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A relative energy inequality for an anisotropic Navier–Stokes–Nernst–Planck–Poisson system — Weak-strong uniqueness and *a posteriori* error estimates

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Abstract

In this work, we build upon the framework of suitable weak solutions to the anisotropic Navier—Stokes–Nernst–Planck–Poisson (NSNPP) system, as developed in [HLP25], and establish a relative energy inequality for these solutions. This inequality serves as the basis for proving the weak-strong uniqueness property. Additionally, we exploit the relative energy inequality as a tool to obtain *a posteriori* error estimates. We are interested in the high-viscosity–low-Reynolds number limit of the NSNPP, which leads to the anisotropic Stokes–Nernst–Planck–Poisson (SNPP) system. Utilizing the relative energy framework, we derive an error estimate for the distance between solutions to the NSNPP and SNPP model in natural energy and dissipation norms. Moreover, we prove the existence of regular local-in-time solutions to the NSNPP as well as the SNPP system, possessing sufficient regularity to be admissible in the relative energy framework.

1 Introduction

Electrokinetics is essential in the design of nano-fluid "lab-on-a-chip" devices [ZY12]. This innovation has the potential to revolutionize and democratize healthcare by making laboratory diagnostics more affordable and widely accessible. Precise control of fluid dynamics is crucial for automating laboratory functions on microfluidic chips. But at such small scales, mechanical stirring becomes impractical due to, amongst other reasons, the high viscosity of the fluid. To address this challenge, one innovative approach involves dissolving ions in the fluid and using an electric field to manipulate the flow for effective mixing [ZY12]. Notably, an anisotropic medium has been observed to produce persistent electrokinetic flows, particularly in the presence of oscillating electric fields [Pen+15].

To model these effects, multiple physical effects on multiple different scales have to be taken into account. When charged particles $c^{\pm}$ are dissolved in an incompressible fluid with velocity $\boldsymbol{v}$ and under the influence of an external electric field $-\nabla\psi$, three major effects govern the movement of the charges. The charges diffuse, they are transported by the surrounding fluid and the electric field induces a directed movement, called electromigration. Further, space charge exerts an electric body force on the fluid velocity. Moreover, there is a complex interaction of the electric field, the charges and the velocity field with the underlying anisotropy. An attempt to model this physical interaction leads us to a coupled Navier–Stokes–Nernst–Planck–Poisson (NSNPP) system, [Pro94, Chap. 3.4]. More explicitly, we are considering,

$$\mathrm{Re}\,(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) - \Delta \boldsymbol{v} + \nabla p + (c^+ - c^-)\nabla\psi = 0 \qquad \text{in } (0,T)\times\Omega, \tag{1a}$$

$$\nabla\cdot\boldsymbol{v} = 0 \qquad \text{in } (0,T)\times\Omega, \tag{1b}$$

$$\partial_t c^{\pm} + \nabla\cdot(c^{\pm}\boldsymbol{v}) - \nabla\cdot\big(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla c^{\pm} \pm c^{\pm}\nabla\psi)\big) = 0 \qquad \text{in } (0,T)\times\Omega, \tag{1c}$$

$$-\nabla\cdot(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla\psi) - (c^+ - c^-) = 0 \qquad \text{in } (0,T)\times\Omega, \tag{1d}$$

where $\Omega\subseteq\mathbb{R}^3$ is a smooth and bounded domain, $\mathrm{Re},\lambda,\varepsilon>0$ are positive constants, $\boldsymbol{\mathcal{E}}(\boldsymbol{d}) := \boldsymbol{I} + \varepsilon\,\boldsymbol{d}\otimes\boldsymbol{d}$ and $\boldsymbol{\Lambda}(\boldsymbol{d}) := \boldsymbol{I} + \lambda\,\boldsymbol{d}\otimes\boldsymbol{d}$, where $\boldsymbol{d}$ is the so-called director with $\boldsymbol{d}(\boldsymbol{x})\in\mathbb{R}^3$ and $\boldsymbol{d}\cdot\boldsymbol{n} = 0$ on $\Gamma := \partial\Omega$. The evolution of the fluid's velocity field is described by the Navier–Stokes equations for incompressible fluids (1a)–(1b). The charge densities evolve according to the Nernst–Planck equations (1c) including an anisotropic diffusion term $-\nabla\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla c^{\pm})$ as well as two transport terms, one due to the velocity field $\boldsymbol{v}$ and one due to the electric field

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$-\nabla\psi$. The notation $c^{\pm}$ means an enumeration, so that (1c) in fact denotes two equations: one for the positive charges c^{+} with a plus in front of the $\nabla\psi$ term and one for the negative charges c^{-} with a minus in front of the $\nabla\psi$. Finally, the electric potential is given by the Poisson equation (1d). The existence of suitable weak solutions to system (1) was shown in [HLP25] and we refer to the introduction therein for a discussion of the relevant literature on the existence of weak solutions.

In isotropic fluids the movement of the charges and the formation of the electric field are independent of the direction. Not so in anisotropic fluids, where the movement of the ions due to diffusion and electromigration, as well as the diffusion of the electric potential may depend on the direction. In our case they depend on the director $\boldsymbol{d}$, which gives the preferred direction of motion. Our choice for the matrices $\boldsymbol{\Lambda}(\boldsymbol{d})$ and $\boldsymbol{\mathcal{E}}(\boldsymbol{d})$ stems from the modeling of liquid crystals, most famously known for their application in LCDs. One possible way to model liquid crystals are the Erickson–Leslie equations [EKL18], including a time-evolution for the director $\boldsymbol{d}$. The mobilities of the charges vary depending on the motion being parallel or perpendicular to the director, see [Cal+16] for an extensive model derivation for nematic electrolytes. As it was done in [Cal+16] we choose the anisotropy matrices to be of the form $\boldsymbol{\mathcal{E}}(\boldsymbol{d})$ and $\boldsymbol{\Lambda}(\boldsymbol{d})$. The difference between the anisotropy matrices $\boldsymbol{\mathcal{E}}(\boldsymbol{d})$ and $\boldsymbol{\Lambda}(\boldsymbol{d})$ is crucial for capturing the distinct physical properties of electric conductivity and electric permittivity, which are inherently different physical properties.

Additionally, the system is equipped with the following boundary conditions

$$\boldsymbol{v} = 0 \text{ and } \boldsymbol{\Lambda}(\boldsymbol{d})\left(\nabla c^{\pm} \pm c^{\pm}\nabla\psi\right)\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\times[0,T],$$
$$\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla\psi\cdot\boldsymbol{n} + \tau\psi = \xi \text{ on } \Gamma\times[0,T],$$

where τ is a positive constant and ξ the externally applied electric potential, which may vary in space and time. These are the standard no-slip boundary conditions for the velocity field, no-flux boundary condition for the charge densities, which correspond to the assumption that the charges cannot cross the boundary of the domain and Robin boundary conditions for the electric potential. As in [BFS14], we choose Robin boundary conditions to model the electric double layer, which usually forms at an electrolyte-solid-interface [NT04, Chap. 7].

A main difficulty beside the multiple effects in the system are the multiple scales at which these effects are present. The highly oscillating electric field is present on way faster time scales than the induced fluid flow. To achieve accurate simulations at reasonable computational cost, model reduction techniques adapted to different parameter regimes are essential [Haa+23]. Besides the data assimilation, this is the most important building block for *digital twins*, which model physical systems at multiple scales using multiple mathematical descriptions. According to Forbes [McK24], “The global market for digital-twin technology will grow about 60% annually over the next five years, reaching 73.5 billion dollars by 2027.” To enable model switching and adaptation during simulations, a general framework is needed to compare models within a given class and assess the influence of changing or vanishing parameters. This can be achieved by appropriate *a posteriori* error estimators. While a theory for developing *a posteriori* error estimators exist for linear PDEs [ESV10], they have been deduced for certain specific nonlinear evolution equations [Fis15]. In this work, we apply the general framework of relative-energy estimates as an *a posteriori* error estimate in order to infer modeling errors for the case of vanishing Reynolds number in the NSNPP system.

The physical scaling in [Cal+16] suggests that the Reynolds number is very small, around 10^{-6}. For such small Reynolds numbers the Stokes equation can be a good approximation for the Navier–Stokes equation. So additionally to the NSNPP system (1), we consider the Stokes–Nernst–Planck–Poisson (SNPP) system, where (1a) is replaced by

$$-\Delta\boldsymbol{v} + \nabla p + (c^{+} - c^{-})\nabla\psi = 0 \qquad \text{in } (0,T)\times\Omega. \tag{2a}$$

The aim of this work is to derive a relative energy inequality for the suitable weak solutions to (1) which provides further properties of these solutions and compares a suitable weak solution to a smooth test function. A first and simple consequence is the weak-strong uniqueness. Additionally, convergence rates for suitable weak solutions to the NSNPP system for vanishing Reynolds number can be derived as well as continuous dependence of these solutions on the initial and boundary data under the assumption that there is a strong reference solution. The exact regularity will be stated below. To give these estimates meaning we show that at least locally in time we have such strong solutions to the NSNPP as well as the SNPP system.

Main contribution: The basis for the proof of all these estimates and the main novelty of this work is the derived relative energy inequality. The relative energy is a non-negative function $\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})$ measuring the distance between a suitable weak solution $\boldsymbol{u}$ and a smooth test function $\tilde{\boldsymbol{u}}$. In the context of the incompressible Navier–Stokes equations, *i.e.,* for a quadratic energy, the relative energy approach was already used by Leray in his seminal work [Ler34] and later on by Serrin [Ser63] to prove weak-strong uniqueness. For more general energy or entropy functionals, this approach can be traced back to Dafermos, see [Daf79; DiP79], in the context of conservation laws. This technique has since been generalized in different directions, for instance to the case of renormalized solutions to reaction-diffusion systems [Fis17] or energy-variational solutions to general conservation laws [EL24] or damped Hamiltonian systems [ALR24]. The relative energy serves nowadays as a general tool in the analysis of PDEs and is used to consider along side the weak-strong uniqueness of solutions [HL21; CJ19; Fis17] also long-time behavior [Las19b], singular limits [FN09], convergence of numerical schemes [BLP21] or comparison with reduced models [Fis15], and even optimal control [Las19a].

For systems with convex energy $\mathcal{E}$ the relative energy can be defined by the first order Taylor expansion of the energy at the smooth test function $\tilde{\boldsymbol{u}}$:

$$\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) = \mathcal{E}(\tilde{\boldsymbol{u}}) - \mathcal{E}(\boldsymbol{u}) + \langle D\mathcal{E}(\tilde{\boldsymbol{u}}), \boldsymbol{u} - \tilde{\boldsymbol{u}}\rangle \geq 0.$$

The non-negativity follows by well-known properties of convex functionals. For system (1) the relative energy has the following form:

$$\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) := \int_\Omega \frac{\mathrm{Re}}{2}|\boldsymbol{v} - \tilde{\boldsymbol{v}}|^2 + \left[\tilde{c}^\pm - c^\pm - c^\pm(\ln \tilde{c}^\pm - \ln c^\pm)\right]^\pm + \frac{1}{2}|\nabla(\psi - \tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})} \,\mathrm{d}\boldsymbol{x} + \frac{\tau}{2}\int_\Gamma |\psi - \tilde{\psi}|^2 \,\mathrm{d}\sigma. \tag{3}$$

The derivation is based on the fact that suitable weak solutions already fulfill an energy-dissipation-inequality. The main idea then is to add this energy-dissipation-inequality, the weak formulation tested with the smooth test function, and system (1) for the smooth function tested in suitable way. As in the existence proof [HLP25] the main challenge was introduced by the difference in anisotropy matrices $\boldsymbol{\mathcal{E}}(\boldsymbol{d})$ and $\boldsymbol{\Lambda}(\boldsymbol{d})$ and we tackle this by careful integration by parts in the bulk as well as on the boundary, that is for $f \in W^{1,p}(\Gamma)$ and $\boldsymbol{v} \in W^{1,p'}(\Gamma)$ we use:

$$\int_\Gamma f\, \nabla_\Gamma \cdot \boldsymbol{v} - f(\boldsymbol{v}\cdot\boldsymbol{n})\nabla_\Gamma \cdot \boldsymbol{n} + \boldsymbol{v}\cdot\nabla_\Gamma f \,\mathrm{d}\sigma = 0$$

where $\nabla_\Gamma \cdot \boldsymbol{v}$ denotes the surface differential divergence and $\nabla_\Gamma f$ the surface differential gradient, see [HLP25, Cor. A.25].

The article is organized as follows: We first introduce some notation and present the main results in Section 2. In Section 3 we state and prove the relative energy inequality and use it to prove the weak-strong uniqueness result. We then proceed to show the local-in-time existence of strong solutions to the NSNPP as well as the SNPP system in Section 4 and conclude this work with the proof of the error estimates in Section 5.

2 Main results

We begin by introducing some basic notation. By Ω, we denote a smooth bounded domain in $\mathbb{R}^3$ and $\Gamma := \partial\Omega$. For all $r \in (1,\infty)$ we define the function spaces

$$L^r_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^r}} \quad \text{and} \quad W^{1,r}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{1,r}}}.$$

The function space $L^p(\Omega)_+$ is defined via $L^p(\Omega)_+ := \{f \in L^p(\Omega) \mid f \geq 0 \text{ a.e. in } \Omega\}$. The norm $|\cdot|_{\boldsymbol{\Lambda}(\boldsymbol{d})}$ is defined by $|\boldsymbol{a}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} = |\boldsymbol{a}|^2 + \lambda(\boldsymbol{d}\cdot\boldsymbol{a})^2$ and similarly for $|\cdot|_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}$. The outer normal of Ω is denoted by $\boldsymbol{n}$. The standard matrix and matrix-vector multiplication is written without an extra sign for brevity,

$$\boldsymbol{A}\boldsymbol{B} = \left[\sum_{j=1}^{3} \boldsymbol{A}_{ij}\boldsymbol{B}_{jk}\right]_{i,k=1}^{3}, \quad \boldsymbol{A}\boldsymbol{a} = \left[\sum_{j=1}^{3} \boldsymbol{A}_{ij}\boldsymbol{a}_{j}\right]_{i=1}^{3}, \quad \boldsymbol{A} \in \mathbb{R}^{3\times 3},\, \boldsymbol{B} \in \mathbb{R}^{3\times 3},\, \boldsymbol{a} \in \mathbb{R}^{3}.$$

The outer vector product is given by $\boldsymbol{a}\otimes\boldsymbol{b} := \boldsymbol{a}\boldsymbol{b}^T = \left[\boldsymbol{a}_i\boldsymbol{b}_j\right]_{i,j=1}^{3}$ for two vectors $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^3$ and by $\boldsymbol{A}\otimes\boldsymbol{a} := \boldsymbol{A}\boldsymbol{a}^T = \left[\boldsymbol{A}_{ij}\boldsymbol{a}_k\right]_{i,j,k=1}^{3}$ for a matrix $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ and a vector $\boldsymbol{a} \in \mathbb{R}^3$. We use the Nabla symbol ∇ for real-valued functions $f : \mathbb{R}^3 \to \mathbb{R}$, vector-valued functions $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ as well as matrix-valued functions $\boldsymbol{A} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ denoting

$$\nabla f := \left[\frac{\partial f}{\partial \boldsymbol{x}_i}\right]_{i=1}^{3}, \quad \nabla \boldsymbol{f} := \left[\frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{x}_j}\right]_{i,j=1}^{3}, \quad \nabla \boldsymbol{A} := \left[\frac{\partial \boldsymbol{A}_{ij}}{\partial \boldsymbol{x}_k}\right]_{i,j,k=1}^{3}.$$

The divergence of a vector-valued and a matrix-valued function is defined by

$$\nabla\cdot\boldsymbol{f} := \sum_{i=1}^{3}\frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{x}_i}, \quad \nabla\cdot\boldsymbol{A} := \left[\sum_{j=1}^{3}\frac{\partial \boldsymbol{A}_{ij}}{\partial \boldsymbol{x}_j}\right]_{i=1}^{3}.$$

These definitions give rise to different calculation rules, *e.g.* $\nabla\cdot(\boldsymbol{a}\otimes\boldsymbol{b}) = \nabla\boldsymbol{a}\,\boldsymbol{b} + \nabla\cdot\boldsymbol{b}\,\boldsymbol{a}$ for $\boldsymbol{a},\boldsymbol{b} \in W^{1,2}(\Omega)$. The use of $\big[\cdot\big]^{\pm}$ implies a summation: for example $\big[\pm c^{\pm}\big]^{\pm} = c^+ - c^-$, $\big[\mp c^{\pm}\big]^{\pm} = -c^+ + c^-$, and

$$\big[c^{\pm}(\ln c^{\pm} + 1)\big]^{\pm} = c^+(\ln c^+ + 1) + c^-(\ln c^- + 1).$$

So the sum $\big[\cdot\big]^{\pm}$ consists of two terms: one where only the top signs of $\pm$ and $\mp$ are used, and one where only the bottom signs are used. Outside of these brackets $\pm$ is used as an enumeration, *e.g.* $c^{\pm} \in L^1(\Omega)$ means $c^+ \in L^1(\Omega)$ and $c^- \in L^1(\Omega)$.

For a given Banach space V, the space $C_w([0,T];V)$ denotes the functions on $[0,T]$ taking values in V that are continuous with respect to the weak topology of V.

In the case $\Gamma \in C^m$, $m \in \mathbb{N}$ we denote for $p \in (1,\infty)$ by $S : W^{m,p}(\Omega) \to W^{m-\frac{1}{p},p}(\Gamma)$, the usual trace operator and by $E : W^{m-\frac{1}{p},p}(\Gamma) \to W^{m,p}(\Omega)$ its right-inverse such that $S(E(f)) = f$ for all $f \in W^{m-\frac{1}{p},p}(\Gamma)$.

Throughout this work $C > 0$ denotes a generic constant, which may change its value without an indication in the notation. We sometimes use $C(\cdot)$ to indicate dependencies of this constant.

Now that we have introduced this notation we can state our definition of suitable weak solutions, which are simply weak solutions, which additionally fulfill an energy inequality. The existence of such suitable weak solutions for $\mathrm{Re} = 1$ is shown in [HLP25] under the following assumptions, *cf.* Assumption 2.1. This existence proof directly transfers to the case $\mathrm{Re} > 0$ so we will not prove the existence of suitable weak solutions again for arbitrary $\mathrm{Re} > 0$.

Assumption 2.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded smooth domain, $\xi \in W^{1,2}([0,T];W^{2,2}(\Gamma))$, $\boldsymbol{d} \in W^{1,\infty}(\Omega)$ with $\boldsymbol{d}\cdot\boldsymbol{n} = 0$ on Γ, $\varepsilon,\lambda > 0$, and the initial data $(\boldsymbol{v}_0, c_0^{\pm}) \in L^2_\sigma(\Omega)\times L^2(\Omega)_+$.*

Definition 2.2 (Suitable weak solutions)**.** *Let Assumption 2.1 be fulfilled. We call $(\boldsymbol{v}, c^{\pm}, \psi)$ a suitable weak solution if*

$$\begin{aligned}
\boldsymbol{v} &\in C_w([0,T];L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_0(\Omega)) \cap L^{10/3}(0,T;L^{10/3}(\Omega)),\\
c^{\pm} &\in C_w([0,T];L^1(\Omega)_+) \cap L^{5/4}(0,T;W^{1,5/4}(\Omega)) \cap L^{5/3}(0,T;L^{5/3}(\Omega)),\\
\sqrt{c^{\pm}} &\in L^2(0,T;W^{1,2}(\Omega)),\\
\psi &\in C_w([0,T];W^{1,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)),\\
\sqrt{c^{\pm}}\nabla\psi &\in L^2(0,T;L^2(\Omega)),
\end{aligned}$$

and (1) *is fulfilled in the weak sense, that is, for all test functions* $\tilde{\boldsymbol{v}} \in W^{1,2}(0,T;L^2_\sigma(\Omega)) \cap L^\infty((0,T)\times\Omega) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$, $\tilde{c}^\pm \in C([0,T]\times\overline{\Omega}) \cap W^{1,5/2}(0,T;L^{5/2}(\Omega)) \cap L^{10}(0,T;W^{1,10}(\Omega))$ *and* $\tilde{\psi} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$*, it holds*

$$\operatorname{Re}\int_\Omega \boldsymbol{v}(t)\cdot\tilde{\boldsymbol{v}}(t) - \boldsymbol{v}_0\cdot\tilde{\boldsymbol{v}}(0)\,\mathrm{d}\boldsymbol{x} + \int_0^t\int_\Omega \nabla\boldsymbol{v}:\nabla\tilde{\boldsymbol{v}} - \operatorname{Re}\boldsymbol{v}\cdot\partial_t\tilde{\boldsymbol{v}} + \big(\operatorname{Re}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + (c^+-c^-)\nabla\psi\big)\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = 0 \tag{4a}$$

$$\int_\Omega c^\pm(t)\,\tilde{c}^\pm(t) - c_0^\pm\,\tilde{c}^\pm(0)\,\mathrm{d}\boldsymbol{x} + \int_0^t\int_\Omega \big(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla c^\pm \pm c^\pm\nabla\psi) - c^\pm\boldsymbol{v}\big)\cdot\nabla\tilde{c}^\pm - c^\pm\partial_t\tilde{c}^\pm\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = 0 \tag{4b}$$

$$\int_\Omega \boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla\psi\cdot\nabla\tilde{\psi}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma(\tau\psi-\xi)\tilde{\psi}\,\mathrm{d}\sigma - \int_\Omega(c^+-c^-)\tilde{\psi}\,\mathrm{d}\boldsymbol{x} = 0 \tag{4c}$$

for <u>all</u> $t\in(0,T)$ *and, additionally, the energy inequality*

$$\left[\int_\Omega \frac{\operatorname{Re}}{2}|\boldsymbol{v}|^2 + \Big[c^\pm(\ln c^\pm + 1)\Big]^\pm + \frac{1}{2}|\nabla\psi|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\boldsymbol{x} + \frac{\tau}{2}\int_\Gamma|\psi|^2\,\mathrm{d}\sigma\right]\Bigg|_0^t + \int_0^t\int_\Omega|\nabla\boldsymbol{v}|^2 + \left[\left|2\nabla\sqrt{c^\pm}\pm\sqrt{c^\pm}\nabla\psi\right|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le \int_0^t\int_\Gamma\psi\partial_t\xi\,\mathrm{d}\sigma\,\mathrm{d}s \tag{5}$$

holds for <u>all</u> $t\in(0,T)$.

Remark 2.3 The electric potential at the initial time is given by the solution of the Poisson equation (1d) with right-hand side $c_0^+ - c_0^-$ and boundary condition $\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla\psi_0\cdot\boldsymbol{n} + \tau\psi_0 = \xi(0)$.

Next, we define the space of test functions for the relative energy inequality, which will also serve as our regularity space for strong solutions. We say $\tilde{\boldsymbol{u}} = (\tilde{\boldsymbol{v}},\tilde{c}^\pm,\tilde{\psi}) \in \widetilde{\mathbb{U}}$ if:

$\tilde{\boldsymbol{v}} \in W^{1,2}(0,T;L^2_\sigma(\Omega)) \cap L^{5/2}(0,T;W^{1,5/2}_{0,\sigma}(\Omega)) \cap L^\infty((0,T)\times\Omega) \cap L^2(0,T;W^{2,2}(\Omega))$,
$\tilde{c}^\pm \in C([0,T]\times\overline{\Omega}) \cap L^{10}(0,T;W^{1,10}(\Omega)) \cap L^4(0,T;W^{2,4}(\Omega)) \cap W^{1,5/2}(0,T;L^{5/2}(\Omega))$,
$\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^\pm \pm \tilde{c}^\pm\nabla\tilde{\psi})\cdot\boldsymbol{n} = 0$ on Γ,
and $\tilde{\psi}$ is the weak solution to the Poisson equation (1d) with right-hand side $\tilde{c}^+ - \tilde{c}^-$,
fulfilling the Robin boundary condition $\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\boldsymbol{n} + \tau\tilde{\psi} = \tilde{\xi}$ on Γ with $\tilde{\xi} \in W^{1,4}(0,T;W^{2,4}(\Gamma))$.

By elliptic regularity [Gri85, Thm. 2.4.2.6] we then have $\tilde{\psi} \in W^{1,5/2}(0,T;W^{2,5/2}(\Omega)) \cap C([0,T];W^{2,10}(\Omega))$ since $\tilde{\xi} \in W^{1,4}(0,T;W^{2,4}(\Gamma)) \hookrightarrow W^{1,4}(0,T;W^{1,10}(\Gamma))$ [DD12, Thm. 3.81]. To deduce more regularity of $\tilde{\psi}$ in space by the regularity of $\tilde{c}^\pm$ with the help of elliptic regularity we would need a more regular director $\boldsymbol{d}$ and external electric potential $\tilde{\xi}$. For later usage we collect the assumptions needed for the existence of strong solutions.

Assumption 2.4. *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded smooth domain,* $\tilde{\xi} \in W^{1,4}([0,T];W^{2,4}(\Gamma))$, $\boldsymbol{d} \in C^{4,1}(\overline{\Omega})$ *with* $\boldsymbol{d}\cdot\boldsymbol{n} = 0$ *on* Γ, $\varepsilon,\lambda > 0$*, and the initial data* $(\tilde{\boldsymbol{v}}_0,\tilde{c}_0^\pm) \in W^{2,6}_{0,\sigma}(\Omega)\times W^{2,10}(\Omega)$, $\tilde{c}_0^\pm \ge l > 0$ *and* $\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}_0^\pm \pm \tilde{c}_0^\pm\nabla\psi_0)\cdot\boldsymbol{n} = 0$.

Here the regularity of the initial data is chosen in a way, that allows us to use maximal L^p–regularity of the Stokes operator for $p = 6$ and of the diffusive part of the Nernst–Planck equation for $p = 10$.

Definition 2.5 (Strong solutions)**.** *We call* $\tilde{\boldsymbol{u}} = (\tilde{\boldsymbol{v}},\tilde{c}^\pm,\tilde{\psi}) \in \widetilde{\mathbb{U}}$ *strong solution to the NSNPP system* (1) *or the SNPP system* (2)*, respectively, if the equations of the NSNPP or SNPP system are fulfilled in the weak sense.*

The first main result of this work is the weak-strong uniqueness of suitable weak solutions to the NSNPP system (1), which is stated in the following proposition.

Proposition 2.6 (Weak-strong uniqueness). *Let Assumption 2.4 hold, let $(\boldsymbol{v}, c^{\pm}, \psi)$ be a suitable weak solution to* (1) *according to Definition 2.2 and let $(\tilde{\boldsymbol{v}}, \tilde{c}^{\pm}, \tilde{\psi}) \in \widetilde{\mathbb{U}}$ be a strong solution to* (1) *emanating from the same initial and boundary values. Then $(\boldsymbol{v}, c^{\pm}, \psi)$ and $(\tilde{\boldsymbol{v}}, \tilde{c}^{\pm}, \tilde{\psi})$ coincide.*

The proof is an easy consequence of the relative energy inequality, which will be derived in Section 3, where we will also provide the proof of Proposition 2.6. We refrain from introducing the relative energy inequality here, even though it is the basis of all our main results, for the sake of readability. The derivation of the relative energy inequality is rather technical and we need some time to introduce all the operators needed for the formulation. Here, in this main section, we will rather focus on the results we obtain with the help of this relative energy inequality.

The relative energy inequality always compares a suitable weak solution with a rather smooth test function, for us a function $\tilde{\boldsymbol{u}} \in \widetilde{\mathbb{U}}$. In order to give the weak-strong uniqueness result, which only holds conditionally, under the assumption that a strong solution exists, more meaning and to be able to compare suitable weak solutions to solutions of the SNPP system, we show that strong solutions to both the NSNPP as well the SNPP system exist at least locally in time. For the NSNPP system the need to restrict the existence of strong solutions to local existence intervals is not surprising, as the existence of global strong solutions to the Navier–Stokes equations is a well-known open problem in space dimension three. But even for the seemingly nicer behaved SNPP system, we were not able to show global existence of strong solutions and this is to the best of our knowledge still an open problem. Similar existence result work for scalar valued diffusion coefficients in the Nernst–Planck equations [CI19] but the techniques used there cannot be extended to our case due to the difference in the anisotropy matrices $\boldsymbol{\mathcal{E}}(\boldsymbol{d})$ and $\boldsymbol{\Lambda}(\boldsymbol{d})$. So we also have to restrict the existence of strong solutions to the SNPP system to local time intervals.

Theorem 2.7 (Existence of a unique, local, strong solution to NSNPP and SNPP). *Let Assumption 2.4 hold, then there exists a $T^* > 0$ such that there is a unique strong solution in $\widetilde{\mathbb{U}}$ according to Definition 2.5 to* (1) *and* (2) *on* $[0, T^*)$.

Remark 2.8 To be able to plug in the strong solution into the relative energy inequality, we need positivity of $\tilde{c}^{\pm}$. The positivity of the initial data $\tilde{c}_0^{\pm}$ easily extends to the whole existence interval of the solutions, which can be seen by the following comparison principle for this more regular solution.

Proposition 2.9 (Comparison principle for $\tilde{c}^{\pm}$). *Let $\tilde{\boldsymbol{v}}$ and $\tilde{\psi}$ fulfill the regularity in $\widetilde{\mathbb{U}}$ and let $\overline{c}^{\pm}$ and $\underline{c}^{\pm}$ be a super- and a sub-solution to the Nernst–Planck equation* (1c) *which also fulfill the regularity in $\widetilde{\mathbb{U}}$, that is* (4b) *is fulfilled for all non-negative test functions $\theta^{\pm} \in W^{1,5/2}(0,T; L^{5/2}(\Omega)) \cap L^{10}(0,T; W^{1,10}(\Omega))$ with $\geq$ for the super solution $\overline{c}^{\pm}$ and with $\leq$ for $\underline{c}^{\pm}$. We then have that $\underline{c}_0^{\pm} \leq \overline{c}_0^{\pm}$ implies $\underline{c}^{\pm} \leq \overline{c}^{\pm}$ everywhere.*

The proof of this proposition is based on a straight forward testing scheme and a Gronwall argument and is performed at the end of Section 4. The positivity of $\tilde{c}^{\pm}$ can now be seen as an easy consequence of this comparison principle.

Corollary 2.10. *Let the assumptions of Proposition 2.9 hold. If the charge densities are initially bounded away from zero, that is $\tilde{c}_0^{\pm} \geq l > 0$, then they stay bounded away from zero $\tilde{c}^{\pm} \geq l_T > 0$, where l_T depends on the final time T.*

Proof. This is an easy consequence of Proposition 2.9. We take $(c^{\pm})^*$ to be the constant in space solution to the ODE

$$\partial_t (c^{\pm})^* + (c^{\pm})^* \left\| \nabla \cdot (\boldsymbol{\Lambda}(\boldsymbol{d}) \nabla \tilde{\psi}) \right\|_{L^\infty(\Omega)} = 0, \qquad (c^{\pm})^*(0) = l$$

which is given by $(c^{\pm})^*(t) = l \exp\left(- \int_0^t \left\| \nabla \cdot (\boldsymbol{\Lambda}(\boldsymbol{d}) \nabla \tilde{\psi})(s) \right\|_{L^\infty(\Omega)} \mathrm{d}s \right)$. The fact that $\nabla \cdot (\boldsymbol{\Lambda}(\boldsymbol{d}) \nabla \tilde{\psi})$ is in $L^1(0,T; L^\infty(\Omega))$ follows from the definition of $\widetilde{\mathbb{U}}$ and elliptic regularity, which implies $\tilde{\psi} \in L^4(0,T; W^{3,4}(\Omega))$ since $\tilde{\xi} \in W^{1,4}(0,T; W^{2,4}(\Gamma))$ and $\tilde{c}^{\pm} \in L^4(0,T; W^{2,4}(\Omega)) \hookrightarrow L^4(0,T; W^{1,4}(\Omega))$. This $(c^{\pm})^*$ is a sub-solution to (1c) and thus by the comparison principle from Proposition 2.9 we know that since $(c^{\pm})^*(0) = l \leq \tilde{c}_0^{\pm}$

that we have $(c^{\pm})^* \leq \tilde{c}^{\pm}$ everywhere. And thus, with the explicit form for $(c^{\pm})^*$ we find

$$l_T := l \exp\left(- \int_0^T \left\| \nabla \cdot (\boldsymbol{\Lambda}(\boldsymbol{d}) \nabla \tilde{\psi})(s) \right\|_{L^\infty(\Omega)} \mathrm{d}s \right) = (c^{\pm})^*(T) \leq (c^{\pm})^* \leq \tilde{c}^{\pm}.$$

□

Now that we have established existence and positivity of strong solutions to the NSNPP and the SNPP system at least locally in time we turn the convergence of suitable weak solutions to the NSNPP system for vanishing Reynolds number.

Theorem 2.11. *Let Assumptions 2.1 and 2.4 hold, let $\tilde{\boldsymbol{u}} \in \widetilde{\mathbb{U}}$ be a strong solution to the SNPP system and let $\{\boldsymbol{u}_{\mathrm{Re}}\}$ be a family of suitable weak solutions to the NSNPP system for bounded Reynolds number emanating from the same initial and boundary values. Then we can estimate the difference of $\tilde{\boldsymbol{u}}$ and $\boldsymbol{u}_{\mathrm{Re}}$ by:*

$$\begin{aligned} \sqrt{\mathrm{Re}}\, \|\boldsymbol{v}_{\mathrm{Re}} - \tilde{\boldsymbol{v}}\|_{L^\infty(0,T;L^2(\Omega))} + \left\| \sqrt{c^{\pm}_{\mathrm{Re}}} - \sqrt{\tilde{c}^{\pm}} \right\|_{L^\infty(0,T;L^2(\Omega))} + \left\| \psi_{\mathrm{Re}} - \tilde{\psi} \right\|_{L^\infty(0,T;W^{1,2}(\Omega))} \\ + \|\boldsymbol{v}_{\mathrm{Re}} - \tilde{\boldsymbol{v}}\|_{L^2(0,T;W^{1,2}(\Omega))} + \left\| \sqrt{c^{\pm}_{\mathrm{Re}}} - \sqrt{\tilde{c}^{\pm}} \right\|_{L^2(0,T;W^{1,2}(\Omega))} \\ + \left\| \sqrt{c^{\pm}_{\mathrm{Re}}} (\nabla \psi_{\mathrm{Re}} - \nabla \tilde{\psi}) \right\|_{L^2(0,T;L^2(\Omega))} \leq C \sqrt{\mathrm{Re}}. \end{aligned} \tag{6}$$

Remark 2.12 (Convergence for vanishing Reynolds number) For vanishing Reynolds number $\mathrm{Re} > 0$ the estimate from the theorem above imply that all suitable weak solutions to the NSNPP system converge to the strong solution of the SNPP, as long as it exists, such that the weak formulation and the energy inequality converge with rate $C\sqrt{\mathrm{Re}}$.

Under some smoothness assumption we can also prove the continuous dependence of suitable weak solutions on data.

Corollary 2.13 (Continuous dependence on boundary and initial data)**.** *Let Assumptions 2.1 and 2.4 hold, let $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ be suitable weak solutions to the NSNPP system with initial data $\boldsymbol{u}_0$ and $\tilde{\boldsymbol{u}}_0$ and external electric potential ξ and $\tilde{\xi}$ respectively. For $\tilde{\boldsymbol{u}} \in \widetilde{\mathbb{U}}$ we have*

$$\begin{aligned} &\sqrt{\mathrm{Re}}\, \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{L^\infty(0,T;L^2(\Omega))} + \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{L^\infty(0,T;L^2(\Omega))} + \left\| \psi - \tilde{\psi} \right\|_{L^\infty(0,T;W^{1,2}(\Omega))} \\ &\quad + \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{L^2(0,T;W^{1,2}(\Omega))} \\ &\quad \leq C \left(\sqrt{\mathrm{Re}}\, \|\boldsymbol{v}_0 - \tilde{\boldsymbol{v}}_0\|_{L^2(\Omega)} + \left\| \sqrt{c^{\pm}_0} - \sqrt{\tilde{c}^{\pm}_0} \right\|_{L^2(\Omega)} + \left\| \psi_0 - \tilde{\psi}_0 \right\|_{W^{1,2}(\Omega)} \right) e^{C(\tilde{\boldsymbol{u}},T)} \\ &\qquad + C e^{C(\tilde{\boldsymbol{u}},T)} \left(\left\| \partial_t (\xi - \tilde{\xi}) \right\|_{L^2(0,T;L^2(\Gamma))} + \sqrt{1 + \left[\left\| \tilde{c}^{\pm} \right\|_{L^\infty((0,T)\times\Omega)} \right]^{\pm}} \left\| \xi - \tilde{\xi} \right\|_{L^2(0,T;W^{1,2}(\Gamma))} \right), \end{aligned}$$

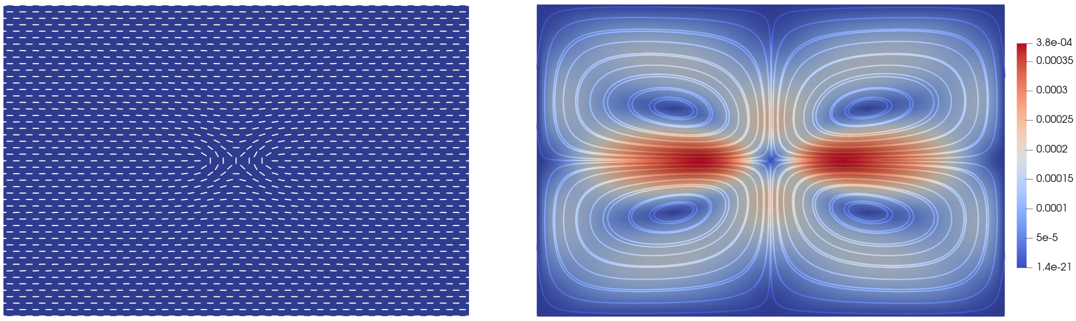
where $C(\tilde{\boldsymbol{u}}, T)$ is some constant depending on the strong reference solution $\tilde{\boldsymbol{u}}$ and the finial time T.

The proofs of this theorem and the corollary are again based on the relative energy inequality and can be found in Section 5. Besides proving the convergence rate of Theorem 2.11 analytically we could also reproduce it in numerical experiments.

2.1 Numerical Experiments

We perform numerical experiments on a two-dimensional rectangular domain using a finite element approximation implemented with the Python package FEniCS [Aln+15; LMW+12]. The simulation code is available on GitHub [Pla24a]. A regular triangular mesh, equidistant time steps, and a fixed-point solver are used to solve a decoupled version of the system (1). Further implementation details can be found in [Pla24a] and [Pla24b].

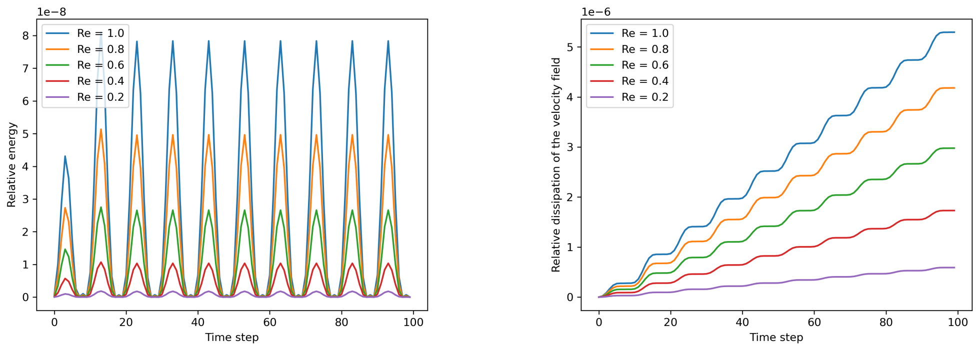
We consider an alternating electric field acting from left to right and use the director field $\boldsymbol{d}$ shown in Figure 1a. The streamline plot of the time-averaged velocity field for $\mathrm{Re} = 1$, shown in Figure 1b, qualitatively agrees with



(a) Director field. (b) Streamline plot of the simulated velocity field.

Figure 1: Simulated velocity field for a given director field.

the experimentally observed flow pattern described in [Pen+15]. The initial condition assumes a quiescent fluid and a uniform distribution of positive and negative charges, *i.e.*, $c_0^+ \equiv c_0^- \equiv 0.5$. While it is far from trivial—and we make no general claim—that the analytical convergence rate from Theorem 2.11 carries over to our numerical scheme, it is nonetheless promising to observe that the expected convergence rate appears to be realized in our computations. This is demonstrated in the following figures. Figure 2 presents the evolution of the relative energy *c.f.* (3) and the relative dissipation of the velocity field, defined as $\|\nabla(\boldsymbol{v}_{\mathrm{Re}} - \tilde{\boldsymbol{v}})\|^2_{L^2(0,t;L^2(\Omega))}$, for various values of the Reynolds number. Here, $\boldsymbol{v}_{\mathrm{Re}}$ is the solution to the full NSNPP system (1), while $\tilde{\boldsymbol{v}}$ denotes the solution of the simplified SNPP system (2). The oscillatory behavior in the relative energy and the wavy pattern



(a) Relative energy over time. (b) Relative dissipation over time.

Figure 2: Relative error metrics over time for different Reynolds numbers.

in the dissipation are due to the sinusoidal nature of the chosen external electric potential, which changes sign periodically. Nevertheless, we observe a clear monotonic decrease in both error measures as the Reynolds number decreases. This trend is further confirmed in Figure 3, which plots the maximum relative energy error against the Reynolds number. The observed slope matches the theoretical convergence rate predicted by (6).

3 Relative energy inequality

Before we introduce the relative energy for our system we shortly comment on the construction of the relative energy and on the strategy of how to prove the relative energy inequality. This shall give a better insight on the choice of the appropriate calculations in the following proof. We note that a relative energy-inequality is proven in a general context for energy-variational solutions in [ALR24]. For a convex Gâteaux-differentiable energy $\mathcal{E}$ on a Banach space $\mathbb{U}$, $\mathcal{E} : \mathbb{U} \to [0, \infty]$, we define the relative energy $\mathcal{R} : \mathbb{U} \times \widetilde{\mathbb{U}} \to [0, \infty]$ via

$$\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) := \mathcal{E}(\boldsymbol{u}) - \mathcal{E}(\tilde{\boldsymbol{u}}) - \langle D\mathcal{E}(\tilde{\boldsymbol{u}}), \boldsymbol{u} - \tilde{\boldsymbol{u}} \rangle \,,$$

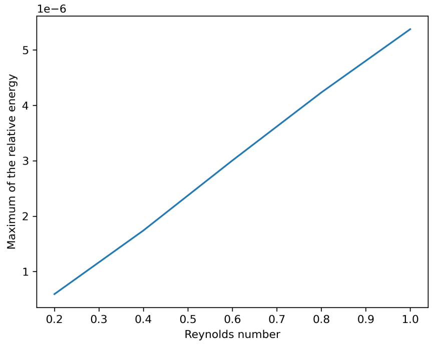


Figure 3: Maximum relative energy error as a function of the Reynolds number.

where D denotes the Gâteaux-derivative. Taking formally the time derivative of the relative energy, we infer

$$\begin{aligned}\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})\Big|_0^t &= \mathcal{E}(\boldsymbol{u})\Big|_0^t - \int_0^t \langle D\mathcal{E}(\tilde{\boldsymbol{u}}), \partial_t \tilde{\boldsymbol{u}}\rangle + \langle D\mathcal{E}(\tilde{\boldsymbol{u}}), \partial_t \boldsymbol{u} - \partial_t \tilde{\boldsymbol{u}}\rangle - \langle D^2\mathcal{E}(\tilde{\boldsymbol{u}})\partial_t \tilde{\boldsymbol{u}}, \boldsymbol{u} - \tilde{\boldsymbol{u}}\rangle \,\mathrm{d}s \\ &= \mathcal{E}(\boldsymbol{u})\Big|_0^t - \int_0^t \langle D\mathcal{E}(\tilde{\boldsymbol{u}}), \partial_t \boldsymbol{u}\rangle - \langle D^2\mathcal{E}(\tilde{\boldsymbol{u}})\partial_t \tilde{\boldsymbol{u}}, \boldsymbol{u} - \tilde{\boldsymbol{u}}\rangle \,\mathrm{d}s\,.\end{aligned}$$

In order to evaluate the right-hand side further, we need to use the energy-inequality (5), we need the weak formulation (4) tested with the derivative of the energy evaluated at the smooth functions $D\mathcal{E}(\tilde{\boldsymbol{u}})$, as well as the equations (1) for the smooth function tested by $D^2\mathcal{E}(\tilde{\boldsymbol{u}})(\boldsymbol{u} - \tilde{\boldsymbol{u}})$. In order to complete this short motivation, we will calculate these terms for the considered system. Recalling that the energy takes the form,

$$\mathcal{E}(\boldsymbol{u}) := \int_\Omega \frac{\mathrm{Re}}{2}|\boldsymbol{v}|^2 + \llbracket c^\pm(\ln c^\pm + 1)\rrbracket^\pm + \frac{1}{2}|\nabla\psi|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\boldsymbol{x} + \frac{\tau}{2}\int_\Gamma |\psi|^2\,\mathrm{d}\sigma\,,$$

where the variable ψ has to be understood as a nonlocal term depending on $c^\pm$, *i.e.*, $\psi := B(c^+ - c^-)$, where $B : L^1(\Omega) \to L^1(\Omega)$ solving (1d) with inhomogeneous Robin-boundary conditions. Calculating the first two variations, we end up with

$$\begin{aligned}\langle D\mathcal{E}(\boldsymbol{u}), \boldsymbol{w}\rangle &= \int_\Omega \mathrm{Re}\,\boldsymbol{v}\cdot\boldsymbol{w}_{\boldsymbol{v}} + \llbracket \boldsymbol{w}_{c^\pm}(\ln c^\pm + 2)\rrbracket^\pm + B(c^+ - c^-)(\boldsymbol{w}_{c^+} - \boldsymbol{w}_{c^-})\,\mathrm{d}\boldsymbol{x} \\ \langle D^2\mathcal{E}(\boldsymbol{u}), (\boldsymbol{w}, \boldsymbol{z})\rangle &= \int_\Omega \mathrm{Re}\,\boldsymbol{w}_{\boldsymbol{v}}\cdot\boldsymbol{z}_{\boldsymbol{v}} + \left\llbracket \frac{1}{c^\pm}\boldsymbol{w}_{c^\pm}\boldsymbol{z}_{c^\pm}\right\rrbracket^\pm + B(\boldsymbol{w}_{c^+} - \boldsymbol{w}_{c^-})(\boldsymbol{z}_{c^+} - \boldsymbol{z}_{c^-})\,\mathrm{d}\boldsymbol{x}\,.\end{aligned}$$

Finally, we observe for the terms in question

$$\begin{aligned}D\mathcal{E}(\tilde{\boldsymbol{u}}) &= (\mathrm{Re}\,\tilde{\boldsymbol{v}}, \ln\tilde{c}^\pm \pm \tilde{\psi})^T \\ D^2\mathcal{E}(\tilde{\boldsymbol{u}})(\boldsymbol{u} - \tilde{\boldsymbol{u}}) &= \left(\mathrm{Re}\,(\boldsymbol{v} - \tilde{\boldsymbol{v}}), \frac{1}{\tilde{c}^\pm}(c^\pm - \tilde{c}^\pm) \pm \llbracket B(\pm c^\pm \mp \tilde{c}^\pm)\rrbracket^\pm\right)^T.\end{aligned}$$

By the definition of ψ and similar $\tilde{\psi}$, we may identify $\pm\llbracket B(\pm c^\pm \mp \tilde{c}^\pm)\rrbracket^\pm = \pm(B(c^+ - c^-) - B(\tilde{c}^+ - \tilde{c}^-)) = \pm(\psi - \tilde{\psi})$. Thus in order to derive the relative energy inequality, we have to subtract the weak formulation (4a) and (4b) tested with $\tilde{\boldsymbol{v}}$ and $\ln\tilde{c}^\pm + 2 \pm \tilde{\psi}$, respectively as well as the strong formulation (1a) and (1c) for the smooth function $\tilde{\boldsymbol{u}}$ tested with $\boldsymbol{v} - \tilde{\boldsymbol{v}}$ and $\frac{1}{\tilde{c}^\pm}(c^\pm - \tilde{c}^\pm) \pm (\psi - \tilde{\psi})$, respectively from the energy inequality (5). Which is exactly the strategy we are going to follow in the next subsection.

3.1 Formulation of the relative energy inequality

Let $\mathbb{U}$ be the solution space of weak solutions $(\boldsymbol{v}, c^{\pm}, \psi)$ fulfilling the regularity assumptions from Definition 2.2 and recall $\widetilde{\mathbb{U}}$ the space of test function. We define the relative energy $\mathcal{R} : \mathbb{U} \times \widetilde{\mathbb{U}} \to L^{\infty}(0,T)$ by

$$\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) := \int_{\Omega} \frac{\mathrm{Re}}{2}|\boldsymbol{v}-\tilde{\boldsymbol{v}}|^2 + \left[\tilde{c}^{\pm} - c^{\pm} - c^{\pm}(\ln \tilde{c}^{\pm} - \ln c^{\pm})\right]^{\pm} + \frac{1}{2}|\nabla(\psi - \tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})} \,\mathrm{d}\boldsymbol{x} + \frac{\tau}{2}\int_{\Gamma} |\psi - \tilde{\psi}|^2 \,\mathrm{d}\sigma. \tag{7}$$

With the regularities from $\mathbb{U}$ and $\widetilde{\mathbb{U}}$ all terms appearing in the definition of the relative energy are indeed in $L^{\infty}(0,T)$, which follows from the energy inequality (5) for the $c^{\pm}$–terms. Moreover, we define the relative dissipation potential $\mathcal{W} : \mathbb{U} \times \widetilde{\mathbb{U}} \to L^1(0,T)$ by

$$\begin{aligned}\mathcal{W}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) := &\int_{\Omega} |\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})|^2 + \left[4\left|\nabla\sqrt{c^{\pm}} - \nabla\sqrt{\tilde{c}^{\pm}}\right|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} + c^{\pm}|\nabla(\psi-\tilde{\psi})|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^{\pm} \mathrm{d}\boldsymbol{x} \\ &+ 2\int_{\Omega} |\nabla^2(\psi-\tilde{\psi})|^2 + \varepsilon\lambda|\nabla^2(\psi-\tilde{\psi})\boldsymbol{d}\cdot\boldsymbol{d}|^2 + \varepsilon\lambda|\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{d}|^2 \,\mathrm{d}\boldsymbol{x} \\ &+ 2\int_{\Omega} (\varepsilon+\lambda)|\nabla^2(\psi-\tilde{\psi})\boldsymbol{d}|^2 \,\mathrm{d}\boldsymbol{x} \\ &+ 2\tau\int_{\Gamma} |\nabla_{\Gamma}(\psi-\tilde{\psi})|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} + |\nabla_{\Gamma}(\psi-\tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})} \,\mathrm{d}\sigma,\end{aligned} \tag{8}$$

where ∇_{Γ} denotes the surface gradient [HLP25, Sec. A.1] and the regularity weight $\mathcal{K} : \widetilde{\mathbb{U}} \to L^1(0,T)$ by

$$\begin{aligned}\mathcal{K}(\tilde{\boldsymbol{u}}) := C\left(1 + \|\boldsymbol{d}\|^8_{W^{1,\infty}(\Omega)}\right)\Big(&\left(1+\mathrm{Re}^3\right)\|\tilde{\boldsymbol{v}}\|^4_{L^{\infty}(\Omega)} + \left[\|\tilde{c}^{\pm}\|^4_{W^{1,\infty}(\Omega)}\right]^{\pm} \\ &+ \|\tilde{\psi}\|^4_{W^{1,\infty}(\Omega)} + \|\tilde{\psi}\|^2_{L^{\infty}(0,T;W^{1,2}(\Omega))} + \|\tilde{\xi}\|^2_{W^{1,2}(\Gamma)} + 1\Big).\end{aligned} \tag{9}$$

Additionally, we introduce the operator $\mathcal{Q}_{\tilde{\boldsymbol{u}}} : \left(W^{1,2}(0,T;L^2(\Gamma)) \cap L^2(0,T;W^{1,2}(\Gamma))\right)^2 \to L^1(0,T)$ measuring the distance between ξ and $\tilde{\xi}$ through

$$\mathcal{Q}_{\tilde{\boldsymbol{u}}}(\xi|\tilde{\xi}) := \frac{1}{2}\|\partial_t(\xi-\tilde{\xi})\|^2_{L^2(\Gamma)} + C\left(1 + \left[\|\tilde{c}^{\pm}\|_{L^{\infty}(\Omega)}\right]^{\pm}\right)\|\xi-\tilde{\xi}\|^2_{W^{1,2}(\Gamma)}. \tag{10}$$

We note that $\mathcal{Q}_{\tilde{\boldsymbol{u}}}(\xi|\tilde{\xi}) = 0$ if and only if $\xi = \tilde{\xi}$. Finally, we define the system operator by

$$\begin{aligned}&\mathcal{A} : \widetilde{\mathbb{U}} \to \left(L^2((0,T)\times\Omega) \times L^{5/2}((0,T)\times\Omega)\right) \\ &\mathcal{A}(\tilde{\boldsymbol{u}}) := \begin{pmatrix} \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}}) \\ \mathcal{A}_{\tilde{c}^{\pm}}(\tilde{\boldsymbol{u}}) \end{pmatrix} := \begin{pmatrix} \mathrm{Re}\,(\partial_t\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}) - \Delta\tilde{\boldsymbol{v}} + (\tilde{c}^{+} - \tilde{c}^{-})\nabla\tilde{\psi} \\ \partial_t\tilde{c}^{\pm} + \nabla\cdot(\tilde{c}^{\pm}\tilde{\boldsymbol{v}}) - \nabla\cdot\left(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi})\right) \end{pmatrix}.\end{aligned} \tag{11}$$

It is clear from the definition of $\widetilde{\mathbb{U}}$ that $\mathcal{A}$ indeed maps to $L^2((0,T)\times\Omega) \times L^{5/2}((0,T)\times\Omega)$. Now, we are in a position to state the relative energy inequality.

Proposition 3.1 (Relative energy inequality)**.** *Under the Assumption 2.1, it holds for all suitable weak solutions* $\boldsymbol{u} = (\boldsymbol{v}, c^{\pm}, \psi) \in \mathbb{U}$*, cf. Definition 2.2, and all test functions* $\tilde{\boldsymbol{u}} = (\tilde{\boldsymbol{v}}, \tilde{c}^{\pm}, \tilde{\psi}) \in \widetilde{\mathbb{U}}$ *with* $\tilde{c}^{\pm} \geq l > 0$ *that*

$$\begin{aligned}&\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})(s)e^{\int_s^t \mathcal{K}(\tilde{\boldsymbol{u}})\,\mathrm{d}\tau}\Big|_{s=0}^{s=t} + \int_0^t \frac{1}{2}\mathcal{W}(\boldsymbol{u}|\tilde{\boldsymbol{u}})e^{\int_s^t \mathcal{K}(\tilde{\boldsymbol{u}})\,\mathrm{d}\tau}\,\mathrm{d}s \\ &\leq \int_0^t \left(\int_{\Omega} \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}})\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) + \left[\mathcal{A}_{\tilde{c}^{\pm}}(\tilde{\boldsymbol{u}})\left(\frac{1}{\tilde{c}^{\pm}}(\tilde{c}^{\pm}-c^{\pm}) \pm (\tilde{\psi}-\psi)\right)\right]^{\pm} \mathrm{d}\boldsymbol{x}\right) e^{\int_s^t \mathcal{K}(\tilde{\boldsymbol{u}})\,\mathrm{d}\tau}\,\mathrm{d}s \\ &\quad + \int_0^t \mathcal{Q}_{\tilde{\boldsymbol{u}}}(\xi|\tilde{\xi})e^{\int_s^t \mathcal{K}(\tilde{\boldsymbol{u}})\,\mathrm{d}\tau}\,\mathrm{d}s\end{aligned} \tag{12}$$

for all $t \in [0,T]$.

Before we turn to the rather lengthy proof of this proposition, we quickly prove the weak-strong uniqueness of suitable weak solutions to system (1).

Proof (of Proposition 2.6). We use the strong solution $\tilde{\boldsymbol{u}} = (\tilde{\boldsymbol{v}}, \tilde{c}^{\pm}, \tilde{\psi})$ as a test function in (12). Since $\tilde{\psi}$ fulfills (1d) with boundary condition $\tilde{\xi} = \xi$ and $\tilde{\boldsymbol{u}}$ solves the system (1) the $\mathcal{A}$– and $\mathcal{Q}_{\tilde{\boldsymbol{u}}}$–terms on the right-hand side of (12) vanish. Additionally, since $(\tilde{\boldsymbol{v}}, \tilde{c}^{\pm}, \tilde{\psi})$ and $(\boldsymbol{v}, c^{\pm}, \psi)$ emanate from the same initial conditions, we have $\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})(0) = 0$. By the non-negativity of the relative dissipation potential (8), we obtain

$$\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})(t) \leq 0$$

for all $t \in [0, T]$. From that, we can deduce that additionally to $\tilde{c}^{\pm} \geq l > 0$ also $c^{\pm} > 0$ almost everywhere and thus by [HLP23, Rem. 5.4], we obtain $\boldsymbol{u} = \tilde{\boldsymbol{u}}$ and our proof is complete. □

3.2 Proof of the relative energy inequality (12)

Now, we turn to the proof of the relative energy inequality in Theorem 3.1. We first collect some useful lemmata.

3.2.1 Mathematical toolbox

We start off by recalling some estimates needed throughout the proof. First, we prove a simple estimate for the squared difference of two square roots, this is a special case of [Las21, Lem. 2.8].

Lemma 3.2. *For all $x, y \in \mathbb{R}$ with $x > 0$ and $y \geq 0$, we have*

$$\left(\sqrt{x} - \sqrt{y}\right)^2 \leq x - y - y(\ln x - \ln y). \tag{13}$$

Proof. For $x, y > 0$ the inequality (13) follows by a simple expansion of the left-hand side using that $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ and $a\ln(x) = \ln(x^a)$ for $a > 0$:

$$\begin{aligned}
\left(\sqrt{x} - \sqrt{y}\right)^2 &= x - 2\sqrt{x}\sqrt{y} + y = x - y - 2\sqrt{y}\left(\sqrt{x} - \sqrt{y}\right) \\
&= x - y - y\ln\left(\frac{x}{y}\right) - 2\sqrt{y}\left(\sqrt{x} - \sqrt{y} - \frac{\sqrt{y}}{2}\ln\left(\frac{x}{y}\right)\right) \\
&= x - y - y\left(\ln x - \ln y\right) - 2\sqrt{y}\left(\sqrt{x} - \sqrt{y} - \sqrt{y}\left(\ln\sqrt{x} - \ln\sqrt{y}\right)\right) \\
&\leq x - y - y\left(\ln x - \ln y\right),
\end{aligned}$$

since $\sqrt{x} - \sqrt{y} - \sqrt{y}\left(\ln\sqrt{x} - \ln\sqrt{y}\right) \geq 0$. The latter follows from the convexity of the exponential. Since $y \mapsto y\ln y$ is continuous on $[0, \infty)$ inequality (13) remains true for all $y \geq 0$. □

Next, we show a neat gradient reformulation.

Lemma 3.3 (Gradient trick)**.** *Let $f \in W^{1,1}(\Omega)$ and $f \geq 0$ be such that $\sqrt{f} \in W^{1,2}(\Omega)$ holds, then*

$$2\,\nabla\sqrt{f} - 2\sqrt{\frac{f}{\tilde{f}}}\nabla\sqrt{\tilde{f}} - 2\sqrt{\frac{f}{\tilde{f}}}\nabla\sqrt{f} + 2\,\frac{f}{\tilde{f}}\nabla\sqrt{\tilde{f}} = -\sqrt{\tilde{f}}\,\nabla\left(1 - \sqrt{\frac{f}{\tilde{f}}}\right)^2 \tag{14}$$

holds for all $\tilde{f} \in C^1(\overline{\Omega})$ with $\tilde{f} \geq l > 0$.

Proof. An algebraic transformation of the left-hand side of (14) yields

$$\begin{aligned}
&2\,\nabla\sqrt{f} - 2\sqrt{\frac{f}{\tilde{f}}}\nabla\sqrt{\tilde{f}} - 2\sqrt{\frac{f}{\tilde{f}}}\nabla\sqrt{f} + 2\,\frac{f}{\tilde{f}}\nabla\sqrt{\tilde{f}} \\
&\quad = 2\left(\nabla\sqrt{f} - \sqrt{\frac{f}{\tilde{f}}}\nabla\sqrt{\tilde{f}}\right)\left(1 - \sqrt{\frac{f}{\tilde{f}}}\right) = 2\,\sqrt{\tilde{f}}\left(\frac{\nabla\sqrt{f}}{\sqrt{\tilde{f}}} - \frac{\sqrt{f}}{\tilde{f}}\nabla\sqrt{\tilde{f}}\right)\left(1 - \sqrt{\frac{f}{\tilde{f}}}\right).
\end{aligned}$$

Using the product rule for weak derivatives [Zei90, Prob. 21.3d], we have $\nabla\left(\sqrt{\frac{f}{\tilde{f}}}\right)=\left(\frac{\nabla\sqrt{f}}{\sqrt{\tilde{f}}}-\frac{\sqrt{f}}{\tilde{f}}\nabla\sqrt{\tilde{f}}\right)$ and thus obtain (14). □

Finally, we recall an estimate of the trilinear part of the Navier–Stokes equations. Here, we follow an estimate from the proof of [Las22, Prop. 3.1].

Lemma 3.4 (Estimate for the trilinear term of the Navier–Stokes equations)**.** *For $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ fulfilling the regularity of $\mathbb{U}$ and $\widetilde{\mathbb{U}}$ respectively, it holds that*

$$\mathrm{Re}\int_\Omega ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v})\cdot\tilde{\boldsymbol{v}}+((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}})\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\leq\frac{1}{4}\left\|\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})\right\|^2_{L^2(\Omega)}+C\mathrm{Re}^4\left\|\tilde{\boldsymbol{v}}\right\|^4_{L^6(\Omega)}\left\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\right\|^2_{L^2(\Omega)}. \tag{15}$$

Proof. Using the fact that the trilinear term is skew-symmetric in the last two entries, *i.e.* that $\int_\Omega(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}=0$ for all $\boldsymbol{v},\boldsymbol{w}\in W^{1,2}_{0,\sigma}(\Omega)$, we can expand the left-hand side of (15)

$$\begin{aligned}\int_\Omega(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}}+(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}&=\int_\Omega((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}+(\tilde{\boldsymbol{v}}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}}+(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\\&=\int_\Omega((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

We now apply Hölder's inequality, the Gagliardo–Nirenberg inequality with $\frac{1}{3}=\lambda\left(\frac{1}{2}-\frac{1}{3}\right)+(1-\lambda)\frac{1}{2}$, that is with $\lambda=\frac{1}{2}$, the norm inequality $\|\boldsymbol{v}\|_{W^{1,2}(\Omega)}\leq C\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}$ for all $\boldsymbol{v}\in W^{1,2}_{0,\sigma}(\Omega)$, and Young's inequality with $p=\frac{2}{1+\lambda}$ and $p'=\frac{2}{1-\lambda}$ in order to obtain

$$\begin{aligned}\mathrm{Re}\int_\Omega(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}}+(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}&\leq\mathrm{Re}\left\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\right\|_{L^3(\Omega)}\left\|\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})\right\|_{L^2(\Omega)}\left\|\tilde{\boldsymbol{v}}\right\|_{L^6(\Omega)}\\&\leq C\,\mathrm{Re}\left\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\right\|^{1/2}_{L^2(\Omega)}\left\|\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})\right\|^{3/2}_{L^2(\Omega)}\left\|\tilde{\boldsymbol{v}}\right\|_{L^6(\Omega)}\\&\leq\frac{1}{4}\left\|\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})\right\|^2_{L^2(\Omega)}+C\,\mathrm{Re}^4\left\|\tilde{\boldsymbol{v}}\right\|^4_{L^6(\Omega)}\left\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\right\|^2_{L^2(\Omega)}.\end{aligned}$$

□

3.2.2 Proof of the relative energy inequality

With these tools at our fingertips, we turn to the proof of the relative energy inequality (12). This proof is rather technical and long. A reader more interested in *a posteriori* error estimates to the SNPP system may skip this section and continue with Section 4 for the existence proof of strong solutions to the NSNPP and SNPP systems or even Section 5 for the proof of the error estimates.

Proof (of Proposition 3.1). The proof is based on a combination of appropriately testing the weak formulation (4) and the system operator $\mathcal{A}$, adding the energy inequality and then estimating the resulting terms using an integration by parts on the boundary, that is for $f\in W^{1,p}(\Gamma)$ and $\boldsymbol{v}\in W^{1,p'}(\Gamma)$ it holds:

$$\int_\Gamma f\,\nabla_\Gamma\cdot\boldsymbol{v}-f\,(\boldsymbol{v}\cdot\boldsymbol{n})\,\nabla_\Gamma\cdot\boldsymbol{n}+\boldsymbol{v}\cdot\nabla_\Gamma f\,\mathrm{d}\sigma=0 \tag{16}$$

where $\nabla_\Gamma f$ and $\nabla_\Gamma\cdot\boldsymbol{v}$ denote the surface gradient and the surface divergence, respectively, see [HLP25, Cor. A.25]. We refer the interested reader to [HLP25, Appendix A.1] for a short introduction into surface differential operators. We test the weak formulation (4a) of $\boldsymbol{v}$ and the weak formulation (4b) of $c^\pm$ with $-\tilde{\boldsymbol{v}}$ and $-(\tilde{\mu}^\pm+2)=-(\ln\tilde{c}^\pm\pm\tilde{\psi}+2)$ respectively, where the chemical potential $\tilde{\mu}^\pm$ is given by $\tilde{\mu}^\pm:=\ln\tilde{c}^\pm\pm\tilde{\psi}$.

Adding the energy inequality (5) and adding and subtracting the system operator $\mathcal{A}(\tilde{\boldsymbol{u}})$ tested with an appropriate test function, namely $\left(\tilde{\boldsymbol{v}}-\boldsymbol{v},\ \frac{1}{\tilde{c}^\pm}(\tilde{c}^\pm-c^\pm)\pm(\tilde{\psi}-\psi)\right)^T$, we obtain

$$
\begin{aligned}
&\left[\int_\Omega \frac{\mathrm{Re}}{2}|\boldsymbol{v}|^2 + \left[c^\pm(\ln c^\pm+1)\right]^\pm + \frac{1}{2}|\nabla\psi|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\boldsymbol{x} + \frac{\tau}{2}\int_\Gamma |\psi|^2\,\mathrm{d}\sigma\right]\Bigg|_0^t \\
&\quad + \int_0^t\int_\Omega |\nabla\boldsymbol{v}|^2 + \left[\left|2\nabla\sqrt{c^\pm}\pm\sqrt{c^\pm}\nabla\psi\right|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad - \left(\mathrm{Re}\int_\Omega \boldsymbol{v}\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\Big|_0^t + \int_0^t\int_\Omega -\mathrm{Re}\,\boldsymbol{v}\partial_t\tilde{\boldsymbol{v}} + \nabla\boldsymbol{v}:\nabla\tilde{\boldsymbol{v}} + \mathrm{Re}\,(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}} + (c^+-c^-)\nabla\psi\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\right) \\
&\quad - \left[\int_\Omega c^\pm(\tilde{\mu}^\pm+2)\,\mathrm{d}\boldsymbol{x}\Big|_0^t + \int_0^t\int_\Omega -c^\pm\partial_t\tilde{\mu}^\pm - c^\pm\boldsymbol{v}\cdot\nabla\tilde{\mu}^\pm + \boldsymbol{\Lambda}(\boldsymbol{d})(\nabla c^\pm \pm c^\pm\nabla\psi)\cdot\nabla\tilde{\mu}^\pm\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\right]^\pm \\
&\quad + \int_0^t\int_\Omega \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}})\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) + \left[\mathcal{A}_{\tilde{c}^\pm}(\tilde{\boldsymbol{u}})\left(\frac{1}{\tilde{c}^\pm}\left(\tilde{c}^\pm-c^\pm\right)\pm(\tilde{\psi}-\psi)\right)\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s - \int_0^t\int_\Gamma \partial_t\xi\psi\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\le \int_0^t\int_\Omega \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}})\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) + \left[\mathcal{A}_{\tilde{c}^\pm}(\tilde{\boldsymbol{u}})\left(\frac{1}{\tilde{c}^\pm}\left(\tilde{c}^\pm-c^\pm\right)\pm(\tilde{\psi}-\psi)\right)\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s
\end{aligned} \tag{17}
$$

for all $t\in[0,T]$. Here, we used the assumption, that the test function $\tilde{c}^\pm \ge l > 0$ is bounded away from zero, allowing us to test with $(\tilde{c}^\pm)^{-1}$. We now explicitly calculate the left-hand side of (17). We first consider the terms coming from the Navier–Stokes equation, that is

$$
\begin{aligned}
&\left[\int_\Omega \frac{\mathrm{Re}}{2}|\boldsymbol{v}|^2 - \mathrm{Re}\int_\Omega \boldsymbol{v}\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\right]\Bigg|_0^t + \int_0^t\int_\Omega |\nabla\boldsymbol{v}|^2 + \mathrm{Re}\,\boldsymbol{v}\partial_t\tilde{\boldsymbol{v}} - \nabla\boldsymbol{v}:\nabla\tilde{\boldsymbol{v}} - \mathrm{Re}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega -(c^+-c^-)\nabla\psi\cdot\tilde{\boldsymbol{v}} + \left(\mathrm{Re}\,\partial_t\tilde{\boldsymbol{v}} - \Delta\tilde{\boldsymbol{v}} + \mathrm{Re}\,(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}} + \nabla\tilde{p} + (\tilde{c}^+-\tilde{c}^-)\nabla\tilde{\psi}\right)\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&= \left[\int_\Omega \frac{\mathrm{Re}}{2}|\boldsymbol{v}-\tilde{\boldsymbol{v}}|^2\,\mathrm{d}\boldsymbol{x}\right]\Bigg|_0^t + \int_0^t\int_\Omega |\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})|^2 - \mathrm{Re}\,(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}} - \mathrm{Re}\,(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega (\tilde{c}^+-\tilde{c}^-)\nabla\tilde{\psi}\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) - (c^+-c^-)\nabla\psi\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.
\end{aligned} \tag{18}
$$

Now we consider the terms of (17) coming from the Nernst–Planck–Poisson part including a time derivative or no time integral, that is

$$
\begin{aligned}
&\left[\int_\Omega c^\pm\left[(\ln c^\pm+1) - c^\pm(\tilde{\mu}^\pm+2)\right]^\pm + \frac{1}{2}|\nabla\psi|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\boldsymbol{x} + \frac{\tau}{2}\int_\Gamma|\psi|^2\,\mathrm{d}\sigma\right]\Bigg|_0^t - \int_0^t\int_\Gamma \partial_t\xi\psi\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega \left[c^\pm\partial_t\tilde{\mu}^\pm\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \int_0^t\int_\Omega \left[\partial_t\tilde{c}^\pm\left(\frac{1}{\tilde{c}^\pm}\left(\tilde{c}^\pm-c^\pm\right)\pm(\tilde{\psi}-\psi)\right)\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&= \left[\int_\Omega \left[\tilde{c}^\pm - c^\pm - c^\pm(\ln\tilde{c}^\pm - \ln c^\pm)\right]^\pm + \frac{1}{2}|\nabla\psi|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})} - (c^+-c^-)\tilde{\psi}\,\mathrm{d}\boldsymbol{x} + \frac{\tau}{2}\int_\Gamma|\psi|^2\,\mathrm{d}\sigma\right]\Bigg|_0^t \\
&\quad - \int_0^t\int_\Gamma \partial_t\xi\psi\,\mathrm{d}\sigma\,\mathrm{d}s + \int_0^t\int_\Omega (c^+-c^-)\partial_t\tilde{\psi} + \partial_t(\tilde{c}^+-\tilde{c}^-)(\tilde{\psi}-\psi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&= \left[\int_\Omega \left[\tilde{c}^\pm - c^\pm - c^\pm(\ln\tilde{c}^\pm - \ln c^\pm)\right]^\pm + \frac{1}{2}|\nabla\psi|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})} - \boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla\psi\cdot\nabla\tilde{\psi}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \frac{\tau}{2}|\psi|^2 - \tau\psi\tilde{\psi} + \xi\tilde{\psi}\,\mathrm{d}\sigma\right]\Bigg|_0^t \\
&\quad - \int_0^t\int_\Gamma \partial_t\xi\psi\,\mathrm{d}\sigma\,\mathrm{d}s + \int_0^t\int_\Omega \boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla\psi\cdot\nabla\partial_t\tilde{\psi} + \boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla\partial_t\tilde{\psi}\cdot\nabla(\tilde{\psi}-\psi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Gamma \tau\psi\partial_t\tilde{\psi} - \xi\partial_t\tilde{\psi} + \tau\partial_t\tilde{\psi}(\tilde{\psi}-\psi) - \partial_t\tilde{\xi}(\tilde{\psi}-\psi)\,\mathrm{d}\sigma\,\mathrm{d}s \\
&= \left[\int_\Omega \left[\tilde{c}^\pm - c^\pm - c^\pm(\ln\tilde{c}^\pm - \ln c^\pm)\right]^\pm + \frac{1}{2}|\nabla(\psi-\tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \frac{\tau}{2}|\psi-\tilde{\psi}|^2\,\mathrm{d}\sigma\right]\Bigg|_0^t \\
&\quad + \int_0^t\int_\Gamma \partial_t(\tilde{\xi}-\xi)(\psi-\tilde{\psi})\,\mathrm{d}\sigma\,\mathrm{d}s,
\end{aligned} \tag{19}
$$

where we used the Poisson equation (1d) for ψ and $\tilde\psi$ for the second equality. The terms on the first line on the right-hand side fit nicely into the relative energy (7). We now turn to the terms in (17) coming from the Nernst–Planck–Poisson part and including $\boldsymbol{v}$ or $\tilde{\boldsymbol{v}}$,

$$\begin{aligned}
&\int_0^t\int_\Omega \left[c^\pm \boldsymbol{v}\cdot\nabla\tilde\mu^\pm - \tilde c^\pm\tilde{\boldsymbol{v}}\cdot\nabla\left(\frac{-c^\pm}{\tilde c^\pm}\pm(\tilde\psi-\psi)\right)\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&\quad=\int_0^t\int_\Omega \left[c^\pm\boldsymbol{v}\cdot\nabla\ln\tilde c^\pm\right]^\pm + (c^+-c^-)\boldsymbol{v}\cdot\nabla\tilde\psi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&\qquad+\int_0^t\int_\Omega \left[\tilde{\boldsymbol{v}}\cdot\nabla c^\pm - c^\pm\tilde{\boldsymbol{v}}\cdot\nabla\ln\tilde c^\pm\right]^\pm - (\tilde c^+-\tilde c^-)\tilde{\boldsymbol{v}}\cdot\nabla(\tilde\psi-\psi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.
\end{aligned}\tag{20}$$

Next, we consider the terms in (17) coming from the Nernst–Planck–Poisson part including dissipation terms,

$$\begin{aligned}
&\int_0^t\int_\Omega\left[\left|2\nabla\sqrt{c^\pm}\pm\sqrt{c^\pm}\nabla\psi\right|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} - \boldsymbol{\Lambda}(\boldsymbol{d})(\nabla c^\pm \pm c^\pm\nabla\psi)\cdot\nabla\tilde\mu^\pm\right]^\pm\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&+\int_0^t\int_\Omega\left[\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde c^\pm\pm\tilde c^\pm\nabla\tilde\psi)\cdot\nabla\left(\frac{-c^\pm}{\tilde c^\pm}\pm(\tilde\psi-\psi)\right)\right]^\pm\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&=\int_0^t\int_\Omega\left[4|\nabla\sqrt{c^\pm}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm + 4\sqrt{c^+}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{c^+}\cdot\nabla\psi - 4\sqrt{c^-}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{c^-}\cdot\nabla\psi + \left[c^\pm|\nabla\psi|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&+\int_0^t\int_\Omega -\left[\boldsymbol{\Lambda}(\boldsymbol{d})\nabla c^\pm\cdot\nabla\ln\tilde c^\pm\right]^\pm - (c^++c^-)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\tilde\psi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&+\int_0^t\int_\Omega -\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(c^+-c^-)\cdot\nabla\tilde\psi - c^+\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\ln\tilde c^+ + c^-\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\ln\tilde c^-\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&+\int_0^t\int_\Omega\left[-\frac{1}{\tilde c^\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde c^\pm\cdot\nabla c^\pm + \frac{c^\pm}{(\tilde c^\pm)^2}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde c^\pm\cdot\nabla\tilde c^\pm\right]^\pm + (\tilde c^++\tilde c^-)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde\psi\cdot\nabla(\tilde\psi-\psi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&+\int_0^t\int_\Omega\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\tilde c^+-\tilde c^-)\cdot\nabla(\tilde\psi-\psi) + \tilde c^+\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde\psi\cdot\nabla\left(\frac{-c^+}{\tilde c^+}\right) - \tilde c^-\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde\psi\cdot\nabla\left(\frac{-c^-}{\tilde c^-}\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&=\mathrm{T}_{\nabla c,\nabla\tilde c}+\mathrm{T}_{\nabla\psi,\nabla\tilde\psi}+\mathrm{T}_{\nabla c,\nabla\psi},
\end{aligned}\tag{21}$$

where we grouped the terms quadratic in $\nabla c^\pm, \nabla\tilde c^\pm$ in $\mathrm{T}_{\nabla c,\nabla\tilde c}$, the terms quadratic in $\nabla\psi,\nabla\tilde\psi$ in $\mathrm{T}_{\nabla\psi,\nabla\tilde\psi}$ and the mixed terms in $\mathrm{T}_{\nabla c,\nabla\psi}$, so that

$$\begin{aligned}
\mathrm{T}_{\nabla c,\nabla\tilde c} &= \int_\Omega\left[4|\nabla\sqrt{c^\pm}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} - 8\sqrt{\frac{c^\pm}{\tilde c^\pm}}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{c^\pm}\cdot\nabla\sqrt{\tilde c^\pm} + 4\frac{c^\pm}{\tilde c^\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde c^\pm}\cdot\nabla\sqrt{\tilde c^\pm}\right]^\pm\mathrm{d}\boldsymbol{x},\\
\mathrm{T}_{\nabla\psi,\nabla\tilde\psi} &= \int_\Omega\left[c^\pm|\nabla\psi|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm - (c^++c^-)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\tilde\psi + (\tilde c^++\tilde c^-)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde\psi\cdot\nabla(\tilde\psi-\psi)\,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{T}_{\nabla c,\nabla\psi} &= \int_\Omega 4\sqrt{c^+}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{c^+}\cdot\nabla\psi - 4\sqrt{c^-}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{c^-}\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x}\\
&\quad+\int_\Omega -\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(c^+-c^-)\cdot\nabla\tilde\psi - c^+\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\ln\tilde c^+ + c^-\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\ln\tilde c^-\,\mathrm{d}\boldsymbol{x}\\
&\quad+\int_\Omega\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\tilde c^+-\tilde c^-)\cdot\nabla(\tilde\psi-\psi) + \tilde c^+\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde\psi\cdot\nabla\left(\frac{-c^+}{\tilde c^+}\right) - \tilde c^-\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde\psi\cdot\nabla\left(\frac{-c^-}{\tilde c^-}\right)\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

We can rewrite $\mathrm{T}_{\nabla c,\nabla\tilde{c}}$ as

$$\begin{aligned}
\mathrm{T}_{\nabla c,\nabla\tilde{c}} &= \int_\Omega \left[4|\nabla\sqrt{c^\pm}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} - 8\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{c^\pm}\cdot\nabla\sqrt{\tilde{c}^\pm} + 4\frac{c^\pm}{\tilde{c}^\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\sqrt{\tilde{c}^\pm}\right]^\pm \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left[4|\nabla\sqrt{c^\pm} - \nabla\sqrt{\tilde{c}^\pm}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} + 8\left(1-\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\right)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\sqrt{c^\pm}\right]^\pm \mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \left[4\left(\frac{c^\pm}{\tilde{c}^\pm}-1\right)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\sqrt{\tilde{c}^\pm}\right]^\pm \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left[4|\nabla\sqrt{c^\pm} - \nabla\sqrt{\tilde{c}^\pm}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm \mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \left[4\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\left(2\nabla\sqrt{c^\pm} - 2\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\nabla\sqrt{c^\pm} + 2\frac{c^\pm}{\tilde{c}^\pm}\nabla\sqrt{\tilde{c}^\pm} - 2\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\nabla\sqrt{\tilde{c}^\pm}\right)\right]^\pm \mathrm{d}\boldsymbol{x} \\
&\quad - \int_\Omega \left[4\left(\frac{c^\pm}{\tilde{c}^\pm}+1-2\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\right)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\sqrt{\tilde{c}^\pm}\right]^\pm \mathrm{d}\boldsymbol{x} \\
&\underset{\text{Lem. 3.3}}{=} \int_\Omega \left[4|\nabla\sqrt{c^\pm} - \nabla\sqrt{\tilde{c}^\pm}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm \mathrm{d}\boldsymbol{x} \\
&\quad - \int_\Omega \left[4\sqrt{\tilde{c}^\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\left(1-\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\right)^2 + 4\left(1-\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\right)^2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\sqrt{\tilde{c}^\pm}\right]^\pm \mathrm{d}\boldsymbol{x}. \qquad (22)
\end{aligned}$$

Now, we rewrite

$$\begin{aligned}
\mathrm{T}_{\nabla\psi,\nabla\tilde{\psi}} &= \int_\Omega \left[c^\pm|\nabla\psi|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm - (c^+ + c^-)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\tilde{\psi} + (\tilde{c}^+ + \tilde{c}^-)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla(\tilde{\psi}-\psi)\,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left[c^\pm|\nabla(\psi-\tilde{\psi})|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} + (\tilde{c}^\pm - c^\pm)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla(\tilde{\psi}-\psi)\right]^\pm \mathrm{d}\boldsymbol{x} \qquad (23)
\end{aligned}$$

and finally for $\mathrm{T}_{\nabla c,\nabla\psi}$, we find

$$\begin{aligned}
\mathrm{T}_{\nabla c,\nabla\psi} &= \int_\Omega 2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(c^+ - c^-)\cdot\nabla\psi - \boldsymbol{\Lambda}(\boldsymbol{d})\nabla(c^+ - c^-)\cdot\nabla\tilde{\psi} - c^+\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\ln\tilde{c}^+\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega c^-\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\ln\tilde{c}^- - \boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\tilde{c}^+ - \tilde{c}^-)\cdot\nabla(\psi-\tilde{\psi}) - \boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla c^+\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla c^- + \frac{c^+}{\tilde{c}^+}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla\tilde{c}^+ - \frac{c^-}{\tilde{c}^-}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla\tilde{c}^-\,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega 2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(c^+ - c^- - (\tilde{c}^+ - \tilde{c}^-))\cdot\nabla(\psi-\tilde{\psi}) + \boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\tilde{c}^+ - \tilde{c}^-)\cdot\nabla(\psi-\tilde{\psi})\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega -\frac{c^+}{\tilde{c}^+}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\tilde{c}^+ + \frac{c^-}{\tilde{c}^-}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\nabla\tilde{c}^- + \frac{c^+}{\tilde{c}^+}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla\tilde{c}^+ - \frac{c^-}{\tilde{c}^-}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla\tilde{c}^-\,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega 2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(c^+ - c^- - (\tilde{c}^+ - \tilde{c}^-))\cdot\nabla(\psi-\tilde{\psi}) + \left[\pm\left(1-\frac{c^\pm}{\tilde{c}^\pm}\right)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{c}^\pm\cdot\nabla(\psi-\tilde{\psi})\right]^\pm \mathrm{d}\boldsymbol{x}. \qquad (24)
\end{aligned}$$

Inserting (18)–(24) back into (17), we obtain

$$
\begin{aligned}
&\left[\int_\Omega \frac{\mathrm{Re}}{2}|\boldsymbol{v}-\tilde{\boldsymbol{v}}|^2 + \left[\tilde{c}^\pm - c^\pm - c^\pm(\ln\tilde{c}^\pm - \ln c^\pm)\right]^\pm + \frac{1}{2}|\nabla(\psi-\tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \frac{\tau}{2}|\psi-\tilde{\psi}|^2\,\mathrm{d}\sigma\right]\Bigg|_0^t \\
&\quad + \int_0^t\int_\Omega |\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})|^2 + \left[4|\nabla\sqrt{c^\pm}-\nabla\sqrt{\tilde{c}^\pm}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} + c^\pm|\nabla(\psi-\tilde{\psi})|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega 2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(c^+-c^- - (\tilde{c}^+-\tilde{c}^-))\cdot\nabla(\psi-\tilde{\psi})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\le \int_0^t\int_\Omega \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}})\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) + \left[\mathcal{A}_{\tilde{c}^\pm}(\tilde{\boldsymbol{u}})\left(\frac{1}{\tilde{c}^\pm}\left(\tilde{c}^\pm - c^\pm\right) \pm (\tilde{\psi}-\psi)\right)\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad + \mathrm{Re}\int_0^t\int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s - \int_0^t\int_\Omega (\tilde{c}^+-\tilde{c}^-)\nabla\tilde{\psi}\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) - (c^+-c^-)\nabla\psi\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad - \int_0^t\int_\Omega \left[c^\pm\boldsymbol{v}\cdot\nabla\ln\tilde{c}^\pm \pm c^\pm\boldsymbol{v}\cdot\nabla\tilde{\psi} + \tilde{\boldsymbol{v}}\cdot\nabla c^\pm - c^\pm\tilde{\boldsymbol{v}}\cdot\nabla\ln\tilde{c}^\pm \mp \tilde{c}^\pm\tilde{\boldsymbol{v}}\cdot\nabla(\tilde{\psi}-\psi)\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad - \int_0^t\int_\Omega \left[\pm\left(1-\frac{c^\pm}{\tilde{c}^\pm}\right)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{c}^\pm\cdot\nabla(\psi-\tilde{\psi}) + (\tilde{c}^\pm - c^\pm)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla(\tilde{\psi}-\psi)\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad - \int_0^t\int_\Omega \left[-4\sqrt{\tilde{c}^\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\left(1-\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\right)^2 - 4\left(1-\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\right)^2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\sqrt{\tilde{c}^\pm}\right]^\pm \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad - \int_0^t\int_\Gamma \partial_t(\tilde{\xi}-\xi)(\psi-\tilde{\psi})\,\mathrm{d}\sigma\,\mathrm{d}s.
\end{aligned}
\tag{25}
$$

To arrive at the dissipation potential (8) on the left-hand side, we perform an integration by parts on the last term on the left-hand side, which is rather involved since we need to handle a lot of boundary terms. For some of the following transformations, we need more regularity for $\psi, \tilde{\psi}$ and $\boldsymbol{d}$. To make these calculation rigorous, we can take a smooth approximation $(\boldsymbol{d}_n)_n \subseteq C^\infty(\overline{\Omega})$ [Pla24b, Lem. A.13] of the director field $\boldsymbol{d}$, such that $\boldsymbol{d}_n\cdot\boldsymbol{n} = 0$ on Γ for all $n\in\mathbb{N}$, as well as ψ_n and $\tilde{\psi}_n$ such that

$$
\begin{aligned}
-\nabla\cdot(\boldsymbol{\mathcal{E}}(\boldsymbol{d}_n)\nabla\psi_n) &= c^+ - c^- \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\mathcal{E}}(\boldsymbol{d}_n)\nabla\psi_n\cdot\boldsymbol{n} + \tau\psi_n = \xi \quad \text{on } \Gamma \\
-\nabla\cdot(\boldsymbol{\mathcal{E}}(\boldsymbol{d}_n)\nabla\tilde{\psi}_n) &= \tilde{c}^+ - \tilde{c}^- \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\mathcal{E}}(\boldsymbol{d}_n)\nabla\tilde{\psi}_n\cdot\boldsymbol{n} + \tau\tilde{\psi}_n = \tilde{\xi} \quad \text{on } \Gamma
\end{aligned}
$$

almost everywhere in $(0,T)$. Elliptic regularity yields that $\psi_n(t), \tilde{\psi}_n(t) \subseteq W^{3,3/2}(\Omega)$ for all $n\in\mathbb{N}$, since $c^\pm(t), \tilde{c}^\pm(t) \in W^{1,3/2}(\Omega)$ for almost all t, which follows from $\sqrt{c^\pm}\in L^2(0,T;W^{1,2}(\Omega))$ and the product rule for weak derivatives. To keep things simple, we will not do this approximation. An interested reader can find the details in [Pla24b, Chap. 4]. We now integrate the last term on the left-hand side of (25) by parts:

$$
\begin{aligned}
&\int_\Omega \nabla(c^+-c^- - (\tilde{c}^+-\tilde{c}^-))\cdot\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\,\mathrm{d}\boldsymbol{x} \\
&\quad = -\int_\Omega (c^+-c^- - (\tilde{c}^+-\tilde{c}^-))\nabla\cdot\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\,\mathrm{d}\boldsymbol{x} \\
&\qquad + \int_\Gamma (c^+-c^- - (\tilde{c}^+-\tilde{c}^-))\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\,\mathrm{d}\sigma \\
&\quad = \int_\Omega \nabla\cdot\left(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\nabla\cdot\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\,\mathrm{d}\boldsymbol{x} \\
&\qquad + \int_\Gamma (c^+-c^- - (\tilde{c}^+-\tilde{c}^-))\left(\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi})\right)\,\mathrm{d}\sigma,
\end{aligned}
\tag{26}
$$

where the first equality follows from an integration by parts and the second by using $\nabla\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi}))$ as a test function in the Poisson equation (1d). Integrating the volume term by parts twice, we obtain

$$\begin{aligned}
&\int_\Omega \nabla\cdot\Big(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)\,\nabla\cdot\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)\,\mathrm{d}\boldsymbol{x}\\
&\quad= -\int_\Omega \boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\cdot\nabla\Big(\nabla\cdot\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)\Big)\,\mathrm{d}\boldsymbol{x}\\
&\qquad+\int_\Gamma \nabla\cdot\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)\,\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\cdot\boldsymbol{n}\,\mathrm{d}\sigma\\
&\quad= -\int_\Omega \boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\cdot\nabla\cdot\Big(\nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)^T\Big)\,\mathrm{d}\boldsymbol{x}\\
&\qquad+\int_\Gamma \nabla\cdot\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)\Big(\xi-\tilde\xi-\tau(\psi-\tilde\psi)\Big)\,\mathrm{d}\sigma\\
&\quad= \int_\Omega \nabla\Big(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big):\nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)^T\,\mathrm{d}\boldsymbol{x}\\
&\qquad-\int_\Gamma \Big(\nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)^T\boldsymbol{n}\Big)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\,\mathrm{d}\sigma\\
&\qquad+\int_\Gamma \nabla\cdot\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)\Big(\xi-\tilde\xi-\tau(\psi-\tilde\psi)\Big)\,\mathrm{d}\sigma.
\end{aligned}\tag{27}$$

Expanding the matrix scalar product in the volume integral of (27) and using the symmetry of $\nabla^2(\psi-\tilde\psi)$ together with $\boldsymbol{A}:\boldsymbol{B}^T=\boldsymbol{A}^T:\boldsymbol{B}$, we note that

$$\begin{aligned}
&\nabla\Big(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big):\nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\Big)^T\\
&\quad=|\nabla^2(\psi-\tilde\psi)|^2+\varepsilon\lambda\underbrace{\nabla((\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))\boldsymbol{d})^T:\nabla((\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))\boldsymbol{d})}_{=:i}\\
&\qquad+(\varepsilon+\lambda)\underbrace{\nabla^2(\psi-\tilde\psi):\nabla((\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))\boldsymbol{d})}_{=:ii}.
\end{aligned}\tag{28}$$

The first term already gives us a second order term of $(\psi-\tilde\psi)$ with a good sign. Using

$$\nabla((\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))\boldsymbol{d})=(\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))\nabla\boldsymbol{d}+\boldsymbol{d}\otimes(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde\psi))+\boldsymbol{d}\otimes(\nabla^2(\psi-\tilde\psi)\boldsymbol{d}),\tag{29}$$

we can rewrite the remaining terms i and ii. First, using $(\boldsymbol{a}\otimes\boldsymbol{b}):\boldsymbol{A}=\boldsymbol{a}\cdot\boldsymbol{A}\boldsymbol{b}$ and $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c}=(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$, we rewrite

$$\begin{aligned}
i&=\Big((\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))\nabla\boldsymbol{d}^T+\Big(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde\psi)\Big)\otimes\boldsymbol{d}+\Big(\nabla^2(\psi-\tilde\psi)\boldsymbol{d}\Big)\otimes\boldsymbol{d}\Big)\\
&\quad:\Big((\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))\nabla\boldsymbol{d}+\boldsymbol{d}\otimes(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde\psi))+\boldsymbol{d}\otimes(\nabla^2(\psi-\tilde\psi)\boldsymbol{d})\Big)\\
&=(\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))^2\nabla\boldsymbol{d}^T:\nabla\boldsymbol{d}+2(\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))(\nabla\boldsymbol{d}\boldsymbol{d})\cdot(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde\psi))\\
&\quad+2(\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))(\nabla^2(\psi-\tilde\psi)\boldsymbol{d})\cdot(\nabla\boldsymbol{d}\boldsymbol{d})+|\nabla\boldsymbol{d}^T\nabla(\psi-\tilde\psi)\cdot\boldsymbol{d}|^2\\
&\quad+2(\boldsymbol{d}\cdot\nabla^2(\psi-\tilde\psi)\boldsymbol{d})(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde\psi)\cdot\boldsymbol{d})+|(\nabla^2(\psi-\tilde\psi)\boldsymbol{d})\cdot\boldsymbol{d}|^2.
\end{aligned}$$

The only term quadratic in the second order derivative of $(\psi-\tilde\psi)$ is the last one, which has a good sign. Secondly, we rewrite ii again using (29):

$$\begin{aligned}
ii&=(\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))\nabla^2(\psi-\tilde\psi):\nabla\boldsymbol{d}\\
&\quad+\nabla^2(\psi-\tilde\psi):\Big(\boldsymbol{d}\otimes(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde\psi))\Big)+\nabla^2(\psi-\tilde\psi):\Big(\boldsymbol{d}\otimes(\nabla^2(\psi-\tilde\psi)\boldsymbol{d})\Big)\\
&=(\boldsymbol{d}\cdot\nabla(\psi-\tilde\psi))\nabla^2(\psi-\tilde\psi):\nabla\boldsymbol{d}+\nabla\boldsymbol{d}^T\nabla(\psi-\tilde\psi)\cdot(\nabla^2(\psi-\tilde\psi)\boldsymbol{d})+|\nabla^2(\psi-\tilde\psi)\boldsymbol{d}|^2.
\end{aligned}$$

Inserting the expansions of i and ii back into (28), we obtain

$$\begin{aligned}
&\nabla\left(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right):\nabla\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)^T\\
&=|\nabla^2(\psi-\tilde{\psi})|^2+\varepsilon\lambda|(\nabla^2(\psi-\tilde{\psi})\boldsymbol{d})\cdot\boldsymbol{d}|^2+\varepsilon\lambda|\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{d}|^2+(\varepsilon+\lambda)|\nabla^2(\psi-\tilde{\psi})\boldsymbol{d}|^2\\
&\quad+\varepsilon\lambda(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))^2\nabla\boldsymbol{d}^T:\nabla\boldsymbol{d}+2\varepsilon\lambda(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))(\nabla\boldsymbol{d}\boldsymbol{d})\cdot(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi}))\\
&\quad+2\varepsilon\lambda(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))(\nabla^2(\psi-\tilde{\psi})\boldsymbol{d})\cdot(\nabla\boldsymbol{d}\boldsymbol{d})+2\varepsilon\lambda(\boldsymbol{d}\cdot\nabla^2(\psi-\tilde{\psi})\boldsymbol{d})(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{d})\\
&\quad+(\varepsilon+\lambda)(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))\nabla^2(\psi-\tilde{\psi}):\nabla\boldsymbol{d}+(\varepsilon+\lambda)\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\left(\nabla^2(\psi-\tilde{\psi})\boldsymbol{d}\right).
\end{aligned}\tag{30}$$

The volume term on the right-hand side of (27) now again only contains second order derivatives of ψ and $\tilde{\psi}$, but the boundary integrals also contain second order derivatives, which we cannot control using $W^{2,2}(\Omega)\hookrightarrow W^{1,2}(\Gamma)$. In order to have only first order derivatives on the boundary, we integrate the boundary integrals in (27) by parts. Using the formula for the surface gradient and surface divergence, see [HLP25, Thm. A.18 and Thm. A.21],

$$\nabla\cdot\boldsymbol{v}=\nabla_\Gamma\cdot\boldsymbol{v}+\nabla(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n},\quad \nabla_\Gamma f=\nabla f-(\nabla f\cdot\boldsymbol{n})\boldsymbol{n},\tag{31}$$

the boundary integral on the last line on the right-hand side of (27) becomes

$$\begin{aligned}
&\int_\Gamma\nabla\cdot\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\left(\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi})\right)\,\mathrm{d}\sigma\\
&=\int_\Gamma\left(\nabla_\Gamma\cdot\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)+\nabla\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\boldsymbol{n}\cdot\boldsymbol{n}\right)\left(\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi})\right)\,\mathrm{d}\sigma\\
&\underset{(16)}{=}-\int_\Gamma\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\cdot\nabla_\Gamma\left(\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi})\right)\,\mathrm{d}\sigma\\
&\quad+\int_\Gamma\left(\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi})\right)\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\right)(\nabla_\Gamma\cdot\boldsymbol{n})\,\mathrm{d}\sigma\\
&\quad+\int_\Gamma\left(\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi})\right)\nabla\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\boldsymbol{n}\cdot\boldsymbol{n}\,\mathrm{d}\sigma\\
&\underset{(31)}{=}\tau\int_\Gamma\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla_\Gamma(\psi-\tilde{\psi})\right)\cdot\nabla_\Gamma(\psi-\tilde{\psi})\,\mathrm{d}\sigma-\int_\Gamma\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla_\Gamma(\psi-\tilde{\psi})\right)\cdot\nabla_\Gamma(\xi-\tilde{\xi})\,\mathrm{d}\sigma\\
&\quad+\int_\Gamma(\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi}))^2(\nabla_\Gamma\cdot\boldsymbol{n})+(\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi}))\nabla\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\boldsymbol{n}\cdot\boldsymbol{n}\,\mathrm{d}\sigma,
\end{aligned}\tag{32}$$

where we used that the normal parts of the gradient of $(\psi-\tilde{\psi})$ vanish in the scalar product with $\boldsymbol{d}$, since $\boldsymbol{d}$ fulfills the boundary conditions $\boldsymbol{d}\cdot\boldsymbol{n}=0$ on Γ. The first term on the right-hand side has a good sign and is part of the dissipation potential, the second and third integral will be part of the $\mathcal{K}(\tilde{\boldsymbol{u}})\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})$ and $\mathcal{Q}_{\tilde{\boldsymbol{u}}}(\xi|\tilde{\xi})$ terms and the last integral will cancel with part of the remaining boundary integral in the second line of the right-hand side of (27). So far we did not use the trace operator S or the trace extension operator E for the sake of readability since all boundary integrals so far could intuitively be understood in the sense of traces. Now we would like to take the full gradient of the outer normal vector $\boldsymbol{n}$. To give this meaning we could use the trace extension operator E, then $E(\boldsymbol{n})$, is a function on the whole domain Ω. By assumptions from Definition 2.2, we have that Γ is smooth and thus $\boldsymbol{n}$ and $E(\boldsymbol{n})$ are arbitrarily smooth. Out of context it should be clear whether we need $\boldsymbol{n}$ to be defined in Ω or on Γ and thus for the sake of readability we refrain from writing S and E. We find for the boundary integral on the second line on the right-hand side of (27)

$$\begin{aligned}
&-\int_\Gamma\left(\nabla\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)^T\boldsymbol{n}\right)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\,\mathrm{d}\sigma\\
&\quad=-\int_\Gamma\left[\nabla\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\right)-\nabla\boldsymbol{n}^T\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right]\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\,\mathrm{d}\sigma.
\end{aligned}\tag{33}$$

The second term already is of lower order, that is only first order derivatives of ψ appear, and thus parts can be absorbed into the dissipation potential and the rest fits into the term $\mathcal{K}(\tilde{\boldsymbol{u}})\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})$. To reduce the order of derivatives in the first term we use the surface gradient (31) and integrate by parts.

$$
\begin{aligned}
&-\int_\Gamma \nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\Big)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\,\mathrm{d}\sigma\\
&\underset{(31)}{=} -\int_\Gamma \Big[\nabla_\Gamma\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\Big)+\Big(\nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\Big)\cdot\boldsymbol{n}\Big)\boldsymbol{n}\Big]\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\,\mathrm{d}\sigma\\
&\underset{(16)}{=} \int_\Gamma \Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\Big)\nabla_\Gamma\cdot\Big(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\Big)\,\mathrm{d}\sigma\\
&\quad-\int_\Gamma\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\Big)(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n})(\nabla_\Gamma\cdot\boldsymbol{n})\,\mathrm{d}\sigma\\
&\quad+\int_\Gamma\Big(\nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\Big)\cdot\boldsymbol{n}\Big)\Big(\tau(\psi-\tilde{\psi})-(\xi-\tilde{\xi})\Big)\,\mathrm{d}\sigma\\
&=\int_\Gamma\Big(\tau\nabla_\Gamma(\psi-\tilde{\psi})-\nabla_\Gamma(\xi-\tilde{\xi})\Big)\cdot\Big(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\Big)\,\mathrm{d}\sigma\\
&\quad+\int_\Gamma\Big(\nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\Big)\cdot\boldsymbol{n}\Big)\Big(\tau(\psi-\tilde{\psi})-(\xi-\tilde{\xi})\Big)\,\mathrm{d}\sigma\\
&=\tau\int_\Gamma|\nabla_\Gamma(\psi-\tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\sigma-\int_\Gamma\nabla_\Gamma(\xi-\tilde{\xi})\cdot\Big(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla_\Gamma(\psi-\tilde{\psi})\Big)\,\mathrm{d}\sigma\\
&\quad+\int_\Gamma\Big(\nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{n}\Big)\cdot\boldsymbol{n}\Big)\Big(\tau(\psi-\tilde{\psi})-(\xi-\tilde{\xi})\Big)\,\mathrm{d}\sigma\\
&=\tau\int_\Gamma|\nabla_\Gamma(\psi-\tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\sigma-\int_\Gamma\nabla_\Gamma(\xi-\tilde{\xi})\cdot\Big(\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla_\Gamma(\psi-\tilde{\psi})\Big)\,\mathrm{d}\sigma\\
&\quad+\int_\Gamma\Big(\tau(\psi-\tilde{\psi})-(\xi-\tilde{\xi})\Big)\nabla\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\Big)^T\boldsymbol{n}\cdot\boldsymbol{n}\,\mathrm{d}\sigma\\
&\quad+\int_\Gamma\Big(\tau(\psi-\tilde{\psi})-(\xi-\tilde{\xi})\Big)\nabla\boldsymbol{n}^T\Big(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\Big)\cdot\boldsymbol{n}\,\mathrm{d}\sigma,
\end{aligned}\tag{34}
$$

where the curvature term appearing on the first integration by parts cancels with the curvature term on the second integration by parts. As mentioned before, some of the boundary terms, here the terms in the third integral on the right-hand side, cancel with the last boundary integral on the right-hand side of (32). This simply follows from $\boldsymbol{A}\boldsymbol{n}\cdot\boldsymbol{n}=\boldsymbol{A}^T\boldsymbol{n}\cdot\boldsymbol{n}$. Inserting (27) and (30)–(34) and back into (26) we obtain

$$
\begin{aligned}
&\int_\Omega\nabla(c^+-c^--(\tilde{c}^+-\tilde{c}^-))\cdot\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\,\mathrm{d}\boldsymbol{x}\\
&=\int_\Omega|\nabla^2(\psi-\tilde{\psi})|^2+\varepsilon\lambda|(\nabla^2(\psi-\tilde{\psi})\boldsymbol{d})\cdot\boldsymbol{d}|^2+\varepsilon\lambda|\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{d}|^2+(\varepsilon+\lambda)|\nabla^2(\psi-\tilde{\psi})\boldsymbol{d}|^2\,\mathrm{d}\boldsymbol{x}\\
&+\int_\Omega\varepsilon\lambda(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))^2\nabla\boldsymbol{d}^T:\nabla\boldsymbol{d}+2\varepsilon\lambda(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))(\nabla\boldsymbol{d}\boldsymbol{d})\cdot(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi}))\,\mathrm{d}\boldsymbol{x}\\
&+\int_\Omega2\varepsilon\lambda(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))(\nabla^2(\psi-\tilde{\psi})\boldsymbol{d})\cdot(\nabla\boldsymbol{d}\boldsymbol{d})+2\varepsilon\lambda(\boldsymbol{d}\cdot\nabla^2(\psi-\tilde{\psi})\boldsymbol{d})(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{d})\,\mathrm{d}\boldsymbol{x}\\
&+\int_\Omega(\varepsilon+\lambda)(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))\nabla^2(\psi-\tilde{\psi}):\nabla\boldsymbol{d}+(\varepsilon+\lambda)\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\big(\nabla^2(\psi-\tilde{\psi})\boldsymbol{d}\big)\,\mathrm{d}\boldsymbol{x}\\
&-\int_\Gamma\tau(c^+-c^--(\tilde{c}^+-\tilde{c}^-))(\psi-\tilde{\psi})\,\mathrm{d}\sigma+\tau\int_\Gamma|\nabla_\Gamma(\psi-\tilde{\psi})|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\,\mathrm{d}\sigma+\tau\int_\Gamma|\nabla_\Gamma(\psi-\tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\sigma\\
&+\int_\Gamma(c^+-c^--(\tilde{c}^+-\tilde{c}^-))(\xi-\tilde{\xi})-(\boldsymbol{\Lambda}(\boldsymbol{d})+\boldsymbol{\mathcal{E}}(\boldsymbol{d}))\nabla_\Gamma(\psi-\tilde{\psi})\cdot\nabla_\Gamma(\xi-\tilde{\xi})\,\mathrm{d}\sigma\\
&+\int_\Gamma\Big(\tau(\psi-\tilde{\psi})-(\xi-\tilde{\xi})\Big)\Big(\nabla\boldsymbol{n}^T\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\Big)\cdot\boldsymbol{n}\,\mathrm{d}\sigma\\
&+\int_\Gamma\Big(\nabla\boldsymbol{n}^T\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\Big)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\,\mathrm{d}\sigma+\int_\Gamma(\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi}))^2(\nabla_\Gamma\cdot\boldsymbol{n})\,\mathrm{d}\sigma.
\end{aligned}
$$

Inserting this back into (25), we obtain

$$
\begin{aligned}
&\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})\big|_0^t + \int_0^t \mathcal{W}(\boldsymbol{u}|\tilde{\boldsymbol{u}})\,\mathrm{d}s \\
&= \left[\int_\Omega \frac{\mathrm{Re}}{2}|\boldsymbol{v}-\tilde{\boldsymbol{v}}|^2 + \left[\tilde{c}^\pm - c^\pm - c^\pm(\ln\tilde{c}^\pm - \ln c^\pm)\right]^\pm + \frac{1}{2}|\nabla(\psi-\tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \frac{\tau}{2}|\psi-\tilde{\psi}|^2\,\mathrm{d}\sigma\right]\Bigg|_0^t \\
&\quad + \int_0^t\int_\Omega |\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})|^2 + \left[4|\nabla\sqrt{c^\pm} - \nabla\sqrt{\tilde{c}^\pm}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} + c^\pm|\nabla(\psi-\tilde{\psi})|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\right]^\pm + 2|\nabla^2(\psi-\tilde{\psi})|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega 2\varepsilon\lambda|(\nabla^2(\psi-\tilde{\psi})\boldsymbol{d})\cdot\boldsymbol{d}|^2 + 2\varepsilon\lambda|\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{d}|^2 + 2(\varepsilon+\lambda)|\nabla^2(\psi-\tilde{\psi})\boldsymbol{d}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad + 2\tau\int_0^t\int_\Gamma |\nabla_\Gamma(\psi-\tilde{\psi})|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} + |\nabla_\Gamma(\psi-\tilde{\psi})|^2_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\le \int_0^t\int_\Omega \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}})\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) + \left[\mathcal{A}_{\tilde{c}^\pm}(\tilde{\boldsymbol{u}})\left(\frac{1}{\tilde{c}^\pm}\left(\tilde{c}^\pm - c^\pm\right) \pm (\tilde{\psi}-\psi)\right)\right]^\pm\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad + \int_0^t \underbrace{\mathrm{Re}\int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}}_{=:\mathrm{I}} \underbrace{- \int_\Omega (\tilde{c}^+ - \tilde{c}^-)\nabla\tilde{\psi}\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) - (c^+ - c^-)\nabla\psi\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}}_{=:\mathrm{II}}\,\mathrm{d}s \\
&\quad + \int_0^t \underbrace{-\int_\Omega \left[c^\pm\boldsymbol{v}\cdot\nabla\ln\tilde{c}^\pm \pm c^\pm\boldsymbol{v}\cdot\nabla\tilde{\psi} + \tilde{\boldsymbol{v}}\cdot\nabla c^\pm - c^\pm\tilde{\boldsymbol{v}}\cdot\nabla\ln\tilde{c}^\pm \mp \tilde{c}^\pm\tilde{\boldsymbol{v}}\cdot\nabla(\tilde{\psi}-\psi)\right]^\pm\mathrm{d}\boldsymbol{x}}_{+=\mathrm{II}}\,\mathrm{d}s \\
&\quad + \int_0^t \underbrace{-\int_\Omega \left[\pm\left(1-\frac{c^\pm}{\tilde{c}^\pm}\right)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{c}^\pm\cdot\nabla(\psi-\tilde{\psi}) + (\tilde{c}^\pm - c^\pm)\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla(\tilde{\psi}-\psi)\right]^\pm\mathrm{d}\boldsymbol{x}}_{=:\mathrm{III}}\,\mathrm{d}s \\
&\quad + \int_0^t \underbrace{\int_\Omega \left[4\sqrt{\tilde{c}^\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\left(1-\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\right)^2 + 4\left(1-\sqrt{\frac{c^\pm}{\tilde{c}^\pm}}\right)^2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde{c}^\pm}\cdot\nabla\sqrt{\tilde{c}^\pm}\right]^\pm\mathrm{d}\boldsymbol{x}}_{=:\mathrm{IV}}\,\mathrm{d}s \\
&\quad - \int_0^t \underbrace{2\int_\Omega \varepsilon\lambda(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))^2\nabla\boldsymbol{d}^T:\nabla\boldsymbol{d} + 2\varepsilon\lambda(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))(\nabla\boldsymbol{d}\boldsymbol{d})\cdot(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi}))\,\mathrm{d}\boldsymbol{x}}_{=:\mathrm{V}}\,\mathrm{d}s \\
&\quad - \int_0^t \underbrace{4\varepsilon\lambda\int_\Omega (\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))(\nabla^2(\psi-\tilde{\psi})\boldsymbol{d})\cdot(\nabla\boldsymbol{d}\boldsymbol{d}) + (\boldsymbol{d}\cdot\nabla^2(\psi-\tilde{\psi})\boldsymbol{d})(\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\boldsymbol{d})\,\mathrm{d}\boldsymbol{x}}_{+=\mathrm{V}}\,\mathrm{d}s \\
&\quad - \int_0^t \underbrace{2\int_\Omega (\varepsilon+\lambda)(\boldsymbol{d}\cdot\nabla(\psi-\tilde{\psi}))\nabla^2(\psi-\tilde{\psi}):\nabla\boldsymbol{d} + (\varepsilon+\lambda)\nabla\boldsymbol{d}^T\nabla(\psi-\tilde{\psi})\cdot\left(\nabla^2(\psi-\tilde{\psi})\boldsymbol{d}\right)\,\mathrm{d}\boldsymbol{x}}_{+=\mathrm{V}}\,\mathrm{d}s \\
&\quad + \int_0^t \underbrace{2\tau\int_\Gamma (c^+ - c^- - (\tilde{c}^+ - \tilde{c}^-))(\psi-\tilde{\psi})\,\mathrm{d}\sigma}_{=:\mathrm{VI}} \underbrace{- 2\tau\int_\Gamma (\psi-\tilde{\psi})\left(\nabla\boldsymbol{n}^T\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\cdot\boldsymbol{n}\,\mathrm{d}\sigma}_{=:\mathrm{VII}}\,\mathrm{d}s \\
&\quad - \int_0^t \underbrace{2\int_\Gamma \left(\nabla\boldsymbol{n}^T\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\,\mathrm{d}\sigma}_{=:\mathrm{VIII}} + \underbrace{\int_\Gamma \partial_t(\tilde{\xi}-\xi)(\psi-\tilde{\psi})\,\mathrm{d}\sigma}_{=:\mathrm{IX}}\,\mathrm{d}s \\
&\quad - \int_0^t \underbrace{\int_\Gamma 2(c^+ - c^- - (\tilde{c}^+ - \tilde{c}^-))(\xi-\tilde{\xi})\,\mathrm{d}\sigma}_{=:\mathrm{X}} \underbrace{- \int_\Gamma 2(\boldsymbol{\Lambda}(\boldsymbol{d}) + \boldsymbol{\mathcal{E}}(\boldsymbol{d}))\nabla_\Gamma(\psi-\tilde{\psi})\cdot\nabla_\Gamma(\xi-\tilde{\xi})\,\mathrm{d}\sigma}_{=:\mathrm{XI}}\,\mathrm{d}s \\
&\quad + \int_0^t \underbrace{\int_\Gamma 2(\xi-\tilde{\xi})\left(\nabla\boldsymbol{n}^T\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde{\psi})\right)\cdot\boldsymbol{n}\,\mathrm{d}\sigma}_{=:\mathrm{XII}} \underbrace{-2\int_\Gamma (\xi-\tilde{\xi}-\tau(\psi-\tilde{\psi}))^2(\nabla_\Gamma\cdot\boldsymbol{n})\,\mathrm{d}\sigma}_{=:\mathrm{XIII}}\,\mathrm{d}s.
\end{aligned} \tag{35}
$$

Now, we can start estimating the right-hand side to absorb it partly into the dissipation potential $\mathcal{W}(\boldsymbol{u}|\tilde{\boldsymbol{u}})$, partly in $\mathcal{K}(\tilde{\boldsymbol{u}})\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})$, and partly into the operator $\mathcal{Q}_{\tilde{\boldsymbol{u}}}(\xi|\tilde{\xi})$. For I, we use Lemma 3.4 and note that

$$|\,\mathrm{I}\,| \leq \frac{1}{4}\left\|\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})\right\|^2_{L^2(\Omega)} + C\,\mathrm{Re}^3\left\|\tilde{\boldsymbol{v}}\right\|^4_{L^6(\Omega)}\left(\mathrm{Re}\,\left\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\right\|^2_{L^2(\Omega)}\right). \tag{36}$$

For II we have that some terms cancel each other out so that

$$\begin{aligned}
\mathrm{II} &= \int_0^t\int_\Omega (\tilde{c}^+-\tilde{c}^-)\nabla\tilde{\psi}\cdot\boldsymbol{v} + (c^+-c^-)\nabla\psi\cdot\tilde{\boldsymbol{v}} - \Big[c^\pm\boldsymbol{v}\cdot\nabla\ln\tilde{c}^\pm \mp c^\pm\boldsymbol{v}\cdot\nabla\tilde{\psi}\Big]^\pm\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&\quad + \int_0^t\int_\Omega \Big[c^\pm\tilde{\boldsymbol{v}}\cdot\nabla\ln\tilde{c}^\pm \mp \tilde{c}^\pm\tilde{\boldsymbol{v}}\cdot\nabla\psi\Big]^\pm\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&= \int_0^t\int_\Omega (c^+-c^--(\tilde{c}^+-\tilde{c}^-))\left(\nabla\tilde{\psi}\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) + \tilde{\boldsymbol{v}}\cdot\nabla(\psi-\tilde{\psi})\right) + \Big[c^\pm(\tilde{\boldsymbol{v}}-\boldsymbol{v})\cdot\nabla\ln\tilde{c}^\pm\Big]^\pm\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&= \int_0^t\int_\Omega (c^+-c^--(\tilde{c}^+-\tilde{c}^-))\left(\nabla\tilde{\psi}\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) + \tilde{\boldsymbol{v}}\cdot\nabla(\psi-\tilde{\psi})\right) + \Big[(c^\pm-\tilde{c}^\pm)(\tilde{\boldsymbol{v}}-\boldsymbol{v})\cdot\nabla\ln\tilde{c}^\pm\Big]^\pm\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&= \int_0^t\int_\Omega \Big[(c^\pm-\tilde{c}^\pm)(\tilde{\boldsymbol{v}}-\boldsymbol{v})\cdot(\nabla\ln\tilde{c}^\pm \pm \nabla\tilde{\psi}) \pm (c^\pm-\tilde{c}^\pm)\tilde{\boldsymbol{v}}\cdot\nabla(\psi-\tilde{\psi})\Big]^\pm\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s,
\end{aligned} \tag{37}$$

where we used that $\int_\Omega \tilde{c}^\pm\,\tilde{\boldsymbol{v}}\cdot\nabla\ln\tilde{c}^\pm\,\mathrm{d}\boldsymbol{x} = 0$ due to $\tilde{\boldsymbol{v}}\in L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$. To find an upper bound for the first term on the right-hand side, we make use of Lemma 3.2 and the expansion $x-y = (\sqrt{x}-\sqrt{y})^2 + 2\sqrt{y}(\sqrt{x}-\sqrt{y})$,

$$\begin{aligned}
&\left|\int_\Omega \Big[(c^\pm-\tilde{c}^\pm)(\tilde{\boldsymbol{v}}-\boldsymbol{v})\cdot\nabla\left(\ln\tilde{c}^\pm\pm\tilde{\psi}\right)\Big]^\pm\mathrm{d}\boldsymbol{x}\right|\\
&= \left|\int_\Omega \Big[\left(\sqrt{c^\pm}-\sqrt{\tilde{c}^\pm}\right)^2(\tilde{\boldsymbol{v}}-\boldsymbol{v})\cdot\nabla\left(\ln\tilde{c}^\pm\pm\tilde{\psi}\right) + 2\sqrt{\tilde{c}^\pm}\left(\sqrt{c^\pm}-\sqrt{\tilde{c}^\pm}\right)(\tilde{\boldsymbol{v}}-\boldsymbol{v})\cdot\nabla\left(\ln\tilde{c}^\pm\pm\tilde{\psi}\right)\Big]^\pm\mathrm{d}\boldsymbol{x}\right|\\
&\leq \Big[\left\|\nabla\left(\ln\tilde{c}^\pm\pm\tilde{\psi}\right)\right\|_{L^\infty(\Omega)}\left\|\left(\sqrt{c^\pm}-\sqrt{\tilde{c}^\pm}\right)^2\right\|_{L^{\frac{6}{5}}(\Omega)}\left\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\right\|_{L^6(\Omega)}\Big]^\pm\\
&\quad + \Big[\left\|2\sqrt{\tilde{c}^\pm}\nabla\left(\ln\tilde{c}^\pm\pm\tilde{\psi}\right)\right\|_{L^\infty(\Omega)}\left\|\sqrt{c^\pm}-\sqrt{\tilde{c}^\pm}\right\|_{L^2(\Omega)}\left\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)}\Big]^\pm.
\end{aligned} \tag{38}$$

The second line of the right-hand side is already nice and can be absorbed into the term $\mathcal{K}(\tilde{\boldsymbol{u}})\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})$ and $\mathcal{W}(\boldsymbol{u}|\tilde{\boldsymbol{u}})$ by Young's inequality and the norm inequality $\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{L^2(\Omega)} \leq C\,\|\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})\|_{L^2(\Omega)}$, which follows from

$$\begin{aligned}
&\Big[\left\|2\sqrt{\tilde{c}^\pm}\nabla\left(\ln\tilde{c}^\pm\pm\tilde{\psi}\right)\right\|_{L^\infty(\Omega)}\left\|\sqrt{c^\pm}-\sqrt{\tilde{c}^\pm}\right\|_{L^2(\Omega)}\left\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)}\Big]^\pm\\
&\quad\leq \delta\left\|\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})\right\|^2_{L^2(\Omega)} + C\left(\Big[\|\tilde{c}^\pm\|^2_{W^{1,\infty}(\Omega)}\Big]^\pm + \|\tilde{\psi}\|^4_{W^{1,\infty}(\Omega)} + 1\right)\left\|\sqrt{c^\pm}-\sqrt{\tilde{c}^\pm}\right\|^2_{L^2(\Omega)},
\end{aligned}$$

where we use the lower bound l of the test function $\tilde{c}^\pm$ from the definition of $\widetilde{\mathbb{U}}$ to estimate

$$\begin{aligned}
\Big[\left\|2\sqrt{\tilde{c}^\pm}\nabla\left(\ln\tilde{c}^\pm\pm\tilde{\psi}\right)\right\|_{L^\infty(\Omega)}\Big]^\pm &\leq C\Big[\frac{1}{\sqrt{l}}\|\tilde{c}^\pm\|_{W^{1,\infty}(\Omega)} + \|\sqrt{\tilde{c}^\pm}\|^2_{L^\infty(\Omega)} + \|\tilde{\psi}\|^2_{W^{1,\infty}(\Omega)}\Big]^\pm\\
&\leq C\left(\Big[\|\tilde{c}^\pm\|_{W^{1,\infty}(\Omega)}\Big]^\pm + \|\tilde{\psi}\|^2_{W^{1,\infty}(\Omega)}\right)
\end{aligned}$$

to fit this term into $\mathcal{K}(\tilde{\boldsymbol{u}})$. We then apply Lemma 3.2 to upper bound the $L^2(\Omega)$–norm of $\sqrt{c^\pm}-\sqrt{\tilde{c}^\pm}$ by the logarithm term of the relative energy. To estimate the first line of the right-hand side of (38), we use Young's and the Gagliardo–Nirenberg inequalities with $\lambda = \frac{1}{4}$ to obtain

$$
\begin{aligned}
&\left[\left\| \nabla \left(\ln \tilde{c}^{\pm} \pm \tilde{\psi} \right) \right\|_{L^\infty(\Omega)} \left\| \left(\sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right)^2 \right\|_{L^{\frac{6}{5}}(\Omega)} \| \boldsymbol{v} - \tilde{\boldsymbol{v}} \|_{L^6(\Omega)} \right]^{\pm} \\
&\quad \le \delta \| \nabla (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \|_{L^2(\Omega)}^2 + \left[C \left\| \nabla (\ln \tilde{c}^{\pm} \pm \tilde{\psi}) \right\|_{L^\infty(\Omega)}^2 \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{L^{12/5}(\Omega)}^4 \right]^{\pm} \\
&\quad \le \delta \| \nabla (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \|_{L^2(\Omega)}^2 + \left[C \left\| \nabla \left(\ln \tilde{c}^{\pm} \pm \tilde{\psi} \right) \right\|_{L^\infty(\Omega)}^2 \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{L^2(\Omega)}^3 \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{W^{1,2}(\Omega)} \right]^{\pm} \\
&\quad \le \delta \| \nabla (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \|_{L^2(\Omega)}^2 + \left[\frac{1}{4} \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{W^{1,2}(\Omega)}^2 \right]^{\pm} \\
&\qquad + C \left[\left\| \nabla \left(\ln \tilde{c}^{\pm} \pm \tilde{\psi} \right) \right\|_{L^\infty(\Omega)}^4 \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{L^2(\Omega)}^6 \right]^{\pm}
\end{aligned} \tag{39}
$$

To estimate the right-hand side further, we make use of the energy inequality (5), which implies

$$
\left[\int_\Omega \frac{\mathrm{Re}}{2} |\boldsymbol{v}|^2 + \left[c^{\pm} (\ln c^{\pm} + 1) + 1 \right]^{\pm} + \frac{1}{2} |\nabla \psi|_{\boldsymbol{\mathcal{E}}(\boldsymbol{d})}^2 \, \mathrm{d}\boldsymbol{x} + \frac{\tau}{2} \int_\Gamma |\psi|^2 \, \mathrm{d}\sigma \right] (t) \le e^t \left(\mathcal{E}(\boldsymbol{u}_0) + C(\xi) \right)
$$

for some constant $C(\xi)$ depending on ξ, where we added one in the integral to make the charge contribution to the energy non-negative. We can use this estimate to upper bound pth-powers of the relative energy for $p > 1$. More explicitly, using $(a+b)^p \le 2^p \big(|a|^p + |b|^p \big)$ for $a, b \in \mathbb{R}$, we estimate:

$$
\begin{aligned}
\left(\int_\Omega \left(\sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right)^2 \, \mathrm{d}\boldsymbol{x} \right)^p &\le \left(\int_\Omega 4 \left(c^{\pm} + \tilde{c}^{\pm} \right) \, \mathrm{d}\boldsymbol{x} \right)^p \le 4^p 2^p \left(\left(\int_\Omega c^{\pm} \, \mathrm{d}\boldsymbol{x} \right)^p + \left(\int_\Omega \tilde{c}^{\pm} \, \mathrm{d}\boldsymbol{x} \right)^p \right) \\
&\le 2^{3p} \left(\left(\int_\Omega c^{\pm} (\ln c^{\pm} + 1) + 1 \, \mathrm{d}\boldsymbol{x} \right)^p + \| \tilde{c}^{\pm} \|_{L^1(\Omega)}^p \right) \\
&\le 2^{3p} \left(\left(e^t C (\mathcal{E}(\boldsymbol{u}_0) + C(\xi)) \right)^p + \| \tilde{c}^{\pm} \|_{L^1(\Omega)}^p \right) = C \left(1 + \| \tilde{c}^{\pm} \|_{L^1(\Omega)}^p \right)
\end{aligned} \tag{40}
$$

for the charge part of the relative energy, using the pointwise estimate $x \le x(\ln x + 1) + 1$ for $x \ge 0$. This gives an upper bound for the $L^2(\Omega)$–norm for $\sqrt{c^+} - \sqrt{\tilde{c}^+}$ and $\sqrt{c^-} - \sqrt{\tilde{c}^-}$. Using (40) with $p = 2$, we can estimate (39) further by

$$
\begin{aligned}
&\left[\left\| \nabla \left(\ln \tilde{c}^{\pm} \pm \tilde{\psi} \right) \right\|_{L^\infty(\Omega)} \left\| \left(\sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right)^2 \right\|_{L^{\frac{6}{5}}(\Omega)} \| \boldsymbol{v} - \tilde{\boldsymbol{v}} \|_{L^6(\Omega)} \right]^{\pm} \\
&\le \delta \| \nabla (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \|_{L^2(\Omega)}^2 + \left[\frac{1}{4} \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{W^{1,2}(\Omega)}^2 \right]^{\pm} + C \left[\left\| \nabla \left(\ln \tilde{c}^{\pm} \pm \tilde{\psi} \right) \right\|_{L^\infty(\Omega)}^4 \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{L^2(\Omega)}^6 \right]^{\pm} \\
&\le \delta \| \nabla (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \|_{L^2(\Omega)}^2 + \left[\frac{1}{4} \left\| \nabla (\sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}}) \right\|_{L^2(\Omega)}^2 \right]^{\pm} \\
&\quad + C \left[\left(\left\| \nabla \left(\ln \tilde{c}^{\pm} \pm \tilde{\psi} \right) \right\|_{L^\infty(\Omega)}^4 \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{L^2(\Omega)}^4 + \frac{1}{4} \right) \left\| \sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}} \right\|_{L^2(\Omega)}^2 \right]^{\pm} \\
&\le \delta \| \nabla (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \|_{L^2(\Omega)}^2 + \left[\frac{1}{4} \left\| \nabla (\sqrt{c^{\pm}} - \sqrt{\tilde{c}^{\pm}}) \right\|_{L^2(\Omega)}^2 \right]^{\pm} \\
&\quad + C \left(\left[\left\| \nabla \left(\ln \tilde{c}^{\pm} \pm \tilde{\psi} \right) \right\|_{L^\infty(\Omega)}^4 \right]^{\pm} + \left[\| \tilde{c}^{\pm} \|_{L^1(\Omega)}^2 \right]^{\pm} + 1 \right) \mathcal{R}(\boldsymbol{u} | \tilde{\boldsymbol{u}})
\end{aligned} \tag{41}
$$

where we again used Lemma 3.2. The term in front of $\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})$ can be estimated by

$$
C \left(\left[\left\| \nabla \left(\ln \tilde{c}^{\pm} \pm \tilde{\psi} \right) \right\|_{L^\infty(\Omega)}^4 \right]^{\pm} + \left[\| \tilde{c}^{\pm} \|_{L^1(\Omega)}^2 \right]^{\pm} + 1 \right) \le C \left(\left[\| \tilde{c}^{\pm} \|_{W^{1,\infty}(\Omega)}^4 \right]^{\pm} + \| \tilde{\psi} \|_{W^{1,\infty}(\Omega)}^4 + 1 \right).
$$

Now, we are done with the first term from (37). For the remaining term of (37), we note that

$$
\begin{aligned}
&\left| \left[\int_\Omega \pm (c^\pm - \tilde{c}^\pm) \tilde{\boldsymbol{v}} \cdot \nabla(\psi - \tilde{\psi}) \, \mathrm{d}\boldsymbol{x} \right]^\pm \right| \\
&\quad = \left| \left[\pm \int_\Omega \sqrt{c^\pm} \left(\sqrt{c^\pm} - \sqrt{\tilde{c}^\pm} \right) \tilde{\boldsymbol{v}} \cdot \nabla(\psi - \tilde{\psi}) + \sqrt{\tilde{c}^\pm} \left(\sqrt{c^\pm} - \sqrt{\tilde{c}^\pm} \right) \tilde{\boldsymbol{v}} \cdot \nabla(\psi - \tilde{\psi}) \, \mathrm{d}\boldsymbol{x} \right]^\pm \right| \\
&\quad \leq \left[\left\| \sqrt{\tilde{c}^\pm} - \sqrt{c^\pm} \right\|_{L^2(\Omega)} \left\| \sqrt{c^\pm} \nabla(\psi - \tilde{\psi}) \right\|_{L^2(\Omega)} \| \tilde{\boldsymbol{v}} \|_{L^\infty(\Omega)} \right]^\pm \\
&\qquad + \left[\left\| \sqrt{\tilde{c}^\pm} - \sqrt{c^\pm} \right\|_{L^2(\Omega)} \left\| \nabla(\psi - \tilde{\psi}) \right\|_{L^2(\Omega)} \left\| \sqrt{\tilde{c}^\pm} \tilde{\boldsymbol{v}} \right\|_{L^\infty(\Omega)} \right]^\pm ,
\end{aligned} \tag{42}
$$

where we used $x - y = \sqrt{x}(\sqrt{x} - \sqrt{y}) + \sqrt{y}(\sqrt{x} - \sqrt{y})$ for $x, y \geq 0$. Putting (38), (41), and (42) together and again employing Lemma 3.2, we obtain

$$
\begin{aligned}
|\,\mathrm{II}\,| \leq{}& C \left(\|\tilde{\boldsymbol{v}}\|^4_{L^\infty(\Omega)} + \left[\|\tilde{c}^\pm\|^4_{W^{1,\infty}(\Omega)} \right]^\pm + \|\tilde{\psi}\|^4_{W^{1,\infty}(\Omega)} + 1 \right) \mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) \\
&+ \frac{1}{2} \left[\left\| \nabla\sqrt{c^\pm} - \nabla\sqrt{\tilde{c}^\pm} \right\|^2_{L^2(\Omega)} \right]^\pm + \frac{1}{4} \|\nabla(\boldsymbol{v} - \tilde{\boldsymbol{v}})\|^2_{L^2(\Omega)} + \frac{1}{4} \left[\left\| \sqrt{c^\pm} \nabla(\psi - \tilde{\psi}) \right\|^2_{L^2(\Omega)} \right]^\pm .
\end{aligned} \tag{43}
$$

The estimate of III works similar to (42):

$$
\begin{aligned}
|\,\mathrm{III}\,| &= \left| \left[- \int_\Omega \pm \left(1 - \frac{c^\pm}{\tilde{c}^\pm} \right) \boldsymbol{\Lambda}(\boldsymbol{d}) \nabla \tilde{c}^\pm \cdot \nabla(\psi - \tilde{\psi}) + (\tilde{c}^\pm - c^\pm) \boldsymbol{\Lambda}(\boldsymbol{d}) \nabla\tilde{\psi} \cdot \nabla(\tilde{\psi} - \psi) \, \mathrm{d}\boldsymbol{x} \right]^\pm \right| \\
&= \left| \left[\int_\Omega (\tilde{c}^\pm - c^\pm) \boldsymbol{\Lambda}(\boldsymbol{d}) \left(\nabla\tilde{\psi} \pm \nabla \ln \tilde{c}^\pm \right) \cdot \nabla(\tilde{\psi} - \psi) \, \mathrm{d}\boldsymbol{x} \right]^\pm \right| \\
&\leq \left[\left\| \sqrt{\tilde{c}^\pm} - \sqrt{c^\pm} \right\|_{L^2(\Omega)} \left\| \sqrt{c^\pm} \nabla(\tilde{\psi} - \psi) \right\|_{L^2(\Omega)} \left\| \boldsymbol{\Lambda}(\boldsymbol{d}) \nabla \left(\tilde{\psi} \pm \ln \tilde{c}^\pm \right) \right\|_{L^\infty(\Omega)} \right]^\pm \\
&\quad + \left[\left\| \sqrt{\tilde{c}^\pm} - \sqrt{c^\pm} \right\|_{L^2(\Omega)} \left\| \nabla(\psi - \tilde{\psi}) \right\|_{L^2(\Omega)} \left\| \sqrt{\tilde{c}^\pm} \boldsymbol{\Lambda}(\boldsymbol{d}) \nabla \left(\tilde{\psi} \pm \ln \tilde{c}^\pm \right) \right\|_{L^\infty(\Omega)} \right]^\pm \\
&\leq C \left[\left\| \boldsymbol{\Lambda}(\boldsymbol{d}) \nabla \left(\tilde{\psi} \pm \ln \tilde{c}^\pm \right) \right\|^2_{L^\infty(\Omega)} + \left\| \sqrt{\tilde{c}^\pm} \boldsymbol{\Lambda}(\boldsymbol{d}) \nabla \left(\tilde{\psi} \pm \ln \tilde{c}^\pm \right) \right\|_{L^\infty(\Omega)} \right]^\pm \mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) \\
&\quad + \frac{1}{4} \left[\left\| \sqrt{c^\pm} \nabla(\psi - \tilde{\psi}) \right\|^2_{L^2(\Omega)} \right]^\pm .
\end{aligned} \tag{44}
$$

Next, we estimate IV:

$$
\begin{aligned}
|\,\mathrm{IV}\,| &= \left| \left[\int_\Omega 4\sqrt{\tilde c^\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde c^\pm}\cdot\nabla\left(1-\sqrt{\frac{c^\pm}{\tilde c^\pm}}\right)^2 + 4\left(1-\sqrt{\frac{c^\pm}{\tilde c^\pm}}\right)^2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde c^\pm}\cdot\nabla\sqrt{\tilde c^\pm}\,\mathrm{d}\boldsymbol{x}\right]^\pm \right| \\
&\le \left| \left[\int_\Omega 8\left(\sqrt{\tilde c^\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde c^\pm}\right)\cdot\left(\left(\frac{1}{\sqrt{\tilde c^\pm}}(\sqrt{\tilde c^\pm}-\sqrt{c^\pm})\right)\nabla\left(\frac{1}{\sqrt{\tilde c^\pm}}(\sqrt{\tilde c^\pm}-\sqrt{c^\pm})\right)\right)\,\mathrm{d}\boldsymbol{x}\right]^\pm \right| \\
&\quad + \left| \left[\int_\Omega -\left(\sqrt{\tilde c^\pm}-\sqrt{c^\pm}\right)^2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\ln\tilde c^\pm\cdot\nabla\ln\tilde c^\pm\,\mathrm{d}\boldsymbol{x}\right]^\pm \right| \\
&\le \left| \left[\int_\Omega 8\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde c^\pm}\right)\cdot\left(\left(\sqrt{\tilde c^\pm}-\sqrt{c^\pm}\right)^2\nabla\left(\frac{1}{\sqrt{\tilde c^\pm}}\right)+\frac{1}{\sqrt{\tilde c^\pm}}\left(\sqrt{\tilde c^\pm}-\sqrt{c^\pm}\right)\nabla\left(\sqrt{\tilde c^\pm}-\sqrt{c^\pm}\right)\right)\,\mathrm{d}\boldsymbol{x}\right]^\pm \right| \\
&\quad + \left| \left[\int_\Omega -\left(\sqrt{\tilde c^\pm}-\sqrt{c^\pm}\right)^2\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\ln\tilde c^\pm\cdot\nabla\ln\tilde c^\pm\,\mathrm{d}\boldsymbol{x}\right]^\pm \right| \\
&\le \left(\left[\left\| 8\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde c^\pm}\right)\cdot\nabla\left(\frac{1}{\sqrt{\tilde c^\pm}}\right)\right\|_{L^\infty(\Omega)} + C\left\| 8\frac{1}{\sqrt{\tilde c^\pm}}\left(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\sqrt{\tilde c^\pm}\right)\right\|_{L^\infty(\Omega)}^2\right]^\pm \right. \\
&\quad \left. + \left[\left\|\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\ln\tilde c^\pm\cdot\nabla\ln\tilde c^\pm\right\|_{L^\infty(\Omega)}\right]^\pm\right)\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) + \frac14\left[\left\|\nabla\sqrt{\tilde c^\pm}-\nabla\sqrt{c^\pm}\right\|_{L^2(\Omega)}^2\right]^\pm. \qquad (45)
\end{aligned}
$$

For V, we note

$$
\begin{aligned}
|\,\mathrm{V}\,| &\le 6\varepsilon\lambda\,\|\boldsymbol{d}\|_{W^{1,\infty}(\Omega)}^4\left\|\nabla(\psi-\tilde\psi)\right\|_{L^2(\Omega)}^2 \\
&\quad + \left(8\varepsilon\lambda\,\|\boldsymbol{d}\|_{W^{1,\infty}(\Omega)}^4 + 4(\varepsilon+\lambda)\,\|\boldsymbol{d}\|_{W^{1,\infty}(\Omega)}^2\right)\left\|\nabla(\psi-\tilde\psi)\right\|_{L^2(\Omega)}\left\|\nabla^2(\psi-\tilde\psi)\right\|_{L^2(\Omega)} \\
&\le \frac18\left\|\nabla^2(\psi-\tilde\psi)\right\|_{L^2(\Omega)}^2 + C\left(1+\|\boldsymbol{d}\|_{W^{1,\infty}(\Omega)}^8\right)\left\|\nabla(\psi-\tilde\psi)\right\|_{L^2(\Omega)}^2. \qquad (46)
\end{aligned}
$$

For the boundary integral VI, we use the trace estimate [DiB10, Prop. 8.2], which gives for all $p\in[1,3)$ that there exists a $C>0$ such that for all $\delta>0$ and all $u\in W^{1,p}(\Omega)$, we have that

$$
\|u\|_{L^q(\Gamma)}^p \le \delta\,\|\nabla u\|_{L^p(\Omega)}^p + C\left(1+\frac1\delta\right)\|u\|_{L^p(\Omega)}^p \qquad \text{for} \quad q\in\left[1,\frac{2p}{(3-p)}\right].
$$

Thus, using $x-y=(\sqrt{x}-\sqrt{y})^2+2\sqrt{y}\left(\sqrt{x}-\sqrt{y}\right)$, we obtain

$$
\begin{aligned}
|\,\mathrm{VI}\,| &= \left|2\tau\int_\Gamma (c^+-c^- -(\tilde c^+-\tilde c^-))(\psi-\tilde\psi)\,\mathrm{d}\sigma\right| \le 2\tau\left[\left\|c^\pm-\tilde c^\pm\right\|_{L^{4/3}(\Gamma)}\right]^\pm\left\|\psi-\tilde\psi\right\|_{L^4(\Gamma)} \\
&\le \left[2\tau\left\|\left(\sqrt{c^\pm}-\sqrt{\tilde c^\pm}\right)^2\right\|_{L^{4/3}(\Gamma)} + 4\tau\left\|\sqrt{\tilde c^\pm}\left(\sqrt{c^\pm}-\sqrt{\tilde c^\pm}\right)\right\|_{L^{4/3}(\Gamma)}\right]^\pm\left\|\psi-\tilde\psi\right\|_{L^4(\Gamma)} \\
&\le 2\tau\left[\left\|\sqrt{c^\pm}-\sqrt{\tilde c^\pm}\right\|_{L^{8/3}(\Gamma)}^2\right]^\pm\left\|\psi-\tilde\psi\right\|_{L^4(\Gamma)} + 4\tau\left[\left\|\sqrt{\tilde c^\pm}\right\|_{L^\infty(\Gamma)}\left\|\sqrt{c^\pm}-\sqrt{\tilde c^\pm}\right\|_{L^{4/3}(\Gamma)}\right]^\pm\left\|\psi-\tilde\psi\right\|_{L^4(\Gamma)} \\
&\le 2\tau\left[\delta\left\|\nabla\sqrt{c^\pm}-\nabla\sqrt{\tilde c^\pm}\right\|_{L^2(\Omega)}^2 + C\left(1+\frac1\delta\right)\left\|\sqrt{c^\pm}-\sqrt{\tilde c^\pm}\right\|_{L^2(\Omega)}^2\right]^\pm\left\|\psi-\tilde\psi\right\|_{L^\infty(0,T;W^{1,2}(\Omega))} \\
&\quad + C\left[\left\|\sqrt{\tilde c^\pm}\right\|_{L^\infty(\Omega)}\left\|\sqrt{c^\pm}-\sqrt{\tilde c^\pm}\right\|_{W^{1,2}(\Omega)}\right]^\pm\left\|\psi-\tilde\psi\right\|_{W^{1,2}(\Omega)} \\
&\le \frac14\left[\left\|\nabla\sqrt{c^\pm}-\nabla\sqrt{\tilde c^\pm}\right\|_{L^2(\Omega)}^2\right]^\pm + C\left(1+\|\tilde\psi\|_{L^\infty(0,T;W^{1,2}(\Omega))}^2\right)\left[\left\|\sqrt{c^\pm}-\sqrt{\tilde c^\pm}\right\|_{L^2(\Omega)}^2\right]^\pm \\
&\quad + \left[\frac14\left\|\nabla\left(\sqrt{c^\pm}-\sqrt{\tilde c^\pm}\right)\right\|_{L^2(\Omega)}^2 + \frac14\left\|\sqrt{c^\pm}-\sqrt{\tilde c^\pm}\right\|_{L^2(\Omega)}^2 + C\left\|\sqrt{\tilde c^\pm}\right\|_{L^\infty(\Omega)}^2\left\|\psi-\tilde\psi\right\|_{W^{1,2}(\Omega)}^2\right]^\pm, \qquad (47)
\end{aligned}
$$

where we have chosen $\delta = \left(Ce^T(\mathcal{E}(\boldsymbol{u}_0) + C(\xi)) + \|\tilde{\psi}\|_{L^\infty(0,T;W^{1,2}(\Omega))} + 8\tau \right)^{-1}$ and used that

$$\|\psi\|_{L^\infty(0,T;W^{1,2}(\Omega))} \leq C \sup_{t\in(0,T)} (\mathcal{E}(\boldsymbol{u})(t) + 1) \leq Ce^T(\mathcal{E}(\boldsymbol{u}_0) + C(\xi))$$

by the energy inequality. For VII, we note that

$$\begin{aligned}
|\,\mathrm{VII}\,| &= 2\tau \left| \int_\Gamma (\psi - \tilde{\psi}) \left(\nabla \boldsymbol{n}^T \boldsymbol{\Lambda}(\boldsymbol{d}) \nabla(\psi - \tilde{\psi}) \right) \cdot \boldsymbol{n} \,\mathrm{d}\sigma \right| \leq C \|\boldsymbol{\Lambda}(\boldsymbol{d})\|_{L^\infty(\Gamma)} \|\psi - \tilde{\psi}\|_{L^2(\Gamma)} \|\nabla(\psi - \tilde{\psi})\|_{L^2(\Gamma)} \\
&\leq C \|\boldsymbol{\Lambda}(\boldsymbol{d})\|_{W^{1,\infty}(\Omega)} \|\psi - \tilde{\psi}\|_{W^{1,2}(\Omega)} \|\psi - \tilde{\psi}\|_{W^{2,2}(\Omega)} \\
&\leq C \|\boldsymbol{\Lambda}(\boldsymbol{d})\|^2_{W^{1,\infty}(\Omega)} \|\psi - \tilde{\psi}\|^2_{W^{1,2}(\Omega)} + \frac{1}{8} \|\psi - \tilde{\psi}\|^2_{W^{2,2}(\Omega)} \\
&= C \left(\|\boldsymbol{d}\|^2_{W^{1,\infty}(\Omega)} + 1 \right) \left(\|\nabla(\psi - \tilde{\psi})\|^2_{L^2(\Omega)} + \|\psi - \tilde{\psi}\|^2_{L^2(\Gamma)} \right) + \frac{1}{8} \|\nabla^2(\psi - \tilde{\psi})\|^2_{L^2(\Omega)} \\
&\leq C \left(\|\boldsymbol{d}\|^2_{W^{1,\infty}(\Omega)} + 1 \right) \mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) + \frac{1}{8} \|\nabla^2(\psi - \tilde{\psi})\|^2_{L^2(\Omega)},
\end{aligned} \tag{48}$$

and for VIII, again using [DiB10, Prop. 8.2], we have

$$\begin{aligned}
|\,\mathrm{VIII}\,| &= 2 \left| \int_\Gamma \left(\nabla \boldsymbol{n}^T \boldsymbol{\Lambda}(\boldsymbol{d}) \nabla(\psi - \tilde{\psi}) \right) \cdot \boldsymbol{\mathcal{E}}(\boldsymbol{d}) \nabla(\psi - \tilde{\psi}) \,\mathrm{d}\sigma \right| \\
&\leq C \|\boldsymbol{\Lambda}(\boldsymbol{d})\|_{W^{1,\infty}(\Omega)} \|\boldsymbol{\mathcal{E}}(\boldsymbol{d})\|_{W^{1,\infty}(\Omega)} \|\nabla(\psi - \tilde{\psi})\|^2_{L^2(\Gamma)} \\
&\leq C \|\boldsymbol{\Lambda}(\boldsymbol{d})\|_{W^{1,\infty}(\Omega)} \|\boldsymbol{\mathcal{E}}(\boldsymbol{d})\|_{W^{1,\infty}(\Omega)} \left(C \left(1 + \frac{1}{\delta} \right) \|\nabla(\psi - \tilde{\psi})\|^2_{L^2(\Omega)} + \delta \|\nabla^2(\psi - \tilde{\psi})\|^2_{L^2(\Omega)} \right) \\
&\leq C \left(1 + \|\boldsymbol{d}\|^4_{W^{1,\infty}(\Omega)} \right) \mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) + \frac{1}{8} \|\nabla^2(\psi - \tilde{\psi})\|^2_{L^2(\Omega)}.
\end{aligned} \tag{49}$$

For IX, we estimate

$$\begin{aligned}
|\,\mathrm{IX}\,| &= \left| \int_\Gamma \partial_t(\tilde{\xi} - \xi)(\psi - \tilde{\psi}) \,\mathrm{d}\sigma \right| \\
&\leq \frac{1}{2} \|\partial_t(\tilde{\xi} - \xi)\|^2_{L^2(\Gamma)} + \frac{1}{2} \|(\psi - \tilde{\psi})\|^2_{L^2(\Gamma)} \leq \frac{1}{2} \|\partial_t(\tilde{\xi} - \xi)\|^2_{L^2(\Gamma)} + \frac{1}{\tau} \mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})
\end{aligned} \tag{50}$$

and for X similarly to VI again using [DiB10, Prop. 8.2] and $x - y = (\sqrt{x} - \sqrt{y})^2 + 2\sqrt{y}\left(\sqrt{x} - \sqrt{y}\right)$, we find

$$\begin{aligned}
|\,\mathrm{X}\,| &= 2 \left| \int_\Gamma (c^+ - c^- - (\tilde{c}^+ - \tilde{c}^-))(\xi - \tilde{\xi}) \,\mathrm{d}\sigma \right| \\
&\leq 2 \left\| \xi - \tilde{\xi} \right\|_{L^4(\Gamma)} \left(\left[\left\| \sqrt{\tilde{c}^\pm} - \sqrt{\tilde{c}^\pm} \right\|^2_{L^{8/3}(\Gamma)} + 2 \left\| \sqrt{\tilde{c}^\pm} \right\|_{L^\infty(\Gamma)} \left\| \sqrt{\tilde{c}^\pm} - \sqrt{\tilde{c}^\pm} \right\|_{L^{4/3}(\Gamma)} \right]^\pm \right) \\
&\leq 2 \left\| \xi - \tilde{\xi} \right\|_{L^4(\Gamma)} \left[\delta \left\| \nabla\sqrt{\tilde{c}^\pm} - \nabla\sqrt{\tilde{c}^\pm} \right\|^2_{L^2(\Omega)} + C \left(1 + \frac{1}{\delta} \right) \left\| \sqrt{\tilde{c}^\pm} - \sqrt{\tilde{c}^\pm} \right\|^2_{L^2(\Omega)} \right]^\pm \\
&\quad + C \left\| \xi - \tilde{\xi} \right\|_{L^4(\Gamma)} \left[\left\| \sqrt{\tilde{c}^\pm} \right\|_{L^\infty(\Gamma)} \left\| \sqrt{\tilde{c}^\pm} - \sqrt{\tilde{c}^\pm} \right\|_{W^{1,2}(\Omega)} \right]^\pm \\
&\leq \frac{1}{2} \left[\left\| \nabla\sqrt{\tilde{c}^\pm} - \nabla\sqrt{\tilde{c}^\pm} \right\|^2_{L^2(\Omega)} + C \left(1 + \left\| \xi - \tilde{\xi} \right\|^2_{L^4(\Gamma)} \right) \left\| \sqrt{\tilde{c}^\pm} - \sqrt{\tilde{c}^\pm} \right\|^2_{L^2(\Omega)} \right]^\pm \\
&\quad + C \left[\left\| \sqrt{\tilde{c}^\pm} \right\|^2_{L^\infty(\Omega)} \left\| \xi - \tilde{\xi} \right\|^2_{W^{1,2}(\Gamma)} + \frac{1}{4} \left\| \sqrt{\tilde{c}^\pm} - \sqrt{\tilde{c}^\pm} \right\|^2_{L^2(\Omega)} \right]^\pm,
\end{aligned} \tag{51}$$

where we have chosen $\delta = \left(8 \left(1 + \|\xi - \tilde{\xi}\|_{L^4(\Gamma)} \right) \right)^{-1}$ and used the embedding $W^{1,2}(\Gamma) \hookrightarrow L^4(\Gamma)$ [DD12, Thm. 3.81]. For XI, we again use [DiB10, Prop. 8.2] and estimate

$$
\begin{aligned}
|\,\mathrm{XI}\,| &= 2\left|\int_\Gamma -(\boldsymbol{\Lambda}(\boldsymbol{d})+\boldsymbol{\mathcal{E}}(\boldsymbol{d}))\nabla_\Gamma(\psi-\tilde\psi)\cdot\nabla_\Gamma(\xi-\tilde\xi)\,\mathrm{d}\sigma\right| \\
&\le \left\|\nabla_\Gamma(\xi-\tilde\xi)\right\|^2_{L^2(\Gamma)} + \|\boldsymbol{\Lambda}(\boldsymbol{d})+\boldsymbol{\mathcal{E}}(\boldsymbol{d})\|^2_{L^\infty(\Gamma)}\left\|\nabla_\Gamma(\psi-\tilde\psi)\right\|^2_{L^2(\Gamma)} \\
&\le \left\|\nabla_\Gamma(\xi-\tilde\xi)\right\|^2_{L^2(\Gamma)} + C\,\|\boldsymbol{d}\|^2_{W^{1,\infty}(\Omega)}\left\|\nabla(\psi-\tilde\psi)\right\|^2_{L^2(\Gamma)} \\
&\le \left\|\xi-\tilde\xi\right\|^2_{W^{1,2}(\Gamma)} + C\,\|\boldsymbol{d}\|^2_{W^{1,\infty}(\Omega)}\left(C\left(1+\frac{1}{\delta}\right)\left\|\nabla(\psi-\tilde\psi)\right\|^2_{L^2(\Omega)} + \delta\left\|\nabla^2(\psi-\tilde\psi)\right\|^2_{L^2(\Omega)}\right) \\
&\le \left\|\xi-\tilde\xi\right\|^2_{W^{1,2}(\Gamma)} + \frac{1}{16}\left\|\nabla^2(\psi-\tilde\psi)\right\|^2_{L^2(\Omega)} + C\left(1+\|\boldsymbol{d}\|^4_{W^{1,\infty}(\Omega)}\right)\left\|\nabla(\psi-\tilde\psi)\right\|^2_{L^2(\Omega)}, \qquad (52)
\end{aligned}
$$

where we have chosen $\delta = \left(16\left(1+C\,\|\boldsymbol{d}\|^2_{W^{1,\infty}(\Omega)}\right)\right)^{-1}$ and estimated the norm of the surface gradient of ψ by the norm of the full gradient, *cf.* [HLP25, Lem. A.36]. For XII similarly to VIII, again using [DiB10, Prop. 8.2], we find

$$
\begin{aligned}
|\,\mathrm{XII}\,| &= 2\left|\int_\Gamma -(\xi-\tilde\xi)\left(\nabla\boldsymbol{n}^T\boldsymbol{\Lambda}(\boldsymbol{d})\nabla(\psi-\tilde\psi)\right)\cdot\boldsymbol{n}\,\mathrm{d}\sigma\right| \\
&\le \|\xi-\tilde\xi\|^2_{L^2(\Gamma)} + C\,\|\boldsymbol{\Lambda}(\boldsymbol{d})\|^2_{L^\infty(\Gamma)}\,\|\nabla(\psi-\tilde\psi)\|^2_{L^2(\Gamma)} \\
&\le \|\xi-\tilde\xi\|^2_{L^2(\Gamma)} + C\,\|\boldsymbol{d}\|^2_{W^{1,\infty}(\Omega)}\left(\delta\left\|\nabla^2(\psi-\tilde\psi)\right\|^2_{L^2(\Omega)} + C\left(1+\frac{1}{\delta}\right)\|\nabla(\psi-\tilde\psi)\|^2_{L^2(\Omega)}\right) \\
&\le \|\xi-\tilde\xi\|^2_{L^2(\Gamma)} + \frac{1}{16}\|\nabla^2(\psi-\tilde\psi)\|^2_{L^2(\Omega)} + C\left(1+\|\boldsymbol{d}\|^4_{W^{1,\infty}(\Omega)}\right)\|\nabla(\psi-\tilde\psi)\|^2_{L^2(\Omega)}, \qquad (53)
\end{aligned}
$$

where we have chosen $\delta = \left(16\left(1+C\,\|\boldsymbol{d}\|^2_{W^{1,\infty}(\Omega)}\right)\right)^{-1}$. Finally, we turn to the curvature term XIII and find

$$
|\,\mathrm{XIII}\,| \le 2\left|\int_\Gamma(\xi-\tilde\xi-\tau(\psi-\tilde\psi))^2(\nabla_\Gamma\cdot\boldsymbol{n})\,\mathrm{d}\sigma\right| \le C\left(\|\psi-\tilde\psi\|^2_{L^2(\Gamma)} + \|\xi-\tilde\xi\|^2_{L^2(\Gamma)}\right). \qquad (54)
$$

Putting (36) and (43)–(54) back into (35), we obtain

$$
\begin{aligned}
&\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})\big|_0^t + \int_0^t \mathcal{W}(\boldsymbol{u}|\tilde{\boldsymbol{u}})\,\mathrm{d}s \\
&\le \int_0^t\int_\Omega \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}})\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) + \left[\mathcal{A}_{\tilde c^\pm}(\tilde{\boldsymbol{u}})\left(\frac{1}{\tilde c^\pm}\left(\tilde c^\pm - c^\pm\right)\pm(\tilde\psi-\psi)\right)\right]^\pm\mathrm{d}\boldsymbol{x} + \mathcal{K}(\tilde{\boldsymbol{u}})\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) + Q_{\tilde{\boldsymbol{u}}}(\xi|\tilde\xi)\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \|\nabla(\boldsymbol{v}-\tilde{\boldsymbol{v}})\|^2_{L^2(\Omega)} + \left[\left\|\sqrt{c^\pm}\nabla(\psi-\tilde\psi)\right\|^2_{L^2(\Omega)}\right]^\pm + \left\|\nabla^2(\psi-\tilde\psi)\right\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\quad + 2\int_0^t\left[\left\|\nabla\sqrt{\tilde c^\pm}-\nabla\sqrt{\tilde c^\pm}\right\|^2_{L^2(\Omega)}\right]^\pm\mathrm{d}s.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}})\big|_0^t &+ \frac{1}{2}\int_0^t \mathcal{W}(\boldsymbol{u}|\tilde{\boldsymbol{u}})\,\mathrm{d}s \\
&\le \int_0^t\int_\Omega \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}})\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}) + \left[\mathcal{A}_{\tilde c^\pm}(\tilde{\boldsymbol{u}})\left(\frac{1}{\tilde c^\pm}\left(\tilde c^\pm - c^\pm\right)\pm(\tilde\psi-\psi)\right)\right]^\pm\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\quad + \int_0^t \mathcal{K}(\tilde{\boldsymbol{u}})\mathcal{R}(\boldsymbol{u}|\tilde{\boldsymbol{u}}) + \mathcal{Q}_{\tilde{\boldsymbol{u}}}(\xi|\tilde\xi)\,\mathrm{d}s. \qquad (55)
\end{aligned}
$$

An application of Gronwall's inequality yields the relative energy inequality (12). □

4 Existence of local-in-time strong solutions to the anisotropic NSNPP and SNPP systems

In this section, we will prove the existence of local-in-time strong solutions to the NSNPP as well as the SNPP system. The existence of strong solutions to certain isotropic SNPP systems is already known global-in-time, see for example [CIL21]. However, this result makes use of identifying the electromigration term in the Nernst–Planck equations with the Poisson equation. Due to the difference in the anisotropy matrices $\boldsymbol{\mathcal{E}}(\boldsymbol{d})$ and $\boldsymbol{\Lambda}(\boldsymbol{d})$ we cannot derive such an *a priori* bound and we need to restrict our selves to local-in-time strong solutions. In the following we will show this existence of local-in-time strong solutions both for the NSNPP and the SNPP system by deriving *a priori* estimates which allow us to use an ODE comparison principle. To make these rigorous we need to use an appropriate regularized system, which allows us to test the (Navier–)Stokes equation with $\boldsymbol{v} - \Delta \boldsymbol{v}$ and the Nernst–Planck equations with $c^{\pm} - \Delta c^{\pm}$, which we will introduce in the following.

Choosing a Galerkin basis of eigenfunctions of the Stokes operator for the Navier–Stokes equation allowing us to test with the Stokes operator of $\boldsymbol{v}$ and an elliptic regularization for the NPP part of the system we find a global strong solution to this regularized system using a similar fixed point argument as in [HLP25]. For an approximation $\boldsymbol{v}_m \in V_m$, where V_m is the span of the finite eigenfunction basis $\{\boldsymbol{w}_1^m, \dots, \boldsymbol{w}_{N_m}^m\}$, and the regularization coefficient $\kappa_m > 0$ we consider the regularized coupled system:

$$\big(\mathrm{Re}\,(\partial_t \boldsymbol{v}_m + (\boldsymbol{v}_m \cdot \nabla)\boldsymbol{v}_m) - \Delta \boldsymbol{v}_m + \nabla p_m + (c_m^+ - c_m^-)\nabla \psi_m,\, \boldsymbol{w}\big)_{L^2} = 0 \quad \text{in } (0,T) \text{ for all } \boldsymbol{w} \in V_m, \tag{56a}$$

$$\nabla \cdot \boldsymbol{v}_m = 0 \quad \text{in } (0,T) \times \Omega, \tag{56b}$$

$$\partial_t c_m^{\pm} + \nabla \cdot (c_m^{\pm} \boldsymbol{v}_m) - \nabla \cdot \big(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla c_m^{\pm} \pm c_m^{\pm} \nabla \psi_m)\big) = 0 \quad \text{in } (0,T) \times \Omega, \tag{56c}$$

$$-\nabla \cdot (\boldsymbol{\mathcal{E}}(\boldsymbol{d}) \nabla \psi_m) - S_{\kappa_m}(c_m^+ - c_m^-) = 0 \quad \text{in } (0,T) \times \Omega, \tag{56d}$$

where $(\cdot\,,\cdot)_{L^2}$ denotes the $L^2(\Omega)$ scalar product and S_{κ_m} is an elliptic regularization operator [HLP25, Def. 4.1], which is given by the solution operator to the anisotropic Poisson equation, that is $S_{\kappa_m}(f) = \varphi$, where

$$\varphi - \kappa_m \nabla \cdot (\boldsymbol{\mathcal{E}}(\boldsymbol{d}) \nabla \varphi) = f \quad \text{in } \Omega, \qquad \boldsymbol{\mathcal{E}}(\boldsymbol{d}) \nabla \varphi + \tau \varphi = \xi \quad \text{on } \Gamma.$$

In the following existence proof of local strong solutions the precise form of this elliptic regularization is not used directly, we only make use of the elliptic regularity and that for $\kappa_m \searrow 0$ it approximates the identity operator. That is for $\kappa_m \searrow 0$ and $f_m \to f$ in $L^2(\Omega)$ we find $S_{\kappa_m}(f_m) \to f$ in $L^2(\Omega)$. Additionally, we equip system (56) with the initial conditions

$$\boldsymbol{v}_m(0) = \boldsymbol{v}_{0,m} \in V_m, \qquad c_m^{\pm}(0) = c_0^{\pm} \in L^2(\Omega)_+ \qquad \text{in } \Omega$$

and the boundary conditions

$$\boldsymbol{v}_m = 0, \qquad \boldsymbol{\Lambda}(\boldsymbol{d})(\nabla c_m^{\pm} \pm c_m^{\pm} \nabla \psi_m) \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{\mathcal{E}}(\boldsymbol{d}) \nabla \psi_m \cdot \boldsymbol{n} + \tau \psi_m = \xi \qquad \text{on } [0,T] \times \Gamma.$$

Equation (56a) is a standard Galerkin discretization of the Navier–Stokes equation, which can be seen as a non-linear system of ordinary differential equations (ODE). A solution $\boldsymbol{v}_m$ then is the vector $\boldsymbol{v}_m = (v_1^m, \dots, v_{N_m}^m) \in C^1([0,T]; \mathbb{R}^{N_m})$ such that $\boldsymbol{v}_m = \sum v_i^m \boldsymbol{w}_i^m$ fulfills (56a) for all test functions from the finite dimension test space V_m. As already indicated here we will use $\boldsymbol{v}_m$ both for the vector-valued function with values in $\mathbb{R}^{N_m}$ as well as the abstract function with values in the Sobolev space $W^{2,2}(\Omega)$, which a common abuse of notation, and we equip the space $\mathbb{R}^{N_m}$ with the norm $\|\boldsymbol{v}_m\|_m := \big\|\sum v_i^m \boldsymbol{w}_i^m\big\|_{W^{1,2}(\Omega)}$. In the following proposition we will prove the existence of global strong solutions to the regularized system (56) via a Schauder fixed point argument.

Proposition 4.1. *For every* $m \in \mathbb{N}$, $\mathrm{Re} \geq 0, T > 0$, $\kappa_m > 0$, $\boldsymbol{v}_{0,m} \in V_m$, $c_0^{\pm} \in L^2(\Omega)_+$, $\boldsymbol{d} \in C^{4,1}(\overline{\Omega})$ *with* $\boldsymbol{d} \cdot \boldsymbol{n} = 0$ *on* Γ, *and* $\xi \in W^{1,2}(0,T; W^{4,2}(\Gamma))$, *the regularized system* (56) *has a global strong solution* $\boldsymbol{u}_m = (\boldsymbol{v}_m, c_m^{\pm}, \psi_m)$ *such that*

$$\begin{aligned}
&\boldsymbol{v}_m \in \mathrm{AC}([0,T]; V_m), \quad c^{\pm}, \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)) \cap W^{1,2}(0,T; W^{-1,2}(\Omega)) \\
&\psi_m \in C([0,T]; W^{4,2}(\Omega)) \cap L^2(0,T; W^{5,2}(\Omega)) \cap W^{1,2}(0,T; W^{3,2}(\Omega))
\end{aligned}$$

and the equations in (56) *are fulfilled pointwise almost everywhere.*

Proof. We use a Schauder fixed-point argument. For fixed $\boldsymbol{v}_m \in \operatorname{span}\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_{N_m}\} \subseteq L^\infty(0,T;L^s(\Omega))$ with $s > d$ and $\kappa_m > 0$ we already know that the regularized NPP subsystem (56c)–(56d) has a weak solution $(c_m^\pm, \psi_m)$ with $c_m^\pm \in C([0,T];L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)) \cap W^{1,2}(0,T;W^{-1,2}(\Omega))$ and $\psi_m \in C([0,T];W^{4,2}(\Omega)) \cap L^2(0,T;W^{5,2}(\Omega)) \cap W^{1,2}(0,T;W^{3,2}(\Omega))$, which follows from [HLP25, Prop. 4.3] and elliptic regularity. Here, we used that we have $\boldsymbol{d} \in C^{4,1}(\overline{\Omega})$ and $\xi \in C([0,T];W^{4,2}(\Gamma))$ in order to obtain information on the fifth order derivative of ψ_m by elliptic regularity [GT01, Thm. 9.19]. Now, we have $(c_m^+ - c_m^-)\nabla\psi_m \in C([0,T];L^2(\Omega))$ for the right-hand side of (56a) and thus the solutions to the finite dimensional system of ODEs (56a) has a unique local-in-time solution $\boldsymbol{v}_m \in C^1([0,T^*);\mathbb{R}^{N_m})$ due to the Picard–Lindelöf theorem, since on the discrete level we solve an ODE with polynomial and thus locally Lipschitz continuous right-hand side [Emm04, Thm. 7.2.3]. This local solution can be extended to the whole time interval $[0,T]$ with classical *a priori* estimates for $\boldsymbol{v}_m$, which are derived by testing with the Stokes operator applied to $\boldsymbol{v}_m$ and give the boundedness in $L^\infty(0,T;W^{1,2}(\Omega))$ for $\boldsymbol{v}_{0,m} \in W^{1,2}_{0,\sigma}(\Omega)$ [BF13, Sec. V.2].

We now define the solution operator $\mathcal{T}_m : C([0,T];\mathbb{R}^{N_m}) \to C([0,T];\mathbb{R}^{N_m})$ that decouples the (Navier–)Stokes and Nernst–Planck–Poisson (NPP) part of system (56) by mapping a given $\boldsymbol{v}_m$ to the solution $(c_m^\pm, \psi_m)$ of the NPP sub-system (56c)–(56d) and then giving the solution $\overline{\boldsymbol{v}}_m$ to the (Navier–)Stokes equation (56a) for the right-hand side given by this $(c_m^\pm, \psi_m)$. The operator $\mathcal{T}_m$ maps the closed and convex set: $M := \{\boldsymbol{v} \in C([0,T];\mathbb{R}^{N_m}) | \, \|\boldsymbol{v}\|_m \le C_M\}$ onto itself, where C_M is such that

$$
\begin{aligned}
\|\boldsymbol{v}_m\|_{L^\infty(0,T;W^{1,2}_{0,\sigma}(\Omega))} &\le C \left\|(c_m^+ - c_m^-)\nabla\psi_m\right\|_{L^\infty(0,T;L^2(\Omega))} \le C \left[\left\|c_m^\pm\right\|_{L^\infty(0,T;L^2(\Omega))}\right]^{\pm} \|\nabla\psi_m\|_{L^\infty((0,T)\times\Omega)} \\
&\le C \left[\left\|c_m^\pm\right\|_{L^\infty(0,T;L^2(\Omega))}\right]^{\pm} \|\psi_m\|_{L^\infty(0,T;W^{4,2}(\Omega))} \le C(\kappa_m, T) =: C_M,
\end{aligned} \tag{57}
$$

where the last inequality follows from [HLP25, Lem. 4.5] and is independent of $\boldsymbol{v}_m$. If we show in addition that $\mathcal{T}_m$ is continuous and compact, then, by Schauder's fixed point theorem [Rou13, Thm. 1.9] there exists a fixed point $\boldsymbol{v}_m \in M$ such that $\mathcal{T}_m(\boldsymbol{v}_m) = \boldsymbol{v}_m$.

We first show the compactness of $\mathcal{T}_m$. Let $(\boldsymbol{v}_{m,n})_n \subseteq C([0,T];\mathbb{R}^{N_m})$ be a bounded sequence and let $(c_{m,n}^\pm, \psi_{m,n})_n$ be the corresponding solutions to the Nernst–Planck–Poisson system. The compactness in space simply follows by the boundedness of $\boldsymbol{v}_m$ in $L^\infty(0,T;W^{1,2}(\Omega))$ since $\mathbb{R}^{N_m}$ is finite dimensional. For the compactness in time we need an estimate for the time derivative. For $\mathrm{Re} > 0$ we know that $\partial_t \boldsymbol{v}_{m,n}$ is bounded in $L^2(0,T;L^2(\Omega))$ [BF13, Sec. V.2]. For $\mathrm{Re} = 0$ the solution operator $A_S^{-1} : W^{-1,2}(\Omega) \to W^{1,2}_{0,\sigma}(\Omega)$ to the (discretized/ finite dimensional) stationary Stokes equation is a bounded linear operator and thus the boundedness of $\partial_t((c_{m,n}^+ - c_{m,n}^-)\nabla\psi_{m,n}) \in L^2(0,T;W^{-1,2}(\Omega))$ (which follows from a simple testing of the Nernst–Planck equation (56c)) implies the boundedness of $\partial_t \boldsymbol{v}_{m,n} = \partial_t A_S^{-1}((c_{m,n}^+ - c_{m,n}^-)\nabla\psi_{m,n}) = A_S^{-1}(\partial_t((c_{m,n}^+ - c_{m,n}^-)\nabla\psi_{m,n})) \in L^2(0,T;W^{1,2}(\Omega))$ (by basic properties of the Bochner integral [Emm04, Thm. 7.1.15iii)]). And thus by the compact embedding

$$
L^\infty(0,T;\mathbb{R}^{N_m}) \cap W^{1,2}(0,T;L^2(\Omega)) \hookrightarrow\hookrightarrow C([0,T];\mathbb{R}^{N_m})
$$

see for example [Sim86, Cor. 4], we find a subsequence of $(\mathcal{T}_m(\boldsymbol{v}_{m,n}))_n$ convergent in $C([0,T];\mathbb{R}^{N_m})$ and the compactness of $\mathcal{T}_m$ follows.

Next, we show the continuity of $\mathcal{T}_m$. Let $(\boldsymbol{v}_{m,n})_n \subseteq C([0,T];\mathbb{R}^{N_m})$ be a sequence converging to $\boldsymbol{v}_m \in C([0,T];\mathbb{R}^{N_m})$. Then we find that $(c_{m,n}^\pm)_n$ converges to $c_m^\pm$ in $C([0,T];L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ and consequently $(\psi_{m,n})_n$ converges to ψ_m in $C([0,T];W^{4,2}(\Omega)) \cap L^2(0,T;W^{5,2}(\Omega))$ by elliptic regularity. Before we show the convergence of $(c_{m,n}^\pm)_n$ we note that this already implies the continuity of $\mathcal{T}_m$. Subtracting the equation for $\mathcal{T}(\boldsymbol{v}_{m,n}) = \overline{\boldsymbol{v}}_{m,n}$ and $\mathcal{T}(\boldsymbol{v}_m) = \overline{\boldsymbol{v}}_m$ and testing with the difference we obtain:

$$
\begin{aligned}
&\frac{\mathrm{Re}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\overline{\boldsymbol{v}}_{m,n} - \overline{\boldsymbol{v}}_m\right\|^2_{L^2(\Omega)} + \frac{3}{4}\int_\Omega |\nabla(\overline{\boldsymbol{v}}_{m,n} - \overline{\boldsymbol{v}}_m)|^2 \,\mathrm{d}\boldsymbol{x} \\
&\quad \le C\,\mathrm{Re}^4 \|\overline{\boldsymbol{v}}_m\|^4_{L^6(\Omega)} \left\|\overline{\boldsymbol{v}}_{m,n} - \overline{\boldsymbol{v}}_m\right\|^2_{L^2(\Omega)} \\
&\qquad + \left\|(c_{m,n}^+ - c_{m,n}^-)\nabla\psi_{m,n} - (c_m^+ - c_m^-)\nabla\psi_m\right\|_{L^2(\Omega)} \left\|\overline{\boldsymbol{v}}_{m,n} - \overline{\boldsymbol{v}}_m\right\|_{L^2(\Omega)},
\end{aligned} \tag{58}
$$

where we used Lemma 3.4 to estimate the trilinear term in the Navier–Stokes equation.

For $\mathrm{Re} = 0$, that is for the SNPP system, we can use Young's inequality to absorb the $\|\overline{\boldsymbol{v}}_{m,n} - \overline{\boldsymbol{v}}_m\|_{L^2(\Omega)}$–term from the last term on the right-hand side into the gradient term on the left-hand side using $\|\boldsymbol{v}_m\|_{W_0^{1,2}(\Omega)} \leq C\,\|\nabla \boldsymbol{v}_m\|_{L^2(\Omega)}$. Taking the supremum over the time interval $[0,T]$ we find for $\mathrm{Re} = 0$:

$$
\begin{aligned}
&\|\mathcal{T}(\boldsymbol{v}_{m,n}) - \mathcal{T}(\boldsymbol{v}_m)\|_{C([0,T];\mathbb{R}^{N_m})} = \|\overline{\boldsymbol{v}}_{m,n} - \overline{\boldsymbol{v}}_m\|_{C([0,T];\mathbb{R}^{N_m})} \\
&\quad \leq C\,\|(c^+_{m,n} - c^-_{m,n})\nabla\psi_{m,n} - (c^+_m - c^-_m)\nabla\psi_m\|_{C([0,T];L^2(\Omega))} \\
&\quad \leq C\left\|\left[\pm(c^\pm_{m,n} - c^\pm_m)\right]^\pm \nabla\psi_{m,n}\right\|_{C([0,T];L^2(\Omega))} + C\left\|\left[\pm c^\pm_m\right]^\pm(\nabla\psi_{m,n} - \nabla\psi_m)\right\|_{C([0,T];L^2(\Omega))} \\
&\quad \leq C\left\|\left[\pm(c^\pm_{m,n} - c^\pm_m)\right]^\pm\right\|_{C([0,T];L^2(\Omega))}\|\nabla\psi_{m,n}\|_{C([0,T];L^\infty(\Omega))} \\
&\qquad + C\left\|\left[\pm c^\pm_m\right]^\pm\right\|_{C([0,T];L^2(\Omega))}\|\nabla\psi_{m,n} - \nabla\psi_m\|_{C([0,T;W^{2,2}(\Omega))} \to 0.
\end{aligned}
\tag{59}
$$

For the NSNPP system, so for $\mathrm{Re} > 0$, we use (58) and apply Gronwall's lemma to find:

$$
\begin{aligned}
&\frac{\mathrm{Re}}{2}\|(\overline{\boldsymbol{v}}_{m,n} - \overline{\boldsymbol{v}}_m)(t)\|^2_{L^2(\Omega)} \\
&\qquad \leq \frac{1}{2\mathrm{Re}}\int_0^t e^{\int_s^t (C\mathrm{Re}^3\|\overline{\boldsymbol{v}}_m\|^4_{L^6(\Omega)}+1)\,\mathrm{d}\tau}\,\|(c^+_{m,n} - c^-_{m,n})\nabla\psi_{m,n} - (c^+_m - c^-_m)\nabla\psi_m\|^2_{L^2(\Omega)}\,\mathrm{d}s
\end{aligned}
$$

since $\overline{\boldsymbol{v}}_{m,n}$ and $\overline{\boldsymbol{v}}_m$ emanate from the same initial data. As in (59) we find that the convergence of $(c^\pm_{m,n})_n$ in $C([0,T];L^2(\Omega))$ is enough to show the convergence of $(\overline{\boldsymbol{v}}_{m,n})_n$. Here, only the $L^2(\Omega)$ convergence is shown or the convergence in the norm induced by $\|\cdot\|_{L^2(\Omega)}$ in $\mathbb{R}^{N_m}$. But as all norms are equivalent in finite dimensional vector spaces, we can deduce the convergence of $(\overline{\boldsymbol{v}}_{m,n})_n$ to $\overline{\boldsymbol{v}}_m$ also in $C([0,T];\mathbb{R}^{N_m})$.

The convergence of $(c^\pm_{m,n})_n$ also follows from a Gronwall type argument. We subtract the equations for $c^\pm_{m,n}$ and $c^\pm_m$ and test with $c^\pm_{m,n} - c^\pm_m$ to find:

$$
\begin{aligned}
&\left[\frac{\mathrm{d}}{\mathrm{d}t}\|c^\pm_{m,n} - c^\pm_m\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(c^\pm_{m,n} - c^\pm_m)|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\,\mathrm{d}\boldsymbol{x}\right]^\pm \\
&\quad = \int_\Omega \left[(c^\pm_{m,n}\boldsymbol{v}_{m,n} - c^\pm_m\boldsymbol{v}_m)\cdot\nabla(c^\pm_{m,n} - c^\pm_m) + \boldsymbol{\Lambda}(\boldsymbol{d})(c^\pm_{m,n}\nabla\psi_{m,n} - c^\pm_m\nabla\psi_m)\cdot\nabla(c^\pm_{m,n} - c^\pm_m)\right]^\pm \mathrm{d}\boldsymbol{x} \\
&\quad \leq \left[\frac{1}{2}\|\nabla(c^\pm_{m,n} - c^\pm_m)\|^2_{L^2(\Omega)} + C\,\|c^\pm_m(\boldsymbol{v}_{m,n} - \boldsymbol{v}_m)\|^2_{L^2(\Omega)}\right]^\pm \\
&\qquad + C\left[\|(c^\pm_{m,n} - c^\pm_m)\nabla\psi_{m,n} + c^\pm_m(\nabla\psi_{m,n} - \nabla\psi_m)\|^2_{L^2(\Omega)}\right]^\pm \\
&\quad \leq \frac{1}{2}\left[\|\nabla(c^\pm_{m,n} - c^\pm_m)\|^2_{L^2(\Omega)}\right]^\pm + C\left[\|c^\pm_m\|^2_{L^3(\Omega)}\right]^\pm\|\boldsymbol{v}_{m,n} - \boldsymbol{v}_m\|^2_{L^6(\Omega)} + C\left[\|c^\pm_{m,n} - c^\pm_m\|^2_{L^2(\Omega)}\right]^\pm
\end{aligned}
$$

where we used that $\boldsymbol{v}_{m,n}$ and $\boldsymbol{v}_m$ are divergence free and elliptic regularity to find $\|\psi_{m,n} - \psi_m\|_{W^{4,2}(\Omega)} \leq C\left[\|c^\pm_{m,n} - c^\pm_m\|_{L^2(\Omega)}\right]^\pm$. The convergence of $(c^\pm_{m,n})_n$ follows from absorbing the first term on the right-hand side into the left-hand side and applying Gronwall's lemma to find that

$$
\begin{aligned}
\left[\|(c^\pm_{m,n} - c^\pm_m)(t)\|^2_{L^2(\Omega)}\right]^\pm &\leq C\,e^{Ct}\int_0^t \left[\|c^\pm_m\|^2_{L^3(\Omega)}\right]^\pm\|\boldsymbol{v}_{m,n} - \boldsymbol{v}_m\|^2_m\,\mathrm{d}s \\
&\leq C\,e^{CT}\|\boldsymbol{v}_{m,n} - \boldsymbol{v}_m\|^2_{C([0,T];\mathbb{R}^{N_m})}\left[\|c^\pm_m\|^2_{L^2(0,T;W^{1,2}(\Omega))}\right]^\pm
\end{aligned}
\tag{60}
$$

and using the convergence of $(\boldsymbol{v}_{m,n})_n$ to $\boldsymbol{v}_m$ in $C([0,T];\mathbb{R}^{N_m})$. Finally, we show the uniqueness of such a fixed point, which follows by similar Gronwall argument. We assume that there are two fixed points $\boldsymbol{v}_m = \mathcal{T}_m(\boldsymbol{v}_m)$ and $\tilde{\boldsymbol{v}}_m = \mathcal{T}_m(\tilde{\boldsymbol{v}}_m)$ with corresponding solutions $(c^\pm_m, \psi_m)$ and $(\tilde{c}^\pm_m, \tilde{\psi}_m)$ to the Nernst–Planck–Poisson system.

Plugging in (60) into the estimate (58) for $\boldsymbol{v}_m$ and $\tilde{\boldsymbol{v}}_m$ we find:

$$\frac{\mathrm{Re}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m)(t)\|^2_{L^2(\Omega)}+\frac{3}{4}\int_\Omega|\nabla(\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m)|^2\,\mathrm{d}\boldsymbol{x}$$
$$\leq C\mathrm{Re}^4\|\boldsymbol{v}_m\|^4_{L^6(\Omega)}\|\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m\|^2_{L^2(\Omega)}+Ce^{Ct}\int_0^t\left[\|c_m^\pm\|^2_{L^3(\Omega)}\right]^\pm\|\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m\|^2_{L^6(\Omega)}\,\mathrm{d}s.$$

For the SNPP system, that is for $\mathrm{Re}=0$, the embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ combined with the estimate $\|\boldsymbol{v}\|_{W_0^{1,2}(\Omega)}\leq C\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}$ and an application of Gronwall's inequality yields $\boldsymbol{v}_m=\tilde{\boldsymbol{v}}_m$. For the NSNPP system with $\mathrm{Re}>0$ we integrate in time to find

$$\frac{\mathrm{Re}}{2}\|(\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m)(t)\|^2_{L^2(\Omega)}+\frac{3}{4}\int_0^t\int_\Omega|\nabla(\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m)|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$
$$\leq C\mathrm{Re}^4\int_0^t\|\boldsymbol{v}_m\|^4_{L^6(\Omega)}\|\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m\|^2_{L^2(\Omega)}\,\mathrm{d}s+C\int_0^te^{C\tau}\int_0^\tau\left[\|c_m^\pm\|^2_{L^3(\Omega)}\right]^\pm\|\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m\|^2_{L^6(\Omega)}\,\mathrm{d}s\,\mathrm{d}\tau$$
$$\leq C\mathrm{Re}^4\int_0^t\|\boldsymbol{v}_m\|^4_{L^6(\Omega)}\|\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m\|^2_{L^2(\Omega)}\,\mathrm{d}s+Cte^{Ct}\int_0^t\left[\|c_m^\pm\|^2_{L^3(\Omega)}\right]^\pm\|\boldsymbol{v}_m-\tilde{\boldsymbol{v}}_m\|^2_{L^6(\Omega)}\,\mathrm{d}s,$$

where we used $\|\boldsymbol{v}_{m,0}-\tilde{\boldsymbol{v}}_{m,0}\|^2_{L^2(\Omega)}=0$ and thus by another application of Gronwall's lemma we find that $\boldsymbol{v}_m=\tilde{\boldsymbol{v}}_m$ also for $\mathrm{Re}>0$. □

Our next order of business is to prove the existence of local strong solutions to the original (unregularized) system *cf.* Theorem2.7. This is done in the following proof. Now all the estimates need to be independent of the discretization and regularization parameters m and κ_m (in contrast to the previous proof).

Proof (of Theorem 2.7). We introduce the time dependent function $\eta(t):=\|\tilde{c}^\pm\|^2_{W^{1,2}(\Omega)}+\mathrm{Re}\|\tilde{\boldsymbol{v}}\|^2_{W^{1,2}(\Omega)}+1$ and will derive estimates to show a bound of the form $\partial_t\eta\leq C\eta^n$ for some $n\in\mathbb{N}$ (which is unconnected to the regularization from before) from which we will deduce the local-in-time existence of a solution $\tilde{\boldsymbol{u}}$ for vanishing regularization. A bootstrap argument then allows us to show that these solutions are strong and lie in $\widetilde{\mathbb{U}}$, which we will show at the end of this proof. To make this rigorous we need to use the solution to the regularized system. We omit here the regularization coefficients m and κ_m for readability and just recall that the Galerkin basis for the (Navier–)Stokes equation in 4.1 was chosen such that we are allowed to test with $\boldsymbol{v}_m-\Delta\boldsymbol{v}_m$, which we will proceed to do now. We first test the (Navier–)Stokes equation with $\tilde{\boldsymbol{v}}-\Delta\boldsymbol{v}$ to find:

$$\frac{\mathrm{Re}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\tilde{\boldsymbol{v}}|^2+|\nabla\tilde{\boldsymbol{v}}|^2\,\mathrm{d}\boldsymbol{x}+\int_\Omega|\nabla\tilde{\boldsymbol{v}}|^2+|\Delta\tilde{\boldsymbol{v}}|^2\,\mathrm{d}\boldsymbol{x}$$
$$=\int_\Omega\mathrm{Re}(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\Delta\tilde{\boldsymbol{v}}-(\tilde{c}^+-\tilde{c}^-)\nabla\tilde{\psi}\cdot(\tilde{\boldsymbol{v}}-\Delta\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}. \tag{61}$$

We can estimate the first term on the right-hand side by an application of the Gagliardo–Nirenberg and Young's inequalities for any $\delta>0$:,

$$\left|\mathrm{Re}\int_\Omega(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\Delta\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\right|\leq\mathrm{Re}\|\Delta\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}\|\nabla\tilde{\boldsymbol{v}}\|_{L^3(\Omega)}\|\tilde{\boldsymbol{v}}\|_{L^6(\Omega)}$$
$$\leq C\mathrm{Re}\|\Delta\tilde{\boldsymbol{v}}\|^{3/2}_{L^2(\Omega)}\|\nabla\tilde{\boldsymbol{v}}\|^{1/2}_{L^2(\Omega)}\|\tilde{\boldsymbol{v}}\|_{L^6(\Omega)}\leq\delta\|\Delta\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}+C\mathrm{Re}^4\|\nabla\tilde{\boldsymbol{v}}\|^6_{W^{1,2}(\Omega)}.$$

For the second term on the right-hand side of (61) we find

$$\left|\int_\Omega(\tilde{c}^+-\tilde{c}^-)\nabla\tilde{\psi}\cdot(\tilde{\boldsymbol{v}}-\Delta\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}\right|\leq\delta\left(\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}+\|\Delta\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right)+C\left[\|\tilde{c}^\pm\|^2_{L^3(\Omega)}\right]^\pm\left\|\nabla\tilde{\psi}\right\|^2_{L^6(\Omega)}$$
$$\leq\delta\left(\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}+\|\Delta\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right)+C\left[\|\tilde{c}^\pm\|^4_{W^{1,2}(\Omega)}\right]^\pm,$$

where we used Sobolev's embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ and that by elliptic regularity we have

$$\|\nabla\tilde{\psi}\|_{L^6(\Omega)}\leq\|\tilde{\psi}\|_{W^{2,6}(\Omega)}\leq C\left[\|\tilde{c}^\pm\|_{L^6}(\Omega)\right]^\pm\leq C\left[\|\tilde{c}^\pm\|_{W^{1,2}(\Omega)}\right]^\pm$$

Next, we test the Nernst–Planck equations with $\tilde{c}^{\pm} - \Delta\tilde{c}^{\pm}$ to find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{c}^{\pm}\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tilde{c}^{\pm}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla\tilde{c}^{\pm}|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})} - (\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi}))\cdot\nabla(\Delta\tilde{c}^{\pm})\,\mathrm{d}\boldsymbol{x}$$
$$= \underbrace{\mp\int_\Omega \tilde{c}^{\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla\tilde{c}^{\pm}\,\mathrm{d}\boldsymbol{x}}_{=:\,i} + \underbrace{\int_\Omega(\nabla\tilde{c}^{\pm}\cdot\tilde{\boldsymbol{v}})\Delta\tilde{c}^{\pm}\,\mathrm{d}\boldsymbol{x}}_{=:\,ii}, \tag{62}$$

here we used that $\tilde{\boldsymbol{v}}$ has vanishing divergence and no slip boundary conditions, so that

$$\int_\Omega \tilde{c}^{\pm}\tilde{\boldsymbol{v}}\cdot\nabla\tilde{c}^{\pm}\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Omega \tilde{\boldsymbol{v}}\cdot\nabla(\tilde{c}^{\pm})^2\,\mathrm{d}\boldsymbol{x} = -\frac{1}{2}\int_\Omega(\tilde{c}^{\pm})^2\,\nabla\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma(\tilde{c}^{\pm})^2\,\tilde{\boldsymbol{v}}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0$$

by an integration by parts and the no-flux boundary conditions $\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi})\cdot\boldsymbol{n} = 0$, which are indeed fulfilled on the regularized level *cf.* Proposition 4.1. For the last term on the left-hand side to give us a good term in the second derivative of $\tilde{c}^{\pm}$ we need to integrate by parts:

$$-\int_\Omega(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi}))\cdot\nabla(\Delta\tilde{c}^{\pm})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi}))\cdot\nabla\cdot(\nabla^2\tilde{c}^{\pm})\,\mathrm{d}\boldsymbol{x}$$
$$= \int_\Omega \nabla(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi})) : \nabla^2\tilde{c}^{\pm}\,\mathrm{d}\boldsymbol{x} - \int_\Gamma \nabla^2\tilde{c}^{\pm}\boldsymbol{n}\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi}))\,\mathrm{d}\sigma. \tag{63}$$

The volume term gives us a good term in the second derivative of $\tilde{c}^{\pm}$ and lower order terms, which can be estimated. Using $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$ and $\nabla(f\boldsymbol{a}) = f\nabla\boldsymbol{a} + \boldsymbol{a}\otimes\nabla f$ we find:

$$\nabla(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi})) : \nabla^2\tilde{c}^{\pm}$$
$$= |\nabla^2\tilde{c}^{\pm}|^2 + \lambda(\boldsymbol{d}\otimes(\nabla^2\tilde{c}^{\pm}\boldsymbol{d})) : \nabla^2\tilde{c}^{\pm} + \underbrace{\lambda\left((\boldsymbol{d}\cdot\nabla\tilde{c}^{\pm})\nabla\boldsymbol{d} + \boldsymbol{d}\otimes\nabla\boldsymbol{d}^T\nabla\tilde{c}^{\pm}\right) : \nabla^2\tilde{c}^{\pm}}_{=:iii}$$
$$+ \underbrace{\left(\tilde{c}^{\pm}\nabla^2\tilde{\psi} + \nabla\tilde{\psi}\otimes\nabla\tilde{c}^{\pm} + \lambda\tilde{c}^{\pm}(\nabla\tilde{\psi}\cdot\boldsymbol{d})\nabla\boldsymbol{d} + \lambda\boldsymbol{d}\otimes\nabla(\tilde{c}^{\pm}(\nabla\tilde{\psi}\cdot\boldsymbol{d}))\right) : \nabla^2\tilde{c}^{\pm}}_{+=iii}.$$

The first two terms have a good sign

$$|\nabla^2\tilde{c}^{\pm}|^2 + \lambda(\boldsymbol{d}\otimes(\nabla^2\tilde{c}^{\pm}\boldsymbol{d})) : \nabla^2\tilde{c}^{\pm} = |\nabla^2\tilde{c}^{\pm}|^2 + \lambda|\nabla^2\tilde{c}^{\pm}\boldsymbol{d}|^2$$

and the remaining terms can be estimate for $\boldsymbol{d}$ smooth enough (which is the case here since by Assumption 2.4 we have $\boldsymbol{d}\in C^4(\overline{\Omega})$):

$$\left|\int_\Omega iii\,\mathrm{d}\boldsymbol{x}\right| \le C\|\nabla^2\tilde{c}^{\pm}\|_{L^2(\Omega)}\Big(\|\nabla\tilde{c}^{\pm}\|_{L^2(\Omega)} + \|\tilde{c}^{\pm}\|_{L^4}\|\nabla^2\tilde{\psi}\|_{L^4(\Omega)} + \|\nabla\tilde{c}^{\pm}\|_{L^2(\Omega)}\|\nabla\tilde{\psi}\|_{L^\infty(\Omega)}$$
$$+ \|\tilde{c}^{\pm}\|_{L^4(\Omega)}\|\nabla\tilde{\psi}\|_{L^4(\Omega)}\Big)$$
$$= \delta\|\nabla^2\tilde{c}^{\pm}\|^2_{L^2(\Omega)} + C\left(\|\nabla\tilde{c}^{\pm}\|^4_{L^2(\Omega)} + \|\tilde{c}^{\pm}\|^4_{L^2(\Omega)} + 1\right),$$

where we used that $\|\tilde{\psi}\|_{W^{3,2}(\Omega)} \le C\|\tilde{c}^{\pm}\|_{W^{1,2}(\Omega)}$ and the boundedness of $\boldsymbol{d}$ in $W^{1,\infty}(\Omega)$. To handle the boundary term on the right-hand side of (63), we use the surface gradient and the identity: $\nabla f = \nabla_\Gamma f + (\nabla f\cdot\boldsymbol{n})\boldsymbol{n}$. Plugging this into the boundary term of (63) we obtain

$$-\int_\Gamma \nabla^2\tilde{c}^{\pm}\boldsymbol{n}\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi}))\,\mathrm{d}\sigma$$
$$= -\int_\Gamma(\nabla(\nabla\tilde{c}^{\pm}\cdot\boldsymbol{n}) - \nabla\boldsymbol{n}^T\nabla\tilde{c}^{\pm})\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi}))\,\mathrm{d}\sigma$$
$$= -\int_\Gamma(\nabla_\Gamma(\nabla\tilde{c}^{\pm}\cdot\boldsymbol{n}) + (\nabla((\nabla\tilde{c}^{\pm}\cdot\boldsymbol{n}))\cdot\boldsymbol{n})\boldsymbol{n} - \nabla\boldsymbol{n}^T\nabla\tilde{c}^{\pm})\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi}))\,\mathrm{d}\sigma$$
$$= -\int_\Gamma(\nabla_\Gamma(\nabla\tilde{c}^{\pm}\cdot\boldsymbol{n}) - \nabla\boldsymbol{n}^T\nabla\tilde{c}^{\pm})\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi}))\,\mathrm{d}\sigma$$
$$= \pm\int_\Gamma \nabla_\Gamma(\tilde{c}^{\pm}(\nabla\tilde{\psi}\cdot\boldsymbol{n}))\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi})) + (\nabla\boldsymbol{n}^T\nabla\tilde{c}^{\pm})\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm} \pm \tilde{c}^{\pm}\nabla\tilde{\psi}))\,\mathrm{d}\sigma$$

where we used the no-flux boundary conditions for the second to last equality and $\boldsymbol{d}\cdot\boldsymbol{n}=0$ on Γ, so that $\nabla c^{\pm}\cdot\boldsymbol{n}=\boldsymbol{\Lambda}(\boldsymbol{d})\nabla c^{\pm}\cdot\boldsymbol{n}=\mp c^{\pm}\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\psi\cdot\boldsymbol{n}=\mp c^{\pm}\nabla\psi\cdot\boldsymbol{n}$ for the last equality. To estimate the remaining boundary terms we note:

$$\begin{aligned}\int_\Gamma(\nabla\boldsymbol{n}^T\nabla\tilde{c}^{\pm})\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm}\pm\tilde{c}^{\pm}\nabla\tilde{\psi}))\,\mathrm{d}\sigma &\le C\left\|\nabla\tilde{c}^{\pm}\right\|^2_{L^2(\Gamma)}+C\left\|\nabla\tilde{c}^{\pm}\right\|_{L^2(\Gamma)}\left\|\tilde{c}^{\pm}\right\|_{L^4(\Gamma)}\left\|\nabla\tilde{\psi}\right\|_{L^4(\Gamma)}\\ &\le\delta\left\|\nabla^2\tilde{c}^{\pm}\right\|^2_{L^2(\Omega)}+C\left(\left\|\nabla\tilde{c}^{\pm}\right\|^4_{L^2(\Omega)}+\left\|\tilde{c}^{\pm}\right\|^4_{L^2(\Omega)}+1\right),\end{aligned}$$

where we used the embedding $W^{1,2}(\Omega)\hookrightarrow L^4(\Gamma)$ for space dimension three, the elliptic bound $\|\nabla\tilde{\psi}\|_{L^4(\Gamma)}\le C\|\tilde{\psi}\|_{W^{2,2}(\Omega)}\le C\left\|\tilde{c}^{\pm}\right\|_{L^2(\Omega)}$ and the trace estimate from [DiB10, Prop. 8.2]. For the other boundary integral we need to work a bit more:

$$\begin{aligned}&\int_\Gamma\nabla_\Gamma(\tilde{c}^{\pm}(\nabla\tilde{\psi}\cdot\boldsymbol{n}))\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm}\pm\tilde{c}^{\pm}\nabla\tilde{\psi}))\,\mathrm{d}\sigma\\ &\quad=\int_\Gamma\left((\nabla\tilde{\psi}\cdot\boldsymbol{n})\nabla_\Gamma\tilde{c}^{\pm}+\tilde{c}^{\pm}\nabla_\Gamma(\nabla\tilde{\psi}\cdot\boldsymbol{n})\right)\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})(\nabla\tilde{c}^{\pm}\pm\tilde{c}^{\pm}\nabla\tilde{\psi}))\,\mathrm{d}\sigma\\ &\quad\le C\left(\|\nabla\tilde{c}^{\pm}\|_{L^2(\Gamma)}\|\nabla\tilde{\psi}\|_{L^\infty(\Gamma)}+\|\tilde{c}^{\pm}\|_{L^4(\Gamma)}\|\nabla^2\tilde{\psi}\|_{L^4(\Gamma)}\right)\left(\|\nabla\tilde{c}^{\pm}\|_{L^2(\Gamma)}+\|\nabla\tilde{\psi}\|_{L^2(\Gamma)}\right)\\ &\quad\le C\left\|\nabla\tilde{c}^{\pm}\right\|^2_{L^2(\Gamma)}\|\nabla\tilde{\psi}\|_{L^\infty(\Gamma)}+\delta\left\|\nabla\tilde{c}^{\pm}\right\|^2_{L^2(\Gamma)}+C\left\|\tilde{c}^{\pm}\right\|^2_{L^4(\Gamma)}\|\nabla^2\tilde{\psi}\|^2_{L^4(\Gamma)}\\ &\qquad+C\|\nabla\tilde{\psi}\|^2_{L^2(\Gamma)}\|\nabla\tilde{\psi}\|^2_{L^\infty(\Gamma)}+C\left\|\tilde{c}^{\pm}\right\|_{L^4(\Gamma)}\|\nabla^2\tilde{\psi}\|_{L^4(\Gamma)}\|\nabla\tilde{\psi}\|_{L^2(\Gamma)}\\ &\quad\le\delta\left\|\nabla^2\tilde{c}^{\pm}\right\|^2_{L^2(\Omega)}+C\left(\left\|\tilde{c}^{\pm}\right\|^4_{W^{1,2}(\Omega)}+1\right)\end{aligned}$$

where we used $\|\nabla_\Gamma f\|_{L^2(\Gamma)}\le C\|\nabla f\|_{L^2(\Gamma)}$, $\|\nabla\tilde{\psi}\|_{L^\infty(\Gamma)}\le C\|\nabla\tilde{\psi}\|_{W^{2,2}(\Omega)}\le C\|\tilde{\psi}\|_{W^{3,2}(\Omega)}\le C\left\|\tilde{c}^{\pm}\right\|_{W^{1,2}(\Omega)}$ and $\|\nabla^2\tilde{\psi}\|_{L^4(\Gamma)}\le C\|\nabla^2\tilde{\psi}\|_{W^{1,2}(\Omega)}\le C\|\tilde{\psi}\|_{W^{3,2}(\Omega)}\le C\left\|\tilde{c}^{\pm}\right\|_{W^{1,2}(\Omega)}$. Now, we can use Hölder's and Young's inequalities and elliptic regularity to estimate the right-hand side of (62). For the first term i we note:

$$\begin{aligned}i&\le C\left\|\tilde{c}^{\pm}\right\|_{L^3(\Omega)}\left\|\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\right\|_{L^6(\Omega)}\left\|\nabla\tilde{c}^{\pm}\right\|_{L^2(\Omega)}\\ &\le C\left\|\tilde{c}^{\pm}\right\|^{3/2}_{L^2(\Omega)}\left\|\nabla\tilde{c}^{\pm}\right\|^{3/2}_{L^2(\Omega)}\le C\left\|\tilde{c}^{\pm}\right\|^3_{L^2(\Omega)}+C\left\|\nabla\tilde{c}^{\pm}\right\|^3_{L^2(\Omega)}.\end{aligned}$$

For the second term ii we observe:

$$\begin{aligned}ii&\le C\left\|\nabla\tilde{c}^{\pm}\right\|_{L^3(\Omega)}\|\tilde{\boldsymbol{v}}\|_{L^6(\Omega)}\left\|\Delta\tilde{c}^{\pm}\right\|_{L^2(\Omega)}\le C\left\|\nabla\tilde{c}^{\pm}\right\|^{1/2}_{L^2(\Omega)}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}\left\|\nabla^2\tilde{c}^{\pm}\right\|^{3/2}_{L^2(\Omega)}\\ &\le\delta\left\|\nabla^2\tilde{c}^{\pm}\right\|^2_{L^2(\Omega)}+C\left\|\nabla\tilde{c}^{\pm}\right\|^2_{L^2(\Omega)}\|\nabla\tilde{\boldsymbol{v}}\|^4_{L^2(\Omega)}\le\delta\left\|\nabla^2\tilde{c}^{\pm}\right\|^2_{L^2(\Omega)}+C\left\|\nabla\tilde{c}^{\pm}\right\|^6_{L^2(\Omega)}+C\|\nabla\tilde{\boldsymbol{v}}\|^6_{L^2(\Omega)}.\end{aligned}$$

The last term on the right-hand side can be estimated using (61). Recalling the definition of $\eta:=\left\|c^{\pm}\right\|^2_{W^{1,2}(\Omega)}+\mathrm{Re}\,\|\tilde{\boldsymbol{v}}\|^2_{W^{1,2}(\Omega)}+1$, we can estimate $C\|\nabla\tilde{\boldsymbol{v}}\|^6_{L^2(\Omega)}$ dependent on the Reynolds number Re. For $\mathrm{Re}>0$ we have $C\|\nabla\tilde{\boldsymbol{v}}\|^6_{L^2(\Omega)}\le\frac{C}{\mathrm{Re}^3}\eta^3$ and for $\mathrm{Re}=0$ we have $C\|\nabla\tilde{\boldsymbol{v}}\|^6_{L^2(\Omega)}\le C\left[\left\|c^{\pm}\right\|^{12}_{W^{1,2}(\Omega)}\right]^{\pm}\le C\eta^6$. Putting all of the above together and we obtain:

$$\partial_t\eta+\frac{1}{2}\int_\Omega|\nabla\tilde{\boldsymbol{v}}|^2+|\Delta\tilde{\boldsymbol{v}}|^2+|\nabla^2c^{\pm}|^2+\lambda|\nabla^2c^{\pm}\boldsymbol{d}|^2\,\mathrm{d}\boldsymbol{x}\le C_{\mathrm{Re}}\eta^6$$

with $C_{\mathrm{Re}}=C$ independent of Re for $\mathrm{Re}=0$ and $C_{\mathrm{Re}}=C(\mathrm{Re}^{-1}+\mathrm{Re}+1)$ for $\mathrm{Re}>0$. By an ODE comparison principle we can deduce that there exists a $T^*>0$ such that we have a solution $\tilde{\boldsymbol{v}}\in L^2(0,T;W^{2,2}(\Omega))$, $\tilde{c}^{\pm}\in L^\infty(0,T;W^{1,2}(\Omega))\cap L^2(0,T;W^{2,2}(\Omega))$ and thus $\tilde{\psi}\in L^\infty(0,T;W^{3,2}(\Omega))$. All of the above estimates are independent of the regularization parameters κ_m and m. So we directly get weakly convergent subsequences for vanishing regularization in the appropriate spaces. To pass to the limit in the (nonlinear) weak formulation we additionally need to have some strong convergence. Using the Nernst–Planck equations we find an additional estimate for $\partial_t\tilde{c}^{\pm}$ in $L^2(0,T;L^2(\Omega))$ and thus using the compact Aubin–Lions embedding:

$$L^2(0,T;W^{2,2}(\Omega))\cap W^{1,2}(0,T;L^2(\Omega))\hookrightarrow\hookrightarrow L^2(0,T;W^{1,2}(\Omega))$$

see for example [Rou13, Lem. 7.7], we find $\tilde{c}_m^\pm \to \tilde{c}^\pm$ in $L^2(0,T;W^{1,2}(\Omega))$ and thus by elliptic regularity $\tilde{\psi}_m \to \tilde{\psi}$ in $L^2(0,T;W^{3,2}(\Omega))$, which is enough to pass to the limit in the weak formulation of the fully coupled SNPP system. For the limit passage for vanishing regularization for the fully coupled NSNPP system we additionally need some strong convergence of the regularized velocity field. For that we use thar for $\mathrm{Re} > 0$ we have that $\tilde{\boldsymbol{v}}_m$ is bounded in $L^\infty(0,T;W^{1,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega))$ independently of the regularization by the definition of η. Thus we have that $\boldsymbol{f}_m := \Delta\tilde{\boldsymbol{v}}_m - \mathrm{Re}(\tilde{\boldsymbol{v}}_m\cdot\nabla)\tilde{\boldsymbol{v}}_m - (\tilde{c}_m^+ - \tilde{c}_m^-)\nabla\tilde{\psi}_m$ is bounded in $L^2(0,T;L^2(\Omega))$ independently of m and by (56a) we find

$$\mathrm{Re}\,\partial_t\tilde{\boldsymbol{v}}_m = P_m\left(\Delta\tilde{\boldsymbol{v}}_m - \mathrm{Re}(\tilde{\boldsymbol{v}}_m\cdot\nabla)\tilde{\boldsymbol{v}}_m - (\tilde{c}_m^+ - \tilde{c}_m^-)\nabla\tilde{\psi}_m\right),$$

where $P_m : L^2(\Omega) \to V_m \hookrightarrow L^2(\Omega)$ denotes the L^2-orthogonal projection onto the discrete Galerkin space. This implies that also $\partial_t\tilde{\boldsymbol{v}}_m$ is bounded in $L^2(0,T;L^2(\Omega))$ independently of m by the stability of the projection P_m. Using the compact embedding $L^2(0,T;W^{2,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)) \hookrightarrow\hookrightarrow L^2(0,T;W^{1,2}(\Omega))$ we get the strong convergence of $\tilde{\boldsymbol{v}}_m$ in $L^2(0,T;W^{1,2}(\Omega))$ and we can thus pass to the limit in the non-linear term of the Navier–Stokes equations.

We now turn our attention to additional regularities of this solution to guarantee that $\tilde{\boldsymbol{u}} = (\tilde{\boldsymbol{v}}, \tilde{c}^\pm, \tilde{\psi}) \in \widetilde{\mathbb{U}}$, which follow by a standard bootstrap argument using maximal L^p–regularity.

Regularity of $\tilde{\boldsymbol{v}}$ for the NSNPP system: To obtain the $L^\infty((0,T)\times\Omega)$–bound for the velocity $\tilde{\boldsymbol{v}}$ in the NSNPP system, we use the maximal L^p–regularity of the Stokes operator. We first note that:

$$\mathrm{Re}\;\partial_t\tilde{\boldsymbol{v}} - \Delta\tilde{\boldsymbol{v}} + \nabla\tilde{p} = -\mathrm{Re}\;(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}} - (\tilde{c}^+ - \tilde{c}^-)\nabla\tilde{\psi} \in L^{5/2}(0,T;L^{5/2}(\Omega)), \tag{64}$$

since $\tilde{\boldsymbol{v}} \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)) \hookrightarrow L^{10}(0,T;L^{10}(\Omega)) \cap L^{10/3}(0,T;W^{1,10/3}(\Omega))$. Then, using the maximal L^p–regularity of the Stokes operator and the embedding

$$W^{1,5/2}(0,T;L^{5/2}(\Omega)) \cap L^{5/2}(0,T;W^{2,5/2}(\Omega)) \hookrightarrow L^p(0,T;L^p(\Omega)) \cap L^5(0,T;W^{1,5}(\Omega))$$

for all $p \in [1,\infty)$, we find that the right-hand side of (64) is in $L^4(0,T;L^4(\Omega))$ and thus by the embedding $W^{1,4}(0,T;L^4(\Omega)) \cap L^4(0,T;W^{2,4}(\Omega)) \hookrightarrow C([0,T^*]\times\overline{\Omega})$ we find the required L^∞–bound for $\tilde{\boldsymbol{v}}$. Thus we find $\tilde{\boldsymbol{v}}$ has enough regularity to be in $\widetilde{\mathbb{U}}$.

Regularity for $\tilde{\boldsymbol{v}}$ for the SNPP system: We first note that by using $\tilde{c}^\pm \in L^\infty(0,T;W^{1,2}(\Omega))$, $\tilde{\psi} \in L^\infty(0,T;W^{3,2}(\Omega))$, and the regularity results of the Stokes operator

$$\Delta\tilde{\boldsymbol{v}} + \nabla\tilde{p} = -(\tilde{c}^+ - \tilde{c}^-)\nabla\tilde{\psi} \in L^\infty(0,T;L^6(\Omega))$$

we now have $\tilde{\boldsymbol{v}} \in L^\infty(0,T;W^{2,6}(\Omega)) \hookrightarrow L^\infty((0,T)\times\Omega))$ [BF13, Thm. IV.6.5 and Thm. IV.6.6]. Using maximal L^p–regularity for the diffusive part of the Nernst–Planck equations, that is:

$$\partial_t\tilde{c}^\pm - \nabla\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{c}^\pm) = \tilde{c}^\pm\nabla\cdot\tilde{\boldsymbol{v}} + \nabla\tilde{c}^\pm\cdot\tilde{\boldsymbol{v}} \pm \nabla\tilde{c}^\pm\cdot\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi} \pm \tilde{c}^\pm\nabla\cdot(\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}) \tag{65}$$

where the first term on the right-hand side vanishes, since $\tilde{\boldsymbol{v}}$ is divergence free. Since the right-hand side of (65) is in $L^2(0,T;L^2(\Omega))$, we first find $\tilde{c}^\pm \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)) \hookrightarrow L^{10}(0,T;L^{10}(\Omega)) \cap L^{10/3}(0,T;W^{1,10/3}(\Omega))$. The differentiability in time then extends to the solution $\tilde{\boldsymbol{v}}$ of the Stokes equations by its linearity we have $\partial_t\tilde{\boldsymbol{v}} = \partial_t((\tilde{c}^+ - \tilde{c}^-)\nabla\tilde{\psi})$. Thus we already find that $\tilde{\boldsymbol{v}}$ fulfills the regularity of $\widetilde{\mathbb{U}}$.

Regularity for $\tilde{c}^\pm$ and $\tilde{\psi}$: With the L^∞–regularity for $\tilde{\boldsymbol{v}}$ and the maximal L^p–regularity for $\tilde{c}^\pm$ from (65) we find $\tilde{\psi}$ in $L^{10}(0,T;W^{2,10}(\Omega))$ and consequently we find that the right-hand side of (65) is now even in $L^{10/3}(0,T;L^{10/3}(\Omega))$. Again using maximal L^p–regularity we find:

$$\tilde{c}^\pm \in W^{1,10/3}(0,T;L^{10/3}(\Omega)) \cap L^{10/3}(0,T;W^{2,10/3}(\Omega)) \hookrightarrow L^{10}(0,T;W^{1,10}(\Omega)) \cap C([0,T^*]\times\overline{\Omega})$$

Now, the right hand-side of (65) is in $L^{10}(0,T;L^{10}(\Omega))$ and maximal L^p–regularity gives

$$\tilde{c}^{\pm} \in W^{1,10}(0,T;L^{10}(\Omega)) \cap L^{10}(0,T;W^{2,10}(\Omega)) \hookrightarrow C^{\alpha/2,1+\alpha}([0,T^*]\times\overline{\Omega})$$

for all $\alpha \in [0,1/2)$, which is enough to find that also $\tilde{c}^{\pm}$ and $\tilde{\psi}$ fulfill the regularity of $\widetilde{\mathbb{U}}$.

For these solutions uniqueness follows by the standard procedure of subtracting the equations for two solutions and then testing with difference, where one uses that both $\tilde{\boldsymbol{v}}$ and $\tilde{c}^{\pm}$ are in $L^{\infty}((0,T)\times\Omega)$. Gronwall's lemma then gives us the uniqueness similarly to the uniqueness proof on the regularized level, *cf.* Proof of Proposition 4.1. □

We finish this section by proving the comparison principle for the strong solution to the charge equation (1c).

Proof (of Proposition 2.9). We subtract the equation for the sub-solution $\underline{c}^{\pm}$ from the equations for the super-solution $\overline{c}^{\pm}$ and test with $(\overline{c}^{\pm}-\underline{c}^{\pm})^+$ to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left\|(\overline{c}^{\pm}-\underline{c}^{\pm})^+\right\|_{L^2(\Omega)}^2 &+ \int_\Omega |\nabla(\overline{c}^{\pm}-\underline{c}^{\pm})^+|^2_{\boldsymbol{\Lambda}(\boldsymbol{d})}\,\mathrm{d}\boldsymbol{x}\\
&\le \int_\Omega (\overline{c}^{\pm}-\underline{c}^{\pm})^+\tilde{\boldsymbol{v}}\cdot\nabla(\overline{c}^{\pm}-\underline{c}^{\pm})^+ \mp (\overline{c}^{\pm}-\underline{c}^{\pm})^+\boldsymbol{\Lambda}(\boldsymbol{d})\nabla\tilde{\psi}\cdot\nabla(\overline{c}^{\pm}-\underline{c}^{\pm})^+\,\mathrm{d}\boldsymbol{x}\\
&\le \frac{1}{2}\left\|\nabla(\overline{c}^{\pm}-\underline{c}^{\pm})^+\right\|_{L^2(\Omega)}^2 + C\left\|\boldsymbol{\Lambda}(\boldsymbol{d})\right\|_{L^\infty(\Omega)}^2\left\|\nabla\tilde{\psi}\right\|_{L^\infty(\Omega)}^2\left\|(\overline{c}^{\pm}-\underline{c}^{\pm})^+\right\|_{L^2(\Omega)}^2,
\end{aligned}$$

where we used that the convection term due to the velocity field $\tilde{\boldsymbol{v}}$ vanishes, since $\tilde{\boldsymbol{v}}$ is divergence free. Due to the regularity of $\tilde{\psi}$ in $\widetilde{\mathbb{U}}$ we have $\nabla\tilde{\psi} \in L^{10}(0,T;W^{1,10}(\Omega)) \hookrightarrow L^{10}(0,T;L^{\infty}(\Omega))$ and thus we can apply Gronwall's lemma to find

$$\frac{1}{2}\left\|(\overline{c}^{\pm}-\underline{c}^{\pm})^+\right\|_{L^2(\Omega)}^2(t) \le \frac{1}{2}\left\|(\overline{c}_0^{\pm}-\underline{c}_0^{\pm})^+\right\|_{L^2(\Omega)}^2 \exp\left(\int_0^t C\left\|\boldsymbol{\Lambda}(\boldsymbol{d})\right\|_{L^\infty(\Omega)}^2\left\|\nabla\tilde{\psi}\right\|_{L^\infty(\Omega)}^2\,\mathrm{d}s\right) = 0$$

which directly implies $\underline{c}^{\pm} \le \overline{c}^{\pm}$ almost everywhere and then with the continuity of this inequality even holds everywhere. □

5 Proof of the error estimate

We will use the relative energy inequality and test it with the solution $\tilde{\boldsymbol{u}}$ of the SNPP system. We note that the regularity weight $\mathcal{K}$ is bounded independently of the Reynolds number for bounded sequences of Reynolds numbers.

Proof (of Theorem 2.11). By the regularity of the strong solution $\tilde{\boldsymbol{u}} \in \widetilde{\mathbb{U}}$ and the positivity of the charges, *cf.* Corollary 2.10, we can use it as a test function in the relative energy inequality. Since the Nernst–Planck–Poisson part in the SNPP and the NSNPP system are identical, only the term with $\mathcal{A}_{\tilde{\boldsymbol{v}}}$ on the right-hand side of the relative energy inequality (55) (before the application of Gronwall's inequality) remains, that is

$$\mathcal{R}(\boldsymbol{u}_{\mathrm{Re}}|\tilde{\boldsymbol{u}})(s)\Big|_{s=0}^{s=t} + \int_0^t \frac{1}{2}\mathcal{W}(\boldsymbol{u}_{\mathrm{Re}}|\tilde{\boldsymbol{u}})\,\mathrm{d}s \le \int_0^t \left(\int_\Omega \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}})\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}_{\mathrm{Re}})\,\mathrm{d}\boldsymbol{x}\right) + \mathcal{K}(\tilde{\boldsymbol{u}})\mathcal{R}(\boldsymbol{u}_{\mathrm{Re}}|\tilde{\boldsymbol{u}})(s)\,\mathrm{d}s.$$

To estimate the right-hand side we note

$$\begin{aligned}
\int_\Omega \mathcal{A}_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{u}})\cdot(\tilde{\boldsymbol{v}}-\boldsymbol{v}_{\mathrm{Re}})\,\mathrm{d}\boldsymbol{x} &= \mathrm{Re}\int_\Omega \left(\partial_t\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\right)(\tilde{\boldsymbol{v}}-\boldsymbol{v}_{\mathrm{Re}})\,\mathrm{d}\boldsymbol{x}\\
&\le \frac{\mathrm{Re}}{2}\left\|\partial_t\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)}^2 + \frac{\mathrm{Re}}{2}\left\|\boldsymbol{v}_{\mathrm{Re}}-\tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

By the regularity of the Stokes solution *cf.* the definition of $\widetilde{\mathbb{U}}$ we know that $\|\partial_t \tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} \in L^1(0,T)$. Thus, we obtain:

$$\mathcal{R}(\boldsymbol{u}_{\mathrm{Re}}|\tilde{\boldsymbol{u}})(t) + \frac{1}{2}\int_0^t \mathcal{W}(\boldsymbol{u}_{\mathrm{Re}}|\tilde{\boldsymbol{u}})\,\mathrm{d}s \le \int_0^t \mathrm{Re}\,\|\partial_t \tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \mathcal{K}(\tilde{\boldsymbol{u}})\mathcal{R}(\boldsymbol{u}_{\mathrm{Re}}|\tilde{\boldsymbol{u}})\,\mathrm{d}s$$

and an application of Gronwall's lemma yields

$$\begin{aligned}\mathcal{R}(\boldsymbol{u}_{\mathrm{Re}}|\tilde{\boldsymbol{u}})(t) &+ \frac{1}{2}\int_0^t \mathcal{W}(\boldsymbol{u}_{\mathrm{Re}}|\tilde{\boldsymbol{u}})e^{\int_s^t \mathcal{K}(\tilde{\boldsymbol{u}})\,\mathrm{d}\tau}\,\mathrm{d}s \\ &\le C\,\mathrm{Re}\int_0^t \|\partial_t \tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\, e^{\int_s^t \mathcal{K}(\tilde{\boldsymbol{u}})\,\mathrm{d}\tau}\,\mathrm{d}s \le C(\tilde{\boldsymbol{u}})\,\mathrm{Re} \searrow 0 \quad \text{for } \mathrm{Re} \searrow 0. \end{aligned} \tag{66}$$

The convergence rate (6) follows by taking the square root in (66) almost directly from the definition of the relative energy $\mathcal{R}$ in (7) and the relative dissipation potential $\mathcal{W}$ in (8), where the convergence of $\sqrt{c^{\pm}_{\mathrm{Re}}}$ in $L^\infty(0,T;L^2(\Omega))$ follows from

$$(\sqrt{x} - \sqrt{y})^2 \le x - y - y(\ln x - \ln y) \quad \text{for } x, y \in \mathbb{R}, x > 0, y \ge 0,$$

see [Las21, Lem. 2.8]. □

Proof (of Corollary 2.13). The error estimate is a direct consequence of the relative energy inequality from Proposition 3.1. □

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