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Derivation of a thermo-visco-elastic plate model at small strains

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Abstract

We investigate a three-dimensional thermo-visco-elastic model with Kelvin-Voigt rheology under small strains confined to a thin domain. The model comprises a quasistatic linear momentum equation, with viscous stresses adhering to a Kelvin-Voigt viscosity law, coupled with a nonlinear heat equation governing temperature. The heat equation incorporates source terms arising from viscous dissipation and adiabatic heat sources due to thermal expansion. The model ensures thermodynamic consistency, maintaining energy conservation, positive temperature, and entropy production. We analyze the asymptotic behavior of solutions as the domain thickness approaches zero, deriving an effective two-dimensional model. This derivation involves rescaling the domain to a fixed thickness and establishing uniform a priori estimates relative to the plate's thickness. In the limit, the temperature becomes vertically constant, and displacement are of Kirchhoff-Love type, enabling meaningful interpretation of the limiting objects within the plate's two-dimensional cross-section. The mechanical equations consist of two parabolic equations, one for the membrane part and one for the bending part. Notably, the viscosity law in the limiting model departs from the Kelvin-Voigt form, reflecting nontrivial kinematic constraints on the rescaled out-of-plane strains. The bending of the plate does not depend on the temperature in the limit.

1 Introduction

The derivation of lower-dimensional theories for thin structures such as rods, plates, or shells is a classical problem in continuum mechanics. The first rigorous result for the plane membrane system and Kirchhoff's plate equation was given in [Mor59]. Subsequent contributions, both in the linear and nonlinear setting, addressed rods, plates, and shells; see, for example, [CiD79, Mie88, CiL89, Cia22, LeR00]. For static problems, rigorous dimension reduction results based on Γ-convergence were established in the early 1990s (see e.g. [ABP91]). These have since been refined and extended to encompass a wide range of mechanical behaviors, material responses, and coupled problems. For a non-exhaustive list of such developments, we refer the interested reader to [FJM02, FJM06, BLS16, ALL19, Pad22, BG*23]. In the time-dependent setting, the derivation of reduced models typically relies either on the fact that the system can be reformulated as a (generalized) gradient system—as is the case in viscoelasticity and elastoplasticity [FrK20, LiM11]—thereby enabling the use of evolutionary Γ-convergence, or on the availability of explicit solution formulas [Lic13].

In this work, we study the behavior of solutions to a thermo-visco-elastic model with Kelvin–Voigt rheology at small strains, formulated on a thin three-dimensional plate occupying the domain $\Omega_\varepsilon = \omega \times (-\varepsilon/2, +\varepsilon/2)$, as the thickness $\varepsilon > 0$ tends to zero. Our goal is to derive an effective two-dimensional model in the limit $\varepsilon \to 0$. The three-dimensional system of partial differential equations (PDEs) is based on a thermodynamically consistent model discussed extensively in the works of ROUBÍČEK; see in particular [KrR19, Chapter 8].

Neglecting inertial effects, the system consists of two coupled parabolic equations: a momentum bal-

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ance for the displacement u,

$$-\mathrm{div}(\mathbb{C}\mathsf{e}(u)+\mathbb{D}\mathsf{e}(\dot{u})+\theta\mathbb{B})=f(t,x),\quad \text{in }\Omega_\varepsilon$$

and the heat equation for the temperature θ

$$c_{\mathrm{v}}(\theta)\dot{\theta}-\mathrm{div}(\mathbb{K}(\theta)\nabla\theta)=Q+\mathbb{D}\mathsf{e}(\dot{u}):\mathsf{e}(\dot{u})+\theta\mathbb{B}:\mathsf{e}(\dot{u}),\quad \text{in }\Omega_\varepsilon.$$

The system is supplemented by mixed boundary conditions for u, inhomogeneous Neumann boundary conditions for θ, and appropriate initial data. Our main objective is to establish convergence of the family of solutions $(u_\varepsilon,\theta_\varepsilon)_{\varepsilon>0}$ to limit quantities that can be interpreted on the two-dimensional midplane ω of the plate. Moreover, we aim to characterize the resulting effective model in the case of general anisotropic materials. We emphasize that the model does not possess a gradient-flow structure (cf. [Mie11]), so that techniques based on evolutionary Γ-convergence for gradient systems, such as those in [Mie16], are not directly applicable in this setting.

While the existence theory for such systems is well developed–even in the finite-strain setting–see [Rou09, Rou10, BaR11, Rou13, MiR20, BFK23], a rigorous dimension reduction remains an open problem. In [BIF87], dimension reduction was addressed for a simplified, fully linear model by neglecting viscous effects and replacing the temperature in the adiabatic terms with a fixed one. Without these simplifications, we encounter a central difficulty. From the momentum balance, we know that the symmetric gradient $\mathsf{e}(\dot{u})$ is L^2-integrable, since u is assumed to lie in $\mathrm{H}^1(0,T;\mathrm{H}^1(\Omega;\mathbb{R}^d))$. However, this only yields L^1-integrability of the term $\mathbb{D}\mathsf{e}(\dot{u}):\mathsf{e}(\dot{u})$, which acts as an additional heat source. Consequently, we must rely on L^1-theory for the heat equation. In general, such theory provides integrability of θ only up to the exponent $\frac{d+2}{d+1}=\frac{5}{4}$ (with $d=3$ being the space dimension), which is insufficient to control the adiabatic term $\theta\mathbb{B}:\mathsf{e}(\dot{u})$, where at least L^2-integrability of θ is needed. To obtain the desired L^2-regularity for the temperature, we reformulate the system using the enthalpy transformation, as proposed in [BaR11] (see also [KrR19, Section 8.1]), and impose suitable growth conditions on the model coefficients.

The paper is organized as follows: Section 2 introduces the model and notation, and states the main results. In particular, we list the assumptions on the data and coefficients that will be used throughout the text. Furthermore, we introduce a reformulation of the system based on the thermal part of the internal energy (also referred to as *enthalpy* in [KrR19]), which we denote by w. For the dimension reduction, we also perform a rescaling of the domain and the displacement fields. In Section 3, we discuss the existence of solutions to the rescaled thermo-visco-elastic system, following the approach in [KrR19]. To this end, we first study a regularized version of the system, depending on a parameter $\delta>0$, where the heat sources are replaced by more regular approximations that admit better integrability. This allows us to apply L^2-theory for nonlinear parabolic equations. We establish uniform bounds that are independent of both the plate thickness ε and the regularization parameter δ. In particular, we derive a priori estimates for the enthalpy w following the strategy of [Rou09, BaR11], originally developed in [BoG89, BD*97]. The limit passage $\varepsilon\to 0$ is carried out in Section 4. A central difficulty here is that the Aubin–Lions lemma cannot be applied directly, as standard estimates for $\dot{w}_\varepsilon$ are not available due to the lack of admissible test functions in the heat equation. To overcome this, we decompose w_ε into the average across the plate thickness and a remainder. While the time derivative of the averaged part can be estimated uniformly, the remainder is shown to converge strongly to zero in suitable topologies.

In the limit, the thermal variable w becomes independent of the vertical coordinate, and the rescaled displacements converge to so-called Kirchhoff–Love type displacements:

$$u(x',x_3)=\big(U_1(x')-x_3\partial_{x_1}U_3(x'),\ U_2(x')-x_3\partial_{x_2}U_3(x'),\ U_3(x')\big)^\top.$$

The limit system is formulated in terms of the in-plane displacement $\overline{U} = (U_1, U_2)$, the out-of-plane displacement U_3, the thermal part of the internal energy w, and the out-of-plane strain variables $\varkappa_{\rm even}$ and $\varkappa_{\rm odd}$, all defined on the two-dimensional mid-surface domain $[0,T] \times \omega$.

The resulting two-dimensional limit model captures the interplay of membrane and bending effects in the elastic response, coupled with a nonlinear heat equation driven by dissipative and adiabatic contributions. It consists of the second-order membrane and fourth-order plate equations,

$$
\begin{aligned}
-\mathrm{div}'(\overline{\mathbb{C}}\,\mathsf{e}(\overline{U}) + \overline{\mathbb{D}}\,\mathsf{e}(\dot{\overline{U}}) + \mathbb{P}_{\mathbb{C}}^*\,\varkappa_{\rm even} + \mathbb{P}_{\mathbb{D}}^*\,\dot\varkappa_{\rm even} + \Theta(w)\,\overline{\mathbb{B}}) &= F_{1,2}(t,x'),\\
-\mathrm{div}'\mathrm{div}'(\overline{\mathbb{C}}\,\nabla^2 U_3 + \overline{\mathbb{D}}\,\nabla^2 \dot U_3 + \mathbb{P}_{\mathbb{C}}^*\,\varkappa_{\rm odd} + \mathbb{P}_{\mathbb{D}}^*\,\dot\varkappa_{\rm odd}) &= F_3(t,x') - \mathrm{div}'\,\widetilde{F}_3(t,x'),
\end{aligned}
$$

the effective heat equation,

$$
\dot w - \mathrm{div}'\big(\mathcal{K}_{\rm eff}(w)\nabla' w\big) = \mathcal{Q}_{\rm tot}\big(t, x';\, \mathsf{e}(\dot{\overline{U}}), \nabla^2 U_3, \dot\varkappa_{\rm even}, \dot\varkappa_{\rm odd}, \Theta(w)\big),
$$

and the evolution equations for the out-of-plane strains,

$$
\begin{aligned}
\dot\varkappa_{\rm even} + \mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}}\,\varkappa_{\rm even} &= -\mathbb{M}_{\mathbb{D}}^{-1}\big(\mathbb{P}_{\mathbb{C}}\,\overline{\mathsf{e}}(\overline{U}) + \mathbb{P}_{\mathbb{D}}\,\overline{\mathsf{e}}(\dot U_1, \dot U_2) + \Theta(w)\,\mathsf{b}_{\mathbb{B}}\big),\\
\dot\varkappa_{\rm odd} + \mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}}\,\varkappa_{\rm odd} &= -\tfrac{1}{12}\mathbb{M}_{\mathbb{D}}^{-1}\big(\mathbb{P}_{\mathbb{C}}\,\nabla^2 U_3 + \mathbb{P}_{\mathbb{D}}\,\nabla^2 \dot U_3\big).
\end{aligned}
$$

Here, the tensors $\overline{\mathbb{C}}$, $\overline{\mathbb{D}}$, $\mathbb{M}_{\mathbb{C}}$, $\mathbb{M}_{\mathbb{D}}$, $\mathbb{P}_{\mathbb{C}}$, and $\mathbb{P}_{\mathbb{D}}$ represent a decomposition of the original elasticity and viscosity tensors $\mathbb{C}$ and $\mathbb{D}$ into in-plane, out-of-plane, and mixed components; cf. (14). The body force densities $F_{1,2}$, F_3, and $\widetilde{F}_3$ (along with the corresponding surface force densities) are defined via suitable averages of the original three-dimensional body force f and heat source h. Finally, in Section 5, we discuss the special case of isotropic elasticity and address the thermodynamic consistency of the effective system. Moreover, we relate our limit model to the effective viscoelastic plate model derived in [Lic13], by rewriting the system using a memory kernel formulation.

To the best of our knowledge, this is the first rigorous derivation of a dimension-reduced model for a fully coupled, thermodynamically consistent thermo-visco-elastic system without gradient structure.

2 Setting and notation

Given a visco-elastic body that occupies the domain $\Omega \subset \mathbb{R}^d$, for $d = 3$, and a time horizon $0 < T < \infty$, we are interested in the evolution of the displacement $u : [0,T] \times \Omega \to \mathbb{R}^d$ and temperature $\theta : [0,T] \times \Omega \to \mathbb{R}$ under external forces and heat sources. The governing equations of our model consist of the linear momentum equation for the displacement

$$
-\mathrm{div}(\mathbb{C}\mathsf{e}(u) + \mathbb{D}\mathsf{e}(\dot u) + \theta\mathbb{B}) = f(t,x), \tag{1a}
$$

with $\mathsf{e}(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ denoting the linearized strain tensor, $\mathbb{C}$ denoting the fourth-order elasticity tensor, $\mathbb{D}$ is the fourth-order viscosity tensor, $\mathbb{B}$ is the second-order thermal expansion tensor, and $f : [0,T] \times \Omega \to \mathbb{R}^d$ being a volume force density acting in Ω. Note that we neglected inertial effects, i.e., the acceleration term $\rho_0 \ddot u$, with mass density ρ_0, does not appear on the left-hand side.

The equation for the displacement u is coupled to the heat equation for the temperature θ, viz.

$$
c_{\rm v}(\theta)\dot\theta - \mathrm{div}(\mathbb{K}(\theta)\nabla\theta) = Q + \mathbb{D}\mathsf{e}(\dot u) : \mathsf{e}(\dot u) + \theta\mathbb{B} : \mathsf{e}(\dot u), \tag{1b}
$$

where $c_\mathrm{v}(\theta)$ and $\mathbb{K}(\theta)$ denote the (temperature-dependent) heat capacity and conductivity, respectively, and Q is a fixed heat source density. Moreover, we used the notation $A : B = \sum_{i,j=1}^d A_{ij}B_{ij}$ for the scalar product of matrices $A, B \in \mathbb{R}^{d\times d}$.

The system is complemented by Dirichlet and Neumann boundary conditions for the displacement u

$$u = u_\mathrm{D}(x) \quad \text{on } (0,T)\times\Gamma_\mathrm{D}, \quad (\mathbb{C}\mathsf{e}(u) + \mathbb{D}\mathsf{e}(\dot u) + \theta\mathbb{B})\nu = h(t,x) \quad \text{on } (0,T)\times\Gamma_\mathrm{N}, \tag{1c}$$

where $\Gamma := \partial\Omega$, $\Gamma_\mathrm{D} \subset \Gamma$, $\Gamma_\mathrm{N} := \Gamma\setminus\Gamma_\mathrm{D}$ and $\nu : \Gamma \to S^{d-1}$ is the outer normal unit, with $u_\mathrm{D} : [0,T]\times\Gamma_\mathrm{D} \to \mathbb{R}^d$ being a fixed displacement on the Dirichlet part and $h : [0,T]\times\Gamma_\mathrm{N}\to\mathbb{R}^d$ denotes a surface force density.

For the heat equation, we assume inhomogeneous Neumann boundary conditions, namely

$$\mathbb{K}(\theta)\nabla\theta\cdot\nu = q(t,x) \quad \text{on } (0,T)\times\Gamma \tag{1d}$$

for some given boundary flux q.

Finally, we impose the initial conditions

$$u(0,\cdot) = u^0 \quad \text{in } \Omega, \qquad \theta(0,\cdot) = \theta^0 \quad \text{in } \Omega. \tag{1e}$$

2.1 Thermodynamic consistency

The model in (1a)–(1e) is thermodynamically consistent. Indeed, we introduce the specific Helmholtz free energy density $\psi : \mathbb{R}^{d\times d}_\mathsf{sym}\times(0,\infty)\to\mathbb{R}$ via the decomposition into purely mechanical, coupling, and purely thermal contributions, viz.

$$\psi(e,\theta) := \frac{1}{2}\mathbb{C}e : e + \theta\mathbb{B} : e - \psi_0(\theta). \tag{2}$$

The heat capacity is then given via $c_\mathrm{v}(\theta) = -\theta\partial^2_{\theta\theta}\psi(e,\theta) = \theta\psi_0''(\theta)$, which is only defined in terms of ψ_0 due to the linear dependence on θ of the coupling term and therefore only depends on θ.

The entropy density η and the internal energy E are given via

$$\eta(e,\theta) = -\partial_\theta\psi(e,\theta), \quad E(e,\theta) = \psi(e,\theta) + \theta\eta(e,\theta).$$

In particular, with ψ as in (2), we arrive at

$$\eta(e,\theta) = -\mathbb{B} : e + \psi_0'(\theta), \quad E(e,\theta) = \frac{1}{2}\mathbb{C}e : e - \psi_0(\theta) + \theta\psi_0'(\theta).$$

We call $W_0(\theta) = \theta\psi_0'(\theta) - \psi_0(\theta)$ the thermal part of the internal energy. Note that we have that $\psi_0(0)-\psi_0(\theta)+\theta\psi_0'(\theta) = \int_0^\theta c_\mathrm{v}(r)\,\mathrm{d}r$. The potential of dissipative forces due to viscous deformation is given via $\zeta(\dot e) := \frac{1}{2}\mathbb{D}\dot e : \dot e$. The system in (1a) and (1b) can then be written equivalently as

$$\begin{aligned} -\mathrm{div}\big[\partial_e E - \theta\partial_e\eta + \partial_{\dot e}\zeta\big] &= f(t,x), \\ \frac{\mathrm{d}}{\mathrm{d}t}E - \mathrm{div}(\mathbb{K}(\theta)\nabla\theta) &= Q + \big[\partial_e E - \theta\partial_e\eta + \partial_{\dot e}\zeta\big] : \mathsf{e}(\dot u). \end{aligned} \tag{3}$$

We derive an energy balance by testing the equation for linear momentum in (1a) with $\dot u$, to obtain after integrating by parts the mechanical energy balance

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathbb{C}\mathsf{e}(u) : \mathsf{e}(u) + \partial_{\dot e}\zeta(\mathsf{e}(\dot u)) : \mathsf{e}(\dot u)\,\mathrm{d}x = \int_\Omega f\cdot\dot u\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} h\cdot\dot u\,\mathrm{da} - \int_\Omega \theta\mathbb{B} : \mathsf{e}(\dot u)\,\mathrm{d}x. \tag{4}$$

Using the heat equation in (1b) for the last term on the right-hand side, we arrive at the energy balance

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\mathbb{C}\mathsf{e}(u) : \mathsf{e}(u) + W_0(\theta)\Big\}\,\mathrm{d}x = \int_\Omega f\cdot\dot u\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} h\cdot\dot u\,\mathrm{da} + \int_\Omega Q\,\mathrm{d}x + \int_\Gamma q\,\mathrm{da},$$

where we used that $\frac{\mathrm{d}}{\mathrm{d}t}W_0(\theta) = c_\mathrm{v}(\theta)\dot\theta$. This equality shows that the total energy $\mathcal{E}(u,\theta) = \int_\Omega E(\mathsf{e}(u),\theta)\,\mathrm{d}x$ of the system is conserved for vanishing external forces, fluxes, and heating.

On the other hand, we have for the entropy that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta(\mathsf{e}(u),\theta)\,\mathrm{d}x &= \int_\Omega \frac{Q + \partial_{\dot e}\zeta(\mathsf{e}(\dot u)) : \mathsf{e}(\dot u) + \mathrm{div}(\mathbb{K}(\theta)\nabla\theta)}{\theta}\,\mathrm{d}x\\ &= \int_\Omega \frac{Q + \partial_{\dot e}\zeta(\mathsf{e}(\dot u)) : \mathsf{e}(\dot u)}{\theta} + \frac{\mathbb{K}(\theta)\nabla\theta\cdot\nabla\theta}{\theta^2}\,\mathrm{d}x + \int_\Gamma \frac{q}{\theta}\,\mathrm{da}.\end{aligned}$$

Thus, for nonnegative Q, q and positive semidefinite $\mathbb{K}(\theta)$, we have positivity of the entropy production.

Remark 1. *From a mathematical perspective it is advantageous to choose either one of the quantities– internal energy E, entropy η, temperature θ, or, as below, the thermal part of the internal energy w–as the independent "thermal" variable τ and express the others in terms of the chosen, i.e., $E = \widehat{E}(e,\tau)$, $\eta = \widehat{\eta}(e,\tau)$, $\theta = \Theta(e,\tau)$ $w = \widehat{W}(e,\tau)$. In principle, all formulations will be equivalent, and the system takes the form*

$$\begin{aligned}-\mathrm{div}\big[\partial_e\widehat{E} - \Theta\partial_e\widehat{\eta} + \partial_{\dot e}\zeta\big] &= f(t,x),\\ \frac{\mathrm{d}}{\mathrm{d}t}\widehat{E} - \mathrm{div}(\mathbb{K}(\Theta)\nabla\Theta) &= Q + \big[\partial_e\widehat{E} - \Theta\partial_e\widehat{\eta} + \partial_{\dot e}\zeta\big] : \mathsf{e}(\dot u).\end{aligned}$$

2.2 Transformation to thermal part of internal energy

To derive suitable a priori estimates for the existence result as well as the limit passage, it is desirable to assume certain growth properties of the heat capacity $\theta \mapsto c_\mathrm{v}(\theta)$ to be able to treat the term $\theta\mathbb{B} : \mathsf{e}(\dot u)$ on the right-hand side of the heat equation. We follow here the ideas in [Rou10] and rewrite the system in terms of the so-called thermal part of internal energy (sometimes also called enthalpy) using the transformation

$$w = W(\theta) = \int_0^\theta c_\mathrm{v}(r)\,\mathrm{d}r, \quad \theta = \Theta(w) := W^{-1}(w) \text{ for } w \geq 0,$$

where we assumed positivity $c_\mathrm{v} > 0$ such that the primitive W of c_v (normalized such that $W(0) = 0$) is strictly increasing and therefore invertible. In particular, W differs from W_0 just by a constant, namely, $\psi_0(0)$. Note that we have $\frac{\mathrm{d}}{\mathrm{d}t}W(\theta(t)) = c_\mathrm{v}(\theta(t))\dot\theta(t)$. Therefore, by further setting $\mathcal{K}(w) := \mathbb{K}(\Theta(w))/c_\mathrm{v}(\Theta(w))$, we can rewrite the system in (1a)–(1e) as

$$-\mathrm{div}(\mathbb{C}(\mathsf{e}(u)) + \mathbb{D}(\mathsf{e}(\dot u)) + \Theta(w)\mathbb{B}) = f(t,x) \tag{5a}$$

$$\dot w - \mathrm{div}(\mathcal{K}(w)\nabla w) = Q(t,x) + \mathbb{D}\mathsf{e}(\dot u) : \mathsf{e}(\dot u) + \Theta(w)\mathbb{B} : \mathsf{e}(\dot u). \tag{5b}$$

The boundary conditions for the displacement are

$$u = u_\mathrm{D}(x) \text{ on } (0,T)\times\Gamma_\mathrm{D}, \quad (\mathbb{C}\mathsf{e}(u) + \mathbb{D}\mathsf{e}(\dot u) + \Theta(w)\mathbb{B})\nu = h(t,x) \text{ on } (0,T)\times\Gamma_\mathrm{N}, \tag{5c}$$

and the Neumann boundary condition in (1d) translate to

$$\mathcal{K}(w)\nabla w \cdot \nu = q(t,x) \quad \text{on } (0,T)\times\Gamma. \tag{5d}$$

Finally, the initial conditions now read

$$u(0,\cdot) = u^0 \quad \text{in } \Omega, \qquad w(0,\cdot) = w^0 \quad \text{in } \Omega. \tag{5e}$$

2.3 Existence of solutions

The following assumptions are sufficient for the existence of solutions to the thermo-visco-elastic system in (5):

(A1) The domain $\Omega \subset \mathbb{R}^3$ is open, bounded, connected and has a Lipschitz boundary $\Gamma = \partial\Omega$. It is decomposed disjointly into Dirichlet part Γ_D and Neumann part Γ_N such that $\Gamma_\mathrm{N} = \Gamma \setminus \Gamma_\mathrm{D}$. The Dirichlet part has positive surface measure $\mathcal{H}^2(\Gamma_\mathrm{D}) > 0$.

(A2) The elasticity and viscosity tensors $\mathbb{C} \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{3\times3\times3\times3})$ and $\mathbb{D} \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{3\times3\times3\times3})$ satisfy the symmetries $\mathbb{C}_{jikl} = \mathbb{C}_{ijkl} = \mathbb{C}_{klij}$ and $\mathbb{D}_{jikl} = \mathbb{D}_{ijkl} = \mathbb{D}_{klij}$ for $i,j,k,l = 1,\ldots,3$. Moreover, they are uniformly elliptic on $\mathbb{R}^{3\times3}_{\mathsf{sym}}$ such that

$$\exists \gamma_\mathbb{C} > 0, \gamma_\mathbb{D} > 0 \ : \quad \mathbb{C}(x)e : e \geq \gamma_\mathbb{C}|e|^2, \quad \mathbb{D}(x)\dot{e} : \dot{e} \geq \gamma_\mathbb{D}|\dot{e}|^2 \quad \text{a.e. in } \Omega.$$

(A3) The thermal expansion tensor $\mathbb{B} \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{3\times3})$ is symmetric;

(A4) The heat capacity is such that $c_\mathrm{v} \in \mathrm{C}^0(\mathbb{R};\mathbb{R})$, and there exist $c_0 > 0$, $s \geq 2$ with

$$\forall w \in \mathbb{R} : \quad c_0(1+|w|)^{s-1} \leq c_\mathrm{v}(w) \tag{6}$$

(A5) The heat conductivity satisfies $\mathbb{K} \in \mathrm{C}^0(\mathbb{R};\mathbb{R}^{d\times d})$ with

$$\mathcal{K} : \mathbb{R} \to \mathbb{R}^{d\times d}, \quad \text{where} \quad \mathcal{K}(w) := \frac{\mathbb{K}(\Theta(w))}{c_\mathrm{v}(\Theta(w))}$$

being bounded and uniformly elliptic;

(A6) The bulk and surface force densities in (5a) and (5c) satisfy $f \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^3))$ and $h \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma_\mathrm{N};\mathbb{R}^3))$. We define the loading $\ell \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathbb{R}^3)^*)$ via

$$\langle \ell(t), u\rangle := \int_\Omega f(t,x)\cdot u \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} h(t,x)\cdot u \,\mathrm{da}$$

(A7) The Dirichlet data satisfies $u_\mathrm{D} \in \mathrm{H}^1(\Omega;\mathbb{R}^3)$.

(A8) The heat source density and the boundary heat flux in (5b) and (5d) are non-negative and satisfy $Q \in \mathrm{L}^1(0,T;\mathrm{L}^1(\Omega))$ and $q \in \mathrm{L}^1(0,T;\mathrm{L}^1(\Gamma))$.

(A9) The initial values are such that $u_0 \in \mathrm{H}^1(\Omega;\mathbb{R}^d)$ with $u_0 - u_\mathrm{D} \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^d)$ and $w^0 \in \mathrm{L}^1(\Omega)$ with $w^0 \geq w^0_{\mathsf{min}} > 0$ a.e. in Ω.

Remark 2. *1 The growth condition for $\theta \mapsto c_{\mathrm{v}}(\theta)$ in Assumption (A4) implies that $\Theta(w) \leq C_0(1+w)^{1/s}$ for $w \geq 0$, some $C_0 \in [1,\infty)$. Moreover, the condition $s \geq 2$ can be relaxed to $s > 6/5$, see [KrR19]. However, more refined estimates are required in this case.*

2 Taking a time-independent Dirichlet data as in (A7) has the advantage that $\dot{u}$ of a weak solution (u,w) according to Definition 3 vanishes on $(0,T)\times\Gamma_{\mathrm{D}}$. In particular, $\dot{u}$ is an admissible test function for (7)*, i.e., the tests in Step 1 of Proposition 9 and* (31) *are valid. However, one may also consider a time-dependent Dirichlet data $u_{\mathrm{D}} \in \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega;\mathbb{R}^d))$ and then $\dot{u}-\dot{u}_{\mathrm{D}}$ has to be used instead.*

Definition 3 (Weak formulation of (5))**.** Let $r \in (1,\frac{5}{4})$ and assume that the Assumptions (A1)–(A9) are satisfied. A pair (u,w) of functions $u : [0,T]\times\Omega \to \mathbb{R}^d$, $w : [0,T]\times\Omega \to \mathbb{R}$ is said to be a *weak solution of the initial-boundary-value problem in* (5a)–(5e) if $u \in \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega;\mathbb{R}^d))$ and $u - u_{\mathrm{D}} \in \mathrm{H}^1(0,T;\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d))$ and $w \in \mathrm{L}^r(0,T;\mathrm{W}^{1,r}(\Omega))$ with $\dot{w} \in \mathrm{L}^1(0,T;(\mathrm{W}^{1,r'}(\Omega))^*)$, and if it satisfies the integral identity

$$\int_0^T\int_\Omega \big(\mathbb{C}\mathsf{e}(u)+\mathbb{D}\mathsf{e}(\dot{u})+\Theta(w)\mathbb{B}\big):\mathsf{e}(v)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\langle \ell(t),v\rangle\,\mathrm{d}t, \tag{7}$$

for all $v \in \mathrm{L}^2(0,T;\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d))$ together with $u(0,\cdot) = u_0$ in $\mathrm{H}^1(\Omega;\mathbb{R}^d)$, and if for all $\phi \in \mathrm{W}^{1,r'}(0,T;\mathrm{W}^{1,r'}(\Omega))$ with $\phi(T)=0$

$$\begin{aligned}\int_0^T\int_\Omega \big(-w\dot{\phi}+\mathcal{K}(w)\nabla w\cdot\nabla\phi\big)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega w^0(x)\phi(0,x)\,\mathrm{d}x \\ = \int_0^T\int_\Omega \big(Q+\mathbb{D}\mathsf{e}(\dot{u}):\mathsf{e}(\dot{u})+\Theta(w)\mathbb{B}:\mathsf{e}(\dot{u})\big)\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Gamma q\phi\,\mathrm{d}a\,\mathrm{d}t.\end{aligned} \tag{8}$$

Theorem 4 (Existence of solutions)**.** *Let the Assumptions (A1)–(A9) hold. Then the initial-boundary-value problem given by the system in* (5a)–(5e) *admits a weak solution in the sense of Definition 3. Moreover, we have that $\Theta(w) > 0$ and $w > 0$ a.e. in $[0,T]\times\Omega_1$.*

The proof is postponed to Section 3.

2.4 Convergence to effective plate model

For the dimension reduction to a plate model, we consider domains with a plate geometry, namely,

$$\Omega_\varepsilon = \omega\times(-\varepsilon/2,+\varepsilon/2) \text{ for } 0<\varepsilon\ll 1 \text{ and } \omega\subset\mathbb{R}^2.$$

In addition to the decomposition of the boundary into Dirichlet and Neumann part, we also use abbreviations to distinguish between vertical and horizontal part of the boundary in this setting. We set $\Gamma_\varepsilon^\perp := \partial\omega\times(-\varepsilon/2,+\varepsilon/2)$ for the vertical part and $\Gamma_\varepsilon^\parallel := \omega\times\{-\varepsilon/2,+\varepsilon/2\}$ for the upper and lower part of $\partial\Omega_\varepsilon$, and we further define the lower part $\Gamma_\varepsilon^- := \omega\times\{-\varepsilon/2\}$ and the upper part $\Gamma_\varepsilon^+ := \omega\times\{+\varepsilon/2\}$. Hence, altogether we have the disjoint decomposition of the boundary

$$\partial\Omega_\varepsilon = \Gamma_\varepsilon^\perp\cup\Gamma_\varepsilon^\parallel\cup\big(\partial\omega\times\{-\varepsilon/2,+\varepsilon/2\}\big) = \Gamma_\varepsilon^\perp\cup\Gamma_\varepsilon^-\cup\Gamma_\varepsilon^+\cup\big(\partial\omega\times\{-\varepsilon/2,+\varepsilon/2\}\big).$$

Finally, we assume that the Dirichlet part of the boundary is given in the form $\Gamma_\varepsilon^{\mathsf{D}} = \gamma_{\mathsf{D}}\times(-\varepsilon/2,+\varepsilon/2)$ for some subset $\gamma_{\mathsf{D}}\subset\partial\omega$ with positive 1-dimensional Hausdorff measure.

To pass to the limit $\varepsilon \to 0$ in a meaningful way, we need to interpret all functions as quantities defined on one and the same domain. To this end, we introduce the transformation

$$\Omega_\varepsilon \ni x = (x', x_3) \mapsto \tilde{x} := \mathbb{S}_\varepsilon x \in \omega \times (-1/2, +1/2) =: \Omega_1, \text{ where } \mathbb{S}_\varepsilon := \mathrm{diag}(1, 1, \varepsilon^{-1}).$$

Correspondingly, we have the following parts of the boundary

- vertical boundary $\Gamma_1^\perp := \partial\omega \times (-1/2, +1/2)$;
- horizontal boundary $\Gamma_1^\pm := \omega \times \{-1/2, +1/2\}$;
- upper boundary $\Gamma_1^+ := \omega \times \{+1/2\}$;
- lower boundary $\Gamma_1^- := \omega \times \{-1/2\}$;
- Dirichlet boundary $\Gamma_1^{\mathsf{D}} := \gamma_{\mathsf{D}} \times (-1/2, +1/2)$;
- Neumann boundary $\Gamma_1^{\mathsf{N}} := \partial\Omega_1 \backslash \Gamma_1^{\mathsf{D}}$ with horizontal and vertical parts $\Gamma_1^{\mathsf{N},\|} := \omega \times \{-1/2, 1/2\}$ and $\Gamma_1^{\mathrm{N},\perp} := (\partial\omega \setminus \gamma_{\mathrm{D}}) \times (-1/2, +1/2)$, respectively.

For a function $\phi : \Omega_\varepsilon \to \mathbb{R}$, we introduce the 'tilde' notation to denote by $\widetilde{\phi} : \Omega_1 \to \mathbb{R}$ the function in the new variables, i.e.

$$\widetilde{\phi}(\tilde{x}) := \phi(\mathbb{S}_\varepsilon^{-1}\tilde{x}) = \phi(x) \quad \text{such that} \quad \nabla_x \phi = \mathbb{S}_\varepsilon \nabla_{\tilde{x}} \widetilde{\phi}(\tilde{x}) =: \nabla_\varepsilon \widetilde{\phi}(\tilde{x}).$$

Furthermore, for vector-valued displacements $u : \Omega_\varepsilon \to \mathbb{R}^3$, we introduce the 'hat' notation to define rescaled displacements via

$$\widehat{u}(\tilde{x}) := \mathbb{S}_\varepsilon^{-1}\widetilde{u}(\tilde{x}) = \mathbb{S}_\varepsilon^{-1} u(\mathbb{S}_\varepsilon^{-1}\tilde{x}) \quad \text{such that} \quad \mathsf{e}_x(u) = \mathbb{S}_\varepsilon \mathsf{e}_{\tilde{x}}(\widehat{u})\mathbb{S}_\varepsilon =: \kappa_\varepsilon(\widehat{u})$$

Due to the form of the rescaled strain tensors $\kappa_\varepsilon(\widehat{u})$, it is known that limits of rescaled displacements $\widehat{u}_\varepsilon$ with bounded elastic energy have to lie in the space of Kirchhoff-Love displacements (see e.g. [Cia97]), which is defined by

$$\mathcal{V}_{\mathsf{KL}}(\Omega_1) := \big\{ v \in \mathrm{H}^1(\Omega_1; \mathbb{R}^d) \,\big|\, \mathsf{e}_{13}(v) = \mathsf{e}_{23}(v) = \mathsf{e}_{33}(v) = 0 \big\} \tag{9}$$

The space admits the following alternative characterization (cf. [Cia97, Part A, Section 1.4, Theorem 1.4-1 (c)])

Lemma 5 (Characterization of Kirchhoff–Love displacements)**.**

$$\mathcal{V}_{\mathrm{KL}}(\Omega_1) = \Big\{ \widehat{u} = (\widehat{u}_i)_{1\le i\le 3} \,\Big|\, \widehat{u}_i(\tilde{x}) = U_i(x_1, x_2) - \tilde{x}_3 \partial_{x_i} U_3(x_1, x_2) \textit{ for } i = 1, 2 \textit{ and } \\ \widehat{u}_3(\tilde{x}) = U_3(x_1, x_2) \textit{ for } U_i \in \mathrm{H}^1_{\gamma_{\mathrm{D}}}(\omega), U_3 \in \mathrm{H}^2_{\gamma_{\mathrm{D}}}(\omega) \Big\} \tag{10}$$

where $\mathrm{H}^2_{\gamma_{\mathrm{D}}}(\omega) := \{ U \in \mathrm{H}^2(\omega) \mid U = \nabla U \cdot \nu_\gamma = 0 \textit{ on } \gamma_{\mathrm{D}} \}$.

Moreover, due to the uniform ellipticity of $\mathcal{K}$, we expect limits of the solutions $\widetilde{w}_\varepsilon$ to live in the set

$$\mathcal{W}_0 := \{ \widetilde{w} \in \mathrm{W}^{1,r}(\Omega_1) \,|\, \partial_{x_3}\widetilde{w} = 0 \text{ a.e. } \Omega_1 \} \simeq \mathrm{W}^{1,r}(\omega). \tag{11}$$

We can now state the additional assumptions that we will use for carrying out the limit passage $\varepsilon \to 0$.

(B1) The domain $\omega \subset \mathbb{R}^2$ is open, bounded, connected, and has a Lipschitz boundary $\gamma = \gamma_\mathrm{D} \cup \gamma_\mathrm{N}$ such that $\mathcal{H}^1(\gamma_\mathrm{D}) > 0$.

(B2) The elasticity and the viscosity tensor $\mathbb{C}$ and $\mathbb{D}$ are spatially constant.

(B3) The thermal expansion tensor $\mathbb{B}$ is constant.

(B4) The bulk force density satisfies $f_\varepsilon(t,x) = \mathbb{S}_\varepsilon^{-1}\mathsf{f}_*(t,\tilde{x})$ for some $\mathsf{f}_* \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_1;\mathbb{R}^d))$ and the surface force density is such that

$$h_\varepsilon(t,x) = \begin{cases} \mathbb{S}_\varepsilon^{-1}\mathsf{h}_*^\perp(t,\tilde{x}) & \text{for } \tilde{x} \in \Gamma_1^{\mathrm{N},\perp}. \\ \varepsilon\mathbb{S}_\varepsilon^{-1}\mathsf{h}_*^\parallel(t,\tilde{x}) & \text{for } \tilde{x} \in \Gamma_1^{\mathrm{N},\parallel}, \end{cases}$$

where $\mathsf{h}_*^\perp \in \mathrm{L}^2((0,T)\times\Gamma_1^{\mathrm{N},\perp})$ and $\mathsf{h}_*^\parallel \in \mathrm{L}^2((0,T)\times\Gamma_1^{\mathrm{N},\parallel})$. We define the rescaled loading $\widehat{\ell}_* \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega_1;\mathbb{R}^3)^*)$ via

$$\langle\widehat{\ell}_*(t),\widehat{u}\rangle := \int_{\Omega_1}\mathsf{f}_*(t)\cdot\widehat{u}\,\mathrm{d}\tilde{x} + \int_{\Gamma_1^{\mathrm{N},\parallel}}\mathsf{h}_*^\parallel(t)\cdot\widehat{u}\,\mathrm{da} + \int_{\Gamma_1^{\mathrm{N},\perp}}\mathsf{h}_*^\perp(t)\cdot\widehat{u}\,\mathrm{da}.$$

(B5) The Dirichlet data is such that $u_{\mathrm{D},\varepsilon}(x) = \mathbb{S}_\varepsilon\widehat{u}_{\mathrm{D},\varepsilon}(\tilde{x}) := \mathbb{S}_\varepsilon\widehat{u}_\mathsf{D}^{(0)}(\tilde{x}) + \varepsilon\mathbb{S}_\varepsilon\widehat{u}_\mathsf{D}^{(1)}(\tilde{x}) + \varepsilon^2\mathbb{S}_\varepsilon\widehat{u}_\mathsf{D}^{(2)}(\tilde{x})$ for given functions $\widehat{u}_\mathsf{D}^{(0)} \in \mathcal{V}_\mathrm{KL}(\Omega_1)$, $\widehat{u}_\mathsf{D}^{(1)} \in V_{33}(\Omega_1)$, and $\widehat{u}_\mathsf{D}^{(2)} \in \mathrm{H}^1(\Omega_1;\mathbb{R}^d)$, where $V_{33}(\Omega_1) := \{(\widehat{u}_1,\widehat{u}_2,\widehat{u}_3) \in \mathrm{H}^1(\Omega_1;\mathbb{R}^d) \,|\, \partial_{\tilde{x}_3}\widehat{u}_3 = 0\}$.

(B6) The heat source density is such that $Q_\varepsilon(t,x) = \mathsf{Q}_*(t,\tilde{x})$ for some $\mathsf{Q}_* \in \mathrm{L}^1(0,T;\mathrm{L}^1(\Omega_1))$ and the boundary heat flux satisfies

$$q_\varepsilon(t,x) = \begin{cases} \mathsf{q}_*^\perp(t,\tilde{x}) & \text{for } \tilde{x} \in \Gamma_1^\perp \\ \varepsilon\mathsf{q}_*^\parallel(t,\tilde{x}) & \text{for } \tilde{x} \in \Gamma_1^\parallel, \end{cases}$$

for $\mathsf{q}_*^\perp \in \mathrm{L}^1(0,T;\mathrm{L}^1(\Gamma_1^\perp))$ and $\mathsf{q}_*^\parallel \in \mathrm{L}^1(0,T;\mathrm{L}^1(\Gamma_1^\parallel))$.

(B7) $c_\mathrm{v} \in \mathrm{C}^0(\mathbb{R};\mathbb{R})$ satisfies the growth condition (6) for $c_0 > 0$, $s \geq 2$.

(B8) The heat conductivity is such that $\mathbb{K} \in \mathrm{C}^0(\mathbb{R};\mathbb{R}^{d\times d})$, and $\mathcal{K}(w) = \frac{\mathbb{K}(\Theta(w))}{c_\mathrm{v}(\Theta(w))}$ is uniformly elliptic and bounded, i.e. there exists $\gamma_\mathcal{K} > 0$ such that $\mathcal{K}(w)\xi\cdot\xi \geq \gamma_\mathcal{K}|\xi|^2$ for any $w \in \mathbb{R}$, $\xi \in \mathbb{R}^3$, and $\sup_{w\in\mathbb{R}}|\mathcal{K}(w)| < \infty$.

(B9) The initial values are such that $u_{0,\varepsilon}(x) = \mathbb{S}_\varepsilon\widehat{u}_{0,\varepsilon}$ for $\widehat{u}_{0,\varepsilon} \in \mathrm{H}^1_{\Gamma_{\mathrm{D},1}}(\Omega_1;\mathbb{R}^3)$ with $\widehat{u}_{0,\varepsilon} \to \widehat{u}_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{D},1}}(\Omega_1;\mathbb{R}^3)$ and $\kappa_\varepsilon(\widehat{u}_{0,\varepsilon}) \to \kappa^0$ in $\mathrm{L}^2(\Omega_1;\mathbb{R}^{3\times3}_\mathrm{sym})$. The limit κ^0 is affine with respect to x_3. Finally, we assume that $w_{0,\varepsilon} = \widetilde{w}^0$ for some $\widetilde{w}^0 \in \mathrm{L}^1(\Omega_1)$ with $\partial_{x_3}\widetilde{w}^0 = 0$ a.e. in Ω_1.

Remark 6. *(i) The condition that $x_3 \mapsto \kappa^0(x',x_3)$ is affine for a.e. $x' \in \omega$ will be used to decompose the limiting strains $\kappa(t) = \lim_{\varepsilon\to0}\kappa_\varepsilon(\widehat{u}_\varepsilon(t))$ into an even and an odd part, see* (44)*.*

(ii) It is also possible to consider an initial value $\widehat{u}_{0,\varepsilon}$ that is minimizer of the functional $\mathcal{E}_\varepsilon(\widehat{u}_{0,\varepsilon}) = \int_{\Omega_1}\frac{1}{2}\mathbb{C}\kappa_\varepsilon(\widehat{u}):\kappa_\varepsilon(\widehat{u})\,\mathrm{d}x - \langle\ell_0,\widehat{u}\rangle$ for some given external loading ℓ_0. Then, it is well known, that minimizers $\widehat{u}_\varepsilon$ of $\mathcal{E}_\varepsilon$ converge to minimizers $\widehat{u}$ of the effective functional $\mathcal{E}_0$, where $\mathcal{E}_0(\widehat{u}) = \int_{\Omega_1}\frac{1}{2}\mathbb{C}_{KL}\overline{\mathsf{e}}(\widehat{u}):\overline{\mathsf{e}}(\widehat{u})\,\mathrm{d}x - \langle\ell_0,\widehat{u}\rangle$ if $\widehat{u}$ is a Kirchhoff–Love displacements and $\mathcal{E}_0(\widehat{u}) = +\infty$ otherwise, see [Cia97]. Here, $\overline{\mathsf{e}}(\widehat{u}) \in \mathbb{R}^{2\times2}_{sym}$ denotes the in-plane strain tensor and, using the notation from (14)*, the effective elasticity tensor is given as $\mathbb{C}_{KL} = \overline{\mathbb{C}} - \mathbb{P}_\mathbb{C}^*\mathbb{M}_\mathbb{C}^{-1}\mathbb{P}_\mathbb{C}$. Moreover, we have that $\kappa_\varepsilon(\widehat{u}_\varepsilon) \to \kappa$ in $\mathrm{L}^2(\Omega_1;\mathbb{R}^{3\times3})$, where $\kappa_{ij} = \overline{\mathsf{e}}_{ij}(\widehat{u})$ for $i,j = 1,2$ and the vector $\varkappa \in \mathrm{L}^2(\Omega_1;\mathbb{R}^3)$, defined via $\varkappa_i = \kappa_{i3}$ for $i = 1,2,3$, satisfies the relation $\mathbb{M}_\mathbb{C}\varkappa + \mathbb{P}_\mathbb{C}\overline{\mathsf{e}}(\widehat{u}) = 0$.*

With the transformation to the fixed domain Ω_1 and the rescaling of the displacements, it is not hard to confirm that the rescaled pair $(\widehat{u}_\varepsilon, \widetilde{w}_\varepsilon)$, corresponding to solutions to the system in (5), satisfy the rescaled equation of linear momentum

$$\int_{\Omega_1} \big(\mathbb{C}\kappa_\varepsilon(\widehat{u}_\varepsilon(t)) + \mathbb{D}\kappa_\varepsilon(\dot{\widehat{u}}_\varepsilon(t)) + \Theta(\widetilde{w}_\varepsilon(t))\mathbb{B}\big) : \kappa_\varepsilon(\widehat{v})\,\mathrm{d}\tilde{x} = \langle \widehat{\ell}_*(t), \widehat{v}\rangle, \tag{12}$$

for almost every $t \in [0,T]$ and for all $\widehat{v} \in \mathrm{H}^1_{\Gamma_{\mathrm{D},1}}(\Omega_1;\mathbb{R}^3)$ and the rescaled heat equation

$$\begin{aligned}\int_0^T \int_{\Omega_1} \big(-\widetilde{w}_\varepsilon\dot{\widetilde{\xi}} + \mathcal{K}(\widetilde{w}_\varepsilon)\nabla_\varepsilon\widetilde{w}_\varepsilon \cdot \nabla_\varepsilon\widetilde{\xi}\big)\,\mathrm{d}\tilde{x}\,\mathrm{d}t - \int_{\Omega_1} \tilde{w}^0\widetilde{\xi}(0,\cdot)\,\mathrm{d}\tilde{x} \\ = \int_0^T \int_{\Omega_1} \big(\mathsf{Q}_* + \mathbb{D}\kappa_\varepsilon(\dot{\widehat{u}}) : \kappa_\varepsilon(\dot{\widehat{u}}) + \Theta(\widetilde{w}_\varepsilon)\mathbb{B} : \kappa_\varepsilon(\dot{\widehat{u}})\big)\widetilde{\xi}\,\mathrm{d}\tilde{x}\,\mathrm{d}t \\ + \int_0^T \int_{\Gamma_1^{\|}} \mathsf{q}_*^{\|}\widetilde{\xi}\,\mathrm{d}a\,\mathrm{d}t + \int_0^T \int_{\Gamma_1^{\perp}} \mathsf{q}_*^{\perp}\widetilde{\xi}\,\mathrm{d}a\,\mathrm{d}t\end{aligned} \tag{13}$$

for all $\widetilde{\xi} \in \mathrm{W}^{1,r'}(0,T;\mathrm{W}^{1,r'}(\Omega_1))$ with $\widetilde{\xi}(T) = 0$.

To identify the effective lower-dimensional system, it is convenient to introduce the tensors $\overline{\mathbb{C}} \in \mathrm{Lin}(\mathbb{R}^{2\times2}_{\mathsf{sym}}, \mathbb{R}^{2\times2}_{\mathsf{sym}})$ $\mathbb{P}_\mathbb{C} \in \mathrm{Lin}(\mathbb{R}^{2\times2}_{\mathsf{sym}}, \mathbb{R}^3)$ and the matrix $\mathbb{M}_\mathbb{C} \in \mathbb{R}^{3\times3}_{\mathsf{sym}}$ via

$$\begin{aligned}\overline{\mathbb{C}}\,\overline{A} : \overline{B} &:= \mathbb{C}\begin{pmatrix}\overline{A} & 0\\ 0 & 0\end{pmatrix} : \begin{pmatrix}\overline{B} & 0\\ 0 & 0\end{pmatrix}, \quad \mathbb{P}_\mathbb{C}\overline{A}\cdot\begin{pmatrix}a\\ z\end{pmatrix} := \mathbb{C}\begin{pmatrix}\overline{A} & 0\\ 0 & 0\end{pmatrix} : \begin{pmatrix}\mathbf{0} & a\\ a^\top & z\end{pmatrix} \\ \mathbb{M}_\mathbb{C}\begin{pmatrix}a\\ z\end{pmatrix}\cdot\begin{pmatrix}b\\ y\end{pmatrix} &:= \mathbb{C}\begin{pmatrix}\mathbf{0} & a\\ a^\top & z\end{pmatrix} : \begin{pmatrix}\mathbf{0} & b\\ b^\top & y\end{pmatrix}, \quad \text{for } \overline{A}, \overline{B} \in \mathbb{R}^{2\times2}_{\mathsf{sym}},\ a, b \in \mathbb{R}^2,\ y, z \in \mathbb{R}.\end{aligned} \tag{14}$$

Analogously, we define the quantities $\overline{\mathbb{D}} \in \mathrm{Lin}(\mathbb{R}^{2\times2}_{\mathsf{sym}}, \mathbb{R}^{2\times2}_{\mathsf{sym}})$, $\mathbb{P}_\mathbb{D} \in \mathrm{Lin}(\mathbb{R}^{2\times2}_{\mathsf{sym}}, \mathbb{R}^3)$, $\mathbb{M}_\mathbb{D} \in \mathbb{R}^{3\times3}_{\mathsf{sym}}$ for the viscosity tensor $\mathbb{D}$. We will also denote $\overline{\mathbb{B}} = (\mathbb{B}_{ij})_{i,j=1,2}$ and $\mathsf{b}_\mathbb{B} = (2\mathbb{B}_{13}, 2\mathbb{B}_{23}, \mathbb{B}_{33})^\top \in \mathbb{R}^3$. We decompose the heat conductivity tensor as

$$\mathcal{K}(w) = \begin{pmatrix}\overline{\mathcal{K}}(w) & k_1(w)\\ k_1(w)^\top & k_2(w)\end{pmatrix}, \quad \text{where } \overline{\mathcal{K}}(w) \in \mathbb{R}^{2\times2}_{\mathsf{sym}},\ k_1(w) \in \mathbb{R}^2,\ k_2(w) \in \mathbb{R}_+. \tag{15}$$

The effective two-dimensional heat conductivity tensor $w \mapsto \overline{\mathcal{K}}_{\mathrm{eff}}(w) \in \mathbb{R}^{2\times2}_{\mathsf{sym}}$ is given via $\overline{\mathcal{K}}_{\mathrm{eff}}(w) = \overline{\mathcal{K}}(w) - \frac{1}{k_2(w)}k_1(w)\otimes k_1(w)$. The form of $\overline{\mathcal{K}}_{\mathrm{eff}}(w)$ is not surprising. Indeed, let us consider the quadratic form $z \mapsto \frac{1}{2}\mathcal{K}(w)z\cdot z$ for $z \in \mathbb{R}^3$. The effective two-dimensional heat conductivity tensor arises from minimizing out the vertical direction, viz.

$$\frac{1}{2}\overline{\mathcal{K}}_{\mathrm{eff}}(w)z'\cdot z' = \min_{z_3\in\mathbb{R}} \frac{1}{2}\mathcal{K}(w)\begin{pmatrix}z'\\ z_3\end{pmatrix}\cdot\begin{pmatrix}z'\\ z_3\end{pmatrix} \quad \text{with } Z_3^*(w,z') = -\frac{k_1(w)\cdot z'}{k_2(w)} \tag{16}$$

denoting the unique minimizer for given $w \in \mathbb{R}$ and $z' \in \mathbb{R}^2$. Note that $\overline{\mathcal{K}}_{\mathrm{eff}}(w)$ is the Schur complement of $k_2(w)$ in $\mathcal{K}(w)$.

Moreover, we define the planar strain $\overline{\mathsf{e}}(u) \in \mathbb{R}^{2\times2}_{\mathsf{sym}}$ as

$$\overline{\mathsf{e}}_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = 1,2.$$

Theorem 7 (Limit passage to plate model)**.** *Let us assume that the Assumptions (A1)–(A9) and (B1)–(B9) hold. Then, the rescaled weak solutions* $(\widehat{u}_\varepsilon, \widetilde{w}_\varepsilon)$ *to the system* (5) *converge (up to subsequences) to limits* $(\widehat{u}, \widetilde{w})$ *with* $\widehat{u} \in \mathrm{H}^1(0,T;\mathcal{V}_{\mathrm{KL}}(\Omega_1)))$*,* $\widetilde{w} \in \mathrm{L}^r(0,T;\mathcal{W}_0)$ *such that*

$$\widehat{u}_\varepsilon \rightharpoonup \widehat{u} \text{ in } \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega;\mathbb{R}^3)), \qquad \widetilde{w}_\varepsilon \rightharpoonup \widetilde{w} \text{ in } \mathrm{L}^r(0,T;\mathrm{W}^{1,r}(\Omega)). \tag{17}$$

In particular, the limits $\widehat{u}$ *and* $\widetilde{w}$ *can be identified with functions* $\overline{U} = (U_1, U_2)$*,* U_3 *and* W*, respectively, defined on* $[0,T] \times \omega$ *that are weak solutions to the following effective system consisting of the membrane and plate equations*

$$\int_\omega \big(\overline{\mathbb{C}}\mathsf{e}(\overline{U}) + \overline{\mathbb{D}}\mathsf{e}(\dot{\overline{U}}) + \mathbb{P}^*_{\mathbb{C}}\varkappa_{\mathrm{even}} + \mathbb{P}^*_{\mathbb{D}}\dot{\varkappa}_{\mathrm{even}} + \Theta(W)\overline{\mathbb{B}}\big) : \mathsf{e}(\overline{V})\,\mathrm{d}x' = \langle L_{1,2}(t), \overline{V}\rangle$$
$$\frac{1}{12}\int_\omega \big(\overline{\mathbb{C}}\nabla^2 U_3 + \overline{\mathbb{D}}\nabla^2 \dot{U}_3 + \mathbb{P}^*_{\mathbb{C}}\varkappa_{\mathrm{odd}} + \mathbb{P}^*_{\mathbb{D}}\dot{\varkappa}_{\mathrm{odd}}\big) : \nabla^2 V_3\,\mathrm{d}x' = \langle L_3(t), V_3\rangle$$

for all $\overline{V} = (V_1, V_2) \in \mathrm{H}^1_{\gamma_\mathrm{D}}(\omega) \times \mathrm{H}^1_{\gamma_\mathrm{D}}(\omega)$ *and* $V_3 \in \mathrm{H}^2_{\gamma_\mathrm{D}}(\omega)$ *almost everywhere in* $[0,T]$*, the out-of-plane strains satisfy*

$$\dot{\varkappa}_{\mathrm{even}} + \mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}}\varkappa_{\mathrm{even}} = -\mathbb{M}_{\mathbb{D}}^{-1}(\mathbb{P}_{\mathbb{C}}\mathsf{e}(\overline{U}) + \mathbb{P}_{\mathbb{D}}\mathsf{e}(\dot{U}_1, \dot{U}_2) + \Theta(W)\mathsf{b}_{\mathbb{B}}),$$
$$\dot{\varkappa}_{\mathrm{odd}} + \mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}}\varkappa_{\mathrm{odd}} = -\mathbb{M}_{\mathbb{D}}^{-1}(\mathbb{P}_{\mathbb{C}}\nabla^2 U_3 + \mathbb{P}_{\mathbb{D}}\nabla^2 \dot{U}_3).$$

almost everywhere in $[0,T] \times \omega$*, and the effective heat equation for* W *reads*

$$\int_0^T \int_\omega \big(-W\dot{\xi} + \mathcal{K}_{\mathrm{eff}}(W)\nabla' W \cdot \nabla'\xi\big)\,\mathrm{d}x\,\mathrm{d}t - \int_\omega W^0 \xi(0,\cdot)\,\mathrm{d}x$$
$$= \int_0^T \int_\omega \mathcal{Q}_{\mathrm{tot}}\big(t, x'; \mathsf{e}(\dot{\overline{U}}), \nabla^2 U_3, \dot{\varkappa}_{\mathrm{even}}, \dot{\varkappa}_{\mathrm{odd}}, \Theta(W)\big)\xi\,\mathrm{d}x'\,\mathrm{d}t + \int_0^T \int_{\partial\omega} \overline{\mathsf{q}}^\perp_* \xi\,\mathrm{d}a\,\mathrm{d}t \tag{18}$$

for all $\xi \in \mathrm{W}^{1,r'}(0,T;\mathrm{W}^{1,r'}(\omega))$ *with* $\xi(T) = 0$*, where the total heat source* $\mathcal{Q}_{\mathrm{tot}}$ *is defined in* (50)*.*

3 Existence proof

For the remaining part of the text, we drop 'hat' and 'tilde' over the quantities to keep the notation simple. In order to prove the existence of solutions, we first consider a regularized system, by introducing for $\delta > 0$ the regularized heat source

$$\mathcal{Q}_\delta(\dot{\kappa}, w; t, x) = \mathsf{Q}_\delta(t,x) + \frac{\Theta(w)\mathbb{B} : \dot{\kappa} + \mathbb{D}\dot{\kappa} : \dot{\kappa}}{1 + \delta|\dot{\kappa}|^2}, \tag{19}$$

where $\mathsf{Q}_\delta \in \mathrm{L}^2((0,T) \times \Omega_1)$ is such that $\mathsf{Q}_\delta \to \mathsf{Q}$ in $\mathrm{L}^1(0,T;\mathrm{L}^1(\Omega_1))$. Moreover, we consider $\mathsf{q}^\perp_\delta \in \mathrm{L}^2((0,T) \times \Gamma^\perp_1)$ and $\mathsf{q}^\parallel_\delta \in \mathrm{L}^2((0,T) \times \Gamma^\parallel_1)$ such that $\mathsf{q}^\perp_\delta \to \mathsf{q}^\perp_*$ in $\mathrm{L}^1(0,T;\mathrm{L}^1(\Gamma^\perp_1))$ and $\mathsf{q}^\parallel_\delta \to \mathsf{q}^\parallel_*$ in $\mathrm{L}^1(0,T;\mathrm{L}^1(\Gamma^\parallel_1))$, as well as, regularized initial values $w^0_\delta \in \mathrm{L}^2(\Omega_1)$ such that $w^0_\delta \to w^0$ in $\mathrm{L}^1(\Omega_1)$

Thus, instead of the heat equation in (13), we consider its regularized version

$$\dot{w} - \mathrm{div}_\varepsilon(\mathcal{K}(w)\nabla_\varepsilon w) = \mathcal{Q}_\delta(\kappa_\varepsilon(\dot{u}), w), \tag{20}$$

while the mechanical equation stays the same. The existence of solutions to the regularized system (1a) and (20) can be deduced either via time-discretization or Galerkin approximation, we refer the interested reader to [KrR19, Sect. 8.3] and [Rou09], respectively.

Theorem 8 (Existence for regularized model). *For $\varepsilon > 0$ and $\delta > 0$ fixed, there exists a weak solution $(u_{\delta,\varepsilon}, w_{\delta,\varepsilon}) \in \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega_1;\mathbb{R}^d)) \times (\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega_1)) \cap \mathrm{H}^1(0,T;(\mathrm{H}^1(\Omega_1))^*)$ to the regularized system, meaning that $u_{\delta,\varepsilon} - u_{\mathrm{D},\varepsilon} \in \mathrm{H}^1(0,T;\mathrm{H}^1_{\Gamma_1^{\mathrm{D}}}(\Omega_1;\mathbb{R}^d))$, $w_{\delta,\varepsilon} > 0$, that for all $v \in \mathrm{L}^2(0,T;\mathrm{H}^1_{\Gamma_1^{\mathrm{D}}}(\Omega_1;\mathbb{R}^d))$ it holds*

$$\int_0^T \int_\Omega \big(\mathbb{C}\kappa_\varepsilon(u_{\delta,\varepsilon}) + \mathbb{D}\kappa_\varepsilon(\dot u_{\delta,\varepsilon}) + \Theta(w_{\delta,\varepsilon})\mathbb{B}\big) : \kappa_\varepsilon(v)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle \ell_*(t), v\rangle\,\mathrm{d}t, \tag{21}$$

for any $\xi \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega_1))$

$$\begin{aligned}\int_0^T \langle \dot w_{\delta,\varepsilon}, \xi\rangle\,\mathrm{d}t + \int_0^T \int_\Omega \mathcal{K}(w_{\delta,\varepsilon})\nabla_\varepsilon w_{\delta,\varepsilon}\cdot\nabla_\varepsilon \xi\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\int_\Omega \mathcal{Q}_\delta(\kappa_\varepsilon(\dot u_{\delta,\varepsilon}), w_{\delta,\varepsilon})\xi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Gamma_1^{\parallel}} \mathsf{q}_\delta^{\parallel}\xi\,\mathrm{d}a\,\mathrm{d}t + \int_0^T\int_{\Gamma_1^{\perp}} \mathsf{q}_\delta^{\perp}\xi\,\mathrm{d}a\,\mathrm{d}t,\end{aligned} \tag{22}$$

and the initial conditions $u_{\delta,\varepsilon}(0) = u^0_\varepsilon$, $w_{\delta,\varepsilon}(0) = w^0_\delta$ are fulfilled.

Concerning the positivity of temperature, we note that

$$\frac{\Theta(w)\mathbb{B} : \dot\kappa + \mathbb{D}\dot\kappa : \dot\kappa}{1+\delta|\dot\kappa|^2} \geq -\frac{C_0^2|\mathbb{B}|^2}{2\gamma_{\mathbb{D}}}|w|^2$$

due to Remark 2. Hence, we can carry out similar arguments as in [Rou13, Remark 12.10] to obtain positivity $w_{\delta,\varepsilon} \geq \frac{2}{C_0^2|\mathbb{B}|^2T\gamma_{\mathbb{D}}^{-1}+2/w^0_{\min}} > 0$, which holds uniformly in ε and δ. It follows that also $\Theta(w_{\delta,\varepsilon}) > 0$.

Proposition 9 (A priori estimates for the regularized system). *Let us assume that Assumptions (A1)-(A9) and (B1)–(B9) hold. Fix $r \in (1, \frac{5}{4})$. Let $(u_{\delta,\varepsilon}, w_{\delta,\varepsilon})$ denote the solutions for the regularized system in Theorem 8, then there exists a constant $C > 0$ independent of $\delta > 0$ and $0 < \varepsilon < 1$ such that*

$$\|\kappa_\varepsilon(u_{\delta,\varepsilon})\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega_1))} + \|\kappa_\varepsilon(\dot u_{\delta,\varepsilon})\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_1))} + \|u_{\delta,\varepsilon}\|_{\mathrm{H}^1(0,T;\mathrm{H}^1(\Omega))} \leq C, \tag{23}$$

$$\|w_{\delta,\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega_1))} + \|w_{\delta,\varepsilon}\|_{\mathrm{L}^r(0,T;\mathrm{W}^{1,r}(\Omega_1))} + \frac{1}{\varepsilon}\|\partial_{x_3} w_{\delta,\varepsilon}\|_{\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))} \leq C. \tag{24}$$

Proof. Step 1. We fix $\tau \in [0,T]$ and test (21) with $v = \dot u_{\delta,\varepsilon}(t)\chi_{[0,\tau]}(t)$, and after some rearranging we obtain

$$\begin{aligned}\int_0^\tau\int_\Omega \big(\mathbb{C}\kappa_\varepsilon(u_{\delta,\varepsilon}) + \mathbb{D}\kappa_\varepsilon(\dot u_{\delta,\varepsilon})\big) : \kappa_\varepsilon(\dot u_{\delta,\varepsilon})\,\mathrm{d}x\,\mathrm{d}t &= \int_0^\tau \langle \ell_*(t), \dot u_{\delta,\varepsilon}\rangle\,\mathrm{d}t \\ &\quad - \int_0^\tau\int_\Omega \Theta(w_{\delta,\varepsilon})\mathbb{B} : \kappa_\varepsilon(\dot u_{\delta,\varepsilon})\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

In the left-hand side, we use uniform ellipticity of $\mathbb{C}$ and $\mathbb{D}$ in (A2) to obtain after using the chain rule

$$\begin{aligned}\frac{\gamma_{\mathbb{C}}}{2}\|\kappa_\varepsilon(u_{\delta,\varepsilon}(\tau))\|^2_{\mathrm{L}^2(\Omega_1)} + \gamma_{\mathbb{D}}\int_0^\tau \|\kappa_\varepsilon(\dot u_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(\Omega_1)}\,\mathrm{d}t \leq \frac{C_{\mathbb{C}}}{2}\|\kappa_\varepsilon(u^0_\varepsilon)\|^2_{\mathrm{L}^2(\Omega)} \\ + \int_0^\tau \|\ell_*(t)\|_{\mathrm{H}^1(\Omega_1)^*}\|\dot u_{\delta,\varepsilon}\|_{\mathrm{H}^1(\Omega_1)}\,\mathrm{d}t + C_{\mathbb{B}}\int_0^\tau \|\Theta(w_{\delta,\varepsilon})\|_{\mathrm{L}^2(\Omega_1)}\|\kappa_\varepsilon(\dot u_{\delta,\varepsilon})\|_{\mathrm{L}^2(\Omega_1)}\,\mathrm{d}t.\end{aligned}$$

In the right-hand side, we apply Young's inequality several times and use that due to Korn's inequality there exists a constant $C_{\mathsf{K}} > 0$ (depending only on the fixed domain Ω_1) such that $\|\dot{u}_{\delta,\varepsilon}\|_{\mathrm{H}^1(\Omega_1)} \le C_{\mathsf{K}}\|\mathrm{e}(\dot{u}_{\delta,\varepsilon})\|_{\mathrm{L}^2(\Omega_1)} \le \|\kappa_\varepsilon(\dot{u}_{\delta,\varepsilon})\|_{\mathrm{L}^2(\Omega_1)}$ as $0<\varepsilon<1$. We are then left with

$$\|\kappa_\varepsilon(u_{\delta,\varepsilon}(\tau))\|^2_{\mathrm{L}^2(\Omega_1)} + \|\kappa_\varepsilon(\dot{u}_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))} \\ \le C\Big(\|\kappa_\varepsilon(u^0_\varepsilon)\|^2_{\mathrm{L}^2(\Omega_1)} + \|\ell_*\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega_1)^*)} + \|\Theta(w_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))},\Big)$$

where $C>0$ does not depend on δ, τ and ε. Using the Assumption in (B9) for the initial value u^0_ε and (B4) for the external forces, it follows for all $\tau \in (0,T]$ that

$$\|\kappa_\varepsilon(u_{\delta,\varepsilon}(\tau))\|^2_{\mathrm{L}^2(\Omega_1)} + \|\kappa_\varepsilon(\dot{u}_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))} \le C_1\big(1 + \|\Theta(w_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))}\big) \tag{25}$$

for some constant C_1 independent of ε and δ.

Step 2. For the derivation of the estimates in (24), we use the techniques established by Boccardo and Gallouët [BoG89, BD*97] and follow the depiction in [Rou09, Proposition 4.3] and [Rou10, Proposition 4.2].

Again, keeping $\tau \in (0,T]$ fixed, we consider for $0<k<1$ the C^2-function $\phi_k(z) := z - \frac{1}{1-k}\frac{1}{(1+z)^{k-1}}$ for $z\in[0,\infty)$. We have $\phi_k'(z) = 1 - \frac{1}{(1+z)^k}$ and $\phi_k''(z) = \frac{k}{(1+z)^{k+1}}$ and perform the test in (22) with $\xi = \phi_k'(w_{\delta,\varepsilon})\chi_{[0,\tau]}$. By chain rule, we then get

$$\int_{\Omega_1}\phi_k(w_{\delta,\varepsilon}(\tau))\,\mathrm{d}x - \int_{\Omega_1}\phi_k(w^0_\varepsilon)\,\mathrm{d}x + \int_0^\tau\int_{\Omega_1}\phi_k''(w_{\delta,\varepsilon})\mathcal{K}(w_{\delta,\varepsilon})\nabla_\varepsilon w_{\delta,\varepsilon}\cdot\nabla_\varepsilon w_{\delta,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^\tau\int_{\Omega_1}\mathcal{Q}_\delta(\kappa_\varepsilon(\dot{u}_{\delta,\varepsilon}), w_{\delta,\varepsilon})\phi_k'(w_{\delta,\varepsilon})\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^\tau\int_{\Gamma_1^\parallel}\mathsf{q}_\delta^\parallel\phi_k'(w_{\delta,\varepsilon})\,\mathrm{d}a\,\mathrm{d}t + \int_0^\tau\int_{\Gamma_1^\perp}\mathsf{q}_\delta^\perp\phi_k'(w_{\delta,\varepsilon})\,\mathrm{d}a\,\mathrm{d}t. \tag{26}$$

Note that ϕ_k has linear growth, i.e., $-\frac{1}{1-k}+z \le \phi_k(z) \le z$, while $0\le\phi_k'(z)\le 1$ in the following. In particular, exploiting that $|\mathcal{Q}_\delta(\dot\kappa, w; t,x)| \le \mathsf{Q}_\delta(t,x) + |\mathbb{B}||\Theta(w)||\dot\kappa| + \mathbb{D}\dot\kappa:\dot\kappa$, we estimate the right-hand side using Hölder's and Young's inequalities

$$\int_0^\tau\int_{\Omega_1}\mathcal{Q}_\delta(\kappa_\varepsilon(\dot{u}_{\delta,\varepsilon}), w_{\delta,\varepsilon})\phi_k'(w_{\delta,\varepsilon})\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Gamma_1^\parallel}\mathsf{q}_\delta^\parallel\phi_k'(w_{\delta,\varepsilon})\,\mathrm{d}a\,\mathrm{d}t + \int_0^\tau\int_{\Gamma_1^\perp}\mathsf{q}_\delta^\perp\phi_k'(w_{\delta,\varepsilon})\,\mathrm{d}a\,\mathrm{d}t \\ \le \|\mathsf{Q}_\delta\|_{\mathrm{L}^1(0,\tau;\mathrm{L}^1(\Omega_1))} + C\big(\|\kappa_\varepsilon(\dot{u}_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))} + \|\Theta(w_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))}\big) \\ + \|\mathsf{q}_\delta^\parallel\|_{\mathrm{L}^1(0,\tau;\mathrm{L}^1(\Gamma_1^\parallel))} + \|\mathsf{q}_\delta^\perp\|_{\mathrm{L}^1(0,\tau;\mathrm{L}^1(\Gamma_1^\perp))}$$

for all $\tau\in(0,T]$ and a constant $C>0$ independent of ε and δ. In the left-hand side of (26), we use the bounds for ϕ_k and uniform ellipticity of $\mathcal{K}$ in (B3) and the convergences $\mathsf{Q}_\delta \to \mathsf{Q}_*$ in $\mathrm{L}^1(0,T;\mathrm{L}^1(\Omega_1))$ and $\mathsf{q}_\delta^\parallel \to \mathsf{q}_*^\parallel$ in $\mathrm{L}^1(0,T;\mathrm{L}^1(\Gamma_1^\parallel))$ and $\mathsf{q}_\delta^\perp \to \mathsf{q}_*^\perp$ in $\mathrm{L}^1(0,T;\mathrm{L}^1(\Gamma_1^\perp))$, to obtain for any $\varepsilon>1$, $\delta>1$ and $\tau\in(0,T]$

$$\|w_{\delta,\varepsilon}(\tau)\|_{\mathrm{L}^1(\Omega_1)} + \int_0^\tau\int_{\Omega_1}\frac{|\nabla_\varepsilon w_{\delta,\varepsilon}|^2}{(1+w_{\delta,\varepsilon})^{k+1}}\,\mathrm{d}x\,\mathrm{d}t \\ \le C_2\big(1 + \|\kappa_\varepsilon(\dot{u}_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))} + \|\Theta(w_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))}\big) \tag{27}$$

for some constant $C_2>0$ independent of ε and δ.

Step 3. Multiplying (27) by $1/(2C_2)$ and adding the result to (25) gives

$$\|\kappa_\varepsilon(u_{\delta,\varepsilon}(\tau))\|^2_{\mathrm{L}^2(\Omega_1)} + \frac{1}{2}\|\kappa_\varepsilon(\dot u_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))} + \frac{1}{2C_2}\|w_{\delta,\varepsilon}(\tau)\|_{\mathrm{L}^1(\Omega_1)} + \frac{1}{2C_2}\int_0^\tau\int_{\Omega_1}\frac{|\nabla_\varepsilon w_{\delta,\varepsilon}|^2}{(1+w_{\delta,\varepsilon})^{k+1}}\,\mathrm{d}x\,\mathrm{d}t \le \Big(C_1+\frac{1}{2}\Big)\big(1+\|\Theta(w_{\delta,\varepsilon})\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega_1))}\big) \quad (28)$$

for all $\delta>0,\ \varepsilon\in(0,1],\ \tau\in[0,T]$. Using now the estimate $\Theta(w)\le C_0(w)^{1/s}$ for $s>2$ (see Remark 2) gives a uniform estimate for $\|w_{\delta,\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega_1))}$ via Grönwall's inequality. From this estimate, we immediately obtain the uniform bound for $\|\kappa_\varepsilon(u_{\delta,\varepsilon})\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega_1))}$ and $\|\kappa_\varepsilon(\dot u_{\delta,\varepsilon})\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_1))}$. Eventually, the uniform bound for $\|u_{\delta,\varepsilon}\|_{\mathrm{H}^1(0,T;\mathrm{H}^1(\Omega_1))}$ follows from Korn's inequality and $\|\mathrm{e}(v)\|_{\mathrm{L}^2(\Omega_1)}\le\|\kappa_\varepsilon(v)\|_{\mathrm{L}^2(\Omega_1)}$ (since $\varepsilon<1$), which provide us

$$\|u_{\delta,\varepsilon}\|_{\mathrm{H}^1(0,T;\mathrm{H}^1(\Omega_1))}\le C_{\mathrm{K}}\|\kappa_\varepsilon(u_{\delta,\varepsilon})\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_1))}+C_{\mathrm{K}}\|\kappa_\varepsilon(u_{\mathrm{D},\varepsilon})\|_{\mathrm{L}^2(\Omega_1)}+\|u_{\mathrm{D},\varepsilon}\|_{\mathrm{H}^1(\Omega_1)}.$$

The norms containing the Dirichlet data are uniformly bounded in ε and δ by decomposition (B5), and the norm of $\kappa_\varepsilon(u_{\delta,\varepsilon})$ is bounded by the already shown estimate for $\|\kappa_\varepsilon(u_{\delta,\varepsilon})\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_1))}$.

Step 4. It remains to prove the uniform estimate for $\nabla_\varepsilon w_{\delta,\varepsilon}$ in $\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega;\mathbb{R}^3))$. We start by applying Hölder's inequality with the exponents $2/r$ and $2/(2-r)$ (recall that $r\in(1,5/4)$) to obtain

$$\begin{aligned}\int_0^T\int_\Omega|\nabla_\varepsilon w_{\delta,\varepsilon}|^r\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega\frac{|\nabla_\varepsilon w_{\delta,\varepsilon}|^r}{(1+w_{\delta,\varepsilon})^{\frac{r(k+1)}{2}}}(1+w_{\delta,\varepsilon})^{\frac{r(k+1)}{2}}\,\mathrm{d}x\,\mathrm{d}t\\ &\le\Big(\int_0^T\int_\Omega\frac{|\nabla_\varepsilon w_{\delta,\varepsilon}|^2}{(1+w_{\delta,\varepsilon})^{k+1}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{r}{2}}\Big(\int_0^T\int_\Omega(1+w_{\delta,\varepsilon})^{\frac{r(k+1)}{2-r}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{2-r}{2}}\\ &= I_1\times I_2.\end{aligned}$$

We observe that I_1 is already uniformly controlled via (28). For I_2, we use the Gagliardo–Nirenberg estimate, in the following form

$$\|z\|_{\mathrm{L}^{\tilde q}(\Omega_1)}\le C_{\mathsf{GN}}\|z\|^{1-\lambda}_{\mathrm{L}^1(\Omega_1)}\big(\|z\|_{\mathrm{L}^1(\Omega_1)}+\|\nabla z\|_{\mathrm{L}^r(\Omega_1)}\big)^\lambda$$
$$\text{for}\quad \frac{1}{\tilde q}\ge\frac{1-\lambda}{1}+\lambda\Big(\frac{1}{r}-\frac{1}{d}\Big),\quad 0<\lambda\le1,\quad z\in\mathrm{W}^{1,r}(\Omega_1), \quad (29)$$

where $d=3$ is the dimension of the domain Ω_1. In particular, for $z=1+w_{\delta,\varepsilon}(t)$ and $\tilde q=r(1+k)/(2-r)$ and $k=\frac{1}{d}(d+2-r(d+1))$, we get $k>0$ since $r<(d+2)/(d+1)=5/4$, and we can set $\lambda=r/\tilde q=(2-r)/(1+k)\in(0,1)$ to obtain

$$\begin{aligned}\|\nabla_\varepsilon w_{\delta,\varepsilon}\|^r_{\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))}\le I_1\times I_2 &\le C\Big(\int_0^T\|1+w_{\delta,\varepsilon}(t)\|^{\tilde q}_{\mathrm{L}^{\tilde q}(\Omega_1)}\,\mathrm{d}t\Big)^{1-r/2}\\ &\le C\Big(\int_0^T(1+\|\nabla w_{\delta,\varepsilon}\|^r_{\mathrm{L}^r(\Omega_1)})\,\mathrm{d}t\Big)^{1-r/2} \quad (30)\end{aligned}$$

where we also used that $\|w_{\delta,\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega_1))}\le C$. Note that all constants $C>0$ are independent of δ and ε. Finally, we note that $\|\nabla w_{\delta,\varepsilon}\|_{\mathrm{L}^r(\Omega_1)}\le\|\nabla_\varepsilon w_{\delta,\varepsilon}\|_{\mathrm{L}^r(\Omega_1)}$ as $0<\varepsilon<1$. Thus, using Young's inequality on the right-hand side to conclude that $\nabla_\varepsilon w_{\delta,\varepsilon}$ is uniformly bounded in $\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))$. □

We are now in position to pass to the limit $\delta \to 0$ for ε fixed. Note that for ε fixed, we also obtain a uniform estimate for the time-derivative $\dot{w}_{\delta,\varepsilon}$. Indeed, testing the heat equation with an arbitrary test function $\xi \in \mathrm{L}^\infty(0,T;\mathrm{W}^{1,r'}(\Omega_1))$ and using the a priori estimates established above as well as $\|\mathcal{Q}_\delta(\kappa_\varepsilon, w_{\delta,\varepsilon})\|_{\mathrm{L}^1(0,T;\mathrm{L}^1\Omega_1)}$ and $\|q_\delta^\parallel\|_{\mathrm{L}^1(0,T;\mathrm{L}^1(\Gamma_1^\parallel))} + \|q_\delta^\perp\|_{\mathrm{L}^1(0,T;\mathrm{L}^1(\Gamma_1^\perp))} \le C$ gives that $\|\dot{w}_{\delta,\varepsilon}\|_{\mathrm{L}^1(0,T;(\mathrm{W}^{1,r'}(\Omega))^*)}$ is bounded uniformly with respect to δ (but not with respect to ε).

Theorem 10 (Existence of rescaled solutions; limit passage $\delta \to 0$). *Let $(u_{\delta,\varepsilon}, w_{\delta,\varepsilon})$ be from Theorem 8 and keep $\varepsilon \in (0,1]$ fixed. Then, there exists a sequence $(\delta_k)_{k\in\mathbb{N}} \subset (0,1)$, $\delta_k \searrow 0$, and limits $(u_\varepsilon, w_\varepsilon)$ such that*

$$\begin{cases} u_{\delta_k,\varepsilon} \rightharpoonup u_\varepsilon & \text{in } \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega;\mathbb{R}^3)), \\ w_{\delta_k,\varepsilon} \rightharpoonup w_\varepsilon & \text{in } \mathrm{L}^r(0,T;\mathrm{W}^{1,r}(\Omega)), \end{cases}$$

and $(u_\varepsilon, w_\varepsilon)$ is a weak solution of (5) *in the sense of Definition 3. Moreover, all estimates in Proposition 9 remain true for the limits $(u_\varepsilon, w_\varepsilon)$ via weak lower semicontinuity of the norm.*

Proof. By Proposition 9, we find a sequence $(\delta_k)_{k\in\mathbb{N}}$, $\delta_k \searrow 0$, such that the stated convergences hold. We will now mainly follow the ideas from [Rou09, Proposition 4.6] and [BaR11, Lemma 4.2]. One crucial point is to get strong convergence for $(w_{\delta,\varepsilon})_{\delta>0}$ and $(\kappa_\varepsilon(\dot{u}_{\delta,\varepsilon}))_{\delta>0}$ in order to deal with the nonlinearities in (21) and (22).

First, we note that $\dot{w}_{\delta,\varepsilon}$ still enjoys a bound in $\mathrm{L}^1(0,T;(\mathrm{W}^{1,r'}(\Omega_1))^*)$ that is uniformly in δ (for ε fixed). Thus, we obtain from Aubin–Lions' compactness theorem (see [NeR01, Lemma 3]) the strong convergence $w_{\delta_k,\varepsilon} \to w_\varepsilon$ in $\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))$ and, up to further subsequence (not relabelled), convergence a.e. in $(0,T)\times\Omega_1$. As Θ has at most $\frac{1}{2}$-growth, continuity implies $\Theta(w_{\delta_k,\varepsilon}) \to \Theta(w_\varepsilon)$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_1))$. In particular, the mechanical equation (7) can be readily derived from (21). Also, the initial condition is fulfilled as $u_\varepsilon^0 = u_{\delta_k,\varepsilon}(0) \rightharpoonup u_\varepsilon(0)$.

The limit passage in (21) is used to show $\kappa_\varepsilon(\dot{u}_{\delta_k,\varepsilon}) \to \kappa_\varepsilon(\dot{u}_\varepsilon)$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_1))$. We compute

$$\begin{aligned} &\int_0^T\!\!\int_{\Omega_1} \mathbb{D}\kappa_\varepsilon(\dot{u}_\varepsilon){:}\kappa_\varepsilon(\dot{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \le \limsup_{k\to\infty} \int_0^T\!\!\int_{\Omega_1} \mathbb{D}\kappa_\varepsilon(\dot{u}_{\delta_k,\varepsilon}){:}\kappa_\varepsilon(\dot{u}_{\delta_k,\varepsilon})\,\mathrm{d}x\,\mathrm{d}t \\ &= \limsup_{k\to\infty}\Big(\int_0^T\!\!\int_{\Omega_1} -\mathbb{C}\kappa_\varepsilon(u_{\delta_k,\varepsilon}){:}\kappa_\varepsilon(\dot{u}_{\delta_k,\varepsilon}) - \Theta(w_{\delta_k,\varepsilon})\mathbb{B}{:}\kappa_\varepsilon(\dot{u}_{\delta_k,\varepsilon})\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle \widehat{\ell}_*, \dot{u}_{\delta_k,\varepsilon}\rangle\,\mathrm{d}t\Big) \\ &= \limsup_{k\to\infty}\Big(\int_{\Omega_1} \frac{1}{2}\mathbb{C}\kappa_\varepsilon(u_\varepsilon(0)){:}\kappa_\varepsilon(u_\varepsilon(0))\,\mathrm{d}x - \int_{\Omega_1}\frac{1}{2}\mathbb{C}\kappa_\varepsilon(u_{\delta_k,\varepsilon}(T)) : \kappa_\varepsilon(u_{\delta_k,\varepsilon}(T))\,\mathrm{d}x \\ &- \int_0^T\!\!\int_{\Omega_1} \Theta(w_{\delta_k,\varepsilon})\mathbb{B} : \kappa_\varepsilon(\dot{u}_{\delta_k,\varepsilon})\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle \widehat{\ell}_*, \dot{u}_{\delta_k,\varepsilon}\rangle\,\mathrm{d}t\Big) = \int_0^T\!\!\int_{\Omega_1} \mathbb{D}\kappa_\varepsilon(\dot{u}_\varepsilon){:}\kappa_\varepsilon(\dot{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{31}$$

where we used weak lower semi-continuity in the first, the mechanical equation (21) in the second, the chain rule in the third, and finally the mechanical equation (7) in the fourth line. Thus, since $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_1;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$ is a Radon-Riesz space, we conclude that $\kappa_\varepsilon(\dot{u}_{\delta_k,\varepsilon}) \to \kappa_\varepsilon(\dot{u}_\varepsilon)$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_1;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$ and (up to further subsequence; not relabelled) a.e. in $(0,T)\times\Omega_1$.

We pass to the limit $\delta_k \to 0$ in (22) for test functions $\xi \in \mathrm{W}^{1,r'}(0,T;\mathrm{W}^{1,r'}(\Omega_1))$ with $\xi(T) = 0$ as follows: In the left-hand side, we integrate by parts in the time derivative, and in the right-hand side we use the strong convergences $\Theta(w_{\delta_k,\varepsilon}) \to \Theta(w_\varepsilon)$ and $\kappa_\varepsilon(\dot{u}_{\delta_k,\varepsilon}) \to \kappa_\varepsilon(\dot{u}_\varepsilon)$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_1;\mathbb{R}^{3\times3}_{\mathrm{sym}}))$

for the adiabatic terms. For $\mathcal{Q}_\delta$, as defined in (19), we note that almost everywhere in $[0,T]\times\Omega_1$

$$\begin{aligned}\big|\mathcal{Q}_\delta(\kappa_\varepsilon(\dot u_{\delta_k,\varepsilon}),w_{\delta_k,\varepsilon})-Q-\Theta(w_\varepsilon)\mathbb{B}:\kappa_\varepsilon(\dot u_\varepsilon)-\mathbb{D}\kappa_\varepsilon(\dot u_\varepsilon):\kappa_\varepsilon(\dot u_\varepsilon)\big|&\\ \le|Q_\delta-Q|+\mathbb{D}\kappa_\varepsilon(\dot u_{\delta_k,\varepsilon}):\kappa_\varepsilon(\dot u_{\delta,\varepsilon})+\mathbb{D}\kappa_\varepsilon(\dot u_\varepsilon):\kappa_\varepsilon(\dot u_\varepsilon)&\\ +|\Theta(w_{\delta_k,\varepsilon})\mathbb{B}:\kappa_\varepsilon(\dot u_{\delta_k,\varepsilon})|+|\Theta(w_\varepsilon)\mathbb{B}:\kappa_\varepsilon(\dot u_\varepsilon)|.&\end{aligned}$$

Hence, we can use Pratt's theorem and the strong- and a.e.-convergences again to eventually conclude (8). Finally, using equation (8), one shows a posteriori that w_ε has a time derivative in the space $\mathrm{L}^1(0,T;(\mathrm{W}^{1,r'}(\Omega))^*)$, see also Lemma 11. □

For the limit passage $\varepsilon\to0$, the above arguments are not applicable since we cannot establish a uniform bound for $\dot w_\varepsilon$ in $\mathrm{L}^1(0,T;(\mathrm{W}^{1,r'}(\Omega))^*)$. However, we can exploit the following decomposition of w_ε into its vertical average and a remainder. For the former, the Aubin–Lions lemma can be applied, and the latter strongly converges to 0 due to the boundedness of $\varepsilon^{-1}\partial_{x_3}w_\varepsilon$. Indeed, let us denote by W_ε the average in vertical direction, i.e., we set

$$W_\varepsilon(t,x'):=\int_0^1 w_\varepsilon(t,x',z)\,\mathrm{d}z,\qquad\text{and}\qquad\vartheta_\varepsilon:=w_\varepsilon-W_\varepsilon.\tag{32}$$

The remainder $\vartheta_\varepsilon\in\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))$ is given via

$$\vartheta_\varepsilon(t,x',x_3)=\int_0^1\int_z^{x_3}\partial_{x_3}w_\varepsilon(t,x',y)\,\mathrm{d}y\,\mathrm{d}z.$$

Hence, with $\varepsilon^{-1}\|\partial_{x_3}w_\varepsilon\|_{\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))}\le$ being bounded (see (24) in Proposition 9), we obtain $\vartheta_\varepsilon\to0$ strongly in $\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))$. Thus, it remains to prove the strong convergence of W_ε.

Lemma 11 (Estimate for $\dot W_\varepsilon$)**.** *Let $(w_\varepsilon,u_\varepsilon)$ be the rescaled solutions in Theorem 10 and W_ε defined as in* (32)*. Then, there exists $C>0$, independent of $0<\varepsilon<1$, such that $\|\dot W_\varepsilon\|_{\mathrm{L}^1(0,T;(\mathrm{W}^{1,r'}(\omega)))}\le C$.*

Proof. Note that we can identify functions in $\mathrm{W}^{1,r'}(\omega)$ with functions in $\mathcal{W}_0$, see (11). For a.e. $t\in[0,T]$ and $\xi\in\mathcal{W}_0$, it holds

$$\begin{aligned}\langle\dot W_\varepsilon(t),\xi\rangle=\langle\dot w_\varepsilon(t),\xi\rangle=-\int_{\Omega_1}\mathcal{K}(w_\varepsilon(t))\nabla_\varepsilon w_\varepsilon(t)\cdot\begin{pmatrix}\nabla'\xi\\0\end{pmatrix}\mathrm{d}x+\int_{\Gamma_1}\mathsf{q}_*(t)\xi\,\mathrm{da}&\\ +\int_{\Omega_1}\big(\mathsf{Q}_*(t)+\mathbb{D}\kappa_\varepsilon(\dot u_\varepsilon(t)):\kappa_\varepsilon(\dot u_\varepsilon(t))+\Theta(w_\varepsilon(t))\mathbb{B}:\kappa_\varepsilon(\dot u_\varepsilon(t))\big)\xi\,\mathrm{d}x&\end{aligned}$$

In particular, we obtain

$$\begin{aligned}|\langle\dot W_\varepsilon(t),\xi\rangle|\le\big\{C_{\mathcal{K}}\|\nabla_\varepsilon w_\varepsilon(t)\|_{\mathrm{L}^r(\Omega_1)}+\|\mathsf{q}_*(t)\|_{\mathrm{L}^1(\Gamma_1)}+\|\mathsf{Q}_*(t)\|_{\mathrm{L}^1(\Omega_1)}&\\ +C_{\mathbb{D}}\|\kappa_\varepsilon(\dot u_\varepsilon(t))\|^2_{\mathrm{L}^2(\Omega_1)}+C_{\mathbb{B}}\|\Theta(w_\varepsilon(t))\|_{\mathrm{L}^2(\Omega_1)}\|\kappa_\varepsilon(\dot u_\varepsilon(t))\|_{\mathrm{L}^2(\Omega_1)}\big\}\|\xi\|_{\mathrm{W}^{1,r'}(\Omega_1)},&\end{aligned}$$

where we have used the continuous embedding $\mathrm{W}^{1,r'}(\Omega_1)\hookrightarrow\mathrm{L}^\infty(\Omega_1)$ (note that $r'>d+2$ with $d=3$) and the continuity of the trace operator $\mathrm{W}^{1,r'}(\Omega_1)\to\mathrm{L}^\infty(\Gamma_1)$. Thus, we have proven the claim.

□

Lemma 12. *Let $(w_\varepsilon,u_\varepsilon)$ be the rescaled solutions in Theorem 10, then, the sequence w_ε is precompact in $\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))$.*

Proof. With the decomposition in (32), the claim follows from the strong convergence of $\vartheta_\varepsilon\to0$ in $\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))$ and using Lemma 11 and the Aubin–Lions lemma for W_ε, see [Rou13, Ch. 7]. □

4 Proof of Theorem 7

We are now in position to pass to the limit in the rescaled system (12) and (13).

Step 1. Let $(u_\varepsilon, w_\varepsilon)_{\varepsilon>0}$ be the rescaled solutions from Theorem 10, then, due to the uniform estimates (with respect to $\varepsilon > 0$) in Proposition 9, we can extract (non-relabelled) subsequences such that

$$u_\varepsilon \rightharpoonup u \text{ in } \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega_1;\mathbb{R}^3)), \quad \kappa_\varepsilon(u_\varepsilon) \rightharpoonup \kappa \text{ in } \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_1;\mathbb{R}^{3\times3})), \tag{33}$$

with limits $u \in \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega;\mathbb{R}^3))$ and $\kappa \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}))$. Moreover, since $\kappa_\varepsilon(u_\varepsilon) = \mathbb{S}_\varepsilon \mathsf{e}(u_\varepsilon)\mathbb{S}_\varepsilon$ is bounded in $\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}))$, we obtain $\mathsf{e}_{i3}(u) = 0$ a.e. in $(0,T)\times\Omega_1$ such that $u(t,\cdot) \in \mathcal{V}_{\mathrm{KL}}(\Omega_1)$ for almost every $t \in [0,T]$. Thus, following characterization of Kirchhoff–Love displacements in Lemma 5, we can identify the limit u with a triple $(U_1,U_2,U_3) \in \mathrm{H}^1(0,T;\mathrm{H}^1(\omega)) \times \mathrm{H}^1(0,T;\mathrm{H}^1(\omega)) \times \mathrm{H}^1(0,T;\mathrm{H}^2(\omega))$. We also obtain that $\kappa_{ij} = \mathsf{e}_{ij}(u)$ and $\dot\kappa_{ij} = \mathsf{e}_{ij}(\dot u)$ for $i,j \in \{1,2\}$. For the remaining components, we will use the notation $\varkappa = (\kappa_{13},\kappa_{23},\kappa_{33})^\top \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_1;\mathbb{R}^3))$.

Concerning the thermal part of the internal energy density, we have that

$$w_\varepsilon \rightharpoonup w \text{ in } \mathrm{L}^r(0,T;\mathrm{W}^{1,r}(\Omega_1)), \quad \tfrac{1}{\varepsilon}\partial_{x_3} w_\varepsilon \rightharpoonup \zeta \text{ in } \mathrm{L}^r(0,T;\mathrm{L}^r(\Omega)), \tag{34}$$

from which $w(t,\cdot) \in \mathcal{W}_0$ for almost every $t \in [0,T]$ immediately follows, cf. (11) for the definition of the space $\mathcal{W}_0$. In particular, we identify $w(t) \in \mathcal{W}_0$ with a function $w(t) \in \mathrm{W}^{1,r}(\omega)$. With Lemma 12, we conclude that also $w_\varepsilon \to w$ strongly in $\mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))$. Hence, using the growth condition in Assumption (A4) (see also Remark 2), we conclude that $\Theta(w_\varepsilon) \to \Theta(w)$ in $\mathrm{L}^2((0,T)\times\Omega_1)$.

Step 2. We pass to the limit in the mechanical equation (12) testing with $v \in \mathrm{L}^2(0,T;\mathcal{V}_{\mathrm{KL}}(\Omega_1))$ so that $\mathsf{e}_{i3}(v) = 0$ for $i = 1,2,3$. With the convergences above and the decomposition of $\mathbb{C}$ and $\mathbb{D}$ in (14), we arrive at

$$\int_0^T \int_{\Omega_1} \big(\overline{\mathbb{C}}\bar{\mathsf{e}}(u) + \overline{\mathbb{D}}\bar{\mathsf{e}}(\dot u) + \mathbb{P}_{\mathbb{C}}^*\varkappa + \mathbb{P}_{\mathbb{D}}^*\dot\varkappa + \Theta(w)\overline{\mathbb{B}}\big) : \bar{\mathsf{e}}(v)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle \ell_*(t), v\rangle\,\mathrm{d}t. \tag{35}$$

Next, we test the mechanical equation (12) with $v_\varepsilon = (\varepsilon v', \varepsilon^2 v_3)$, where the horizontal part satisfies $v' \in \mathrm{L}^2(0,T;\mathrm{H}^1_{\Gamma_{\mathrm{D},1}}(\Omega_1;\mathbb{R}^2))$ and the vertical component is such that $v_3 \in \mathrm{L}^2(0,T;\mathrm{H}^1_{\Gamma_{\mathrm{D},1}}(\Omega_1))$. In the limit $\varepsilon \to 0$, we obtain

$$\int_0^T \int_{\Omega_1} \big(\mathbb{P}_{\mathbb{C}}\bar{\mathsf{e}}(u) + \mathbb{P}_{\mathbb{D}}\bar{\mathsf{e}}(\dot u) + \mathbb{M}_{\mathbb{C}}\varkappa + \mathbb{M}_{\mathbb{D}}\dot\varkappa + \Theta(w)\mathsf{b}_{\mathbb{B}}\big) \cdot \begin{pmatrix} \frac12 \partial_{x_3} v' \\ \partial_{x_3} v_3 \end{pmatrix}\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{36}$$

By a variant of the fundamental lemma of calculus of variations [Cia97, Step (iii) of the proof of Theorem 1.4-1.], we deduce that

$$\dot\varkappa + \mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}}\varkappa = -\mathbb{M}_{\mathbb{D}}^{-1}(\mathbb{P}_{\mathbb{C}}\bar{\mathsf{e}}(u) + \mathbb{P}_{\mathbb{D}}\bar{\mathsf{e}}(\dot u) + \Theta(w)\mathsf{b}_{\mathbb{B}}) =: \mathbb{F}[\bar{\mathsf{e}}(u), \bar{\mathsf{e}}(\dot u), \Theta(w)] \text{ a.e. in } (0,T)\times\Omega_1. \tag{37}$$

Note that $\mathbb{M}_{\mathbb{D}} \in \mathbb{R}^{3\times3}_{\mathsf{sym}}$ is invertible, since $\mathbb{D}$ is positive definite. Let $\mathbb{E} : [0,T] \to \mathbb{R}^{3\times3}$ denote the matrix exponential $\mathbb{E}(\tau) := \exp(-\tau\mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}})$ such that $\dot{\mathbb{E}} = -\mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}}\mathbb{E}$ and $\mathbb{E}(0) = \mathbb{I}$. Thus, we find that the unique solution of the ODE (parametrized by $x \in \Omega_1$) in (37) is given by

$$\begin{aligned}\varkappa(t,x) &= \mathcal{A}[\bar{\mathsf{e}}(u), \bar{\mathsf{e}}(\dot u), \Theta(w), \varkappa^0](t,x) \\ &:= \int_0^t \mathbb{E}(t-s)\mathbb{F}\big[\bar{\mathsf{e}}(u), \bar{\mathsf{e}}(\dot u), \Theta(w)\big](s,x)\,\mathrm{d}s + \mathbb{E}(t)\varkappa^0(x),\end{aligned} \tag{38}$$

where the initial value satisfies $\varkappa^0 = (\kappa^0_{13}, \kappa^0_{23}, \kappa^0_{33})^\top \in \mathrm{L}^2(\Omega_1; \mathbb{R}^3)$ with κ^0 from Assumption (B9). We highlight that $\mathcal{A} : \mathrm{L}^2(0,T; \mathrm{L}^2(\Omega_1; \mathbb{R}^{2\times 2}\times\mathbb{R}^{2\times 2}\times\mathbb{R})) \times \mathrm{L}^2(\Omega_1;\mathbb{R}^3) \to \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_1;\mathbb{R}^3))$ is affine in $u, \dot{u}$, and $\Theta(w)$.

Step 3. For the limit passage in (13), we choose a test function $\xi \in \mathrm{W}^{1,r'}(0,T;\mathrm{W}^{1,r'}(\Omega_1))$ with $\xi(T) = 0$ and multiply the equation by $\varepsilon > 0$. Using the convergences above, we can pass to the limit $\varepsilon \to 0$ and arrive at

$$\int_0^T \int_{\Omega_1} \big(k_1(w) \cdot \nabla' w + k_2(w)\zeta\big)\partial_{x_3}\xi \,\mathrm{d}x\,\mathrm{d}t = 0,$$

where $\zeta \in \mathrm{L}^r(0,T;\mathrm{L}^r(\Omega_1))$ is the limit of $\frac{1}{\varepsilon}\partial_{x_3} w_\varepsilon$. We conclude that $\zeta = Z_3^*(w, \nabla' w) = -(k_1(w)\cdot \nabla' w)/k_2(w)$ (cf. (16)).

Arguing as in the proof of Theorem 10, we can assume that $\kappa_\varepsilon(\dot{u}_\varepsilon) \to \dot{\kappa}$ in $\mathrm{L}^2((0,T)\times\Omega_1;\mathbb{R}^{3\times 3})$. Thus, testing (13) with ξ as before and additionally such that $\partial_{x_3}\xi = 0$ gives in the limit

$$\begin{aligned}
&\int_0^T \int_{\Omega_1} \big(-w\dot{\xi} + \mathcal{K}_{\mathrm{eff}}(w)\nabla' w \cdot \nabla'\xi\big)\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_1} w^0 \xi(0,\cdot)\,\mathrm{d}x \\
&\quad = \int_0^T \int_{\Omega_1} \big(\mathsf{Q}_* + \mathbb{D}\dot{\kappa} : \dot{\kappa} + \Theta(w)\mathbb{B} : \dot{\kappa}\big)\xi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Gamma_1^\parallel} \mathsf{q}_*^\parallel \xi \,\mathrm{d}a\,\mathrm{d}t + \int_0^T \int_{\Gamma_1^\perp} \mathsf{q}_*^\perp \xi\,\mathrm{d}a\,\mathrm{d}t.
\end{aligned} \tag{39}$$

Note that for the viscous and the adiabatic heat sources, we can write

$$\begin{aligned}
\mathbb{D}\dot{\kappa} : \dot{\kappa} &= \overline{\mathbb{D}}\bar{\mathsf{e}}(\dot{u}) : \bar{\mathsf{e}}(\dot{u}) + 2\mathbb{P}_{\mathbb{D}}\bar{\mathsf{e}}(\dot{u}) \cdot \dot{\varkappa} + \mathbb{M}_{\mathbb{D}}\dot{\varkappa}\cdot\dot{\varkappa}, \\
\Theta(w)\mathbb{B} : \dot{\kappa} &= \Theta(w)\overline{\mathbb{B}} : \bar{\mathsf{e}}(\dot{u}) + \Theta(w)\mathsf{b}_{\mathbb{B}}\cdot\dot{\varkappa}.
\end{aligned} \tag{40}$$

Step 4. Finally, we perform the actual dimension reduction. Starting point are the limiting equations in (35) and (39) and the characterization of Kirchhoff–Love displacements in Lemma 5.

Step 4.1 Reduction of mechanical equation. Let us first look at the mechanical equation in (35). We denote by $(U_1(t), U_2(t), U_3(t)) \in \mathrm{H}^1(\omega)\times\mathrm{H}^1(\omega)\times\mathrm{H}^2(\omega)$ the displacements associated with the limit $u(t)$, i.e., $u_1 = U_1 - x_3\partial_{x_1}U_3$, $u_2 = U_2 - x_3\partial_{x_2}U_3$, and $u_3 = U_3$. Analogously, we consider a test function $v \in \mathrm{L}^2(0,T;\mathcal{V}_{\mathrm{KL}}(\Omega_1))$ with associated (V_1,V_2,V_3), see Lemma 5. Then, using that $\int_{-1/2}^{1/2} x_3\,\mathrm{d}x_3 = 0$ and $\int_{-1/2}^{1/2} x_3^2\,\mathrm{d}x_3 = \frac{1}{12}$, we get the viscous membrane model for the in-plane displacements $\overline{U}$

$$\int_0^T \int_\omega \big(\overline{\mathbb{C}}\bar{\mathsf{e}}(\overline{U}) + \overline{\mathbb{D}}\bar{\mathsf{e}}(\dot{\overline{U}}) + \mathbb{P}^*_{\mathbb{C}}\varkappa_{\mathrm{even}} + \mathbb{P}^*_{\mathbb{D}}\dot{\varkappa}_{\mathrm{even}} + \Theta(W)\overline{\mathbb{B}}\big) : \bar{\mathsf{e}}(\overline{V})\,\mathrm{d}x'\,\mathrm{d}t = \int_0^T \langle L_{1,2}(t), \overline{V}\rangle\,\mathrm{d}t, \tag{41}$$

and Kirchhoff's plate equation for the out-of-plane displacement U_3 with viscous evolution

$$\frac{1}{12}\int_0^T \int_\omega \big(\overline{\mathbb{C}}\nabla^2 U_3 + \overline{\mathbb{D}}\nabla^2\dot{U}_3 + \mathbb{P}^*_{\mathbb{C}}\varkappa_{\mathrm{odd}} + \mathbb{P}^*_{\mathbb{D}}\dot{\varkappa}_{\mathrm{odd}}\big) : \nabla^2 V_3\,\mathrm{d}x'\,\mathrm{d}t = \int_0^T \langle L_3(t), V_3\rangle\,\mathrm{d}t, \tag{42}$$

where $\bar{\mathsf{e}}_{ij}(\overline{U}) = \frac{1}{2}(\partial_{x_i}U_j + \partial_{x_j}U_i)$ for $i,j = 1,2$. Moreover, the external loading for in-plane displacements $L_{1,2} \in \mathrm{L}^2(0,T;\mathrm{H}^1(\omega)^2)$ and the out-of-plane displacements $L_3 \in \mathrm{L}^2(0,T;\mathrm{H}^2(\omega))$ are defined via

$$\begin{aligned}\langle L_{1,2}(t), \overline{V}\rangle &= \int_\omega F_{1,2}(t)\cdot \overline{V}\,\mathrm{d}x' + \int_{\partial\omega\setminus\gamma_{\mathrm{D}}} H_{1,2}(t)\cdot\overline{V}\,\mathrm{da}' \\ \langle L_3(t), V_3\rangle &= \int_\omega F_3(t)V_3 + \widetilde{F}_3(t)\cdot\nabla' V_3\,\mathrm{d}x' + \int_{\partial\omega\setminus\gamma_{\mathrm{D}}} H_3(t)V_3 + \widetilde{H}_3(t)\cdot\nabla' V_3\,\mathrm{da}',\end{aligned} \tag{43}$$

where the densities $F_{1,2}, F_3, \widetilde{F}_3, H_{1,2}, H_3, \widetilde{H}_3$ are defined via averages of f_* and h_*, see [Cia97, Theorem 1.4-1.]. We have also introduced the averaged quantities

$$\varkappa_{\mathrm{even}}(t,x') = \int_{-1/2}^{1/2}\varkappa(t,x',z)\,\mathrm{d}z \quad\text{and}\quad \varkappa_{\mathrm{odd}}(t,x') = 12\int_{-1/2}^{1/2} z\varkappa(t,x',z)\,\mathrm{d}z. \tag{44}$$

Using (37), we check that $(\varkappa_{\mathrm{even}},\varkappa_{\mathrm{odd}}) \in \mathrm{H}^1(0,T;\mathrm{L}^2(\omega;\mathbb{R}^3))\times\mathrm{H}^1(0,T;\mathrm{L}^2(\omega;\mathbb{R}^3))$ satisfies

$$\dot\varkappa_{\mathrm{even}} + \mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}}\varkappa_{\mathrm{even}} = -\mathbb{M}_{\mathbb{D}}^{-1}(\mathbb{P}_{\mathbb{C}}\bar{\mathsf{e}}(\overline{U}) + \mathbb{P}_{\mathbb{D}}\bar{\mathsf{e}}(\dot U_1,\dot U_2) + \Theta(W)\mathsf{b}_{\mathbb{B}}), \tag{45}$$

$$\dot\varkappa_{\mathrm{odd}} + \mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}}\varkappa_{\mathrm{odd}} = -\mathbb{M}_{\mathbb{D}}^{-1}(\mathbb{P}_{\mathbb{C}}\nabla^2 U_3 + \mathbb{P}_{\mathbb{D}}\nabla^2\dot U_3). \tag{46}$$

The initial conditions are $\varkappa^0_{\mathrm{even}}(x') = \int_{-1/2}^{1/2}\varkappa^0(x',z)\,\mathrm{d}z$ and $\varkappa^0_{\mathrm{odd}}(x') = 12\int_{-1/2}^{1/2} z\varkappa^0(x',z)\,\mathrm{d}z$, respectively. Note that by Assumption (B9), $x_3\mapsto\varkappa^0(x',x_3)$ is affine for almost every $x'\in\omega$. Thus, we can write $\varkappa^0(x',x_3) = \varkappa^0_{\mathrm{even}}(x') + x_3\varkappa^0_{\mathrm{odd}}(x')$. Obviously, it holds that

$$\varkappa_{\mathrm{even}} = \mathcal{A}_{\mathrm{even}}\big[\bar{\mathsf{e}}(\overline{U}),\bar{\mathsf{e}}(\dot U_1,\dot U_2),\Theta(W),\varkappa^0_{\mathrm{even}}\big],\quad \varkappa_{\mathrm{odd}} = \mathcal{A}_{\mathrm{odd}}\big[\nabla^2 U_3,\nabla^2\dot U_3,\varkappa^0_{\mathrm{odd}}\big],$$

where $\mathcal{A}_{\mathrm{even}}$ and $\mathcal{A}_{\mathrm{odd}}$ are defined via $\mathcal{A}$. In fact, we have the identity $\varkappa(t,x',z) = \varkappa_{\mathrm{even}}(t,x') + z\varkappa_{\mathrm{odd}}(t,x')$.

Step 4.2 Reduction of heat equation. For the reduced heat equation, we first consider the heat sources on the right-hand side of (39). We first define the fixed bulk heat source

$$\overline{\mathsf{Q}}_*(t,x') = \int_{-1/2}^{1/2}\mathsf{Q}_*(t,x',z)\,\mathrm{d}z + \mathsf{q}_*^{\parallel}(t,x',\tfrac{1}{2}) + \mathsf{q}_*^{\parallel}(t,x',-\tfrac{1}{2}),\quad \text{for } x'\in\omega, \tag{47}$$

and the surface heat source

$$\bar{\mathsf{q}}_*^{\perp}(t,x') = \int_{-1/2}^{1/2}\mathsf{q}_*^{\perp}(t,x',z)\,\mathrm{d}z,\quad \text{for } x'\in\partial\omega.$$

Using the identities in (40), we compute for the viscous heating terms

$$\begin{aligned}\mathcal{Q}_{\mathsf{visc}}(\bar{\mathsf{e}}(\dot{\overline{U}}),\nabla^2\dot U_3,\dot\varkappa_{\mathrm{odd}},\dot\varkappa_{\mathrm{even}}) &:= \int_{-1/2}^{1/2}\mathbb{D}\kappa : \kappa\,\mathrm{d}z \\ &= \overline{\mathbb{D}}\bar{\mathsf{e}}(\dot{\overline{U}}) : \bar{\mathsf{e}}(\dot{\overline{U}}) + \frac{1}{12}\overline{\mathbb{D}}\nabla^2\dot U_3 : \nabla^2\dot U_3 + 2\mathbb{P}_{\mathbb{D}}\bar{\mathsf{e}}(\dot{\overline{U}})\cdot\dot\varkappa_{\mathrm{even}} \\ &\quad + \frac{1}{6}\mathbb{P}_{\mathbb{D}}\nabla^2 U_3\cdot\dot\varkappa_{\mathrm{odd}} + \mathbb{M}_{\mathbb{D}}\dot\varkappa_{\mathrm{even}}\cdot\dot\varkappa_{\mathrm{even}} + \frac{1}{12}\mathbb{M}_{\mathbb{D}}\dot\varkappa_{\mathrm{odd}}\cdot\dot\varkappa_{\mathrm{odd}}.\end{aligned} \tag{48}$$

We emphasize that for this decomposition, it was essential that $\varkappa^0$ (see (B9)) and hence also $\varkappa(t)$ are affine in x_3.

Similarly, we obtain the adiabatic heat sources

$$\mathcal{Q}_{\mathsf{adia}}\big(\overline{\mathsf{e}}(\dot{\overline{U}}), \dot{\varkappa}_{\text{even}}, \Theta(W)\big) := \int_{-1/2}^{1/2} \Theta(W)\overline{\mathbb{B}} : \dot{\kappa}\,\mathrm{d}z = \Theta(W)\overline{\mathbb{B}} : \overline{\mathsf{e}}(\dot{\overline{U}}) + \Theta(W)\mathsf{b}_{\mathbb{B}} \cdot \dot{\varkappa}_{\text{even}}. \tag{49}$$

To simplify the notation, we introduce the total heat source

$$\begin{aligned} &\mathcal{Q}_{\text{tot}}\big(t, x'; \overline{\mathsf{e}}(\dot{\overline{U}}), \nabla^2 U_3, \dot{\varkappa}_{\text{even}}, \dot{\varkappa}_{\text{odd}}, \Theta(W)\big) = \\ &\qquad \overline{\mathsf{Q}}_*(t, x') + \mathcal{Q}_{\mathsf{visc}}(\overline{\mathsf{e}}(\dot{\overline{U}}), \nabla^2 \dot{U}_3, \dot{\varkappa}_{\text{odd}}, \dot{\varkappa}_{\text{even}}) + \mathcal{Q}_{\mathsf{adia}}\big(\overline{\mathsf{e}}(\dot{\overline{U}}), \dot{\varkappa}_{\text{even}}, \Theta(W)\big). \end{aligned} \tag{50}$$

Thus, we find that the effective heat equation in the weak form reads

$$\begin{aligned} &\int_0^T \int_\omega \big(-W\dot{\xi} + \mathcal{K}_{\text{eff}}(W)\nabla' W \cdot \nabla'\xi\big)\,\mathrm{d}x\,\mathrm{d}t - \int_\omega W^0 \xi(0,\cdot)\,\mathrm{d}x \\ &\quad = \int_0^T \int_\omega \mathcal{Q}_{\text{tot}}\big(t, x'; \overline{\mathsf{e}}(\dot{\overline{U}}), \nabla^2 U_3, \dot{\varkappa}_{\text{even}}, \dot{\varkappa}_{\text{odd}}, \Theta(W)\big)\xi\,\mathrm{d}x'\,\mathrm{d}t + \int_0^T \int_{\partial\omega} \overline{\mathsf{q}}_*^{\perp} \xi\,\mathrm{d}a\,\mathrm{d}t. \end{aligned} \tag{51}$$

Thus, we have proven Theorem 7.

5 Discussion of the limit system

5.1 The isotropic case

Let us consider the isotropic case, where $\mathbb{C}$ and $\mathbb{D}$ are given in terms of the elastic and viscous Lamé parameters $\lambda_{\mathsf{el}}, \mu_{\mathsf{el}}$ and $\lambda_{\mathsf{vi}}, \mu_{\mathsf{vi}}$, respectively, namely

$$\mathbb{C}e = \lambda_{\mathsf{el}}\mathrm{tr}(e)\mathbb{I} + 2\mu_{\mathsf{el}}e, \quad \text{and} \quad \mathbb{D}\dot{e} = \lambda_{\mathsf{vi}}\mathrm{tr}(\dot{e})\mathbb{I} + 2\mu_{\mathsf{vi}}\dot{e}.$$

Moreover, we assume that $\mathbb{B} = b\mathbb{I}$ and $\mathcal{K}(w) = k\mathbb{I}$ for constants $b, k > 0$.

In this case, we obtain for $\overline{e} \in \mathbb{R}^{2\times 2}$

$$\overline{\mathbb{C}}\overline{e} = \lambda_{\mathsf{el}}\mathrm{tr}(\overline{e})\mathbb{I}_{2\times 2} + 2\mu_{\mathsf{el}}\overline{e}, \quad \mathbb{P}_{\mathbb{C}}\overline{e} = \begin{pmatrix} 0 \\ 0 \\ \lambda_{\mathsf{el}}\mathrm{tr}\,\overline{e} \end{pmatrix}, \quad \mathbb{M}_{\mathbb{C}} = \begin{pmatrix} 4\mu_{\mathsf{el}} & 0 & 0 \\ 0 & 4\mu_{\mathsf{el}} & 0 \\ 0 & 0 & \lambda_{\mathsf{el}} + 2\mu_{\mathsf{el}} \end{pmatrix}.$$

Analogous formulas hold for $\overline{\mathbb{D}}, \mathbb{P}_{\mathbb{D}}$, and $\mathbb{M}_{\mathbb{D}}$. Thus, the evolution equations in (45) and (46) for the out-of-plane strains $\varkappa_{\text{even}} = (\varkappa_1^{\text{even}}, \varkappa_2^{\text{even}}, \varkappa_3^{\text{even}})$ and $\varkappa_{\text{odd}} = (\varkappa_1^{\text{odd}}, \varkappa_2^{\text{odd}}, \varkappa_3^{\text{odd}})$ take the simpler form

$$\begin{aligned} \dot{\varkappa}_i^{\text{even}} + \frac{\mu_{\mathsf{el}}}{\mu_{\mathsf{vi}}}\varkappa_i^{\text{even}} &= 0 \quad \text{for } i = 1, 2, \\ \dot{\varkappa}_i^{\text{odd}} + \frac{\mu_{\mathsf{el}}}{\mu_{\mathsf{vi}}}\varkappa_i^{\text{odd}} &= 0 \quad \text{for } i = 1, 2, \\ \dot{\varkappa}_3^{\text{even}} + \frac{\lambda_{\mathsf{el}} + 2\mu_{\mathsf{el}}}{\lambda_{\mathsf{vi}} + 2\mu_{\mathsf{vi}}}\varkappa_3^{\text{even}} &= -\frac{\lambda_{\mathsf{el}}\mathrm{div}'\overline{U} + \lambda_{\mathsf{vi}}\mathrm{div}'\dot{\overline{U}} + \Theta(w)\mathsf{b}_{\mathbb{B}}}{\lambda_{\mathsf{vi}}+2\mu_{\mathsf{vi}}}, \\ \dot{\varkappa}_3^{\text{odd}} + \frac{\lambda_{\mathsf{el}} + 2\mu_{\mathsf{el}}}{\lambda_{\mathsf{vi}} + 2\mu_{\mathsf{vi}}}\varkappa_3^{\text{odd}} &= -\frac{\lambda_{\mathsf{el}}\Delta' U_3 + \lambda_{\mathsf{vi}}\Delta'\dot{U}_3}{\lambda_{\mathsf{vi}}+2\mu_{\mathsf{vi}}}. \end{aligned}$$

In particular, we have exponential decay $\varkappa_i^{\text{even}}(t), \varkappa_i^{\text{odd}}(t) \sim \mathrm{e}^{-(\mu_{\text{el}}/\mu_{\text{vi}})t}$ for $i = 1, 2$, independently of $\overline{U}, U_3$ or W. In particular, $\varkappa_i^{\text{even}}$, $\varkappa_i^{\text{even}}$ only act as additional external loadings and sources in the equations for U_1, U_2, U_3 and W, respectively.

5.2 Thermodynamic consistency of the effective model

The effective system derived in the previous system is thermodynamically consistent in that it satisfies conservation of internal energy and positivity of entropy production.

For a given symmetric matrix $A \in \mathbb{R}^{2\times 2}_{\text{sym}}$ and a vector $\varkappa \in \mathbb{R}^3$, we define the symmetric matrix $(A|\varkappa) \in \mathbb{R}^{3\times 3}_{\text{sym}}$ via

$$(A|\varkappa) := \begin{pmatrix} A_{11} & A_{12} & \varkappa_1 \\ A_{21} & A_{22} & \varkappa_2 \\ \varkappa_1 & \varkappa_2 & \varkappa_3 \end{pmatrix}.$$

For the effective two-dimensional system derived in the previous section, we introduce the effective free energy that is decomposed into a membrane and a bending part

$$\begin{aligned} \psi_{\text{eff}}(\overline{e}_1, \overline{e}_2, \varkappa_1, \varkappa_2, \theta) &= \psi^{\text{memb}}_{\text{eff}}(\overline{e}_1, \varkappa_1, \theta) + \psi^{\text{bend}}_{\text{eff}}(\overline{e}_2, \varkappa_2) - \psi_0(\theta), \quad \text{where} \\ \psi^{\text{memb}}_{\text{eff}}(\overline{e}_1, \varkappa_1, \theta) &= \frac{1}{2}\mathbb{C}(\overline{e}_1|\varkappa_1) : (\overline{e}_1|\varkappa_1) + \theta\mathbb{B} : (\overline{e}_1|\varkappa_1), \\ \psi^{\text{bend}}_{\text{eff}}(\overline{e}_2, \varkappa_2) &= 6\mathbb{C}(\overline{e}_2|\varkappa_2) : (\overline{e}_2|\varkappa_2), \end{aligned}$$

where $\overline{e}_1$ and $\overline{e}_2$ are placeholders for $\mathsf{e}(\overline{U})$ and $\nabla^2 U_3$, and $\varkappa_1$ and $\varkappa_2$ are placeholders for $\varkappa_{\text{even}}$ and $\varkappa_{\text{odd}}$, respectively. Analogously, we define the effective dissipation potential

$$\begin{aligned} \zeta_{\text{eff}}(\dot{\overline{e}}_1, \dot{\overline{e}}_2, \dot{\varkappa}_1, \dot{\varkappa}_2) &= \zeta^{\text{memb}}_{\text{eff}}(\dot{\overline{e}}_1, \dot{\varkappa}_1) + \zeta^{\text{bend}}_{\text{eff}}(\dot{\overline{e}}_2, \dot{\varkappa}_2), \quad \text{where} \\ \zeta^{\text{memb}}_{\text{eff}}(\dot{\overline{e}}_1, \dot{\varkappa}_1) &= \frac{1}{2}\mathbb{D}(\dot{\overline{e}}_1|\varkappa_1) : (\dot{\overline{e}}_1|\varkappa_1), \\ \zeta^{\text{bend}}_{\text{eff}}(\dot{\overline{e}}_2, \dot{\varkappa}_2) &= 6\mathbb{D}(\dot{\overline{e}}_2|\dot{\varkappa}_2) : (\dot{\overline{e}}_2|\dot{\varkappa}_2). \end{aligned}$$

The effective entropy density and the effective internal energy density are given by $\eta_{\text{eff}} = -\partial_\theta \psi_{\text{eff}}$ and $E_{\text{eff}} = \psi_{\text{eff}} + \theta\eta_{\text{eff}}$, respectively, i.e.,

$$\begin{aligned} \eta_{\text{eff}}(\overline{e}_1, \varkappa_1, \theta) &= \eta^{\text{memb}}_{\text{eff}}(\overline{e}_1, \varkappa_1, \theta) = -\mathbb{B} : (\overline{e}_1|\varkappa_1) + \psi_0'(\theta) \\ E_{\text{eff}}(\overline{e}_1, \overline{e}_2, \varkappa_1, \varkappa_2, \theta) &= E^{\text{memb}}_{\text{eff}}(\overline{e}_1, \varkappa_1, \theta) + E^{\text{bend}}_{\text{eff}}(\overline{e}_2, \varkappa_2) + \theta\psi_0'(\theta) - \psi_0(\theta), \quad \text{where} \\ E^{\text{memb}}_{\text{eff}}(\overline{e}_1, \varkappa_1) &= \frac{1}{2}\mathbb{C}(\overline{e}_1|\varkappa_1) : (\overline{e}_1|\varkappa_1), \quad E^{\text{bend}}_{\text{eff}}(\overline{e}_2, \varkappa_2) = 6\mathbb{C}(\overline{e}_2|\varkappa_2) : (\overline{e}_2|\varkappa_2). \end{aligned}$$

In particular, with the change of variables $\theta = \Theta(W)$, we have

$$\widehat{E}_{\text{eff}}(\overline{e}_1, \overline{e}_2, \varkappa_1, \varkappa_2, W) = E_{\text{eff}}(\overline{e}_1, \overline{e}_2, \varkappa_1, \varkappa_2, \Theta(W)) = E^{\text{memb}}_{\text{eff}}(\overline{e}_1, \varkappa_1) + E^{\text{bend}}_{\text{eff}}(\overline{e}_2, \varkappa_2) + W.$$

With these definitions, we can rewrite the effective system in (41), (42), (45), (46), and (51) formally as

$$\begin{aligned} -\mathrm{div}'\big[\partial_{\overline{e}_1}\psi^{\text{memb}}_{\text{eff}}(\mathsf{e}(\overline{U}), \varkappa_{\text{even}}, \Theta(W)) + \partial_{\dot{\overline{e}}_1}\zeta^{\text{memb}}_{\text{eff}}(\mathsf{e}(\dot{\overline{U}}), \dot{\varkappa}_{\text{even}})\big] &= F_{1,2}(t, x'), \\ \mathrm{div}'\mathrm{div}'\big[\partial_{\overline{e}_2}\psi^{\text{bend}}_{\text{eff}}(\nabla^2 U_3, \varkappa_{\text{odd}}) + \partial_{\dot{\overline{e}}_2}\zeta^{\text{bend}}_{\text{eff}}(\nabla^2 \dot{U}_3, \dot{\varkappa}_{\text{odd}})\big] &= F_3(t, x') - \mathrm{div}'\widetilde{F}_3(t, x'), \\ \partial_{\varkappa_1}\psi^{\text{memb}}_{\text{eff}}(\mathsf{e}(\overline{U}), \varkappa_{\text{even}}, \Theta(W)) + \partial_{\dot{\varkappa}_1}\zeta^{\text{memb}}_{\text{eff}}(\mathsf{e}(\dot{\overline{U}}), \dot{\varkappa}_{\text{even}}) &= 0, \\ \partial_{\varkappa_2}\psi^{\text{bend}}_{\text{eff}}(\nabla^2 U_3, \varkappa_{\text{odd}}) + \partial_{\dot{\varkappa}_2}\zeta^{\text{bend}}_{\text{eff}}(\nabla^2 \dot{U}_3, \dot{\varkappa}_{\text{odd}}) &= 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\widehat{E}_{\text{eff}} - \mathrm{div}'(\mathcal{K}_{\text{eff}}(W)\nabla' W) &= \overline{\mathsf{Q}}_*(t, x') \\ &\quad + \partial_{(\dot{\overline{e}}_1, \dot{\varkappa}_1)}\zeta^{\text{memb}}_{\text{eff}} : (\mathsf{e}(\dot{\overline{U}})|\dot{\varkappa}_{\text{even}}) + \partial_{(\dot{\overline{e}}_2, \dot{\varkappa}_2)}\zeta^{\text{bend}}_{\text{eff}} : (\nabla^2 \dot{U}_3|\dot{\varkappa}_{\text{odd}}) \\ &\quad + [\partial_{(\overline{e}_1, \varkappa_1)}\widehat{E}_{\text{eff}} - \Theta(W)\partial_{(\overline{e}_1, \varkappa_1)}\widehat{\eta}_{\text{eff}}] : (\mathsf{e}(\dot{\overline{U}})|\dot{\varkappa}_{\text{even}}) + \partial_{(\overline{e}_2, \varkappa_2)}\widehat{E}_{\text{eff}} : (\nabla^2 \dot{U}_3|\dot{\varkappa}_{\text{odd}}). \end{aligned}$$

Note that in the $\partial_{(\overline{e}_2,\varkappa_2)}\widehat{\eta}_{\mathsf{eff}}$ does not appear in the right-hand side since $\widehat{\eta}_{\mathsf{eff}}$ does not depend on $(\overline{e}_2,\varkappa_2)$. Testing, the first four equations with $\dot{\overline{U}}, \dot{U}_3$, $\dot{\varkappa}_{\mathsf{even}}$, and $\varkappa_{\mathsf{odd}}$, respectively, gives with the chain rule the mechanical energy balance

$$\int_\omega \frac{\mathrm{d}}{\mathrm{d}t}(E^{\mathsf{memb}}_{\mathsf{eff}}+E^{\mathsf{bend}}_{\mathsf{eff}}) + 2\zeta_{\mathsf{eff}} + \Theta(W)\mathbb{B} : (\overline{\mathsf{e}}(\dot{\overline{U}})|\dot{\varkappa}_{\mathsf{even}})\,\mathrm{d}x = \langle L_{1,2}(t), \dot{\overline{U}}\rangle + \langle L_3(t), \dot{U}_3\rangle.$$

Using the heat equation in the above form, finally gives the total energy balance for the effective system, namely,

$$\int_\omega \frac{\mathrm{d}}{\mathrm{d}t}\widehat{E}_{\mathsf{eff}}\,\mathrm{d}x' = \langle L_{1,2}(t), \dot{\overline{U}}\rangle + \langle L_3(t), \dot{U}_3\rangle + \int_\omega \overline{\mathsf{Q}}_*(t,x')\,\mathrm{d}x' + \int_{\partial\omega} \overline{\mathsf{q}}^\perp_*\,\mathrm{da}.$$

Similarly, the positivity of the entropy production is shown.

5.3 Reduced model with memory kernel

Since the evolution equations for the out-of-plane strains $\varkappa_{\mathrm{even}}$ and $\varkappa_{\mathrm{odd}}$ can be solved explicitly in terms of U_1, U_2, U_3 and $\Theta(W)$, we can give an even more reduced formulation. Indeed, let us denote by $\varkappa_*$ either $\varkappa_{\mathrm{even}}$ or $\varkappa_{\mathrm{odd}}$ solving (45) or (46), respectively. We have the explicit formula

$$\varkappa_*(t,\cdot) = -\int_0^t \mathbb{E}(t-s)\mathbb{M}_{\mathbb{D}}^{-1}\big(\mathbb{P}_{\mathbb{C}}\sigma_*(s) + \mathbb{P}_{\mathbb{D}}\dot{\sigma}_*(s) + \xi_*(s)\big)\,\mathrm{d}s + \mathbb{E}(t)\varkappa_0^*,$$

where σ_* either denotes $\overline{\mathsf{e}}(\overline{U})$ or $\nabla^2 U_3$ and ξ_* is either $\Theta(W)\mathsf{b}_{\mathbb{B}}$ or 0.

After integrating by parts and using that $\dot{\mathbb{E}} = -\mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}}\mathbb{E}$, we get that

$$\begin{aligned}\varkappa_*(t,\cdot) = \int_0^t \mathbb{E}(t-s)\big(\{\mathbb{M}_{\mathbb{C}}^{-1}\mathbb{P}_{\mathbb{C}} - \mathbb{M}_{\mathbb{D}}^{-1}\mathbb{P}_{\mathbb{D}}\}\dot{\sigma}_*(s) - \mathbb{M}_{\mathbb{D}}^{-1}\xi_*(s)\big)\,\mathrm{d}s - \mathbb{M}_{\mathbb{C}}^{-1}\mathbb{P}_{\mathbb{C}}\sigma_*(t)\\ + \mathbb{E}(t)(\mathbb{M}_{\mathbb{C}}^{-1}\mathbb{P}_{\mathbb{C}}\sigma_*(0) + \varkappa_0^*).\end{aligned}$$

Assuming that the initial value is given as $\varkappa_0^* = -\mathbb{M}_{\mathbb{C}}^{-1}\mathbb{P}_{\mathbb{C}}\sigma_*(0)$, cf. Remark 6(ii), allows us to drop the last term. With this formula, we can rewrite the membrane equation in (41) in the following way

$$\begin{aligned}\int_\omega \Big(\mathbb{C}^{\mathrm{KL}}\overline{\mathsf{e}}(\overline{U}) + \mathbb{D}^{\mathrm{KL}}\overline{\mathsf{e}}(\dot{\overline{U}}) + \int_0^t K(t-s)[\overline{\mathsf{e}}(\dot{\overline{U}})(s), \Theta(W(s))\mathsf{b}_{\mathbb{B}}]\,\mathrm{d}s\\ + \Theta(W)\overline{\mathbb{B}} - \mathbb{P}_{\mathbb{D}}^*\mathbb{M}_{\mathbb{D}}^{-1}\Theta(W(s))\mathsf{b}_{\mathbb{B}}\Big) : \overline{\mathsf{e}}(\overline{V})\,\mathrm{d}x' = \langle L_{1,2}(t), \overline{V}\rangle?\end{aligned}$$

and plate equation in (42) as

$$\frac{1}{12}\int_\omega \big(\mathbb{C}^{\mathrm{KL}}\nabla^2 U_3 + \mathbb{D}^{\mathrm{KL}}\nabla^2\dot{U}_3 + \int_0^t K(t-s)[\nabla^2\dot{U}_3(s), 0]\,\mathrm{d}s\Big) : \nabla^2 V_3\,\mathrm{d}x' = \langle L_3(t), V_3\rangle$$

where the effective Kirchhoff–Love tensors $\mathbb{C}^{\mathrm{KL}} = \overline{\mathbb{C}} - \mathbb{P}_{\mathbb{C}}^*\mathbb{M}_{\mathbb{C}}^{-1}\mathbb{P}_{\mathbb{C}}$ and $\mathbb{D}^{\mathrm{KL}} = \overline{\mathbb{D}} - \mathbb{P}_{\mathbb{D}}^*\mathbb{M}_{\mathbb{D}}^{-1}\mathbb{P}_{\mathbb{D}}$ are the Schur complements of $\mathbb{C}$ and $\mathbb{D}$ with respect to $\mathbb{M}_{\mathbb{C}}$ and $\mathbb{M}_{\mathbb{D}}$, respectively. The memory kernel K is given as in [Lic13] via the formula

$$K(\tau)[\dot{\sigma}_*, \xi_*] = (\mathbb{P}_{\mathbb{C}}^* - \mathbb{P}_{\mathbb{D}}^*\mathbb{M}_{\mathbb{D}}^{-1}\mathbb{M}_{\mathbb{C}})\mathbb{E}(\tau)\big((\mathbb{M}_{\mathbb{C}}^{-1}\mathbb{P}_{\mathbb{C}} - \mathbb{M}_{\mathbb{D}}^{-1}\mathbb{P}_{\mathbb{D}})\dot{\sigma}_* - \mathbb{M}_{\mathbb{D}}^{-1}\xi_*\big). \qquad (52)$$

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