \text{and} \quad P_h := \begin{pmatrix} P_h' & & 0\\ & & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{50}$$

Corollary 2. *Under the assumptions of Theorem 2.2 the constant rotation $P_h' \in SO(2)$ from Corollary 1 satisfies*

$$\|R_h'(x') - P_h'\|^2_{L^2(S)} \leq C\left(\frac{E_2}{h^2} + E_1\right) \tag{51}$$

and

$$\|R^h_{33} - 1\|^2_{L^2(S)} \leq C\left(\frac{E_2}{h^2} + E_1\right). \tag{52}$$

Proof. We obtain from (22) and Hölder's inequality that

$$\|R_h'(x') - \int_I \nabla'y'(x)dx_3\|^2_{L^2(S)} \leq \int_S\int_I |R_h'(x') - \nabla'y'(x)|^2\, dx_3dx',$$

and hence with (48), we deduce

$$\begin{aligned}
\|R_h'(x') - P_h'\|^2_{L^2(S)} \leq &C\left(\|R_h'(x') - \int_I \nabla'y'(x)dx_3\|^2_{L^2(S)} + \|\int_I \nabla'y'(x)dx_3 - P_h'\|^2_{L^2(S)}\right)\\
\leq &C\left(\frac{E_2}{h^2} + E_1\right),
\end{aligned}$$

which completes the proof of (51).
We now turn to the proof of (52). We denote the entries of the matrices R_h and P_h' by R^h_{ij} and P^h_{ij}, respectively. Since R_h and P_h' are rotations it holds for $j=1,2$ that

$$|R^h_{j1}|^2+|R^h_{j2}|^2+|R^h_{j3}|^2=1=|P^h_{j1}|^2+|P^h_{j2}|^2.$$

Consequently, for $j=1,2$

$$\begin{aligned}|R^h_{j3}|^2&=|P^h_{j1}|^2-|R^h_{j1}|^2+|P^h_{j2}|^2-|R^h_{j2}|^2\\&=(|P^h_{j1}|+|R^h_{j1}|)(|P^h_{j1}|-|R^h_{j1}|)+(|P^h_{j2}|+|R^h_{j2}|)(|P^h_{j2}|-|R^h_{j2}|)\\&\le C(|P^h_{j1}|+|R^h_{j1}|+|P^h_{j2}|+|R^h_{j2}|)(|P^h_{j1}-R^h_{j1}|+|P^h_{j2}-R^h_{j2}|).\end{aligned}$$

Then, using a similar argument for R_{3j} and Hölder's inequality it follows from (51) that

$$\|R^h_{j3}\|^2_{L^2(S)}+\|R^h_{3j}\|^2_{L^2(S)}\le C\sqrt{\frac{E_2}{h^2}+E_1},\quad j=1,2. \tag{53}$$

Next we use that

$$\begin{aligned}&|\det R_h'-1|\\=&|\det R_h'-\det P_h'|\\=&|R^h_{11}R^h_{22}-R^h_{12}R^h_{21}-P^h_{11}P^h_{22}+P^h_{12}P^h_{21}|\\\le&|R^h_{11}R^h_{22}-P^h_{11}R^h_{22}|+|P^h_{11}R^h_{22}-P^h_{11}P^h_{22}|+|P^h_{12}R^h_{21}-R^h_{12}R^h_{21}|+|P^h_{12}P^h_{21}-P^h_{12}R^h_{21}|\\\le&(3|R_h|+|P_h'|)|R_h'-P_h'|.\end{aligned} \tag{54}$$

On the other hand, it holds

$$1=\det R_h=R^h_{33}\det R_h'-R^h_{32}(R^h_{11}R^h_{23}-R^h_{13}R^h_{21})+R^h_{31}(R^h_{12}R^h_{23}-R^h_{13}R^h_{22}).$$

Therefore,

$$\begin{aligned}|R^h_{33}-1|^2\le&C\left(|R^h_{33}|^2|\det R_h'-1|^2+|R^h_{32}|^2(|R^h_{11}|^2+|R^h_{21}|^2)(|R^h_{23}|^2+|R^h_{13}|^2)\right.\\&\left.+|R^h_{31}|^2(|R^h_{12}|^2+|R^h_{22}|^2)(|R^h_{23}|^2+|R^h_{13}|^2)\right).\end{aligned} \tag{55}$$

Hence, integrating and using $|R^h_{33}|\le 1$, (54), (55) and (53) we conclude

$$\|R^h_{33}-1\|^2_{L^2(S)}\le C\left(\frac{E_2}{h^2}+E_1\right), \tag{56}$$

which concludes the proof of (52). □

Let us now assume that $I^h(y^h)\le CE_h$, where

$$\lim_{h\to 0}\frac{E_h}{h^2}=0.$$

Then it follows from the structure of the functional (9) and properties (A3), (A4) that $E_2\le CE_h$ and $E_1\le C\frac{E_h}{h^2}$, where E_i are defined in (21). In order to normalize the functions y_h, we prove the following lemma, c.f. [14].

Lemma 2.3. *Let $S \subseteq \mathbb{R}^2$ be a Lipschitz domain and $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$. Let a sequence y^h satisfy $I^h(y^h) \le CE_h$, where*

$$\lim_{h\to 0} \frac{E_h}{h^2} = 0. \tag{57}$$

Let assumptions (A3), (A4) be satisfied. Then there exist a rotation $B_h \in SO(3)$ and $c^h \in \mathbb{R}^3$ such that the following assertions hold for B_h and the functions

$$\tilde{y}^h := (B_h)^T (P_h)^T (y^h - c^h). \tag{58}$$

The rotation B_h has the form $B_h = \begin{pmatrix} B_h' & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$|B_h' - Id| \le Ch^{-1}\sqrt{E_h}, \tag{59}$$

$$\int_\Omega (\tilde{y}^h_{1,2} - \tilde{y}^h_{2,1}) dx = 0, \qquad \text{and} \tag{60}$$

$$\int_\Omega \tilde{y}^h - \begin{pmatrix} x' \\ hx_3 \end{pmatrix} dx = 0. \tag{61}$$

Proof. We suppose that h is small enough such that $\frac{E_h}{h} < 1$. It is easy to see that (61) is satisfied after a proper choice of the constants c^h. We use the notation

$$\bar{y}^h = (P_h)^T y^h.$$

It follows from (48) with Hölder's inequality and $E_2 \le CE_h$ that

$$\left| \frac{1}{|\Omega|} \int_\Omega \nabla' \bar{y}'^h dx - Id \right| \le C\frac{\sqrt{E_h}}{h}. \tag{62}$$

It remains to find $\theta \in (-\pi, \pi]$ such that the rotation $B_h{}' \in SO(2)$,

$$B_h{}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

satisfies (59) and (60) . It follows from (62) that for $h > 0$ small enough

$$\begin{aligned} \left(\int_\Omega (\bar{y}^h_{1,1} + \bar{y}^h_{2,2}) dx \right)^2 \ge& C \left(1 - \left(\int_\Omega (\bar{y}^h_{1,1} - 1) dx \right)^2 - \left(\int_\Omega (\bar{y}^h_{2,2} - 1) dx \right)^2 \right) \\ \ge& C \left(1 - \frac{E_h}{h^2} \right) > 0. \end{aligned} \tag{63}$$

In case $\int_\Omega (\bar{y}^h_{1,2} - \bar{y}^h_{2,1}) dx = 0$, we can choose $B_h{}' = Id$. If this is not the case, in order to satisfy property (60), we choose θ such that

$$\cot\theta = \frac{\int_\Omega (\bar{y}^h_{1,1} + \bar{y}^h_{2,2}) dx}{\int_\Omega (\bar{y}^h_{2,1} - \bar{y}^h_{1,2}) dx}.$$

Furthermore, using (62) and (63) we then obtain

$$|\sin\theta| = \sqrt{\frac{1}{1+\cot^2\theta}} \le C\frac{|\int_\Omega(\bar y^h_{2,1}-\bar y^h_{1,2})dx|}{\sqrt{\left(\int_\Omega(\bar y^h_{2,1}-\bar y^h_{1,2})dx\right)^2+\left(\int_\Omega(\bar y^h_{1,1}+\bar y^h_{2,2})dx\right)^2}} \le C\frac{\sqrt{E_h}}{h} \quad (64)$$

and

$$|\cos\theta - 1| = \left|\frac{|\cot\theta|}{\sqrt{1+\cot^2\theta}}-1\right| \le C\frac{\left(\int_\Omega(\bar y^h_{2,1}-\bar y^h_{1,2})dx\right)^2}{\left(\int_\Omega(\bar y^h_{2,1}-\bar y^h_{1,2})dx\right)^2+\left(\int_\Omega(\bar y^h_{1,1}+\bar y^h_{2,2})dx\right)^2} \quad (65)$$
$$\le C\frac{E_h}{h^2}.$$

Combining (64) and (65) we obtain (59).

□

Next, we define

$$\tilde U^h(x') = \int_I \begin{pmatrix}\tilde y^h_1\\ \tilde y^h_2\end{pmatrix}(x',x_3) - x'dx_3 \qquad \text{and} \qquad V^h(x') = \int_I y^h_3 dx_3. \quad (66)$$

Lemma 2.4. *Let $S \subseteq \mathbb{R}^2$ be a Lipschitz domain and $\Omega = S\times(-\frac12,\frac12)$. Let y^h be a sequence with $I^h(y^h) \le CE_h$, where E_h satsifies* (57)*. Let assumptions (A3) and (A4) be satisfied. Then there exists $\tilde u \in W^{1,2}(S;\mathbb{R}^2)$ such that up to a (non-relabeled) subsequence*

$$\tilde u^h := \min\left\{\frac{1}{\sqrt{E_h}}, \frac{h^2}{E_h}\right\}\tilde U_h \rightharpoonup \tilde u \quad \text{in } W^{1,2}(S;\mathbb{R}^2). \quad (67)$$

Proof. We set $\tilde A'_h := \frac{h}{\sqrt{E_h}}((B'_h)^T(P'_h)^TQ_h - Id)$. Using (A4), (47) (49), and (59) we obtain

$$\|\tilde A'_h\|^2_{W^{1,2}(S)} \le \frac{Ch^2}{E_h}(\|\nabla' Q_h\|^2_{L^2(S)} + \|Q_h - P'_h\|^2_{L^2(S)} + \|B'_h - Id\|^2_{L^2(S)}) \le C,$$

and thus by the Sobolev embedding

$$\|\tilde A'_h\|_{L^p(S)} \le C, \qquad \forall\, 1\le p<\infty. \quad (68)$$

Since

$$(\tilde A'_h)^T\tilde A'_h = \frac{h^2}{E_h}(Q_h^TP'_hB'_h - Id)((B'_h)^T(P'_h)^TQ_h - Id) = -\frac{h}{\sqrt{E_h}}((\tilde A'_h)^T + \tilde A'_h)$$
$$= \quad -\frac{h}{2\sqrt{E_h}}\operatorname{sym}\tilde A'_h = -\operatorname{sym}\frac{h^2}{2E_h}((B'_h)^T(P'_h)^TQ_h - Id)$$

we obtain from (68) with Hölder's inequality that

$$\frac{h^2}{E_h}\|\operatorname{sym}((B'_h)^T(P'_h)^TQ_h - Id)\|_{L^2(S)} \le C\|\tilde A'_h\|^2_{L^4(S)} \le C. \quad (69)$$

DOI 10.20347/WIAS.PREPRINT.3216

Then, combining this with (24) we arrive at

$$\begin{aligned}
&\| \operatorname{sym} \nabla' \tilde{U}_h \|_{L^2(S)} \\
=&\| \operatorname{sym} \left(\int_I \nabla' (\tilde{y}^h)' dx_3 - Id \right) \|_{L^2(S)} \\
\leq&\| \operatorname{sym} \left(\int_I \nabla' (\tilde{y}^h)' dx_3 - (B_h')^T (P_h')^T Q_h \right) \|_{L^2(S)} + \| \operatorname{sym}((B_h')^T (P_h')^T Q_h - Id)\|_{L^2(S)} \\
\leq& C(\sqrt{E_h} + \frac{E_h}{h^2}).
\end{aligned}$$

Using Korn's inequality and normalizations (61), (60) we infer the assertion of the lemma. □

Now we consider

$$U^h(x') := \int_I \begin{pmatrix} y_1^h \\ y_2^h \end{pmatrix} (x', x_3) - x' dx_3, \tag{70}$$

and recall that the quantities P_h', B_h' and Q_h were introduced in Corollary 1, Lemma 2.3, and Theorem 2.2, respectively.

Lemma 2.5. *Let a sequence y^h satisfy the assumptions of Lemma 2.4. Then it holds*

$$|P_h' B_h' - Id|^2 \leq C \max \left(\frac{E_h^2}{h^4}, E_h \right), \tag{71}$$

$$|P_h' - Id|^2 \leq C \frac{E_h}{h^2}, \qquad \textit{and} \tag{72}$$

$$\|Q_h(x') - Id\|_{L^2(S)}^2 \leq C \frac{E_h}{h^2}. \tag{73}$$

Moreover, there exists $u \in W^{1,2}(S; \mathbb{R}^2)$ such that up to a (non-relabeled) subsequence

$$u^h := \min \left(\frac{1}{\sqrt{E_h}}, \frac{h^2}{E_h} \right) U_h \rightharpoonup u \quad \textit{in} \ W^{1,2}(S; \mathbb{R}^2). \tag{74}$$

Proof. The proof is similar to that presented in [21]. From (58) we infer the relation

$$y_h' = P_h' B_h' (\tilde{y}^h)' + (c^h)'. \tag{75}$$

It follows from (75) and the definitions of u^h and $\tilde{u}^h$ that

$$\max(h^{-2} E_h, \sqrt{E_h}) u^h = (P_h' B_h' - Id) x' + \max(h^{-2} E_h, \sqrt{E_h}) P_h' B_h' \tilde{u}^h + (c^h)'. \tag{76}$$

Due to the Dirichlet boundary conditions of functions with finite energy we have that $u^h = 0$ on ∂S. Combining (67) and (76) and using the trace theorem we obtain

$$\begin{aligned}
&\|(P_h' B_h' - Id) x' + (c^h)'\|_{L^2(\partial S)}^2 \leq \max(h^{-4} E_h^2, E_h) \|\tilde{u}^h\|_{L^2(\partial S)}^2 \\
\leq\ & C \max(h^{-4} E_h^2, E_h) \|\tilde{u}^h\|_{W^{1,2}(S)}^2 \leq C \max(h^{-4} E_h^2, E_h).
\end{aligned} \tag{77}$$

Without loss of generality we may assume

$$\int_{\partial S} x'\, d\mathcal{H}^1 = 0 \qquad \text{and} \qquad \int_{\partial S} |x'|^2\, d\mathcal{H}^1 > 0.$$

For any $P \in SO(2)$ a straightforward computation shows, c.f. [21],

$$2|(P - Id)x'|^2 = |(P - Id)|^2 |x'|^2. \tag{78}$$

Therefore, (77) yields

$$|(c^h)'| \le C \max(h^{-2} E_h, \sqrt{E_h}). \tag{79}$$

This together with (77) and (78) gives (71). Collecting (71),(76), (79) we arrive at

$$\|\nabla' u_h\|^2_{L^2(S)} \le C.$$

Taking into account the Dirichlet boundary conditions we obtain (74). Combining (59) and (71) we get (72). Finally, (73) is a consequence of (49) and (72). □

Lemma 2.6. *Under the assumptions of Lemma 2.5 and (A3) it holds*

$$\|\operatorname{sym}(R_h - Id)\|^2_{L^2(S)} \le C \frac{E_h}{h^2} \qquad \text{and} \tag{80}$$

$$\|\nabla_h y^h - Id\|^2_{L^2(\Omega)} \le C \frac{E_h}{h^2}. \tag{81}$$

Moreover, there exists $v \in W^{1,2}(S)$ such that it holds up to a (non-relabeled) subsequence that

$$v^h(x') := \frac{h}{\sqrt{E_h}} V^h(x') \rightharpoonup v \quad \text{in } W^{1,2}(S). \tag{82}$$

Proof. Since R_h is a rotation we have for $i = 1, 2$

$$R^h_{i1} R^h_{31} + R^h_{i2} R^h_{32} + R^h_{i3} R^h_{33} = 0.$$

Therefore,

$$\begin{aligned} R^h_{21} R^h_{31} + (R^h_{22} - 1) R^h_{32} + R^h_{23}(R^h_{33} - 1) &= R^h_{32} + R^h_{23}, \\ (R^h_{11} - 1) R^h_{31} + R^h_{12} R^h_{32} + R^h_{13}(R^h_{33} - 1) &= R^h_{31} + R^h_{13}. \end{aligned}$$

Consequently, using (51), (52) and (72) we obtain

$$\begin{aligned} \|R^h_{32} + R^h_{23}\|^2_{L^2(S)} \le \|R^h_{21}\|^2_{L^2(S)} \|R^h_{31}\|^2_{L^\infty(S)} + \|(R^h_{22} - 1)\|^2_{L^2(S)} \|R^h_{32}\|^2_{L^\infty(S)} \\ + \|R^h_{23}\|^2_{L^\infty(S)} \|(R^h_{33} - 1)\|^2_{L^2(S)} \le C \frac{E_h}{h^2}, \end{aligned} \tag{83}$$

$$\begin{aligned} \|R^h_{31} + R^h_{13}\|^2_{L^2(S)} \le \|(R^h_{11} - 1)\|^2_{L^2(S)} \|R^h_{31}\|^2_{L^\infty(S)} + \|R^h_{12}\|^2_{L^2(S)} \|R^h_{32}\|^2_{L^\infty(S)} \\ + \|R^h_{13}\|^2_{L^\infty(S)} \|(R^h_{33} - 1)\|^2_{L^2(S)} \le C \frac{E_h}{h^2}. \end{aligned} \tag{84}$$

Then, it follows from (51) (56), (72), (83), (84) that

$$\|\operatorname{sym}(R_h - Id)\|^2_{L^2(S)} \le C(\|R'_h - P'_h\|^2_{L^2(S)} + \|P'_h - I\|^2_{L^2(S)}$$

$$+\|R^h_{31}+R^h_{13}\|^2_{L^2(S)}+\|R^h_{32}+R^h_{23}\|^2_{L^2(S)}+\|(R^h_{33}-1)\|^2_{L^2(S)})\le C\frac{E_h}{h^2}.$$

Therefore, using (22) we obtain

$$\|\operatorname{sym}(\nabla_h y^h-Id)\|^2_{L^2(\Omega)}\le C(\|\operatorname{sym}(R_h-Id)\|^2_{L^2(S)}$$
$$+\|\nabla_h y^h-R_h\|^2_{L^2(\Omega)})\le C\frac{E_h}{h^2}.$$

Using the Dirichlet boundary conditions and Proposition 1 in [15] we obtain (81). Then (82) is a consequence of (81). □

Theorem 2.7. *Suppose that $S\in\mathbb{R}^2$ is a Lipschitz domain and $\Omega=S\times(-\frac{1}{2},\frac{1}{2})$. Let y^h be a sequence such that $I^h(y^h)\le CE_h$, where E_h satisfies $\lim_{h\to 0}\frac{E_h}{h^2}=0$. Then for any fixed $x_3\in I$ and h small enough there exists a map $T_h(x',x_3):\Omega\to SO(2)$ such that*

$$\|\nabla'(y^h)'(x)-T_h(x)\|^2_{L^2(\Omega)}\le CE_h, \tag{85}$$

$$\|T_h(x',x_3)-Id\|^2_{L^p(\Omega)}\le C\max\left(E_h^{\frac{2}{p}},\frac{E_h}{h^2}\right),\ \ 2\le p<\infty. \tag{86}$$

Proof. 1 First we prove that there exists a map $\tilde T_h(x',x_3):\Omega\to\mathbb{R}^{2\times 2}$ such that

$$\|\nabla'(y^h)'(x)-\tilde T_h(x)\|^2_{L^2(\Omega)}\le CE_h, \tag{87}$$

$$\|\nabla'\tilde T_h(x',x_3)\|^2_{L^2(\Omega)}\le C\frac{E_h}{h^2}. \tag{88}$$

As in the proof of Theorem 2.2 we consider an open subset $U\subseteq\mathbb{R}^2$ and $K\subseteq U$ compact such that $\operatorname{dist}_\infty(K,\partial U)>3h$. For each point $x'\in K$ we consider the square

$$S_{x',h}=x'+(0,h)^2$$

with lower left corner x' and define the map

$$\hat T_h(x',x_3)=\int_{S_{x',h}}h^{-2}\psi\left(\frac{x'-z'}{h}\right)F(z',x_3)dz', \tag{89}$$

where F is defined in (26) and ψ is a standard mollifier. We also use the notation $\psi_h(\cdot)=h^{-2}\psi(\cdot/h)$. Using the Hölder's inequality and the rotation $Q_{x,x_3,h}$ defined in (27) we get

$$\begin{aligned}&\left|\hat T_h(x',x_3)-Q_{x',x_3,h}\right|^2\\&=\left|\int_{S_{x',h}}h^{-2}\psi\left(\frac{x'-z'}{h}\right)F(z',x_3)dz'-\int_{S_{x',h}}h^{-2}\psi\left(\frac{x'-z'}{h}\right)Q_{x',x_3,h}dz'\right|^2\\&\le\frac{C}{h^2}\int_{S_{x',h}}\left|F(z',x_3)-Q_{x',x_3,h}\right|^2dz'\\&\le\frac{C}{h^2}\int_{S_{x',h}}\operatorname{dist}^2(F(z',x_3),SO(2))dz'.\end{aligned} \tag{90}$$

Since $\int \nabla\psi_h = 0$, using (28) and Hölder's inequality we find for any point $\tilde{x}' \in S_{x',h}$

$$
\begin{aligned}
&\left|\nabla'\hat{T}_h(\tilde{x}', x_3)\right|^2 \\
=&\left|\int\limits_{S_{\tilde{x}',h}} h^{-3}(\nabla\psi)\left(\frac{\tilde{x}'-z'}{h}\right)F(z',x_3)dz' - \int\limits_{S_{\tilde{x}',h}} h^{-3}(\nabla\psi)\left(\frac{\tilde{x}'-z'}{h}\right)Q_{x',x_3,h}dz'\right|^2 \\
\le&\frac{C}{h^4}\int\limits_{S_{x',2h}} \left|F(z',x_3)-Q_{x',x_3,h}\right|^2 dz' \\
\le&\frac{C}{h^4}\int\limits_{S_{x',2h}} \operatorname{dist}^2(F(z',x_3),SO(2))dz'. \qquad (91)
\end{aligned}
$$

Integrating this inequality over $S_{x',h}\times I$ yields

$$
\int\limits_{S_{x',h}\times I} \left|\nabla'\hat{T}_h(z',x_3)\right|^2 dz'dx_3 \le \frac{C}{h^2}\int\limits_{S_{x',h}\times I} \operatorname{dist}^2(F(z',x_3),SO(2))dz'dx_3. \qquad (92)
$$

For any point $\tilde{x}' \in S_{x',h}$ we have

$$
\begin{aligned}
&\left|\hat{T}_h(x',x_3)-\hat{T}_h(\tilde{x}',x_3)\right|^2 \\
\le&\left|\int\limits_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right)F(z',x_3)dz' - \int\limits_{S_{\tilde{x}',h}} h^{-2}\psi\left(\frac{\tilde{x}'-z'}{h}\right)F(z',x_3)dz'\right|^2 \\
\le&\left|\int\limits_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right)F(z',x_3)dz' - \int\limits_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right)Q_{x',x_3,h}dz'\right|^2 \\
&+\left|\int\limits_{S_{\tilde{x}',h}} h^{-2}\psi\left(\frac{\tilde{x}'-z'}{h}\right)F(z',x_3)dz' - \int\limits_{S_{\tilde{x}',h}} h^{-2}\psi\left(\frac{\tilde{x}'-z'}{h}\right)Q_{x',x_3,h}dz'\right|^2 \\
\le&\frac{C}{h^2}\int\limits_{S_{x',2h}} \left|F(z',x_3)-Q_{x',x_3,h}\right|^2 dz' \\
\le&\frac{C}{h^2}\int\limits_{S_{x',2h}} \operatorname{dist}^2(F(z',x_3),SO(2))dz'. \qquad (93)
\end{aligned}
$$

Combining this with (27) and (90) yields

$$
\begin{aligned}
&\int\limits_{S_{x',h}} \left|\hat{T}_h(z',x_3)-F(z',z_3)\right|^2 dz' \\
\le&\int\limits_{S_{x',h}} \left|\hat{T}_h(z',x_3)-\hat{T}_h(x',x_3)\right|^2 dz' + \int\limits_{S_{x',h}} \left|\hat{T}_h(x',x_3)-Q_{x',x_3,h}\right|^2 dz'
\end{aligned}
$$

$$
\begin{aligned}
&+ \int\limits_{S_{x',h}} |Q_{x',x_3,h} - F(z',x_3)|^2\,dz' \\
&\leq C \int\limits_{S_{x',h}} \operatorname{dist}^2(F(z',x_3),SO(2))dz' \qquad (94)
\end{aligned}
$$

Finally, setting $g(\zeta) = \operatorname{dist}(\hat{T}_h(x'+h\zeta,x_3),SO(2))$ and taking into account (91), (94) we come to

$$
\begin{aligned}
&\int_{(0,1)^2} |g|^2 d\zeta + \sup_{(0,1)^2} |\nabla' g|^2\,dz' \\
&= \int\limits_{S_{x',h}} \operatorname{dist}^2(\hat{T}_h(z',x_3),SO(2))dz' + \sup_{S_{x',h}} |\nabla' \operatorname{dist}(\hat{T}_h(z',x_3),SO(2))|^2 \\
&\leq C\Bigg(\int\limits_{S_{x',h}} |\hat{T}_h(z',x_3) - F(z',x_3)|^2 dz' + \int\limits_{S_{x',h}} \operatorname{dist}^2(F(z',x_3),SO(2))dz' \\
&\quad + \sup_{S_{x',h}} |\nabla' \hat{T}_h(z',x_3)|^2 \Bigg) \\
&\leq \frac{C}{h^2} \int\limits_{S_{x',2h}} \operatorname{dist}^2(F(z',x_3),SO(2))dz'.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\operatorname{dist}^2(\hat{T}_h(\tilde{x}',x_3),SO(2)) &\leq \sup_{(0,1)^2} |g|^2 \leq C\Big(\int_{(0,1)^2} |g|^2 d\zeta + \sup_{(0,1)^2} |\nabla' g|^2\Big) \\
&\leq \frac{C}{h^2} \int\limits_{S_{x',2h}} \operatorname{dist}^2(F(z',x_3),SO(2))dz',
\end{aligned}
$$

and thus

$$
\begin{aligned}
\int\limits_{S_{x',h}} \operatorname{dist}^2(\hat{T}_h(\tilde{x}',x_3),SO(2))d\tilde{x}' &\leq \frac{C}{h^2} \int\limits_{S_{x',h}} \int\limits_{S_{\tilde{x}',2h}} \operatorname{dist}^2(F(z',x_3),SO(2))dz'd\tilde{x}' \\
&\leq \frac{C}{h^2} \int\limits_{S_{x',h}} \int\limits_{S_{x',4h}} \operatorname{dist}^2(F(z',x_3),SO(2))dz'd\tilde{x}' \\
&\leq C \int\limits_{S_{x',h}} \operatorname{dist}^2(F(z',x_3),SO(2))dz'. \qquad (95)
\end{aligned}
$$

Now we consider a square lattice of size h in $\mathbb{R}^2$ and sum the inequalities (92), (94) over all squares which intersect K. This yields

$$
\begin{aligned}
&\int\limits_{K\times I} \Big(|\hat{T}_h(x',x_3) - F(x',x_3)|^2 + h^2 |\nabla' \hat{T}_h(x',x_3)|^2 \Big)\,dx'dx_3 \qquad (96) \\
&\leq C \int\limits_{U\times I} \operatorname{dist}^2(F(x),SO(2))dx.
\end{aligned}
$$

Following [14] we consider S to be locally the epigraph of a Lipschitz function, then arguing as in [14] and applying the above estimates we get

$$\int\limits_{K\cap S\times I}\left(|\hat{T}_h(x',x_3)-F(x',x_3)|^2+h^2|\nabla'\hat{T}_h(x',x_3)|^2\right)dx'dx_3 \tag{97}$$
$$\leq C\int\limits_{U\times I}\operatorname{dist}^2(F(x),SO(2))dx.$$

Since S is Lipschitz $\bar{S}$ can be covered by a finite number of open sets $U_0,...,U_l$, where $\bar{U}_0\subset S$. Denote $\hat{T}_i$ the maps constructed in previous steps for T_i and consider a partition of unity correspondent to the cover $\{T_i\}$

$$\eta_i\in C_0^\infty(U_i),\qquad \eta_i\geq 0,\qquad \sum_{i=0}^{l}\eta_i(x')=1,\ \forall x'\in S.$$

If we set

$$\tilde{T}_h=\sum_{i=0}^{l}\eta_i\hat{T}_i \tag{98}$$

then

$$\tilde{T}_h-F=\sum_{i=0}^{l}\eta_i(\hat{T}_i-F),\qquad \text{and}\qquad \nabla'\tilde{T}_h=\sum_{i=0}^{l}\eta_i\nabla'\hat{T}_i+\sum_{i=0}^{l}\nabla'\eta_i(\hat{T}_i-F).$$

Consequently, applying (96) and (97) with $K_i=\operatorname{supp}\eta_i$ we obtain

$$\int\limits_{\Omega}\left(|\tilde{T}_h(x',x_3)-F(x',x_3)|^2+h^2|\nabla'\tilde{T}_h(x',x_3)|^2\right)dx\leq CE_h, \tag{99}$$

which proves (87) and (88). It follows from (95) that

$$\int\limits_{\Omega}\operatorname{dist}^2(\tilde{T}_h(x',x_3),SO(2))dx\leq CE_h,$$

and hence

$$\int\limits_{\Omega}|\tilde{T}_h(x',x_3)|^2dx\leq C\int\limits_{\Omega}(\inf_{P\in SO(2)}|\tilde{T}_h(x',x_3)-P|^2+1)\,dx\leq C(E_h+1)\leq C. \tag{100}$$

2 Now the rotation T_h can be obtained by projecting $\tilde{T}_h$ onto $SO(2)$. Since $SO(2)$ is a smooth manifold, there exists a tubular neighborhood $\mathcal{U}$ of $SO(2)$ such that the projection $\pi:\mathcal{U}\to SO(2)$ is smooth. Note that there exists a $\delta>0$ such that for all $M\notin\mathcal{U}$ there holds $\operatorname{dist}(M,SO(2))\geq\delta$. We define

$$T_h(x',x_3)=\tilde{\pi}(\tilde{T}_h(x',x_3))=\begin{cases}c\pi(\tilde{T}_h(x',x_3)) & \text{if } \tilde{T}_h(x',x_3)\in\mathcal{U},\\ Id & \text{if } \tilde{T}_h(x',x_3)\notin\mathcal{U}.\end{cases}$$

Then in case $\tilde{T}_h(x',x_3)\in\mathcal{U}$ we have $|T_h(x',x_3)-\tilde{T}_h(x',x_3)|=\operatorname{dist}(\tilde{T}_h(x',x_3),SO(2))$. If $\tilde{T}_h(x',x_3)\notin\mathcal{U}$ then $\operatorname{dist}^2(\tilde{T}_h(x',x_3),SO(2))dx>\delta^2$, and hence

$$|T_h(x',x_3)-\tilde{T}_h(x',x_3)|^2=|Id-\tilde{T}_h(x',x_3)|^2$$

$$\leq C(1+\operatorname{dist}^2(T_h(x',x_3),SO(2)))$$
$$\leq C\left(\frac{1}{\delta^2}+1\right)\operatorname{dist}^2(T_h(x',x_3),SO(2)).$$

Integration over Ω then yields

$$\int\limits_{\Omega}|T_h(x',x_3)-\tilde{T}_h(x',x_3)|^2dx \leq C\int\limits_{\Omega}\operatorname{dist}^2(\tilde{T}_h(x',x_3),SO(2))dx \leq CE_h,$$

which together with (87) gives (85).

3 Let Ω' be any compact subset of Ω and $|s|<\operatorname{dist}(\Omega',\partial\Omega)$. It follows from (89), integration by parts and Hölder's inequality that for any $s>0$

$$\begin{aligned}
&\left|\frac{\hat{T}_h(x',x_3+s)-\hat{T}_h(x',x_3)}{s}\right|^2\\
=&\left|\int\limits_{S_{x',h}} h^{-2}\psi\left(\frac{x'-z'}{h}\right)\frac{\nabla'(y^h)'(z',x_3+s)-\nabla'(y^h)'(z',x_3)}{s}dz'\right|^2\\
=&\left|\int\limits_{S_{x',h}} h^{-3}\nabla'\psi\left(\frac{x'-z'}{h}\right)\frac{(y^h)'(z',x_3+s)-(y^h)'(z',x_3)}{s}dz'\right|^2\\
\leq& Ch^{-4}\int\limits_{S_{x',h}}\left|\frac{(y^h)'(z',x_3+s)-(y^h)'(z',x_3)}{s}\right|^2dz'.
\end{aligned}\tag{101}$$

After integration of (101) over $S_{x',h}\times I$ we obtain

$$\begin{aligned}
&\int\limits_{S_{x',h}\times I}\left|\frac{\hat{T}_h(x',x_3+s)-\hat{T}_h(x',x_3)}{s}\right|^2dx\\
\leq& Ch^{-2}\int\limits_{S_{x',h}\times I}\left|\frac{(y^h)'(x',x_3+s)-(y^h)'(x',x_3)}{s}\right|^2dx.
\end{aligned}$$

Therefore, since Ω' is arbitrary, we get for the map $\tilde{T}_h$ defined in (98) that

$$\left\|\frac{\tilde{T}_h(x',x_3+s)-\tilde{T}_h(x',x_3)}{s}\right\|^2_{L^2(\Omega)} \leq \frac{C}{h^2}\left\|\frac{(y^h)'(x',x_3+s)-(y^h)'(x',x_3)}{s}\right\|^2_{L^2(\Omega)}.$$

Passing to the limit $s\to 0$ and using (81) we obtain

$$\|\partial_3\tilde{T}_h(x',x_3)\|^2_{L^2(\Omega)} \leq \frac{C}{h^2}\|\partial_3(y^h)'(x',x_3)\|^2_{L^2(\Omega)} \leq C\frac{E_h}{h^2}.$$

Combining this with (99) yields

$$\|\nabla\tilde{T}_h(x',x_3)\|^2_{L^2(\Omega)} \leq C\frac{E_h}{h^2}. \tag{102}$$

This implies that for the constant map $S_h' = \frac{1}{|\Omega|}\int\limits_\Omega \tilde{T}_h(x)dx$ we have

$$\|\int\limits_I \tilde{T}_h(x',x_3)dx_3 - S_h'\|^2_{L^2(S)} \le C\|\tilde{T}_h(x',x_3) - S_h'\|^2_{L^2(\Omega)} \le C\|\nabla\tilde{T}_h(x',x_3)\|^2_{L^2(\Omega)} \le C\frac{E_h}{h^2}. \tag{103}$$

Together with (24), (49) and (87) one may then derive

$$\begin{aligned}&|S_h' - P_h'|^2\\ \le& C(\|S_h' - \int\limits_I \tilde{T}_h(x',x_3)dx_3\|^2_{L^2(S)} + \|\int\limits_I \tilde{T}_h(x',x_3)dx_3 - \int\limits_I \nabla'(y^h)'(x)dx_3\|^2_{L^2(S)}\\ &+ \|\int\limits_I \nabla'(y^h)'(x)dx_3 - Q_h(x')\|^2_{L^2(S)} + \|Q_h(x') - P_h'\|^2_{L^2(S)})\\ \le& C\frac{E_h}{h^2}.\end{aligned} \tag{104}$$

Combining this with (72), (103) implies

$$\|\tilde{T}_h(x',x_3) - Id\|^2_{L^2(\Omega)} \le C(\|\tilde{T}_h(x',x_3) - S_h'\|^2_{L^2(\Omega)} + |S_h' - P_h'|^2 + |P_h' - Id|^2) \le C\frac{E_h}{h^2}. \tag{105}$$

By the Sobolev embedding one may then derive from (102) and (105) for any $1 \le p < \infty$ that

$$\|\tilde{T}_h(x',x_3) - Id\|^2_{L^p(\Omega)} \le C\|\tilde{T}_h(x',x_3) - Id\|^2_{W^{1,2}(\Omega)} \le C\frac{E_h}{h^2}. \tag{106}$$

In case $\tilde{T}_h(x',x_3) \notin U$, it follows from the definition of the term $T_h(x',x_3)$ that $\|T_h(x',x_3) - Id\|^2_{L^p(S)} = 0$. In case $\tilde{T}_h(x',x_3) \in U$ it holds

$$|T_h(x',x_3) - \tilde{T}_h(x',x_3)| = \operatorname{dist}(\tilde{T}_h(x',x_3), SO(2)) < \delta.$$

Hence, it holds for all $2 \le p < \infty$ that

$$|T_h(x',x_3) - Id|^p \le C\left(\operatorname{dist}^p(\tilde{T}_h(x',x_3), SO(2)) + |\tilde{T}(x',x_3) - Id|^p\right).$$

Consequently, it follows for any $2 \le p < \infty$ and any $x_3 \in I$

$$\begin{aligned}&\|T_h(x',x_3) - Id\|^p_{L^p(S)}\\ \le& C\left(\int\limits_S \operatorname{dist}^p(\tilde{T}_h(x',x_3), SO(2))dx' + \|\tilde{T}(x',x_3) - Id\|^p_{L^p(S)}\right).\end{aligned}$$

Integrating this inequality over I with respect to x_3 we infer

$$\begin{aligned}&\|T_h(x',x_3) - Id\|^2_{L^p(\Omega)}\\ \le& C\left(\int\limits_\Omega \operatorname{dist}^p(\tilde{T}_h(x',x_3), SO(2))dx\right)^{\frac{2}{p}} + C\|\tilde{T}_h(x',x_3) - Id\|^2_{L^p(\Omega)}\\ \le& C\max\left(E_h^{\frac{2}{p}}, \frac{E_h}{h^2}\right).\end{aligned} \tag{107}$$

□

3 Dimension reduction

We introduce the rotation

$$L_h = \begin{pmatrix} Q_h & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{108}$$

Lemma 3.1. *Let $S \in \mathbb{R}^2$ be a Lipschitz domain and $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$. Let y^h be a sequence such that $I^h(y^h) \le CE_h$, where*

$$\lim_{h\to 0} \frac{E_h}{h^4} = 1 \tag{109}$$

or

$$\lim_{h\to 0} \frac{E_h}{h^4} = 0. \tag{110}$$

Then there exists $G \in L^2(\Omega;\mathbb{R}^{3\times 3})$ such that it holds up to a non-relabeled subsequence

$$G_h := \frac{L_h^T \nabla_h y^h - Id}{(E_h)^{1/2}/h} \rightharpoonup G \quad \text{in } L^2(\Omega;\mathbb{R}^{3\times 3}) \text{ as } h \to 0. \tag{111}$$

Moreover, G has the form

$$G = \begin{pmatrix} 0 & 0 & G_{31} \\ 0 & 0 & G_{32} \\ \partial_1 v & \partial_2 v & G_{33} \end{pmatrix}$$

where G_{31} and G_{32} do not depend on x_3.

Proof. 1. For h small enough and under assumption (109) or (110) one can infer from (22), (49), (51), and (81) that

$$\begin{aligned} &\|G_h\|^2_{L^2(\Omega)} \\ \le& C\left(\left\| \frac{\nabla'(y^h)' - Q_h}{(E_h)^{1/2}/h} \right\|^2_{L^2(\Omega)} + \left\| \frac{\nabla' y_3}{(E_h)^{1/2}/h} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial_3 y'}{(E_h)^{1/2}} \right\|^2_{L^2(\Omega)} + \left\| \frac{\frac{\partial_3 y_3}{h} - 1}{(E_h)^{1/2}/h} \right\|^2_{L^2(\Omega)} \right) \\ \le& C. \end{aligned}$$

Therefore, (111) holds.

2. Let Ω' be any compact subset of Ω and $|s| < \operatorname{dist}(\Omega', \partial\Omega)$ and consider the difference quotients

$$H_h(x', x_3) := s^{-1}(G_h(x', x_3 + s) - G_h(x', x_3)).$$

Due to (111)

$$H_h \rightharpoonup H = s^{-1}(G(x', x_3 + s) - G(x', x_3)) \quad \text{in } L^2(\Omega';\mathbb{R}^{3\times 3}). \tag{112}$$

On the other hand,

$$H_h = L_h^T \frac{\nabla_h y^h(x', x_3 + s) - \nabla_h y^h(x', x_3)}{s(E_h)^{1/2}/h}.$$

It follows from (47) and (73) combined with the Sobolev embedding theorem that up to taking a subsequence it holds for all $1 \le p < \infty$ that

$$L_h \to Id \quad \text{in } L^p(S;\mathbb{R}^{3\times 3}). \tag{113}$$

Therefore,

$$L_hH_h = \frac{\nabla_h y^h(x', x_3+s) - \nabla_h y^h(x', x_3)}{s(E_h)^{1/2}/h} \rightharpoonup H \text{ in } L^1(\Omega';\mathbb{R}^{3\times3}). \tag{114}$$

On the other hand,

$$\begin{aligned} L_hH_h &= \frac{\nabla_h y^h(x', x_3+s) - \nabla_h y^h(x', x_3)}{s(E_h)^{1/2}/h} \\ &= \left(\frac{h^2}{(E_h)^{1/2}} \nabla' \frac{1}{s} \int_{x_3}^{x_3+s} \frac{1}{h}\partial_3 y^h(x',z)dz \;\middle|\; \frac{1}{(E_h)^{1/2}} \frac{1}{s} \int_{x_3}^{x_3+s} \partial_3^2 y^h(x',z)dz \right). \end{aligned} \tag{115}$$

By definition of I^h it holds $\int_\Omega \frac{1}{h^\varepsilon}|\partial_3^2 (y^h)'|^2\,dz \le CE_h$ and thus

$$\frac{1}{(E_h)^{1/2}}\partial_3^2 y'^h(x',z) \to (0,0)^T \quad \text{in } L^2(\Omega';\mathbb{R}^2).$$

Next, using (81) one deduces that

$$\frac{h^2}{(E_h)^{1/2}}\nabla' \frac{1}{h}\partial_3 y^h(x',z) \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{in } W^{-1,2}(\Omega';\mathbb{R}^{2\times3}).$$

Therefore, since Ω' is arbitrary it follows from (113),(114) and (115) that all the entries of H are equal to zero, except maybe H_{33}, consequently, the entries of $G(x',x_3)$ do not depend on x_3, except maybe G_{33}.

3. Now we notice that $G'_h = \frac{Q_h^T\nabla'(y^h)' - Id}{(E_h)^{1/2}/h}$ and that (111) yields

$$\int_I G'_h(x',x_3)dx_3 \rightharpoonup \int_I G'(x')dx_3 = G'(x'), \quad \text{in } L^2(S;\mathbb{R}^{2\times2}). \tag{116}$$

Then, due to (24)

$$\begin{aligned} &\int_I G'_h(x',x_3)dx_3 \\ &= \int_I \frac{Q_h^T\nabla'(y^h)' - Id}{(E_h)^{1/2}/h}dx_3 = \frac{Q_h^T \int_I \nabla'(y^h)'dx_3 - Id}{(E_h)^{1/2}/h} \to 0 \quad \text{in } L^2(S;\mathbb{R}^{2\times2}), \end{aligned}$$

and therefore,

$$G' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Next, it holds for $\alpha = 1,2$

$$\int_I G^h_{3\alpha}dx_3 = h\int_I \partial_\alpha y_3/(E_h)^{1/2}dx_3 = \partial_\alpha v_h \rightharpoonup \partial_\alpha v \quad \text{in } L^2(S),$$

which together with (116) gives $G_{3\alpha} = \partial_\alpha v(x')$. □

Remark 3. *Let us denote*

$$\varphi^h_\alpha(x',x_3) = \frac{1}{(E_h)^{1/2}}\left(Q_h^T\partial_3 y'(x',x_3)\right)_\alpha \rightharpoonup \varphi_\alpha(x') = G_{\alpha 3} \quad \textit{in } L^2(S;\mathbb{R}^{3\times 3}).$$

Then

$$G = \begin{pmatrix} 0 & 0 & \varphi_1(x') \\ 0 & 0 & \varphi_2(x') \\ \partial_1 v(x') & \partial_2 v(x') & G_{33}(x) \end{pmatrix}.$$

Lemma 3.2. *Let the assumptions of Lemma 3.1 be satisfied. Then there exists $F \in L^2(\Omega;\mathbb{R}^{3\times 3})$ such that it holds up to a non-relabeled subsequence*

$$F_h := \frac{Q_h^T \int_I \nabla'(y^h)' dx_3 - Id}{(E_h)^{1/2}} \rightharpoonup F(x') \quad \textit{in } L^2(\Omega;\mathbb{R}^{3\times 3}) \tag{117}$$

as $h \to 0$ and

$$\operatorname{sym} F_h \rightharpoonup \operatorname{sym}\nabla' u(x') \quad \textit{in } L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{118}$$

where u is defined in (74)*.*

Proof. It follows from (24) that

$$\|F_h\|^2_{L^2(\Omega)} \leq C \left\| \frac{\int_I \nabla'(y^h)' dx_3 - Q_h}{(E_h)^{1/2}} \right\|^2_{L^2(\Omega)} \leq C,$$

which implies(117).

Now we rewrite F_h as

$$F_h = \frac{\int_I \nabla'(y^h)' dx_3 - Id}{(E_h)^{1/2}} - \frac{Q_h - Id}{(E_h)^{1/2}} + (Q_h - Id)^T \frac{\int_I \nabla'(y^h)' dx_3 - Q_h}{(E_h)^{1/2}} \tag{119}$$

and set $A_h := \frac{Q_h - Id}{(E_h)^{1/4}}$. Using that $Q_h \in SO(2)$ we observe that

$$A_h^T A_h = \frac{(Q_h - Id)^T}{(E_h)^{1/4}}\frac{(Q_h - Id)}{(E_h)^{1/4}} = -\frac{2}{(E_h)^{1/4}}\operatorname{sym} A_h = -2\operatorname{sym}\frac{(Q_h - Id)}{(E_h)^{1/2}}. \tag{120}$$

It follows from (47), (73) and the Sobolev embedding that

$$A_h \to A \quad \text{in } L^p(S;\mathbb{R}^{2\times 2}), \quad 1 \leq p < \infty. \tag{121}$$

Combining (121) with (120) we get

$$\operatorname{sym} A_h \to 0 \quad \text{in } L^p(S;\mathbb{R}^{2\times 2}) \quad 1 \leq p < \infty. \tag{122}$$

Therefore, it follows from (121) and (122) that $\operatorname{sym} A = 0$. It remains to identify A_{12}. It follows from (24), (73) and (74) that

$$A_{12} = \lim_{h\to 0}(E_h)^{1/4} u_h = 0 \quad \text{in } L^p(S) \quad 1 \leq p < \infty, \tag{123}$$

which implies that $A = 0$. Then, (121) yields

$$\operatorname{sym}\frac{(Q_h - Id)}{(E_h)^{1/2}} \to -\frac{A^2}{2} = 0 \quad \text{in } L^p(S;\mathbb{R}^{2\times 2}) \quad 1 \le p < \infty. \tag{124}$$

Now we estimate the third term in (119). By (24) and (73) we have

$$\begin{aligned}
&\left\| (Q_h - Id)^T \frac{\int_I \nabla'(y^h)' dx_3 - Q_h}{(E_h)^{1/2}} \right\|_{L_1(\Omega)} \\
&\le \|Q_h - Id\|_{L^2(\Omega)} \left\| \frac{\int_I \nabla'(y^h)' dx_3 - Q_h}{(E_h)^{1/2}} \right\|_{L^2(\Omega)} \le C\frac{\sqrt{E_h}}{h} \to 0,
\end{aligned}$$

which together with (74), (124) and (117) yields (118). □

Lemma 3.3. *Let the assumptions of Lemma 3.1 be satisfied. Define the function $K \in L^2(\Omega;\mathbb{R}^{2\times 2})$ as $K(x', x_3) := x_3\nabla'\varphi(x')$ where φ is defined in* (3)*. Then it holds up to to a non-relabeled subsequence as $h \to 0$*

$$K_h := \frac{Q_h^T(\nabla'(y^h)' - \int_I \nabla'(y^h)' dx_3)}{(E_h)^{1/2}} \rightharpoonup K(x', x_3) \quad \textit{in} \;\; L^2(\Omega;\mathbb{R}^{2\times 2}). \tag{125}$$

Proof. We use the Dirichlet boundary condition in combination with Korn's inequality applied in the two-dimensional domains $S \times \{x_3\}$ to estimate

$$\begin{aligned}
\|K_h\| \le& C \left\| \frac{\nabla'(y^h)' - \int_I \nabla'(y^h)' dx_3}{(E_h)^{1/2}} \right\|^2_{L^2(\Omega)} \\
\le& C \left\| \frac{\operatorname{sym}(\nabla'(y^h)' - \int_I \nabla'(y^h)' dx_3)}{(E_h)^{1/2}} \right\|^2_{L^2(\Omega)} \\
\le& C \left\| \frac{\nabla'(y^h)' - T_h}{(E_h)^{1/2}} \right\|^2_{L^2(\Omega)} + C \left\| \frac{\operatorname{sym}(T_h - \int_I T_h dx_3)}{(E_h)^{1/2}} \right\|^2_{L^2(\Omega)} \\
\le& C \left\| \frac{\nabla'(y^h)' - T_h}{(E_h)^{1/2}} \right\|^2_{L^2(\Omega)} + C \left\| \frac{\operatorname{sym}(T_h - Id)}{(E_h)^{1/2}} \right\|^2_{L^2(\Omega)},
\end{aligned} \tag{126}$$

where T_h is the function from Theorem 2.7. The first term on the right-hand side is bounded due to (85). Therfore, it is left to show that also the second term is bounded. We set $C_h' := \frac{T_h(x',x_3) - Id}{(E_h)^{1/4}}$ and compute

$$\begin{aligned}
(C_h')^T C_h' &= \frac{T_h(x',x_3)^T - Id}{(E_h)^{1/4}} \cdot \frac{T_h(x',x_3) - Id}{(E_h)^{1/4}} \\
&= -\frac{1}{(E_h)^{1/4}} \left(\frac{T_h(x',x_3)^T - Id}{(E_h)^{1/4}} + \frac{T_h(x',x_3) - Id}{(E_h)^{1/4}} \right)
\end{aligned}$$

$$= \quad -2\frac{1}{(E_h)^{1/4}}\,\mathrm{sym}\, C_h'.$$

Hence, we may estimate using (86) and Hölder's inequality

$$\left\|\frac{\mathrm{sym}(T_h - Id)}{(E_h)^{1/2}}\right\|^2_{L^2(\Omega)} \leq \left\|\frac{\mathrm{sym}(T_h - Id)}{(E_h)^{1/4}}\right\|^4_{L^4(\Omega)} \leq C\max\left\{1, \frac{E_h}{h^4}\right\} \leq C.$$

Let now again Ω' be any compact subset of Ω and $|s| < \mathrm{dist}\,(\Omega', \partial\Omega)$. We introduce the difference quotients

$$M_h(x', x_3) := s^{-1}(K_h(x', x_3 + s) - K_h(x', x_3)).$$

Due to (125)

$$M_h \rightharpoonup M = s^{-1}(K(x', x_3 + s) - K(x', x_3)) \quad \text{in} \quad L^2(\Omega'; \mathbb{R}^{2\times 2}). \tag{127}$$

On the other hand, we have by Remark 3

$$\begin{aligned} M_h &= Q_h^T \frac{\nabla'(y^h)'(x', x_3 + s) - \nabla'(y^h)'(x', x_3)}{s(E_h)^{1/2}} \\ &= \frac{1}{(E_h)^{1/2}}\nabla' \frac{1}{s} Q_h^T \int_{x_3}^{x_3+s} \partial_3 (y^h)'(x', z)dz \rightharpoonup \nabla'\varphi(x') \quad \text{in } W^{-1,2}(\Omega'; \mathbb{R}^{2\times 2}). \end{aligned}$$

Since Ω' is arbitrary, the claimed form of K follows. □

Now we prove our main result.

Proof of Theorem 1.1. (i) Let $E_h = h^\sigma$. It follows from Theorem 2.2, Lemmas 2.5, 2.6, 3.3 and Remark 3 that there exist rotations $R_h(x') : S \to SO(3)$ and $Q_h(x') : S \to SO(2)$ such that

$$\begin{aligned} &\|\nabla_h y(x', x_3) - R_h(x')\|^2_{L^2(\Omega)} \leq Ch^{\sigma-2}, \; \|\nabla' R_h(x')\|^2_{L^2(S)} \leq Ch^{\sigma-4}, \\ &\|\int_I \nabla' y'(x)dx_3 - Q_h(x')\|^2_{L^2(S)} \leq Ch^\sigma, \; \|\nabla' Q_h(x')\|^2_{L^2(S)} \leq Ch^{\sigma-2}. \end{aligned}$$

We expand W_i around the identity, $W_i(Id + A) = \frac{1}{2}\mathcal{Q}_3^i(A) + \eta_i(A)$, where $\mathcal{Q}_3^i(A) = \frac{\partial^2 W_i(A)}{\partial A^2}(Id)(A, A)$ and $\eta_i(A)/|A|^2 \to 0$ as $|A| \to 0$. If now $\omega_i(t) = \sup_{|A|\leq t} |\eta_i(A)|$ we have

$$W_i(Id + A) \geq \frac{1}{2}\mathcal{Q}_3^i(A) - \omega_i(|A|). \tag{128}$$

With the notation $r_h := \frac{y_3^h}{(E_h)^{1/4}}$ it follows from (A4) that

$$\nabla' r_h \otimes \nabla' r_h \rightharpoonup b(x) \quad \text{in} \quad L^2(\Omega; \mathbb{R}^{2\times 2}). \tag{129}$$

If $\sigma > 4$, then (81) yields

$$\nabla' r_h \to 0 \quad \text{in} \quad L^2(\Omega; \mathbb{R}^2), \tag{130}$$

and therefore, $\nabla' r_h \otimes \nabla' r_h \to 0$ in $L^1(\Omega;\mathbb{R}^{2\times 2})$ and in view of (129)

$$\nabla' r_h \otimes \nabla' r_h \rightharpoonup 0 \quad \text{in} \quad L^2(\Omega;\mathbb{R}^{2\times 2}), \quad \text{if } \sigma > 4. \tag{131}$$

We estimate

$$\begin{aligned}
&\liminf_{h\to 0} \frac{1}{E_h} I^h(y^h) \\
\geq\ & \liminf_{h\to 0} \Big(\frac{1}{E_h} \int_\Omega W_1(\nabla'(y^h)'^T Q_h Q_h^T \nabla'(y^h)' + \nabla' y_3^h \otimes \nabla' y_3^h) dx \\
&+ \frac{h^2}{E_h} \int_\Omega W_2(L_h^T \nabla_h y^h) dx \Big) \\
=\ & \liminf_{h\to 0} \Big(\frac{h^2}{E_h} \int_\Omega W_2(Id + \frac{\sqrt{E_h}}{h} G_h) dx \\
&+ \frac{1}{E_h} \int_\Omega W_1((Id + \sqrt{E_h} F_h + \sqrt{E_h} K_h)^T (Id + \sqrt{E_h} F_h + \sqrt{E_h} K_h) \\
&+ \sqrt{E_h} \nabla' r_h \otimes \nabla' r_h) dx \Big) \\
=\ & \liminf_{h\to 0} \Big(\frac{h^2}{E_h} \int_\Omega W_2(Id + \frac{\sqrt{E_h}}{h} G_h) dx + \frac{1}{E_h} \int_\Omega W_1((Id + 2\sqrt{E_h}\,\mathrm{sym}\, F_h \\
&+ 2\,\mathrm{sym}\,\sqrt{E_h} K_h + E_h F_h^T F_h + E_h K_h^T K_h + 2E_h\,\mathrm{sym}(K_h^T F_h) \\
&+ \sqrt{E_h} \nabla' r_h \otimes \nabla' r_h) dx \Big).
\end{aligned}$$

Let us define χ_h as a characteristic function of the set

$$\begin{aligned}
\Omega_h = \Big\{ x \in \Omega :& |G_h| \leq 1/(E_h)^{1/8}, |K_h| \leq 1/(E_h)^{1/8}, |F_h| \leq 1/(E_h)^{1/8}, \\
& |\nabla' r_h| \leq 1/(E_h)^{1/8} \Big\}.
\end{aligned}$$

Then χ_h is bounded and $\chi_h \to 1$ in $L^1(\Omega)$, thus we have $\chi_h G_h \rightharpoonup G$, $\chi_h K_h \rightharpoonup K$, $\chi_h F_h \rightharpoonup F$, $\chi_h r_h \otimes r_h \rightharpoonup b(x')$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Therefore,

$$\begin{aligned}
&\liminf_{h\to 0} \frac{1}{E_h} I^h(y^h) \qquad (132)\\
\geq\ & \liminf_{h\to 0} (\frac{h^2}{E_h} \int_\Omega \chi_h W_2(Id + \frac{\sqrt{E_h}}{h} G_h) dx \\
&+ \frac{1}{E_h} \int_\Omega \chi_h W_1((Id + 2\sqrt{E_h}\,\mathrm{sym}\, F_h + 2\,\mathrm{sym}\,\sqrt{E_h} K_h \\
&+ E_h F_h^T F_h + E_h K_h^T K_h + 2E_h\,\mathrm{sym}(K_h^T F_h) + \sqrt{E_h} \nabla' r_h \otimes \nabla' r_h) dx) \\
\geq\ & \liminf_{h\to 0} \Big(\int_\Omega (\chi_h \frac{1}{2} \mathcal{Q}_3^2(G_h) - \frac{h^2}{E_h} \chi_h \omega_2(\frac{\sqrt{E_h}}{h} G_h)) dx
\end{aligned}$$

$$+\int_\Omega (\chi_h \frac{1}{2}\mathcal{Q}_3^1((2\operatorname{sym}F_h + 2\operatorname{sym}K_h + \nabla' r_h \otimes \nabla' r_h) - C\chi_h E_h(|F_h|^4 + |K_h|^4)$$
$$-\frac{1}{E_h}\chi_h\omega_1(2\sqrt{E_h}\operatorname{sym}F_h + 2\sqrt{E_h}\operatorname{sym}K_h + E_hF_h^TF_h + E_hK_h^TK_h$$
$$+2E_h\operatorname{sym}(K_h^TF_h) + \sqrt{E_h}\nabla' r_h \otimes \nabla' r_h))dx\Big). \tag{133}$$

It is easy to see that whenever $\chi_h \neq 0$, $\frac{\sqrt{E_h}}{h}|G_h| \leq \frac{E_h^{3/8}}{h} \to 0$ and

$$\frac{h^2}{E_h}\chi_h\omega_2\left(\left|\frac{\sqrt{E_h}}{h}G_h\right|\right) = \frac{\chi_h\omega_2(|\frac{\sqrt{E_h}}{h}G_h)|)}{|\frac{\sqrt{E_h}}{h}G_h|^2}|G_h|^2 \to 0. \tag{134}$$

Moreover,

$$E_h(|F_h|^4 + |K_h|^4) \leq \sqrt{E_h} \to 0. \tag{135}$$

If we denote

$$U_h = 2\operatorname{sym}F_h + 2\operatorname{sym}K_h + \sqrt{E_h}F_h^TF_h + \sqrt{E_h}K_h^TK_h + 2\sqrt{E_h}\operatorname{sym}K_h^TF_h$$
$$+ \nabla' r_h \otimes \nabla' r_h,$$

then

$$|\sqrt{E_h}U_h| = |2\sqrt{E_h}\operatorname{sym}F_h + 2\sqrt{E_h}\operatorname{sym}K_h + E_hF_h^TF_h + E_hK_h^TK_h$$
$$+2E_h\operatorname{sym}K_h^TF_h + \sqrt{E_h}\nabla' r_h \otimes \nabla' r_h| \leq C(E_h)^{1/4} \to 0$$

and

$$\frac{1}{E_h}\chi_h\omega_1\left(\left|\sqrt{E_h}\operatorname{sym}F_h + 2\sqrt{E_h}\operatorname{sym}K_h + E_hF_h^TF_h + E_hK_h^TK_h\right.\right.$$
$$\left.\left. +2E_h\operatorname{sym}K_h^TF_h + \sqrt{E_h}\nabla' r_h \otimes \nabla' r_h\right|\right)$$
$$\leq \chi_h\frac{\omega_1(|\sqrt{E_h}U_h|)}{|\sqrt{E_h}U_h|^2}(|2\operatorname{sym}F_h + 2\operatorname{sym}K_h + \nabla' r_h \otimes \nabla' r_h|^2 + \sqrt{E_h}) \to 0,$$

which together with (132)–(135) yields (18) in the case $\sigma > 4$.

(ii) We assume that v, u and φ are smooth and consider

$$\hat{y}^h(x', x_3) := \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^{\sigma/2}u(x') \\ h^{\sigma/2-1}v(x') \end{pmatrix} + x_3\begin{pmatrix} h^{\sigma/2}\varphi(x') \\ 1/2h^{\sigma/2}\mathcal{L}(\tilde{G}) \end{pmatrix}, \tag{136}$$

where $c = \mathcal{L}(\tilde{G})$ is the element which realizes the minimum of $\mathcal{Q}_2^2$, i.e.

$$\mathcal{Q}_2^2(\tilde{G}) = \mathcal{Q}_3^2(\tilde{G} + ce_3 \otimes e_3)).$$

Then

$$\nabla_h\hat{y}^h = \begin{pmatrix} Id + h^{\sigma/2}\nabla' u + x_3h^{\sigma/2}\nabla'\varphi & h^{\sigma/2-1}\varphi \\ h^{\sigma/2-1}\nabla' v & 1 + 1/2h^{\sigma/2-1}\mathcal{L}(\tilde{G}) \end{pmatrix}$$
$$+\begin{pmatrix} o(h^{\sigma/2}) & o(h^{\sigma/2-1}) \\ o(h^{\sigma/2-1}) & o(h^{\sigma/2-1}) \end{pmatrix}$$

and

$$(\nabla_h \hat{y}^h)^T \nabla_h \hat{y}^h = \begin{pmatrix} Id + 2h^{\sigma/2}\operatorname{sym}\nabla' u + 2x_3 h^{\sigma/2}\operatorname{sym}\nabla'\varphi & h^{\sigma/2-1}(\varphi+\nabla' v) \\ h^{\sigma/2-1}(\varphi+\nabla' v) & 1 + h^{\sigma/2-1}\mathcal{L}(\tilde{G}) \end{pmatrix} + \begin{pmatrix} o(h^{\sigma/2}) & o(h^{\sigma/2-1}) \\ o(h^{\sigma/2-1}) & o(h^{\sigma/2-1}) \end{pmatrix}.$$

Finally, using the Taylor expansion we obtain

$$\frac{1}{h^\sigma} I^h \to \frac{1}{2}\int_\Omega \mathcal{Q}_2^2(\tilde{G})dx' + \frac{1}{2}\int_S \mathcal{Q}_3^1(2\operatorname{sym}\nabla' u)dx' + \frac{1}{6}\int_S \mathcal{Q}_3^1(\operatorname{sym}\nabla'\varphi(x'))dx'.$$

For general $u, \varphi, v \in W^{1,2}(S)$ the assertion follows by suitable smooth approximations. □

Using Theorem 1.1 and arguing as in [14, Theorem 2] one can prove Theorem 1.2 .

4 The case $\sigma = 4$.

In case $\sigma = 4$ there is a lack of compactness for the sequence v^h. Due to this fact, it seems impossible to perform a limit transition in the nonlinear term $\nabla' r_h \otimes \nabla' r_h$. This creates obstacles in the derivation of a Γ-limit via v, u and φ. In this case we begin with the energy functionals containing second gradient terms (see, e.g. [24])

$$I_2^h(y) = \int_\Omega W_1(\nabla' y'(x)^T \nabla' y'(x) + \nabla' y_3 \otimes \nabla' y_3(x))dx + h^2 \int_\Omega W_2(\nabla_h y(x))dx + l h^2 \int_\Omega |\nabla_h^2 y(x)|^2 dx + c_1 \int_\Omega |\nabla' y_3(x)|^4 dx.$$

Theorem 4.1. *Suppose that the assumptions of Theorem 1.1 are satisfied.*

(i) (Compactness and lower bound)
If

$$\limsup_{h\to 0} \frac{1}{h^4} I_2^h(y^h) < \infty, \tag{137}$$

then properties (14)–(17) *are satisfied with* $\sigma = 4$. *Moreover,*

$$(v^h)(x') = \frac{1}{h}\int_I y_3^h dx_3 \rightharpoonup v \quad \text{in } W^{2,2}(S), \tag{138}$$

and

$$\liminf_{h\to 0} \frac{1}{h^\sigma} I_2^h(y^h) \geq \Bigg(\frac{1}{2}\int_S \mathcal{Q}_2^2(\tilde{G})dx' + \frac{1}{6}\int_S \mathcal{Q}_3^1(\operatorname{sym}\nabla'\varphi(x'))dx_1 + \frac{1}{2}\int_S \mathcal{Q}_3^1((2\operatorname{sym}\nabla' u(x') + \nabla' v \otimes \nabla' v(x')) + l\int_S |\nabla'^2 v(x')|^2 dx' + l\int_S |\nabla'\varphi(x')|^2 dx' + c_1 \int_S |\nabla' v(x')|^4 dx' \Bigg).$$

(ii) (Optimality of lower bound.)
If $u, \varphi \in W^{1,2}(S)$, $v \in W^{2,2}(S)$ *then there exists* $\hat{y}^h$ *such that* (81) *and* (14)–(17) *hold and*

$$\liminf_{h\to 0} \frac{1}{E_h} I^h(\hat{y}^h) = \left(\frac{1}{2} \int\limits_S \mathcal{Q}_2^2(\tilde{G})dx' + \frac{1}{6} \int\limits_S \mathcal{Q}_3^1(\operatorname{sym}\nabla'\varphi(x'))dx_1 \right.$$
$$+\frac{1}{2} \int\limits_S \mathcal{Q}_3^1(2\operatorname{sym}\nabla' u(x') + \nabla' v \otimes \nabla' v(x')) + l \int\limits_S |\nabla'^2 v(x')|^2 dx'$$
$$\left. + l \int\limits_S |\nabla'\varphi(x')|^2 dx' + c_1 \int\limits_S |\nabla' v(x')|^4 dx' \right).$$

Proof. Arguing as in Theorem 1.1 one can show (81) and (14)–(17). It follows from (137) that

$$\int\limits_\Omega |\nabla_h^2 y(x)|^2 dx \le Ch^2.$$

In particular this means for $i = 1, 2$

$$\int\limits_\Omega |\frac{1}{h}\nabla'^2 y'(x)|^2 dx \le C, \qquad \int\limits_\Omega |\frac{1}{h}\nabla'^2 y_3(x)|^2 dx \le C, \tag{139}$$
$$\int\limits_\Omega |\frac{1}{h^2}\partial_i\partial_3 y'(x)|^2 dx \le C, \qquad \int\limits_\Omega |\frac{1}{h^2}\partial_i\partial_3 y_3(x)|^2 dx \le C, \tag{140}$$
$$\int\limits_\Omega |\frac{1}{h^3}\partial_3^2 y'(x)|^2 dx \le C, \qquad \int\limits_\Omega |\frac{1}{h^3}\partial_3^2 y_3(x)|^2 dx \le C. \tag{141}$$

It follows from the second estimate in (140) and the first estimate in (141) that

$$\frac{1}{h}\partial_i\partial_3 y_3(x) \to 0 \text{ in } L^2(\Omega) \qquad \text{and} \tag{142}$$
$$\frac{1}{h^2}\partial_3^2 y'(x) \to 0 \text{ in } L^2(\Omega). \tag{143}$$

Taking into account (143) and arguing as in Theorem 1.1 we get (16) which together with (73) yields

$$\frac{1}{h^2}\partial_3 y'(x) = \frac{1}{h^2}(I - Q_h^T)\partial_3 y'(x) + \frac{1}{h^2} Q_h^T \partial_3 y'(x) \rightharpoonup \varphi(x') \quad \text{in } L^1(\Omega).$$

Consequently,

$$\frac{1}{h^2}\nabla'\partial_3 y'(x) \rightharpoonup \nabla'\varphi(x') \quad \text{in } W^{-1,1}(\Omega).$$

Therefore, (144) together with the first estimate in (140) we obtain

$$\frac{1}{h^2}\nabla'\partial_3 y'(x) \rightharpoonup \nabla'\varphi(x') \quad \text{in } L^2(\Omega). \tag{144}$$

Moreover, from (142) we have

$$\|\frac{1}{h}\nabla' y_3^h - \frac{1}{h}\int\limits_I \nabla' y_3^h dx_3\|_{L^2(\Omega)} \le C\|\frac{1}{h}\nabla'\partial_3 y_3^h\|_{L^2(\Omega)} \to 0. \tag{145}$$

Consequently, (15) and (146) yield

$$\frac{1}{h}\nabla'^2 y_3^h \rightharpoonup \nabla'^2 v(x') \quad \text{in } W^{-1,2}(\Omega),$$

which together with the second estimate in (139) leads to

$$\frac{1}{h}\nabla'^2 y_3^h \rightharpoonup \nabla'^2 v(x') \quad \text{in } L^2(\Omega). \tag{146}$$

Next, (14), (17), (73) imply

$$\frac{1}{h}(\nabla' y'^h - Id) \rightharpoonup 0 \quad \text{in } L^1(\Omega).$$

Consequently, it follows from the first estimate in (139) that

$$\frac{1}{h}\nabla'^2 y'^h \rightharpoonup 0 \quad \text{in } L^2(\Omega). \tag{147}$$

Convergence (138) follows from (146). This yields $\frac{1}{h^2}\nabla' y_3^h \otimes \nabla' y_3^h \to \nabla' v' \otimes \nabla' v'$ in $L^2(\Omega)$. Then we choose as a recovery sequence (136) and take into consideration that the second term in (140) and terms in (141) are equal to zero on the recovery sequence. Arguing as in Theorem 1.1 and using (144), (146), (147) we conclude the statement of the theorem. □

In this case the limiting functional contains a superfluous last term which is not a part of classical nonlinear Reissner–Mindlin model. It is an open question whether the nonlinear model can be derived without this term. In general, the nonlinear Reissner–Mindlin model lacks compactness, i.e. the nonlinearity is supercritical.

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