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Approximation of time-periodic flow past a translating body by flows in bounded domains

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Abstract

We consider a time-periodic incompressible three-dimensional Navier-Stokes flow past a translating rigid body. In the first part of the paper, we establish the existence and uniqueness of strong solutions in the exterior domain that satisfy pointwise estimates for both the velocity and pressure. The fundamental solution of the time-periodic Oseen equations plays a central role in obtaining these estimates. The second part focuses on approximating this exterior flow within truncated domains, incorporating appropriate artificial boundary conditions. For these bounded domain problems, we prove the existence and uniqueness of weak solutions. Finally, we estimate the error in the velocity component as a function of the truncation radius, showing that, as the latter passes to infinity, the velocities of the truncated problems converge, in an appropriate norm, to the velocity of the exterior flow.

Contents

1 Introduction

Consider an incompressible viscous flow around a rigid body translating with a constant velocity $\zeta \in \mathbb{R}^3 \setminus \{0\}$. For simplicity and without loss of generality, we take the kinematic viscosity of the fluid to be equal to 1. To describe the motion of the fluid, we use a reference frame attached to the solid. Additionally, we assume the fluid to be subject to an external body force and a distribution of velocities

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along the fluid-solid boundary, both time-periodic of period $\mathcal{T} > 0$. Under these conditions, the motion of the fluid is governed by the following equations

$$\begin{cases} \partial_t u - \Delta u - \zeta \cdot \nabla u + u \cdot \nabla u + \nabla p = f & \text{in } \mathbb{T} \times \Omega, \\ \nabla \cdot u = 0 & \text{in } \mathbb{T} \times \Omega, \\ u = h & \text{on } \mathbb{T} \times \Sigma, \\ \lim_{|x| \to \infty} u(t,x) = 0 & \text{for } t \in \mathbb{T}. \end{cases} \tag{1.1}$$

Here and throughout the paper, $\Omega \subset \mathbb{R}^3$ denotes the exterior domain occupied by the liquid, while $\Sigma := \partial\Omega$ represents the common boundary between Ω and the compact set corresponding to the rigid body. We assume that $0 \in \mathbb{R}^3 \setminus \overline{\Omega}$. Since we are interested in time-periodic flows, the torus group $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ serves as the time axis in system (1.1), so that all functions therein are time-periodic with period $\mathcal{T} > 0$. The functions $u\colon \mathbb{T} \times \Omega \to \mathbb{R}^3$ and $p\colon \mathbb{T} \times \Omega \to \mathbb{R}$ represent the unknown velocity field and scalar pressure, respectively.

In the context of applications, a crucial question is how to numerically solve the exterior problem (1.1). Truncating the fluid domain in order to discretize the equations using, for instance, finite elements necessarily introduces artificial boundaries, which must be chosen so as to ensure the well-posedness of the mathematical model and the numerical stability of the simulations. Prescribing the so-called "do-nothing" condition [13,18] on the artificial boundaries arises naturally in the variational formulation after multiplication of the term $-\Delta u + \nabla p$ with a test function and integration by parts. However, as shown in [1,15], this Neumann condition does not guarantee the well-posedness of the resulting boundary value problem for the Navier-Stokes equations. In [2], the question of how to numerically solve the Dirichlet problem for the Stokes system in the exterior of a three-dimensional bounded Lipschitz domain is addressed using a modified "do-nothing" condition on the outer boundary of a truncated domain. A similar idea was subsequently exploited for more complex fluid models in [3,4], and we adopt it in this work, as described below.

Formulation of the problem. Our aim is to investigate how to approximate solutions (u, p) to system (1.1), formulated in the unbounded domain Ω, by solutions (u_R, p_R) to problems posed within bounded domains $\Omega_R = \{x \in \Omega : |x| < R\}$ for $R > 0$ sufficiently large. More precisely, we consider solutions $(v, \wp) = (u_R, p_R)$ to the truncated problems

$$\begin{cases} \partial_t v - \Delta v - \zeta \cdot \nabla v + v \cdot \nabla v + \nabla \wp = f & \text{in } \mathbb{T} \times \Omega_R, \\ \nabla \cdot v = 0 & \text{in } \mathbb{T} \times \Omega_R, \\ v = h & \text{on } \mathbb{T} \times \Sigma, \\ \mathcal{B}_{\mathrm{R}}(v, \wp) = 0 & \text{on } \mathbb{T} \times \partial B_R, \end{cases} \tag{1.2}$$

where $\mathcal{B}_{\mathrm{R}}$ is a suitable boundary operator. The artificial boundary condition $\mathcal{B}_{\mathrm{R}}(v, \wp) = 0$ on $\mathbb{T} \times \partial B_R$ must be selected to ensure both the well-posedness of the resulting mixed boundary value problem and the convergence of u_R to u as $R \to \infty$ in an appropriate norm. Our choice

$$\mathcal{B}_{\mathrm{R}}(v, \wp)(t,x) = \frac{x}{R} \cdot \left(\nabla v(t,x) - \wp(t,x)\mathsf{I} - \frac{1}{2} v(t,x) \otimes v(t,x) \right) + \frac{1 + \mathfrak{s}_\zeta(x)}{R} v(t,x) \tag{1.3}$$

where $\mathfrak{s}_\zeta(x) := [|\zeta||x| + (\zeta \cdot x)]/2$, is inspired by [3]. The present work is a generalization to the time-periodic case of the results obtained in [3] for the steady problem (see also [4] for a linearized steady flow around a rotating and translating body). Note that the operator $\mathcal{B}_{\mathrm{R}}$ defined in (1.3) contains the pseudo-stress tensor $\widetilde{\mathsf{T}}(v, \wp) = \nabla v - \wp\mathsf{I}$. However, all results in this paper remain valid if $\widetilde{\mathsf{T}}$ is replaced by the classical Cauchy stress tensor $\mathsf{T}(v,\mathscr{p}) = \nabla v + \nabla v^\top - \mathscr{p}\mathsf{I}$. Here, the gradient of a vector-valued function of several variables is the transpose of the Jacobian matrix: $(\nabla v)_{ij} = \frac{\partial v_j}{\partial x_i}$, $i,j = 1,2,3$.

To present the main results of the paper, we introduce additional notation and recall basic properties of the relevant function spaces and operators.

Notations. Throughout the paper, we will consistently use the same font style to represent scalar, vector, and tensor-valued functions. Standard notations $L^p(\mathcal{O})$, $W^{k,p}(\mathcal{O})$ and $H^k(\mathcal{O})$ for suitable sets $\mathcal{O}$ will be adopted for Lebesgue and Sobolev spaces, and we occasionally write $\|\cdot\|_{p;D} := \|\cdot\|_{L^p(\mathcal{O})}$ and $\|\cdot\|_{k,p;D} := \|\cdot\|_{W^{k,p}(\mathcal{O})}$ for corresponding norms. We further introduce homogeneous Sobolev spaces by denoting $u \in D^{k,p}(\mathcal{O})$ if and only u is locally integrable with $\nabla^k u \in L^p(\mathcal{O})$. We further introduce the homogeneous By $\mathcal{D}(\mathbb{T})$ we denote the class of real-valued, infinitely differentiable, $\mathcal{T}$-periodic functions.

By $\mathsf{I} \in \mathbb{R}^{3\times 3}$ we denote the three-dimensional identity matrix. We denote the Dirac delta distributions on $\mathbb{R}^3$, $\mathbb{T}$, and $\mathbb{Z}$ by $\delta_{\mathbb{R}^3}$, $\delta_{\mathbb{T}}$ and $\delta_{\mathbb{Z}}$, respectively. Here $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$, where the period $\mathcal{T} > 0$ is fixed throughout the paper. The whole-space problem associated with (1.1) will be formulated in the locally compact abelian group $G := \mathbb{T}\times\mathbb{R}^3$, and the Dirac delta distribution on G, δ_G, will be used to define the fundamental solution of the time-periodic problem. In the context of the exterior problem Ω, the symbol δ_Σ will denote the Dirac delta distribution with support $\Sigma = \partial\Omega \subset \mathbb{R}^3$. By $\mathcal{S}'(\mathbb{R}^3)$ and $\mathcal{S}'(G)$ we will denote the spaces of tempered distributions over $\mathbb{R}^3$ and G, respectively.

If X is a Banach space, we denote by $L^r(\mathbb{T};X)$ the space of all Bochner measurable functions $u :$ $\mathbb{T} \to X$ such that $\|u\|_{L^r(\mathbb{T};X)} := \left(\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \|u(t)\|_X^r \, dt\right)^{\frac{1}{r}} < \infty$, for $1 \le r < \infty$, and $\|u\|_{L^\infty(\mathbb{T};X)} :=$ $\operatorname{ess\,sup}_{t\in[0,\mathcal{T}]} \|u(t)\|_X < \infty$, for $r = \infty$. We denote by $C(\mathbb{T};X)$ the space of continuous functions $f : \mathbb{T} \to X$, which corresponds to the continuous functions $f : [0,\mathcal{T}] \to X$ that satisfy $f(0) = f(\mathcal{T})$.

We will utilize a precise decomposition of the solution into a steady-state component and a purely periodic component, as proposed and employed in [5–7, 12, 14]. Specifically, time-periodic functions $v\colon \mathbb{T} \to X$ are split into a steady-state part $v_0 = \mathscr{P}v$ and a purely periodic part $v_\perp = \mathscr{P}_\perp v$, where the projections $\mathscr{P}$ and $\mathscr{P}_\perp$ are defined by

$$\mathscr{P}v := \int_{\mathbb{T}} v(t)\,\mathrm{d}t = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} v(t)\,\mathrm{d}t, \qquad \mathscr{P}_\perp v := v - \mathscr{P}v. \tag{1.4}$$

To specify the class of admissible boundary traces of strong solutions to (1.1) we define

$$\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma) := \left\{ v|_{\mathbb{T}\times\Sigma} : v \in L^p(\mathbb{T}; W^{2,q}(\Omega)^3),\ \partial_t v \in L^p(\mathbb{T}; L^q(\Omega)^3) \right\}$$

for $p, q \in (1,\infty)$, and we equip this space with the norm

$$\|h\|_{\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)} := \inf\left\{ \|v\|_{L^p(\mathbb{T};W^{2,q}(\Omega))} + \|\partial_t v\|_{L^p(\mathbb{T};L^q(\Omega))} : h = v|_{\mathbb{T}\times\Sigma} \right\}.$$

This function space can be decomposed into spaces of steady-state and of purely periodic functions, given by

$$\mathfrak{T}^q(\Sigma) := \left\{ \mathscr{P}h : h \in \mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma) \right\}, \qquad \mathfrak{T}_\perp^{p,q}(\mathbb{T}\times\Sigma) := \left\{ \mathscr{P}_\perp h : h \in \mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma) \right\}.$$

Then $\mathfrak{T}^q(\Sigma)$ coincides with the Sobolev–Slobodeckij space $W^{2-1/q,q}(\Sigma)$. Similarly, one can identify $\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)$ and $\mathfrak{T}_\perp^{p,q}(\mathbb{T}\times\Sigma)$ with interpolation spaces, which are of Triebel–Lizorkin and Besov type. Since these involved constructions are not necessary for our approach, we omit them here.

When studying the exterior problem, to quantify the decay of functions in a suitable way, we introduce the weight function

$$\nu_\beta^\alpha(x;\zeta) := |x|^\alpha(1+\mathfrak{s}_\zeta(x))^\beta, \qquad \mathfrak{s}_\zeta(x) := \frac{1}{2}\big[|\zeta|\,|x| + \zeta\cdot x\big]$$

for $\alpha,\beta\in\mathbb{R}$, and the corresponding weighted norms

$$\|v\|_{\infty,\nu_\beta^\alpha(\cdot;\zeta);D} := \operatorname*{ess\,sup}_{x\in D} \nu_\beta^\alpha(x;\zeta)|v(x)|,$$
$$\|v\|_{\infty,\nu_\beta^\alpha(\cdot;\zeta);\mathbb{T}\times D} := \operatorname*{ess\,sup}_{(t,x)\in\mathbb{T}\times D} \nu_\beta^\alpha(x;\zeta)|v(t,x)|,$$

where $D\subset\mathbb{R}^3$ is an open set. When $\beta=0$, we simply write $\nu^\alpha := \nu_0^\alpha(\cdot,\zeta)$, so that $\nu^\alpha(x) := |x|^\alpha$.

Main results. The paper's first main result, Theorem 3.1, establishes the existence and uniqueness of strong solutions to problem (1.1), assuming that $f\in L^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)^3$ and $h\in\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)$ with

$$\|\mathscr{P}f\|_{\infty,\nu_1^{5/2}(\cdot;\zeta);\Omega} + \|\mathscr{P}_\perp f\|_{\infty,\nu^4(\cdot;\zeta);\mathbb{T}\times\Omega} + \|h\|_{\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)}$$

sufficiently small. The corresponding solution (u,p) possesses the same decay as the time-periodic fundamental solution, exhibiting an anisotropic decay determined by the steady-state part of the fundamental solution. The decay rates of the pressure and the purely periodic part of the velocity field depend on whether the total flux Φ across Σ is constant in time, where by

$$\Phi(t) := \int_\Sigma h(t,x)\cdot n\,\mathrm{d}S(x).$$

Subsequently, we consider problem (1.2) incorporating the artificial boundary condition (1.3) on the outer boundary of the truncated spatial domain. The second main result, Theorem 4.4, establishes the existence and conditional uniqueness of weak solutions (u_R,p_R) to (1.2)-(1.3), under weaker assumptions on the regularity of the boundary data and provided that $\|\Phi\|_{\infty,\mathbb{T}}$ is small.

Assuming the validity of the earlier well-posedness results, as the third main result of the paper, Theorem 5.1, we prove the following convergence for the gradient of the velocity and for its trace on the artificial boundaries:

$$\|\nabla u - \nabla u_R\|_{L^2(\mathbb{T}\times\Omega_R)} + \|u-u_R\|_{L^2(\mathbb{T}\times\partial B_R)} \leq CR^{-1/2}. \tag{1.5}$$

Structure of the paper. A review of the fundamental solutions of the steady-state and time-periodic Oseen equations, along with the estimates useful for our study, is provided in Section 2. Section 3 addresses the well-posedness of the exterior problem (1.1), including the precise spatial decay of the velocity and pressure. In Section 4, we establish existence and conditional uniqueness of weak solutions to the system (1.2)–(1.3). Finally, the estimate for the truncation error of the velocity field is derived in Section 5.

2 Fundamental solutions

In this section, we introduce the fundamental solution of the time-periodic Oseen equations,

$$\begin{cases} \partial_t u - \Delta u - \zeta\cdot\nabla u + \nabla p = f & \text{in } \mathbb{T}\times\Omega,\\ \nabla\cdot u = 0 & \text{in } \mathbb{T}\times\Omega,\\ u = h & \text{on } \mathbb{T}\times\Sigma. \end{cases} \tag{2.1}$$

We begin by recalling several fundamental solutions for steady problems. In $\mathbb{R}^3$, the fundamental solution of the Laplace operator $-\Delta$ is given by

$$E(x) = \frac{1}{4\pi|x|}, \tag{2.2}$$

that is, $-\Delta E = \delta_{\mathbb{R}^3}$ in $\mathscr{S}'(\mathbb{R}^3)$. The fundamental solution of the 3D Stokes system is the pair $(\Gamma_0^0, P) \in \mathscr{S}'(\mathbb{R}^3)^{3\times 3} \times \mathscr{S}'(\mathbb{R}^3)^3$ given by (see, for example, [11])

$$(\Gamma_0^0, P)(x) = \left(\frac{1}{8\pi|x|} \left(\mathsf{I} + \hat{x} \otimes \hat{x} \right), \frac{1}{4\pi|x|^2} \hat{x} \right)$$

where $\hat{x} := x/|x|$ ($x \in \mathbb{R}^3 \setminus \{0\}$), and the pressure component satisfies

$$P(x) = -\nabla E(x). \tag{2.3}$$

The fundamental solution of the 3D Oseen system has the same pressure part, $P(x) = -\nabla E(x)$, and the velocity component is given by (see [11, 19])

$$\begin{aligned} \Gamma_0^\zeta(x) = &\frac{1}{4\pi|x|} \exp\left(-\mathscr{s}_\zeta(x)\right) \mathsf{I} - \frac{|\zeta|}{16\pi \mathscr{s}_\zeta(x)} \exp\left(-\mathscr{s}_\zeta(x)\right) \left(\hat{x} + \hat{\zeta}\right) \otimes \left(\hat{x} + \hat{\zeta}\right) \\ &- \frac{1 - \exp(-\mathscr{s}_\zeta(x))}{8\pi|x|\mathscr{s}_\zeta(x)} \left(\mathsf{I} - \hat{x} \otimes \hat{x}\right) \\ &+ \frac{|\zeta|}{16\pi} \frac{1 - \exp(-\mathscr{s}_\zeta(x))}{\mathscr{s}_\zeta(x)^2} \left(\hat{x} + \hat{\zeta}\right) \otimes \left(\hat{x} + \hat{\zeta}\right). \end{aligned} \tag{2.4}$$

In the time-periodic case, the fundamental solutions can be identified as solutions to a system of partial differential equations on G. Following [5, 12, 14], the fundamental solution of the Stokes ($\zeta = 0$) or Oseen ($\zeta \neq 0$) equations is a pair $(\Gamma^\zeta, Q) \in \mathscr{S}'(G)^{3\times 3} \times \mathscr{S}'(G)^3$ satisfying

$$\begin{cases} \partial_t \Gamma^\zeta - \Delta \Gamma^\zeta + \nabla Q - (\zeta \cdot \nabla)\Gamma^\zeta = \delta_G \mathsf{I}, \\ \nabla \cdot \Gamma^\zeta = 0. \end{cases} \tag{2.5}$$

The pressure component is given by (recall (2.3))

$$Q = \delta_{\mathbb{T}} \otimes P,$$

meaning $Q(t,x) = \delta_{\mathbb{T}}(t)P(x)$. As in the Stokes case [14], the velocity part Γ^ζ is a sum of the steady-state Oseen fundamental solution and a purely time-periodic remainder satisfying good integrability and pointwise decay estimates. The pressure part, as in the steady regime, is identical to that of the Stokes case, that is, Q is independent of ζ. The velocity component Γ^ζ admits the following decomposition

$$\Gamma^\zeta = 1_{\mathbb{T}} \otimes \Gamma_0^\zeta + \Gamma_\perp^\zeta,$$

with Γ_0^ζ the velocity part of the steady fundamental solution, defined in (2.4), and $\Gamma_\perp^\zeta$ the purely periodic part of Γ^ζ, defined by

$$\Gamma_\perp^\zeta(t,x) = \mathscr{F}_G^{-1} \left[\frac{1 - \delta_{\mathbb{Z}}(k)}{|\xi|^2 + i\left(\frac{2\pi k}{\mathcal{T}} - \zeta \cdot \xi\right)} \left(\mathsf{I} - \hat{\xi} \otimes \hat{\xi} \right) \right],$$

where $\mathscr{F}_G : \mathscr{S}'(G) \to \mathscr{S}'(\hat{G})$, $\hat{G} := \mathbb{Z} \times \mathbb{R}^3$, is the Fourier transform on the group G.

We recall pointwise estimates of the different parts of the fundamental solution.

Proposition 2.1. *For all $\alpha \in \mathbb{N}_0^3$, $r \in [1,\infty)$ and $\varepsilon > 0$ there are $C_1, C_2 > 0$ such that for all $x \in \mathbb{R}^3$ with $|x| \geq \varepsilon$ it holds*

$$|D_x^\alpha \Gamma_0^\zeta(x)| \leq C_1 \nu_{-1-|\alpha|/2}^{-1-|\alpha|/2}(x;\zeta), \tag{2.6}$$

$$\|D_x^\alpha \Gamma_\perp^\zeta(\cdot,x)\|_{L^r(\mathbb{T})} \leq C_2 \nu^{-3-|\alpha|}(x). \tag{2.7}$$

Here $C_1 = C_1(\alpha,r,\varepsilon) > 0$ and $C_2 = C_2(\alpha,r,\varepsilon,\theta) > 0$ are independent of ζ and $\mathcal{T}$ if $\mathcal{T}|\zeta|^2 \leq \theta$.

Proof. See [9, Lemma 3.2] and [6, Theorem 1.1]. For the uniformity of the estimates, see also [7, Theorem 5.8]. □

When using the anisotropic estimates of Γ_0^ζ, we will come across integrals of the form

$$\mathcal{J}_R(a,b) := \int_{\partial B_R} |x|^{-a}(1+s_\zeta(x))^{-b}\,\mathrm{d}S(x) = \int_{\partial B_R} \nu_{-b}^{-a}(x;\zeta)\,\mathrm{d}S(x) \tag{2.8}$$

for $a,b \geq 0$, $R > 0$. In [10, Lemma 2.3] (see also [3, Lemma 3.1]), using polar coordinates, it is shown that

$$\mathcal{J}_R(a,b) \leq C(b) R^{2-a-\min\{1,b\}}, \quad b \neq 1. \tag{2.9}$$

When dealing with $\Gamma_\perp^\zeta$, we shall need the following integrability properties of $\Gamma_\perp^\zeta$.

Proposition 2.2. *We have*

$$\forall q \in \Big(1,\frac{5}{3}\Big): \quad \Gamma_\perp^\zeta \in L^q(\mathbb{T}\times\mathbb{R}^3)^{3\times3}, \tag{2.10}$$

$$\forall q \in \Big[1,\frac{5}{4}\Big): \quad \partial_j\Gamma_\perp^\zeta \in L^q(\mathbb{T}\times\mathbb{R}^3)^{3\times3} \quad (j=1,2,3). \tag{2.11}$$

If $0 < |\zeta| \leq \zeta_0$ for some $\zeta_0 > 0$, the respective L^q-norm can be bounded uniformly in ζ.

Proof. See [6, Theorem 1.1] and [7, Theorem 5.8]. □

3 Existence in the exterior domain

We return to the problem (1.1) in the exterior domain Ω and show existence of solutions with suitable decay properties. In what follows, we use the decomposition of time-periodic functions into a steady-state part $f_0 = \mathscr{P}f$ and a purely periodic part $f_\perp = \mathscr{P}_\perp f$ introduced in (1.4). Our aim is to prove:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^2-boundary $\Sigma = \partial\Omega$. Let $\zeta_0 > 0$ and $p,q \in (1,\infty)$. Then there exists $\varepsilon > 0$ such that for all $f \in L^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)^3$ and $h \in \mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)$ satisfying*

$$\|\mathscr{P}f\|_{\infty,\nu_1^{5/2}(\cdot;\zeta);\Omega} + \|\mathscr{P}_\perp f\|_{\infty,\nu^{3+\delta}(\cdot;\zeta);\mathbb{T}\times\Omega} + \|h\|_{\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)} \leq \varepsilon^2 \tag{3.1}$$

for some $\delta \in (0,1]$, and for all $\zeta \in \mathbb{R}^3\setminus\{0\}$ with $|\zeta| \leq \zeta_0$, there exists a unique strong solution (u,p) to (1.1) satisfying

$$u \in L^p(\mathbb{T};D^{2,q}(\Omega)^3), \qquad \partial_t u \in L^p(\mathbb{T};L^q(\Omega)^3), \qquad p \in L^p(\mathbb{T};D^{1,q}(\Omega))$$

and

$$
\begin{aligned}
\|\nabla^2 u, \partial_t u, \nabla p\|_{L^p(\mathbb{T};L^q(\Omega))} + \|\mathscr{P}u\|_{\infty,\nu_1^1(\cdot;\zeta);\Omega} + \|\nabla \mathscr{P}u\|_{\infty,\nu_{3/2}^{3/2}(\cdot;\zeta);\Omega} + \|\mathscr{P}p\|_{\infty,\nu^2;\Omega} \\
+ \|\mathscr{P}_\perp u\|_{\infty,\nu^2;\mathbb{T}\times\Omega} + \|\nabla \mathscr{P}_\perp u\|_{\infty,\nu^3;\mathbb{T}\times\Omega} + \|\mathscr{P}_\perp p\|_{\infty,\nu^1;\mathbb{T}\times\Omega} \leq \varepsilon.
\end{aligned}
\tag{3.2}
$$

If the boundary data satisfies

$$
\forall t \in \mathbb{T}: \quad \int_\Sigma \partial_t h(t,x)\cdot n \,\mathrm{d}S(x) = 0, \tag{3.3}
$$

then

$$
\|\mathscr{P}_\perp u\|_{\infty,\nu^3;\mathbb{T}\times\Omega} + \|\nabla \mathscr{P}_\perp u\|_{\infty,\nu^{3+\delta};\mathbb{T}\times\Omega} + \|\mathscr{P}_\perp p\|_{\infty,\nu^2;\mathbb{T}\times\Omega} \leq \varepsilon. \tag{3.4}
$$

Remark 3.2. Condition (3.3) means that the total boundary flux

$$
\Phi(t) := \int_\Sigma h(t,x)\cdot n \,\mathrm{d}S(x) \tag{3.5}
$$

is constant in time, that is, $\frac{\mathrm{d}}{\mathrm{d}t}\Phi \equiv 0$. If this is satisfied, then the decay rate of the pressure is $|x|^{-2}$, while for non-constant total flux, the pressure only decays like $|x|^{-1}$ as $|x| \to \infty$. Similarly, the decay of the purely periodic part of the velocity field is faster in this case. This observation is in accordance with [8], where the decay rates for time-periodic weak solutions to (1.1) were derived.

A similar existence result was obtained in [7], but with a different spatial decay rate of the solutions. Since the decay assumptions on external forces considered in [7, Theorem 4.2] are weaker, the decay rates of the derived solutions are slower as well. In contrast, for solutions established in Theorem 3.1 the velocity field u has the same decay as the time-periodic fundamental solution, namely the anisotropic decay determined by the steady-state part. Moreover, the purely periodic velocity field $\mathscr{P}_\perp u$ decays faster than the steady-state part $\mathscr{P}u$, and the decay rate is improved if (3.3) is satisfied, that is, for constant total boundary flux. For $\delta = 1$, this pointwise behavior coincides with the decay observed for weak solutions when f has compact support, see also [8], and can thus be considered the optimal decay rate.

Firstly, we will study a linearized version of problem (1.1), with focus on specific pointwise estimates. Then, a fixed point argument yields the result of Theorem 3.1.

3.1 Linear theory

To prove Theorem 3.1, we first study the associated linear problem (2.1). For pointwise decay estimates of the velocity field $u = u_0 + u_\perp$ split into steady-state and purely periodic parts, we extend the velocity and pressure to zero outside the domain Ω and employ the representation formulas (see [8, 19])

$$
\begin{aligned}
u_0 = \Gamma_0^\zeta *_{\mathbb{R}^3} [f_0\chi_\Omega + n\cdot \widetilde{\mathrm{T}}(u_0,p_0)\delta_\Sigma + (\zeta\cdot n)h_0\delta_\Sigma] \\
+ \Gamma_0^\zeta *_{\mathbb{R}^3} \nabla\cdot[(n\otimes h_0)\delta_\Sigma] - P *_{\mathbb{R}^3} [n\cdot h_0\delta_\Sigma],
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
u_\perp = \Gamma_\perp^\zeta *_G [f_\perp\chi_\Omega + n\cdot \widetilde{\mathrm{T}}(u_\perp,p_\perp)\delta_\Sigma + (\zeta\cdot n)h_\perp\delta_\Sigma] \\
+ \Gamma_\perp^\zeta *_G \nabla\cdot[(n\otimes h_\perp)\delta_\Sigma] - Q *_G [n\cdot h_\perp\delta_\Sigma],
\end{aligned}
\tag{3.7}
$$

where $\widetilde{\mathsf{T}}(v,q) = \nabla v - q\mathsf{I}$ denotes the Cauchy pseudo-stress tensor for the velocity-pressure pair (v,q). The corresponding formulas for the pressure are given by

$$\begin{aligned} p_0 = c_0 &+ P *_{\mathbb{R}^3} [f_0\chi_\Omega + n\cdot \widetilde{\mathsf{T}}(u_0,p_0)\delta_\Sigma + (\zeta\cdot n)h_0\delta_\Sigma] \\ &+ P *_{\mathbb{R}^3} \nabla\cdot[(n\otimes h_0)\delta_\Sigma] \\ &+ P *_{\mathbb{R}^3} [\zeta(h_0\cdot n)\delta_\Sigma], \end{aligned} \tag{3.8}$$

$$\begin{aligned} p_\perp = c_\perp &+ Q *_G [f_\perp\chi_\Omega + n\cdot \widetilde{\mathsf{T}}(u_\perp,p_\perp)\delta_\Sigma + (\zeta\cdot n)h_\perp)\delta_\Sigma] \\ &+ Q *_G \nabla\cdot[(n\otimes h_\perp)\delta_\Sigma] \\ &+ Q *_G [\zeta(h_\perp\cdot n)\delta_\Sigma] + (\delta_\mathbb{T}\otimes E) *_G [(\partial_t h\cdot n)\delta_\Sigma] \end{aligned} \tag{3.9}$$

where $c(t) = c_0 + c_\perp(t)$ is a function only depending on t.

We next prepare several estimates of the convolutions appearing in (3.6)–(3.9). We define the Euclidean ball of radius $R>0$ by $B_R = \{x\in\mathbb{R}^3 : |x|<R\}$, along with the exterior domain $B^R = \{x\in\mathbb{R}^3 : |x|>R\}$, and the spherical shell $B_{R_1,R_2} = \{x\in\mathbb{R}^3 : R_1<|x|<R_2\}$.

Firstly, we consider the terms with contributions at the boundary.

Lemma 3.3. *Let $S>0$ such that $\Sigma\subset B_S$. Let $\zeta\in\mathbb{R}^3$ such that $0<|\zeta|\le\zeta_0$ for some $\zeta_0>0$. Then there is $C = C(\Sigma,S,\zeta_0,\mathcal{T})>0$ such that for all $\psi=\psi_1\delta_\Sigma$ with $\psi_1\in L^1(\mathbb{T}\times\Sigma)$, and for $|x|\ge S$ it holds*

$$\begin{aligned} &\nu_1^1(x;\zeta)\left||\Gamma_0^\zeta\otimes 1_\mathbb{T}| * \psi(t,x)\right| + \nu_{3/2}^{3/2}(x;\zeta)\left||\nabla\Gamma_0^\zeta\otimes 1_\mathbb{T}| * \psi(t,x)\right| \\ &\quad + \nu_2^2(x;\zeta)\left||\nabla^2\Gamma_0^\zeta\otimes 1_\mathbb{T}| * \psi(t,x)\right| \\ &\quad + |x|^3\left||\Gamma_\perp^\zeta| * \psi(t,x)\right| + |x|^4\left||\nabla\Gamma_\perp^\zeta| * \psi(t,x)\right| + |x|^5\left||\nabla^2\Gamma_\perp^\zeta| * \psi(t,x)\right| \\ &\quad + |x|\left|(E\otimes\delta_\mathbb{T}) * \psi(t,x)\right| + |x|^2\left||Q| * \psi(t,x)\right| + |x|^3\left||\nabla Q| * \psi(t,x)\right| \\ &\quad \le C\|\psi_1\|_{L^1(\mathbb{T}\times\Omega_R)}. \end{aligned} \tag{3.10}$$

Moreover, if $\int_\Sigma \psi(t,x)\,\mathrm{d}S(x) = 0$, then

$$|x|^2\left|(E\otimes\delta_\mathbb{T}) * \psi(t,x)\right| + |x|^3\left||Q| * \psi(t,x)\right| + |x|^4\left||\nabla Q| * \psi(t,x)\right| \le C\|\psi_1\|_{L^1(\mathbb{T}\times\Sigma)}. \tag{3.11}$$

Proof. Let $R\in(0,S)$ such that $\Sigma\subset B_R$. For $|x|\ge S>R\ge|y|$ we have

$$\begin{aligned} |x-y| &\ge |x|-|y| \ge (1-R/S)|x| \ge S-R, \\ (1+2\zeta_0R)(1+\mathfrak{s}_\zeta(x-y)) &\ge 1+2|\zeta||y| + \mathfrak{s}_\zeta(x-y) \ge 1+\mathfrak{s}_\zeta(x). \end{aligned}$$

This yields $\nu_\beta^\alpha(x;\zeta) \le C\,\nu_\beta^\alpha(x-y;\zeta)$ for $\alpha,\beta\ge 0$ and a constant $C = C(\alpha,\beta,R,S,\zeta_0)>0$. Therefore, for any function Θ with $|\Theta(t,z)|\le C\nu_{-\beta}^{-\alpha}(z;\zeta)$ for $|z|\ge S-R$, we obtain

$$\begin{aligned} \left|\Theta * \psi(t,x)\right| &\le C\int_\mathbb{T}\int_\Sigma \nu_{-\beta}^{-\alpha}(x-y;\zeta)\,|\psi_1(s,y)|\,\mathrm{d}S(y)\mathrm{d}s \\ &\le C\nu_{-\beta}^{-\alpha}(x;\zeta)\|\psi_1\|_{L^1(\mathbb{T}\times\Sigma)}. \end{aligned}$$

In this proof and the ones that follow, C represents a generic constant that may take different values in different steps of the argument. Moreover, if $|\nabla\Theta(t,z)|\le C\nu_{-\beta}^{-\alpha}(z;\zeta)$ and $\int_\Sigma\psi(t,y)\,\mathrm{d}S(y)=0$, then we obtain

$$\begin{aligned}\big|\Theta * \psi(t,x)\big| &= \Big|\int_{\mathbb{T}}\int_{\Sigma}\big(\Theta(t-s,x-y)-\Theta(t-s,x)\big)\psi_1(s,y)\,\mathrm{d}S(y)\mathrm{d}s\Big| \\ &= \Big|\int_{\mathbb{T}}\int_{\Sigma}\int_0^1 y\cdot\nabla\Theta(t-s,x-\theta y)\psi_1(s,y)\,\mathrm{d}\theta\mathrm{d}S(y)\mathrm{d}s\Big| \\ &\leq CR\int_{\mathbb{T}}\int_{\Sigma}\nu_{-\beta}^{-\alpha}(x-y;\zeta)\,|\psi_1(s,y)|\,\mathrm{d}S(y)\mathrm{d}s \\ &\leq C\nu_{-\beta}^{-\alpha}(x;\zeta)\|\psi_1\|_{L^1(\mathbb{T}\times\Sigma)}.\end{aligned}$$

Due to the estimates (2.6), (2.7) and the decay properties of E, Q and ∇Q, the claim follows from this general result. □

We now consider convolutions of the fundamental solution with functions with suitable spatial decay. Since we assume different decay estimates of the steady-state and the purely periodic part, we study them separately. For the steady-state part, we have the following result.

Lemma 3.4. *There is $C>0$ such that for all $\zeta\in\mathbb{R}^3$ with $0<|\zeta|\leq\zeta_0$ for some $\zeta_0>0$, for all $g\in L^{6/5}(\mathbb{R}^3)$ with $\nu_1^{5/2}(\cdot;\zeta)\,g\in L^\infty(\mathbb{R}^3)$, and for all $x\in\mathbb{R}^3\setminus\{0\}$ it holds*

$$\begin{aligned}\nu_1^1(x;\zeta)\,\big||\Gamma_0^\zeta|*g(x)\big| + \nu_{3/2}^{3/2}(x;\zeta)\,\big||\nabla\Gamma_0^\zeta|*g(x)\big| \\ + |x|^2\,\big||P|*g(x)\big| \leq C\|\nu_1^{5/2}(\cdot;\zeta)\,g\|_{L^\infty(\mathbb{R}^3)}.\end{aligned}$$

Proof. This follows from [3, Theorem 4.7]. □

For the purely periodic part, we have the following estimates.

Lemma 3.5. *Let $\varepsilon>0$, $r\in[1,\infty)$ and $\mu>3$. Then there is $C>0$ such that for all $\zeta\in\mathbb{R}^3\setminus\{0\}$ and $g\in L^1_{\mathrm{loc}}(\mathbb{T}\times\mathbb{R}^3)$ with $(1+\nu^\mu)g\in L^r(\mathbb{T};L^\infty(\mathbb{R}^3)^3)$, and for all $x\in\mathbb{R}^3$ with $|x|\geq\varepsilon$ it holds*

$$\begin{aligned}|x|^3\,\big||\Gamma_\perp^\zeta|*g(t,x)\big| + |x|^{\min\{\mu,4\}}\,\big||\nabla\Gamma_\perp^\zeta|*g(t,x)\big| \\ + |x|^2\,\big||Q|*g(t,x)\big| \leq C\|(1+\nu^\mu)g\|_{L^r(\mathbb{T};L^\infty(\mathbb{R}^3))}.\end{aligned}$$

Proof. Set $M:=\|(1+\nu^\mu)g\|_{L^r(\mathbb{T};L^\infty(\mathbb{R}^3))}$. We start with the estimate of $|\Gamma_\perp^\zeta|*g$. We use Hölder's inequality on $\mathbb{T}$ and Minkowski's integral inequality and split the spatial integral into three parts to obtain

$$\begin{aligned}\big||\Gamma_\perp^\zeta|*g(t,x)\big| &\leq \int_{\mathbb{R}^3}\Big(\int_{\mathbb{T}}\big|\Gamma_\perp^\zeta(t-s,x-y)\big|^{r'}\mathrm{d}s\Big)^{1/r'}\Big(\int_{\mathbb{T}}\big|g(t,y)\big|^r\mathrm{d}s\Big)^{1/r}\mathrm{d}y \\ &\leq CM\sum_{j=1}^3\int_{A_j}\Big(\int_{\mathbb{T}}\big|\Gamma_\perp^\zeta(s,x-y)\big|^{r'}\mathrm{d}s\Big)^{1/r'}\big(1+|y|^\mu\big)^{-1}\mathrm{d}y =: CM\sum_{j=1}^3 I_j\end{aligned}$$

where $r'=r/(r-1)$, and we set $A_1=B_R$, $A_2=B^{4R}$ and $A_3=B_{R,4R}$ with $R=|x|/2$. First, since $|y|\leq R$ implies $|x-y|\geq|x|/2$, we can use (2.7) to obtain

$$I_1\leq C\int_{B_R}|x-y|^{-3}(1+|y|^\mu)^{-1}\,\mathrm{d}y\leq C|x|^{-3}\int_{B_R}(1+|y|)^{-\mu}\,\mathrm{d}y\leq C|x|^{-3}$$

since $\mu > 3$. For the second integral, we again use (2.7) and that $|y| \geq 4R$ implies $|x-y| \geq |y|/2$ to obtain

$$I_2 \leq C \int_{B^{4R}} |x-y|^{-3}(1+|y|)^{-\mu} \,\mathrm{d}y \leq C \int_{B^{4R}} |y|^{-3}|y|^{-\mu} \,\mathrm{d}y = C|x|^{-\mu}.$$

For the third integral, we note that $r' > 1$ and $\mu - 3 > 0$, so that we can choose $\tilde{r} \in (1, 5/3)$ such that $\tilde{r} < r'$ and $3/\tilde{r}' < \mu - 3$. Then Hölder's inequality and (2.10) yield

$$\begin{aligned}
I_3 &\leq C|x|^{-\mu}\Big(\int_{B_{R,4R}} 1 \mathrm{d}y\Big)^{1/\tilde{r}'} \Big(\int_{\mathbb{T}}\int_{B_{R,4R}} \big|\Gamma^\zeta_\perp(s,y)\big|^{\tilde{r}} \,\mathrm{d}y\mathrm{d}s\Big)^{1/\tilde{r}} \\
&= C|x|^{-\mu} R^{3/\tilde{r}'}\Big(\int_{\mathbb{T}}\int_{B_{R,4R}} \big|\Gamma^\zeta_\perp(s,y)\big|^{\tilde{r}} \,\mathrm{d}y\mathrm{d}s\Big)^{1/\tilde{r}} \\
&\leq C|x|^{-\mu+3/\tilde{r}'} \leq C|x|^{-3}.
\end{aligned}$$

Collecting the estimates of I_1, I_2 and I_3, we arrive at

$$\big||\Gamma^\zeta_\perp| * g(t,x)\big| \leq C|x|^{-3}$$

as asserted. For the estimate of $|\nabla\Gamma^\zeta_\perp| * g$ we proceed similarly. At first, we obtain

$$\big||\nabla\Gamma^\zeta_\perp| * g(t,x)\big| \leq CM \sum_{j=1}^{3} \int_{A_j} \Big(\int_{\mathbb{T}} \big|\nabla\Gamma^\zeta_\perp(s,x-y)\big|^{r'} \,\mathrm{d}s\Big)^{1/r'} (1+|y|)^{-\mu} \,\mathrm{d}y =: CM \sum_{j=1}^{3} J_j$$

for the sets A_j, $j = 1,2,3$, as before. Repeating the above arguments, we can estimate J_1 and J_2 as

$$\begin{aligned}
J_1 &\leq C \int_{B_R} |x-y|^{-4}(1+|y|)^{-\mu} \,\mathrm{d}y \leq C|x|^{-4} \int_{B_R} (1+|y|)^{-\mu} \,\mathrm{d}y \leq C|x|^{-4}, \\
J_2 &\leq C \int_{B^{4R}} |x-y|^{-4}(1+|y|)^{-\mu} \,\mathrm{d}y \leq C \int_{B^{4R}} |y|^{-4}|y|^{-\mu} \,\mathrm{d}y = C|x|^{-1-\mu},
\end{aligned}$$

and for J_3 we use $\nabla\Gamma^\zeta_\perp \in L^1(\mathbb{T}\times\mathbb{R}^3)$ by (2.11) to deduce

$$J_3 \leq C(1+|x|)^{-\mu} \int_{\mathbb{T}}\int_{B_{R,4R}} \big|\nabla\Gamma^\zeta_\perp(s,y)\big| \,\mathrm{d}y\mathrm{d}s \leq C|x|^{-\mu}.$$

In total, these estimates yield

$$\big||\nabla\Gamma^\zeta_\perp| * g(t,x)\big| \leq C|x|^{-\min\{4,\mu\}}.$$

For the convolutions with $Q = P \otimes \delta_{\mathbb{T}}$, we use $|P(x)| = C|x|^{-2}$ and argue similarly. □

We now combine the derived pointwise estimates with the results on time-periodic maximal regularity established in [7]. This leads to existence of solutions with suitable spatial decay.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^2-boundary, and let $\zeta_0 > 0$ and $p,q \in (1,\infty)$. Let $h \in \mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)$, and let $f \in L^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)^3$ such that $f = f_0 + f_\perp$ satisfies $\nu_1^{5/2}(\cdot;\zeta)\, f_0 \in L^\infty(\Omega)^3$ and $\nu^{3+\delta} f_\perp \in L^p(\mathbb{T}; L^\infty(\Omega)^3)$ for some $\delta > 0$. For any $\zeta \in \mathbb{R}^3 \setminus \{0\}$ there exists a unique solution (u,p) to (2.1) satisfying*

$$u \in L^p(\mathbb{T}; D^{2,q}(\Omega)^3), \qquad \partial_t u \in L^p(\mathbb{T}; L^q(\Omega)^3), \qquad p \in L^p(\mathbb{T}; D^{1,q}(\Omega)), \tag{3.12}$$

and the estimates

$$\|\partial_t u\|_{L^p(\mathbb{T};L^q(\Omega))} + \|\nabla^2 u\|_{L^p(\mathbb{T};L^q(\Omega))} + \|\nabla p\|_{L^p(\mathbb{T};L^q(\Omega))} \\ \leq C\big(\|f\|_{L^p(\mathbb{T};L^q(\Omega))} + \|h\|_{\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)}\big), \tag{3.13}$$

$$\nu_1^1(x;\zeta)\,\big|u_0(x)\big| + \nu_{3/2}^{3/2}(x;\zeta)\,\big|\nabla u_0(x)\big| + |x|^2\,\big|p_0(x)\big| \\ \leq C\big(\|\nu_1^{5/2}(\cdot;\zeta)\,f_0\|_{L^\infty(\Omega)} + \|h_0\|_{\mathfrak{T}^q(\Sigma)}\big), \tag{3.14}$$

$$|x|^2\big|u_\perp(t,x)\big| + |x|^3\big|\nabla u_\perp(t,x)\big| + |x|\,\big|p_\perp(t,x)\big| \\ \leq C\big(\|\nu^{3+\delta} f_\perp\|_{L^p(\mathbb{T};L^\infty(\Omega))} + \|h_\perp\|_{\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)}\big). \tag{3.15}$$

If the total flux through Σ is constant, that is, if (3.3) *holds, then* (3.15) *can be replaced with*

$$|x|^3\big|u_\perp(t,x)\big| + |x|^{\min\{3+\delta,4\}}\big|\nabla u_\perp(t,x)\big| + |x|^2\,\big|p_\perp(t,x)\big| \\ \leq C\big(\|\nu^{3+\delta} f_\perp\|_{L^p(\mathbb{T};L^\infty(\Omega))} + \|h_\perp\|_{\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)}\big). \tag{3.16}$$

Here $C = C(\Omega, p, q, \delta, \zeta_0) > 0$ if $|\zeta| \leq \zeta_0$.

Proof. We first show that $f \in L^p(\mathbb{T}; L^s(\Omega))$ for all $s \in (1,\infty)$. With the integral $\mathcal{J}_R(a,b)$ from (2.8) and the estimate (2.9) we obtain

$$\int_{B^R} |f_0(x)|^s\,\mathrm{d}x \leq \|f_0\|^s_{\infty,\nu_1^{5/2}(\cdot;\zeta);\Omega} \int_R^\infty \mathcal{J}_r\big(\tfrac{5s}{2}, s\big)\,\mathrm{d}r \leq C\|f_0\|^s_{\infty,\nu_1^{5/2}(\cdot;\zeta);\Omega} \int_R^\infty r^{-5s/2+1}\,\mathrm{d}r.$$

Moreover, we have

$$\int_{\mathbb{T}} \bigg(\int_\Omega |f_\perp(t,x)|^s\,\mathrm{d}x\bigg)^{p/s} \mathrm{d}t \leq \|f_\perp\|^p_{\infty,\nu^{3+\delta};\mathbb{T}\times\Omega} \bigg(\int_\Omega |x|^{-(3+\delta)s}\,\mathrm{d}x\bigg)^{p/s}.$$

Since the remaining integrals in both estimates are finite, we obtain $f \in L^p(\mathbb{T}; L^s(\Omega)^3)$ for any $s \in (1,\infty)$. Therefore, the existence of a solution (u,p) in the class given by (3.12) and subject to inequality (3.13) follows from [7, Theorem 4.7]. Since we can choose any $s < 2$, the velocity field solution satisfies $u \in L^p(\mathbb{T}; L^q(\Omega)^3)$ for $q \in (2,\infty)$ and is unique. Moreover, the pressure field is unique up to addition by a function constant in space, which corresponds to the function $c = c_0 + c_\perp$ in the representation formulas (3.8) and (3.9) for the pressure. Fixing $c \equiv 0$, we ensure uniqueness of p.

To derive the pointwise estimates (3.14), (3.15) and (3.16), we use the representation formulas (3.6) and (3.7) for the steady-state and purely periodic parts of the velocity field. Similarly, we use (3.8) and (3.9) to obtain the estimates of the pressure p. Then the asserted estimates follow directly from Lemma 3.3, Lemma 3.4 and Lemma 3.5, where we use

$$\begin{aligned}
\|n\cdot\widetilde{\mathsf{T}}(u_0,p_0)\|_{L^1(\Sigma)} &\leq C\|\widetilde{\mathsf{T}}(u_0,p_0)\|_{W^{1,q}(\Omega_R)} \\
&\leq C\big(\|\nabla^2 u_0\|_{L^q(\Omega)} + \|\nabla p_0\|_{L^q(\Omega)} + \|h_0\|_{L^q(\Omega)}\big) \\
&\leq C\big(\|f_0\|_{L^q(\Omega)} + \|h_0\|_{\mathfrak{T}^q(\Sigma)}\big), \\
\|n\cdot\widetilde{\mathsf{T}}(u_\perp,p_\perp)\|_{L^1(\mathbb{T}\times\Sigma)} &\leq C\|\widetilde{\mathsf{T}}(u_\perp,p_\perp)\|_{L^p(\mathbb{T};W^{1,q}(\Omega_R))} \\
&\leq C\big(\|\nabla^2 u_\perp\|_{L^p(\mathbb{T};L^q(\Omega))} + \|\nabla p_\perp\|_{L^p(\mathbb{T};L^q(\Omega))} + \|h_\perp\|_{\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)}\big) \\
&\leq C\big(\|f_\perp\|_{L^p(\mathbb{T};L^q(\Omega))} + \|h_\perp\|_{\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)}\big)
\end{aligned}$$

due to (3.13), where we choose any $R > 0$ such that $\partial\Omega \subset B_R$. Observe that in the general case, the pointwise asymptotic behavior of $u_\perp$, $\nabla u_\perp$ and $p_\perp$ is determined by the latter term in the representation formulas (3.7) and (3.9), which leads to estimate (3.15) by using estimate (3.10) from Lemma 3.3. If we assume (3.3), we also have

$$\int_\Sigma h_\perp(t,x)\cdot n \, \mathrm{d}S(x) = 0,$$

so that those terms can be estimated with (3.11) from Lemma 3.3 instead, which leads to the better decay rate stated in (3.16). □

3.2 Solutions to the nonlinear problem

For $k=0,1$ and $\delta\in(0,1]$, we introduce the function space

$$\begin{aligned}
\mathscr{X}_k &:= \big\{v\in L^p(\mathbb{T};W^{2,q}_{\mathrm{loc}}(\Omega)^3)\cap W^{1,p}(\mathbb{T};L^q(\Omega)^3) : \operatorname{div} v = 0,\ \|v\|_{\mathscr{X}_k}<\infty\big\},\\
\|v\|_{\mathscr{X}_k} &:= \|\nabla^2 v\|_{L^p(\mathbb{T};L^q(\Omega))} + \|\partial_t v\|_{L^p(\mathbb{T};L^q(\Omega))}\\
&\qquad + \|\mathscr{P}v\|_{\infty,\nu_1^1(\cdot;\zeta);\Omega} + \|\nabla\mathscr{P}v\|_{\infty,\nu_{3/2}^{3/2}(\cdot;\zeta);\Omega} + N_k(\mathscr{P}_\perp v)
\end{aligned}$$

where

$$\begin{aligned}
N_0(w) &:= \|w\|_{\infty,\nu^2;\mathbb{T}\times\Omega} + \|\nabla w\|_{\infty,\nu^3;\mathbb{T}\times\Omega},\\
N_1(w) &:= \|w\|_{\infty,\nu^3;\mathbb{T}\times\Omega} + \|\nabla w\|_{\infty,\nu^{3+\delta};\mathbb{T}\times\Omega}.
\end{aligned}$$

For given $v\in\mathscr{X}_k$, we consider the problem

$$\begin{cases}
\partial_t u - \Delta u - \zeta\cdot\nabla u + \nabla p = f - \mathscr{N}(v,v) & \text{in } \mathbb{T}\times\Omega,\\
\nabla\cdot u = 0 & \text{in } \mathbb{T}\times\Omega,\\
u = h & \text{on } \mathbb{T}\times\Sigma,
\end{cases} \tag{3.17}$$

where the nonlinear term $\mathscr{N}$ is defined as

$$\mathscr{N}(v_1,v_2) := v_1\cdot\nabla v_2.$$

Below we show that the linear theory from Theorem 3.6 provides a solution (u,p) to this problem if $v\in\mathscr{X}_k$. This defines a solution map $\mathcal{S}\colon v\mapsto u$, and (u,p) solves the nonlinear problem (1.1) if u is a fixed point of $\mathcal{S}$. For obtaining such a fixed point, we first prove the following estimates of the convection term, where we again distinguish steady-state and purely periodic part.

Lemma 3.7. *Let $k\in\{0,1\}$ and let $v_1,v_2\in\mathscr{X}_k$. Then*

$$\|\nu_1^{5/2}(\cdot;\zeta)\,\mathscr{P}\mathscr{N}(v_1,v_2)\|_{L^\infty(\Omega)} + \|\nu^{7/2+k/2}\,\mathscr{P}_\perp\mathscr{N}(v_1,v_2)\|_{L^\infty(\mathbb{T}\times\Omega)} \leq C\|v_1\|_{\mathscr{X}_k}\|v_2\|_{\mathscr{X}_k}.$$

Proof. We set $v_j = z_j + w_j$ with $z_j := \mathscr{P}v_j$ and $w_j = \mathscr{P}_\perp v_j$ for $j=1,2$. Then we have

$$\begin{aligned}
\mathscr{P}\mathscr{N}(v_1,v_2) &= z_1\cdot\nabla z_2 + \mathscr{P}(w_1\cdot\nabla w_2),\\
\mathscr{P}_\perp\mathscr{N}(v_1,v_2) &= z_1\cdot\nabla w_2 + w_1\cdot\nabla z_2 + \mathscr{P}_\perp(w_1\cdot\nabla w_2).
\end{aligned}$$

Therefore, for $x\in\Omega$ we can estimate

$$\begin{aligned}
&\nu_1^{5/2}(x;\zeta)\,|\mathscr{P}\mathscr{N}(v_1,v_2)(x)|\\
&\qquad\leq C\left(\nu_1^1(x;\zeta)\,|z_1(x)|\,|x|^{3/2}|\nabla z_2(x)| + \nu_1^0(x;\zeta)\,|w_1(t,x)||x|^{5/2}|\nabla w_2(t,x)|\right)\\
&\qquad\leq C\|v_1\|_{\mathscr{X}_k}\|v_2\|_{\mathscr{X}_k},
\end{aligned}$$

and

$$
\begin{aligned}
&|x|^{7/2+k/2}\,|\mathscr{P}_\perp\mathscr{N}(v_1,v_2)(t,x)|\\
&\qquad\le C\big(|x|\,|z_1(x)|\,|x|^{5/2+k/2}|\nabla w_2(t,x)|+|x|^{2+k/2}\,|w_1(t,x)|\,|x|^{3/2}\,|\nabla z_2(x)|\\
&\qquad\qquad+|x|^{2+k/2}\,|w_1(t,x)|\,|x|^{3/2}\,|\nabla w_2(t,x)|\big)\\
&\qquad\le C\|v_1\|_{\mathscr{X}_k}\|v_2\|_{\mathscr{X}_k}.
\end{aligned}
$$

This shows the asserted estimates. □

We can now show existence of a solution to (1.1) by a fixed-point argument.

Proof of Theorem 3.1. We set $k=0$ in the general case and we set $k=1$ when (3.3) is satisfied. For $\varepsilon>0$ consider the set

$$\mathscr{X}_{k,\varepsilon}:=\big\{v\in\mathscr{X}_k:\|v\|_{\mathscr{X}_k}\le\varepsilon\big\}.$$

In virtue of Lemma 3.7 and Theorem 3.6, for any $v\in\mathscr{X}_k$ there exists a solution (u,p) to (3.17) with the regularity stated in (3.12) and subject to the estimates

$$
\begin{aligned}
&\nu_1^1(x;\zeta)\,\big|u_0(x)\big|+\nu_{3/2}^{3/2}(x;\zeta)\,\big|\nabla u_0(x)\big|+|x|^2\,\big|p_0(x)\big|\\
&\qquad\le C\big(\|\nu_1^{5/2}(\cdot;\zeta)\,\mathscr{P}\mathscr{N}(v,v)\|_{L^\infty(\Omega)}+\|\nu_1^{5/2}(\cdot;\zeta)f_0\|_{L^\infty(\Omega)}+\|h_0\|_{\mathfrak{T}^q(\Sigma)}\big)\\
&\qquad\le C\big(\|v\|_{\mathscr{X}_k}^2+\varepsilon^2\big),\\
&|x|^{2+k}\big|u_\perp(t,x)\big|+|x|^{3+\min\{k,\delta\}}\big|u_\perp(t,x)\big|+|x|^{1+k}\,\big|p_\perp(t,x)\big|\\
&\qquad\le C\big(\|\nu^{3+\delta}\mathscr{P}_\perp\mathscr{N}(v,v)\|_{L^p(\mathbb{T};L^\infty(\Omega))}+\|\nu^{3+\delta}f_\perp\|_{L^p(\mathbb{T};L^\infty(\Omega))}+\|h_\perp\|_{\mathfrak{T}^{p,q}(\mathbb{T}\times\Sigma)}\big)\\
&\qquad\le C\big(\|v\|_{\mathscr{X}_k}^2+\varepsilon^2\big).
\end{aligned}
$$

For $v\in\mathscr{X}_{k,\varepsilon}$, we thus have

$$\|u\|_{\mathscr{X}_k}\le C\varepsilon^2\le\varepsilon$$

if $\varepsilon>0$ is chosen sufficiently small. Then the solution map $\mathcal{S}\colon v\mapsto u$ is a well-defined self mapping $\mathcal{S}\colon\mathscr{X}_{k,\varepsilon}\to\mathscr{X}_{k,\varepsilon}$. Moreover, for $v_1,v_2\in\mathscr{X}_{k,\varepsilon}$, the differences $\overline{u}=u_1-u_2$ and $\overline{p}=p_1-p_2$, where $u_j:=\mathcal{S}(v_j)$ with corresponding pressure p_j, $j=1,2$, satisfy

$$
\begin{cases}
\partial_t\overline{u}-\Delta\overline{u}-\zeta\cdot\nabla\overline{u}+\nabla\overline{p}=-\mathscr{N}(v_1,v_1)+\mathscr{N}(v_2,v_2) & \text{in } \mathbb{T}\times\Omega,\\
\nabla\cdot\overline{u}=0 & \text{in } \mathbb{T}\times\Omega,\\
\overline{u}=0 & \text{on } \mathbb{T}\times\Sigma.
\end{cases}
$$

Noting that

$$\mathscr{N}(v_1,v_1)-\mathscr{N}(v_2,v_2)=\mathscr{N}(v_1-v_2,v_1)+\mathscr{N}(v_2,v_1-v_2),$$

we can adapt the same argument as before to conclude the estimate

$$\|\mathcal{S}(v_1)-\mathcal{S}(v_2)\|_{\mathscr{X}_k}=\|\overline{u}\|_{\mathscr{X}_k}\le C(\|v_1\|_{\mathscr{X}}+\|v_2\|_{\mathscr{X}_k})\|v_1-v_2\|_{\mathscr{X}_k}\le 2C\varepsilon\|v_1-v_2\|_{\mathscr{X}_k}.$$

Hence, choosing $\varepsilon>0$ sufficiently small, we obtain that $\mathcal{S}$ is also a contraction. Finally, the contraction mapping principle yields the existence of a unique fixed point $u=\mathcal{S}(u)\in\mathscr{X}_{k,\varepsilon}$. If p denotes the associated pressure, then (u,p) is a solution to (1.1) with the asserted properties. □

4 Existence in the truncated domains

Our aim is to find a solution $(v, \mathfrak{p})$ to the problem (1.2)–(1.3) defined in the truncated domain Ω_R. We seek a velocity field in the form $v = \widetilde{h} + \vartheta$ with $\widetilde{h}$ an appropriate extension of h to Ω_R and $\vartheta \in L^2(\mathbb{T}; H^1(\Omega_R)^3)$ satisfying $\vartheta|_\Sigma = 0$ and $\nabla \cdot \vartheta = 0$ in $\mathbb{T} \times \Omega_R$.

4.1 Functions spaces over the truncated domains

In what follows, the usual inner products in $L^2(\Omega_R)$ and $L^2(\partial B_R)$ will be denoted by $(\cdot, \cdot)_{\Omega_R}$ and $(\cdot, \cdot)_{\partial B_R}$, respectively. As in [3, 4], we consider $H^1(\Omega_R)$ endowed with inner product and norm

$$(v, w)_{(R)} := (\nabla v, \nabla w)_{\Omega_R} + \frac{1}{R}(v, w)_{\partial B_R}, \;\; \|w\|_{(R)} = \left(\|\nabla w\|^2_{2,\Omega_R} + \frac{1}{R}\|w\|^2_{2,\partial B_R} \right)^{1/2} \tag{4.1}$$

and we equip the space of time-periodic functions $L^2(\mathbb{T}; H^1(\Omega_R))$ with the norm

$$\|w\|_{(\mathbb{T},R)} := \left(\int_{\mathbb{T}} \|w\|^2_{(R)} \mathrm{d}t \right)^{1/2}.$$

Within this framework, the following estimate holds for time-periodic functions:

Lemma 4.1. *Take a fixed $S \in (0, \infty)$ with $\partial\Omega \subset B_S$ and $R > S$. Then there is a constant $C(S) > 0$ such that*

$$\left(\int_{\mathbb{T}\times B_R \setminus \overline{B_S}} \frac{|u(t,x)|^2}{|x|^2} \mathrm{d}x\mathrm{d}t \right)^{1/2} \leq C(S) \|u\|_{(\mathbb{T},R)}$$

for all $u \in L^2(\mathbb{T}; H^1(\Omega_R))$.

Proof. We can directly apply the reasoning from [3, Theorem 3.6]. □

The space

$$W_R := \left\{ w \in H^1(\Omega_R)^3 : \, w|_\Sigma = 0 \right\},$$

with inner product and norm (4.1), will be relevant in the analysis of problem (1.2)–(1.3).

Consider the following subspaces of divergence-free functions of W_R,

$$\mathscr{V}_R := \left\{ \varphi|_{\Omega_R} : \, \varphi \in C_0^\infty(\Omega)^3 \text{ and } \nabla \cdot \varphi = 0 \text{ in } \Omega \right\},$$

$$V_R := \text{ the closure of } \mathscr{V}_R \text{ in } H^1(\Omega_R)^3,$$

and the space

$$H_R := \text{ the closure of } \mathscr{V}_R \text{ in } L^2(\Omega_R)^3.$$

If Ω is a domain with a Lipschitz continuous boundary, then

$$H_R = \left\{ v \in L^2(\Omega_R)^3 : \nabla \cdot v = 0 \text{ in } \Omega_R \text{ and } v \cdot n = 0 \text{ on } \Sigma \right\},$$

where n represents the unit outer normal on Σ, with $\nabla \cdot v = 0$ and $v \cdot n$ interpreted in the weak sense, and

$$V_R = \left\{ v \in H^1(\Omega_R)^3 : \nabla \cdot v = 0 \text{ in } \Omega_R \text{ and } v = 0 \text{ on } \Sigma \right\}.$$

For $6/5 \leq q_1 < 6$ and $4/3 \leq q_2 < 4$, we have the embeddings

$$V_R \overset{c}{\hookrightarrow} L^{q_1}(\Omega) \oplus L^{q_2}(\partial B_R) \hookrightarrow V_R', \tag{4.2}$$

which are compact and continuous, respectively. For $f \in L^2(\Omega_R)^3$, a weak solution to the Stokes problem

$$\begin{cases} -\Delta u + \nabla \mathfrak{p} = f & \text{in } \Omega_R \\ \nabla \cdot u = 0 & \text{in } \Omega_R, \\ u = 0 & \text{on } \Sigma, \\ \dfrac{x}{R} \cdot \nabla u - \mathfrak{p}\dfrac{x}{R} + \dfrac{1}{R} u = 0 & \text{on } \partial B_R, \end{cases}$$

is a field $u \in V_R$ such that

$$(\nabla u, \nabla \varphi)_{\Omega_R} + \frac{1}{R}(u, \varphi)_{\partial B_R} = (f, \varphi)_{\Omega_R} \quad \forall \varphi \in V_R.$$

Based on this Stokes problem, it is possible to construct a special basis for the spaces H_R and V_R.

Lemma 4.2. *The spectral problem*

$$(\nabla \Psi, \nabla \varphi)_{\Omega_R} + \frac{1}{R}(\Psi, \varphi)_{\partial B_R} = \lambda(\Psi, \varphi)_{\Omega_R}, \quad \forall \varphi \in V_R$$

admits a sequence $\{\Psi_k\}_{k\in\mathbb{N}} \subset V_R$ of non-zero solutions corresponding to a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

which satisfies $\lambda_k \to \infty$ as $k \to \infty$.

Moreover, we can choose $\{\Psi_k\}_{k\in\mathbb{N}}$ in such a way that it forms an orthonormal basis of H_R and $\{\Psi_k/\lambda_k^{1/2}\}_{k\in\mathbb{N}}$ is an orthonormal basis of V_R.

Proof. Given $\Psi \in H_R$, by Lax-Milgram Theorem, the problem

$$(\nabla u, \nabla \varphi)_{\Omega_R} + \frac{1}{R}(u, \varphi)_{\partial B_R} = (\Psi, \varphi)_{\Omega_R}, \quad \forall \varphi \in V_R$$

has a unique solution $u \in V_R$. The solution operator $\mathcal{S} : H_R \to H_R$, $\Psi \mapsto u$, is compact, self-adjoint and positive. Hence, H_R admits an orthonormal basis of eigenfunctions $\Psi_k \in V_R$ of $\mathcal{S}$ with corresponding eigenvalues μ_k satisfying $\mu_k > 0$, for all $k \in \mathbb{N}$, and $\mu_k \to 0$ as $k \to \infty$. Thus, defining $\lambda_k = 1/\mu_k$, we obtain

$$(\nabla \Psi_k, \nabla \varphi)_{\Omega_R} + \frac{1}{R}(\Psi_k, \varphi)_{\partial B_R} = \lambda_k(\Psi_k, \varphi)_{\Omega_R} \quad \forall \varphi \in V_R. \tag{4.3}$$

Suppose that $v \in V_R$ satisfies $(\Psi_k, v)_{(R)} = (\nabla \Psi_k, \nabla v)_{\Omega_R} + (\Psi_k, v)_{\partial B_R}/R = 0$ for all $k \in \mathbb{N}$. From (4.3), it follows that $(\Psi_k, v)_{\Omega_R} = 0$ for all $k \in \mathbb{N}$, and since $\{\Psi_k\}_{k\in\mathbb{N}}$ is a basis of H_R, we conclude that $v \equiv 0$. Hence, the linear span of $\{\Psi_k\}_{k\in\mathbb{N}}$ is dense in V_R. From (4.3), we further obtain

$$\begin{aligned} \left(\frac{\Psi_k}{\lambda_k^{1/2}}, \frac{\Psi_j}{\lambda_j^{1/2}}\right)_{(R)} &= \left(\frac{\nabla \Psi_k}{\lambda_k^{1/2}}, \frac{\nabla \Psi_j}{\lambda_j^{1/2}}\right)_{\Omega_R} + \frac{1}{R}\left(\frac{\Psi_k}{\lambda_k^{1/2}}, \frac{\Psi_j}{\lambda_j^{1/2}}\right)_{\partial B_R} \\ &= \frac{\lambda_k}{\lambda_k^{1/2}\lambda_j^{1/2}}(\Psi_k, \Psi_j)_{\Omega_R} = \frac{\lambda_k}{\lambda_k^{1/2}\lambda_j^{1/2}}\delta_{kj} = \delta_{kj}, \qquad \forall j, k \in \mathbb{N}. \end{aligned}$$

Therefore, $\{\Psi_k/\lambda_k^{1/2}\}_{k\in\mathbb{N}}$ is an orthonormal basis of V_R. □

4.2 Weak solutions in the truncated domain

Assume Ω is a Lipschitz domain and recall the total flux of h over Σ, given by $\Phi(t) := \int_\Sigma h(t,x)\cdot n(x)\mathrm{d}S(x)$. To simplify the presentation, for each fixed R, we define

$$c(u,v,w) := \int_{\Omega_R} u\cdot\nabla v\cdot w\,\mathrm{d}x - \frac{1}{2}\int_{\partial B_R}\left(\frac{x}{R}\cdot u\right)(v\cdot w)\,\mathrm{d}S(x), \tag{4.4}$$

which is well defined for $u,v,w\in H^1(\Omega_R)^3$ and satisfies

$$c(u,v,v) = 0, \quad \forall u\in H_R,\ v\in V_R. \tag{4.5}$$

In what follows, $\sigma(x) := \nabla E(x) = -\frac{x}{4\pi|x|^3}$. Observe that $\sigma = -P$ for the pressure part P of fundamental solution, defined in (2.3).

Lemma 4.3. *Given $h\in H^1(\mathbb{T};H^{1/2}(\partial\Omega)^3)$ define Φ as in (3.5). Let R_0 be such that $\partial\Omega\subset B_{R_0}$. For any $\gamma>0$, there exists $\widetilde{h}\in H^1(\mathbb{T};H^1(\Omega)^3)$ satisfying*

$$\begin{cases} \nabla\cdot\widetilde{h} = 0 & \text{in } \mathbb{T}\times\Omega,\\ \widetilde{h} = h & \text{on } \mathbb{T}\times\Sigma, \end{cases} \tag{4.6}$$

and the estimate

$$|c(\vartheta,\vartheta,\tilde{h})| \leq \gamma\|\vartheta\|_{(R)}^2 + \|\Phi\|_{\infty,\mathbb{T}}\left(C_S\|\sigma\|_{3,\Omega_R} + \frac{1}{8\pi R}\right)\|\vartheta\|_{(R)}^2 \text{ in } \mathbb{T} \tag{4.7}$$

for all $R>R_0$, where C_S is a Sobolev embedding constant.

Proof. Decompose

$$h(t,x) = [h(t,x)-\Phi(t)\sigma|_\Sigma(x)] + \Phi(t)\sigma|_\Sigma(x) =: h^{(1)}(t,x) + h^{(2)}(t,x), \quad (t,x)\in\mathbb{T}\times\Sigma.$$

Then $\int_\Sigma h^{(1)}(t,x)\cdot n(x)\mathrm{d}S = 0$ for all $t\in\mathbb{T}$ and $\nabla\cdot\sigma = 0$ in Ω.

For fixed $R_0>0$ such that $\partial\Omega\subset B_{R_0}$, we can find (see [11, Lemma IX.4.1] and [17, Lemma 3.3]) $w:\mathbb{T}\times\Omega_{R_0}\to\mathbb{R}^3$ such that

$$\begin{cases} \nabla\times w = h^{(1)} & \text{on } \mathbb{T}\times\Sigma,\\ \nabla\times w = 0 & \text{on } \mathbb{T}\times\partial B_{R_0},\\ w = 0 & \text{on } \mathbb{T}\times\partial B_{R_0}, \end{cases}$$

and

$$\|w(t,\cdot)\|_{2,2,\Omega_{R_0}} \leq C(\Omega_{R_0})\|h^{(1)}(t,\cdot)\|_{1/2,2,\Sigma}, \quad t\in\mathbb{T},$$

so that $w\in H^1(\mathbb{T};H^2(\Omega_{R_0})^3)$ along with the estimate

$$\|w\|_{H^1(\mathbb{T};H^2(\Omega_{R_0})^3)} \leq C(\Omega_{R_0})\|h^{(1)}(t,\cdot)\|_{H^1(\mathbb{T};H^{1/2}(\Sigma)^3)}.$$

Let $0<\varepsilon<1$ and $\Psi_\varepsilon\in C^\infty(\mathbb{R};\mathbb{R})$ be such that $\Psi_\varepsilon(\theta)=1$ for $\theta<\frac{\exp(-2/\varepsilon)}{2}$, $\Psi_\varepsilon(\theta)=0$ for $\theta\geq 2\exp(-1/\varepsilon)$, $|\Psi_\varepsilon(\theta)|\leq 1$ and $|\Psi_\varepsilon'(\theta)|\leq\varepsilon/\theta$, for all $\theta>0$. Define $d(x)$ as the distance of a point $x\in\Omega_{R_0}$ to the boundary $\partial\Omega_{R_0}$ and let $\rho(x)$ be the corresponding regularized distance (in the sense of Stein). Using these, define the cut-off function for the domain Ω_{R_0} $\psi_\varepsilon(x) := \Psi_\varepsilon(\rho(x))$, and extend it by 1 to the exterior domain Ω. The extension satisfies (see [11, Lemma III.6.2] and [17, Lemma 3.2])

$$\psi_\varepsilon(x) = \begin{cases} 1 & \text{if} \quad d(x) < \frac{\exp(-2/\varepsilon)}{2\kappa_1}, \\ 0 & \text{if} \quad d(x) \geq 2\exp(-1/\varepsilon), \end{cases}$$

and $\nabla\psi_\varepsilon(x) = \Psi'_\varepsilon(\rho(x))\nabla\rho(x)$, $|\nabla\psi_\varepsilon(x)| \leq \varepsilon\kappa_2/d(x)$, where κ_1 and κ_2 are positive constants independent of the domain. Define

$$\begin{aligned} \widetilde{h}(t,x) &= \widetilde{h^{(1)}}(t,x) + \widetilde{h^{(2)}}(t,x) = \nabla\times(w(t,x)\psi_\varepsilon(x)) + \Phi(t)\sigma(x) \\ &= \nabla\psi_\varepsilon(x)\times w(t,x) + \psi_\varepsilon(x)\nabla\times w(t,x) + \Phi(t)\sigma(x), \quad (t,x)\in\mathbb{T}\times\Omega, \end{aligned}$$

where w is extended to 0 outside $\mathbb{T}\times B_{R_0}$. Clearly, the function $\widetilde{h}$ is divergence free. Now assume $R > R_0$. Then, for sufficiently small ε, following [11, Lemma X.4.2] or [17, Lemma 3.3], we can estimate

$$\begin{aligned} |c(\vartheta,\vartheta,\tilde{h})| &= \left| \int_{\Omega_R} \vartheta\cdot\nabla\vartheta\cdot\tilde{h}\,\mathrm{d}x + \frac{\Phi(t)}{8\pi R^2}\|\vartheta\cdot n\|^2_{2,\partial B_R} \right| \\ &\leq \left| \int_{\Omega_R} \vartheta\cdot\nabla\vartheta\cdot\nabla\times(w\psi_\varepsilon)\,\mathrm{d}x \right| + |\Phi(t)| \left| \int_{\Omega_R} \vartheta\cdot\nabla\vartheta\cdot\sigma\,\mathrm{d}x \right| + \frac{|\Phi(t)|}{8\pi R^2}\|\vartheta\cdot n\|^2_{2,\partial B_R} \\ &\leq \gamma\|\nabla\vartheta\|^2_{2,\Omega_R} + C_S\|\Phi\|_{\infty,\mathbb{T}}\|\sigma\|_{3,\Omega_R}\|\nabla\vartheta\|^2_{2,\Omega_R} + \frac{\|\Phi\|_{\infty,\mathbb{T}}}{8\pi R^2}\|\vartheta\|^2_{2,\partial B_R}, \end{aligned}$$

where C_S is a constant related with the Sobolev embedding in Ω, and $\vartheta \in L^2(\mathbb{T}; V_R)$. □

Taking into account the regularity of the external force used to solve the exterior problem, we can assume that $f \in L^2(\mathbb{T}\times\Omega_R)^3$ in in (1.2)–(1.3). Regarding existence and uniqueness of weak solution for (1.2)–(1.3), we fix

$$0 < \gamma < 1/2 - \|\Phi\|_{\infty,\mathbb{T}}\left(C_S\|\sigma\|_{3,\Omega_R} + \frac{1}{8\pi R} \right)$$

and a solenoidal extension $\tilde{h} \in H^1(\mathbb{T}; H^1(\Omega)^3)$ given by Lemma 4.3. Then, we will seek the velocity field for system (1.2) in the form $v := u_R = \vartheta + \tilde{h}$ where $\vartheta \in L^2(\mathbb{T}; V_R) \cap L^\infty(\mathbb{T}; H_R)$. The velocity ϑ and an associated pressure $\wp$ should satisfy

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_R} \vartheta\cdot\Psi\,\mathrm{d}x + \int_{\Omega_R}\nabla\vartheta : \nabla\Psi\,\mathrm{d}x - \int_{\Omega_R}\zeta\cdot\nabla\vartheta\cdot\Psi\,\mathrm{d}x - \int_{\Omega_R}\wp\nabla\cdot\Psi\,\mathrm{d}x \\ \qquad + \int_{\Omega_R}(\vartheta\cdot\nabla)\tilde{h}\cdot\Psi\,\mathrm{d}x + \int_{\Omega}(\tilde{h}\cdot\nabla)\vartheta\cdot\Psi\,\mathrm{d}x + \int_{\Omega_R}(\vartheta\cdot\nabla)\vartheta\cdot\Psi\,\mathrm{d}x \\ \qquad + \int_{\partial B_R}\frac{1}{R}\left(1 + \jmath_\zeta(x)\right)\vartheta\cdot\Psi\,\mathrm{d}S(x) - \int_{\partial B_R}\frac{1}{2}\left(\vartheta\cdot\frac{x}{R}\right)\tilde{h}\cdot\Psi\,\mathrm{d}S(x) \\ \qquad - \int_{\partial B_R}\frac{1}{2}\left(\tilde{h}\cdot\frac{x}{R}\right)\vartheta\cdot\Psi\,\mathrm{d}S(x) - \int_{\partial B_R}\frac{1}{2}\left(\vartheta\cdot\frac{x}{R}\right)\vartheta\cdot\Psi\,\mathrm{d}S(x) \\ \quad = \int_{\Omega_R} f\cdot\Psi\,\mathrm{d}x - \int_{\Omega_R}\partial_t\tilde{h}\cdot\Psi\,\mathrm{d}x - \int_{\Omega}\nabla\tilde{h} : \nabla\Psi\,\mathrm{d}x \\ \qquad + \int_{\Omega_R}\zeta\cdot\nabla\tilde{h}\cdot\Psi\,\mathrm{d}x - \int_{\partial B_R}\frac{1}{R}\left(1 + \jmath_\zeta(x)\right)\tilde{h}\cdot\Psi\,\mathrm{d}S(x) \\ \qquad - \int_{\Omega_R}(\tilde{h}\cdot\nabla)\tilde{h}\cdot\Psi\,\mathrm{d}x + \int_{\partial B_R}\frac{1}{2}\left(\tilde{h}\cdot\frac{x}{R}\right)\tilde{h}\cdot\Psi\,\mathrm{d}S(x), \quad \forall\Psi\in W_R, \\ \int_{\Omega_R}(\nabla\cdot\vartheta)\phi\,\mathrm{d}x = 0, \quad \forall\phi\in L^2(\Omega_R), \end{cases}$$

in the sense of distributions in $\mathbb{T}$.

It is convenient to recall (4.4) and introduce additional notations

$$\begin{aligned} a(v,w) &:= \int_{\Omega_R} \nabla v : \nabla w \,\mathrm{d}x - \int_{\Omega_R} \zeta \cdot \nabla v \cdot w \,\mathrm{d}x + \int_{\partial B_R} \frac{1}{R} \left(1 + \jmath_\zeta(x)\right) v \cdot w \,\mathrm{d}S(x), \\ b(v,p) &:= -\int_{\Omega_R} (\nabla \cdot v) p \,\mathrm{d}x, \end{aligned} \tag{4.8}$$

so that the above system for $(\vartheta, \wp)$ can be reformulated in a more concise manner as

$$\begin{cases} \langle \partial_t \vartheta, \Psi \rangle + a(\vartheta, \Psi) + b(\Psi, \wp) + c(\vartheta, \tilde{h}, \Psi) + c(\tilde{h}, \vartheta, \Psi) + c(\vartheta, \vartheta, \Psi) \\ \qquad = (f, \Psi)_{\Omega_R} - \left(\partial_t \tilde{h}, \Psi\right)_{\Omega_R} - a(\tilde{h}, \Psi) - c(\tilde{h}, \tilde{h}, \Psi), \quad \forall \Psi \in W_R, \\ b(\vartheta, \phi) = 0, \quad \forall \phi \in L^2(\Omega_R) \end{cases} \tag{4.9}$$

in $\mathbb{T}$. Moreover, we introduce a different inner product on the space $H^1(\Omega_R)^3$, namely,

$$\begin{aligned} (v,w)_{(R,|\zeta|)} &:= \int_{\Omega_R} \nabla v : \nabla w \,\mathrm{d}x + \left(\frac{1}{R} + \frac{|\zeta|}{2}\right) \int_{\partial B_R} (v \cdot w) \,\mathrm{d}S \\ &= a(v,w) + \int_{\Omega_R} \zeta \cdot \nabla v \cdot w \,\mathrm{d}x - \int_{\partial B_R} \frac{1}{R} \frac{(\zeta \cdot x)}{2} (v \cdot w) \,\mathrm{d}S, \end{aligned} \tag{4.10}$$

so that $a(v,v) = \|\nabla v\|_{2,\Omega_R}^2 + \int_{\partial B_R} \left(\frac{1}{R} + \frac{|\zeta|}{2}\right) |v|^2 \,\mathrm{d}S(x) = \|v\|_{(R,|\zeta|)}^2$, for $v \in V_R$.

Theorem 4.4. *Let $f \in L^2(\mathbb{T} \times \Omega_R)^3$ and $h \in H^1(\mathbb{T}; H^{1/2}(\partial\Omega)^3)$ satisfying*

$$2\|\Phi\|_{\infty,\mathbb{T}} \left(C_S \|\sigma\|_{3,\Omega_R} + \frac{1}{8\pi R} \right) < 1. \tag{4.11}$$

Then there exist $v \in L^2(\mathbb{T}; H^1(\Omega_R)^3) \cap L^\infty(\mathbb{T}; L^2(\Omega_R)^3)$ and $\wp_0 \in L^\infty(\mathbb{T}, L^2(\Omega_R))$, $\wp_1 \in L^2(\mathbb{T}; L^2(\Omega_R))$ as well as $\wp_2 \in L^{4/3}(\mathbb{T}; L^2(\Omega_R))$ such that, in the sense of distributions in $\mathbb{T}$, it holds

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \left(\displaystyle\int_{\Omega_R} v \cdot \Psi \,\mathrm{d}x + \int_{\Omega_R} \wp_0 \nabla \cdot \Psi \,\mathrm{d}x \right) \\ \quad + \displaystyle\int_{\Omega_R} \nabla v : \nabla \Psi \,\mathrm{d}x - \int_{\Omega_R} \zeta \cdot \nabla v \cdot \Psi \mathrm{d}x + \int_{\Omega_R} (v \cdot \nabla) v \cdot \Psi \,\mathrm{d}x \\ \quad + \displaystyle\int_{\partial B_R} \frac{1}{R} \left(1 + \jmath_\zeta(x)\right) v \cdot \Psi \,\mathrm{d}S(x) - \int_{\partial B_R} \frac{1}{2} \left(v \cdot \frac{x}{R}\right) v \cdot \Psi \,\mathrm{d}S(x) \\ \quad + \displaystyle\int_{\Omega_R} (\wp_1 + \wp_2) \nabla \cdot \Psi \,\mathrm{d}x = \int_{\Omega_R} f \cdot \Psi \mathrm{d}x, \quad \forall \Psi \in W_R, \\ \displaystyle\int_{\Omega_R} (\nabla \cdot v) \phi \,\mathrm{d}x = 0, \quad \forall \phi \in L^2(\Omega_R), \end{cases} \tag{4.12}$$

and $\vartheta := v - \tilde{h}$, where $\tilde{h}$ from Lemma 4.3, satisfies the energy inequality

$$\begin{aligned} &\int_{\mathbb{T}} \int_{\Omega_R} |\nabla \vartheta|^2 \,\mathrm{d}x\mathrm{d}t + \left(\frac{1}{R} + \frac{|\zeta|}{2}\right) \int_{\partial B_R} |\vartheta|^2 \,\mathrm{d}S\mathrm{d}t \\ &\leq -\int_{\mathbb{T}} \int_{\Omega_R} \nabla \tilde{h} : \nabla \vartheta \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}} \int_{\Omega_R} \zeta \cdot \nabla \tilde{h} \cdot \vartheta \,\mathrm{d}x\mathrm{d}t \\ &\quad - \int_{\mathbb{T} \times \partial B_R} \frac{1}{R} \left(1 + \jmath_\zeta(x)\right) (\tilde{h} \cdot \vartheta) \,\mathrm{d}S(x)\mathrm{d}t \\ &\quad + \int_{\mathbb{T}} \int_{\Omega_R} \left(f - \partial_t \tilde{h} - v \cdot \nabla \tilde{h}\right) \cdot \vartheta \,\mathrm{d}x\mathrm{d}t + \frac{1}{2} \int_{\mathbb{T} \times \partial B_R} \left(\frac{x}{R} \cdot v\right) (\tilde{h} \cdot \vartheta) \,\mathrm{d}S(x)\mathrm{d}t \end{aligned} \tag{4.13}$$

Moreover, if another weak solution $(\tilde{v}, \tilde{\mathfrak{p}})$ *with* $\tilde{v} \in H^1(\mathbb{T} \times \Omega_R)^3$ *exists such that*

$$\|\tilde{v}\|_{L^\infty(\mathbb{T};H^1(\Omega_R))} \leq \delta \tag{4.14}$$

with $\delta > 0$ *sufficiently small, then* $\tilde{v} \equiv v$*.*

Proof. We construct a time-periodic weak solution to problem (4.9) using the Galerkin method. In order to find the velocity ϑ, let $\{\Psi_i\}_{i\in\mathbb{N}} \subset V_R$ be the complete orthonormal system in H_R given by Lemma 4.2. For each $M \in \mathbb{N}$, let $H_R^{(M)}$ be the linear space generated by $\{\Psi_1, ..., \Psi_M\}$ endowed with the inner product of H_R, and let $V_R^{(M)}$ be defined in an analogous way with respect to the inner product of V_R.

In a first stage, approximate velocities $\vartheta^{(M)} \in L^\infty(\mathbb{T}; H_R^{(M)}) \cap L^2(\mathbb{T}; V_R^{(M)})$ will be sought in the form

$$\vartheta^{(M)}(t, x) = \sum_{i=1}^{M} \alpha_i(t)\Psi_i(x), \quad \alpha_j \in W^{1,4/3}(\mathbb{T}). \tag{4.15}$$

In order to determine the T-periodic functions $\alpha_1, ..., \alpha_M$, let $\mathcal{F} : \mathbb{T} \times \mathbb{R}^M \to \mathbb{R}^M$ with components

$$\begin{aligned}\mathcal{F}_m(t, \alpha) = &-\sum_{i=1}^{M} \alpha_i \left[a(\Psi_i, \Psi_m) + c(\Psi_i, \tilde{h}, \Psi_m) + c(\tilde{h}, \Psi_i, \Psi_m)\right] \\ &- \sum_{i,j=1}^{M} \alpha_i\alpha_j c(\Psi_i, \Psi_j, \Psi_m) \\ &+ \left(f - \partial_t\tilde{h}, \Psi_m\right)_{\Omega_R} - a(\tilde{h}, \Psi_m) - c(\tilde{h}, \tilde{h}, \Psi_m), \quad m = 1, ..., M,\end{aligned}$$

where

$$\left(f - \partial_t\tilde{h}, \Psi_m\right)_{\Omega_R} \in L^2(\mathbb{T}), \quad a(\tilde{h}, \Psi_m),\ c(\tilde{h}, \tilde{h}, \Psi_m) \in C(\mathbb{T}), \quad m = 1, ..., M.$$

Then (4.15), more specifically $\alpha = (\alpha_1, ..., \alpha_M)$, will be obtained as a T-periodic solution of the systems of ODEs

$$\alpha' = \mathcal{F}(t, \alpha) \text{ in } \mathbb{T}. \tag{4.16}$$

At this stage, $M \in \mathbb{N}$ is fixed. For a fixed $\underline{\alpha} \in W^{1,4/3}(\mathbb{T})^M$, consider the linearized problem

$$\alpha' = \mathcal{L}(t, \alpha; \underline{\alpha}) \text{ in } \mathbb{T}. \tag{4.17}$$

where

$$\begin{aligned}&\mathcal{L}(\,\cdot, \cdot\,; \underline{\alpha}) : \mathbb{T} \times \mathbb{R}^M \to \mathbb{R}^M, \\ &\mathcal{L}_m(t, \alpha; \underline{\alpha}) = -\sum_{i=1}^{M} \alpha_i A_{im} - \sum_{i=1}^{M} \underline{\alpha}_i \left[c(\Psi_i, \tilde{h}, \Psi_m) + c(\tilde{h}, \Psi_i, \Psi_m)\right] \\ &\qquad\qquad - \sum_{i,j=1}^{M} \underline{\alpha}_i\underline{\alpha}_j c(\Psi_i, \Psi_j, \Psi_m) + g_m(t), \qquad m = 1, ..., M,\end{aligned}$$

and

$$\begin{aligned}A_{im} &:= a(\Psi_i, \Psi_m), & i, m = 1, ..., M, \\ g_m(t) &:= \left(f - \partial_t\tilde{h}, \Psi_m\right)_{\Omega_R} - a(\tilde{h}, \Psi_m) - c(\tilde{h}, \tilde{h}, \Psi_m), & m = 1, ..., M.\end{aligned}$$

In order to alleviate the presentation, we put

$$\psi_0(t) := 1, \quad \psi_k^c(t) := \sqrt{2}\cos\left(\frac{2\pi}{\mathcal{T}}kt\right), \quad \psi_k^s(t) := \sqrt{2}\sin\left(\frac{2\pi}{\mathcal{T}}kt\right), \quad k \in \mathbb{N},$$

and recall the orthonormality relations for $\{\psi_0,\, \psi_k^c,\, \psi_k^s : k \in \mathbb{N}\}$ in $L^2(\mathbb{T})$. A solution for the system of ODEs (4.17) can be sought in the form of a Fourier series

$$\alpha_i(t) = \alpha_{i0}\psi_0(t) + \sum_{k=1}^{\infty} \alpha_{ik}^c \psi_k^c(t) + \sum_{k=1}^{\infty} \alpha_{ik}^s \psi_k^s(t), \quad i = 1, ..., M. \tag{4.18}$$

It is convenient to write (4.17) as

$$\alpha' + A\alpha = G(\underline{\alpha}) \text{ in } \mathbb{T} \tag{4.19}$$

with $A := (A_{im})_{1\le i,m\le M}$ and $G(\underline{\alpha}) = (G_m(\underline{\alpha}))_{1\le m\le M}$, where

$$G_m(\underline{\alpha}) := g_m - \sum_{i=1}^{M} \underline{\alpha}_i \left[c(\Psi_i, \tilde{h}, \Psi_m) + c(\tilde{h}, \Psi_i, \Psi_m)\right] - \sum_{i,j=1}^{M} \underline{\alpha}_i \underline{\alpha}_j c(\Psi_i, \Psi_j, \Psi_m) \in L^{4/3}(\mathbb{T}).$$

Based on (4.18), we define

$$\alpha_0 := \begin{bmatrix} \alpha_{10} \\ \alpha_{20} \\ \vdots \\ \alpha_{M0} \end{bmatrix}, \quad \alpha_k^s := \begin{bmatrix} \alpha_{1k}^s \\ \alpha_{2k}^s \\ \vdots \\ \alpha_{Mk}^s \end{bmatrix}, \quad \alpha_k^c := \begin{bmatrix} \alpha_{1k}^c \\ \alpha_{2k}^c \\ \vdots \\ \alpha_{Mk}^c \end{bmatrix}, \quad k \in \mathbb{N}$$

and

$$G_{m0} := \int_{\mathbb{T}} G_m(t)\psi_0^s(t)\mathrm{d}t, \quad G_{mk}^s := \int_{\mathbb{T}} G_m(t)\psi_k^s(t)\mathrm{d}t, \quad G_{mk}^c := \int_{\mathbb{T}} G_m(t)\psi_k^c(t)\mathrm{d}t,$$

$$G_0 := \begin{bmatrix} G_{10} \\ G_{20} \\ \vdots \\ G_{M0} \end{bmatrix}, \quad G_k^s := \begin{bmatrix} G_{1k}^s \\ G_{2k}^s \\ \vdots \\ G_{Mk}^s \end{bmatrix}, \quad G_k^c := \begin{bmatrix} G_{1k}^c \\ G_{2k}^c \\ \vdots \\ G_{Mk}^c \end{bmatrix}, \quad k \in \mathbb{N}.$$

The Fourier coefficients of a solution α to (4.19) can be obtained by solving the sequence of linear systems

$$A\alpha_0 = G_0, \qquad \begin{bmatrix} A & -\frac{2\pi}{\mathcal{T}}k\,\mathbb{I}_M \\ \frac{2\pi}{\mathcal{T}}k\,\mathbb{I}_M & A \end{bmatrix} \begin{bmatrix} \alpha_k^s \\ \alpha_k^c \end{bmatrix} = \begin{bmatrix} G_k^s \\ G_k^c \end{bmatrix}, \quad k \in \mathbb{N}. \tag{4.20}$$

Here, $\mathbb{I}_M$ is the identity matrix in $\mathbb{R}^{M\times M}$. Note that the matrix $A \in \mathbb{R}^{M\times M}$ is positive definite since we have

$$\sum_{i,m=1}^{M} z_i A_{im} z_m = a(\Psi, \Psi) = \|\nabla\Psi\|_{2,\Omega_R}^2 + \int_{\partial B_R} \left(\frac{1}{R} + \frac{|\zeta|}{2}\right) |\Psi|^2 \mathrm{d}S \quad (z \in \mathbb{R}^M,\ \Psi = z_i\Psi_i),$$

and the block matrices in (4.20), defined in terms of A and $\frac{2\pi}{\mathcal{T}}k\,\mathbb{I}_M$, are nonsingular. For each $k \in \mathbb{N}$,

$$\begin{aligned} \begin{bmatrix} \alpha_k^s \\ \alpha_k^c \end{bmatrix} &= \begin{bmatrix} A & -\frac{2\pi k}{\mathcal{T}}\,\mathbb{I}_M \\ \frac{2\pi k}{\mathcal{T}}\,\mathbb{I}_M & A \end{bmatrix}^{-1} \begin{bmatrix} G_k^s \\ G_k^c \end{bmatrix} \\ &= \begin{bmatrix} \left(A^2 + \frac{4\pi^2k^2}{\mathcal{T}^2}\mathbb{I}_M\right)^{-1} A & \frac{2\pi k}{\mathcal{T}}\left(A^2 + \frac{4\pi^2k^2}{\mathcal{T}^2}\mathbb{I}_M\right)^{-1} \\ -\frac{2\pi k}{\mathcal{T}}\left(A^2 + \frac{4\pi^2k^2}{\mathcal{T}^2}\mathbb{I}_M\right)^{-1} & \left(A^2 + \frac{4\pi^2k^2}{\mathcal{T}^2}\mathbb{I}_M\right)^{-1} A \end{bmatrix} \begin{bmatrix} G_k^s \\ G_k^c \end{bmatrix} \end{aligned}$$

and, with $A_k := \frac{\mathcal{T}}{2\pi k}A$, for $k \in \mathbb{N}$, we have

$$\begin{bmatrix} \alpha_k^s \\ \alpha_k^c \end{bmatrix} = \frac{\mathcal{T}}{2\pi k}\begin{bmatrix} (A_k^2+\mathbb{I}_M)^{-1}A_kG_k^s + (A_k^2+\mathbb{I}_M)^{-1}G_k^c \\ -(A_k^2+\mathbb{I}_M)^{-1}G_k^s + (A_k^2+\mathbb{I}_M)^{-1}A_kG_k^c \end{bmatrix}, \, k \in \mathbb{N}.$$

For any matrix norm $\|\cdot\|$, there exists a constant $C > 0$ such that $\|(A_k^2 + \mathbb{I}_M)^{-1}\| \leq C$, for all $k > \|A\|$. By Hausdorff-Young inequality, we have $\{G_k^s\}_{k\in\mathbb{N}}, \{G_k^c\}_{k\in\mathbb{N}} \in \ell^4(\mathbb{N})^M$ and therefore, $\{k\alpha_k^s\}_{k\in\mathbb{N}}, \{k\alpha_k^c\}_{k\in\mathbb{N}} \in \ell^4(\mathbb{N})^M$. From Hölder's inequality, we obtain $\{\alpha_k^s\}_{k\in\mathbb{N}}, \{\alpha_k^c\}_{k\in\mathbb{N}} \in \ell^{4r/(4+r)}(\mathbb{N})^M$, for all $r > 4/3$. Thus $\{\alpha_k^s\}_{k\in\mathbb{N}}, \{\alpha_k^c\}_{k\in\mathbb{N}} \in \ell^r(\mathbb{N})^M$, for all $1 < r < 4$. This, in turn, yields the existence of $\alpha \in L^2(\mathbb{T})^M$ solving (4.17), and from the identity (4.19), it follows that $\alpha' \in L^{4/3}(\mathbb{T})^M$.

We can thus consider the mapping

$$\begin{aligned} &\mathcal{M} : W^{1,4/3}(\mathbb{T})^M \to W^{1,4/3}(\mathbb{T})^M \\ &\mathcal{M}(\underline{\alpha}) = \alpha. \end{aligned}$$

Our aim is to establish existence of a fixed point of $\mathcal{M}$.

In order to use the Leray–Schauder fixed-point Theorem, we first show that the solution of the problem

$$\alpha' + A\alpha = \lambda G(\alpha) \text{ in } \mathbb{T} \tag{4.21}$$

are uniformly bounded with respect to $\lambda \in [0,1]$. By taking the dot product of both sides of equation (4.21) with α, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\alpha|^2 = &-\sum_{i,m=1}^{M}\alpha_i A_{im}\alpha_m - \lambda\sum_{i,m=1}^{M}\alpha_i\alpha_m\left[c(\Psi_i,\tilde{h},\Psi_m) + c(\tilde{h},\Psi_i,\Psi_m)\right] \\ &-\lambda\sum_{i,j,m=1}^{M}\alpha_i\alpha_j\alpha_m c(\Psi_i,\Psi_j,\Psi_m) + \lambda\sum_{m=1}^{M}g_m\alpha_m, \quad \lambda \in [0,1]. \end{aligned}$$

Recalling (4.15) and using the orthonormality conditions that are induced by $\{\psi_0, \psi_k^c, \psi_k^s : k \in \mathbb{N}\}$ and $\{\Psi_1, ..., \Psi_M\}$ in $L^2(\mathbb{T}, H_R^{(M)})$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\vartheta^{(M)}\|_{2,\Omega_R}^2 &+ a(\vartheta^{(M)},\vartheta^{(M)}) + \lambda c(\vartheta^{(M)},\tilde{h},\vartheta^{(M)}) \\ &+ \lambda c(\tilde{h},\vartheta^{(M)},\vartheta^{(M)}) + \lambda c(\vartheta^{(M)},\vartheta^{(M)},\vartheta^{(M)}) \\ &= \lambda\big(f - \partial_t\tilde{h},\vartheta^{(M)}\big)_{\Omega_R} - \lambda a(\tilde{h},\vartheta^{(M)}) - \lambda c(\tilde{h},\tilde{h},\vartheta^{(M)}). \end{aligned} \tag{4.22}$$

Since, by the time-periodicity of $\vartheta^{(M)}$ and by (4.5), it holds

$$\int_{\mathbb{T}}\frac{d}{dt}\|\vartheta^{(M)}\|_{2,\Omega_R}^2\mathrm{dt} = 0, \quad c(\tilde{h},\vartheta^{(M)},\vartheta^{(M)}) = c(\vartheta^{(M)},\vartheta^{(M)},\vartheta^{(M)}) = 0,$$

and, by direct calculation,

$$c(\vartheta^{(M)},\tilde{h},\vartheta^{(M)}) = -c(\vartheta^{(M)},\vartheta^{(M)},\tilde{h}),$$

we obtain

$$
\begin{aligned}
\|\vartheta^{(M)}\|^2_{(\mathbb{T},R)} \leq & \int_{\mathbb{T}} \left(\|\vartheta^{(M)}\|^2_{(R)} + \frac{|\zeta|}{2} \|\vartheta^{(M)}\|^2_{2,\partial B_R} \right) \mathrm{d}t \\
= & \int_{\mathbb{T}} a(\vartheta^{(M)}, \vartheta^{(M)}) \mathrm{d}t \\
= & \lambda \int_{\mathbb{T}} c(\vartheta^{(M)}, \vartheta^{(M)}, \tilde{h}) \mathrm{d}t + \lambda \int_{\mathbb{T}} \left(f - \partial_t \tilde{h}, \vartheta^{(M)} \right)_{\Omega_R} \mathrm{d}t \\
& - \lambda \int_{\mathbb{T}} a(\tilde{h}, \vartheta^{(M)}) \mathrm{d}t - \lambda \int_{\mathbb{T}} c(\tilde{h}, \tilde{h}, \vartheta^{(M)}) \mathrm{d}t.
\end{aligned}
\tag{4.23}
$$

From Lemma 4.6, estimate (4.7), we conclude

$$
\lambda \int_{\mathbb{T}} c(\vartheta^{(M)}, \vartheta^{(M)}, \tilde{h}) \mathrm{d}t \leq \gamma \|\vartheta^{(M)}\|^2_{(\mathbb{T},R)} + \|\Phi\|_{\infty,\mathbb{T}} \left[C_S \|\sigma\|_{3,\Omega_R} + \frac{1}{8\pi R} \right] \|\vartheta^{(M)}\|^2_{(\mathbb{T},R)},
$$

and since $1 - \|\Phi\|_{\infty,\mathbb{T}} \left[C_S \|\sigma\|_{3,\Omega_R} + \frac{1}{8\pi R} \right] > \gamma > 0$, by estimating the remaining terms in the last equality of (4.23), after estimating the remaining terms on the last equality, we arrive at

$$
\|\vartheta^{(M)}\|_{(\mathbb{T},R)} \leq \frac{C(\Omega_R) \left[\|f\|_{L^2(\mathbb{T}\times\Omega)} + (1+|\zeta|) \|\tilde{h}\|_{H^1(\mathbb{T}\times\Omega)} + \|\tilde{h}\|^2_{H^1(\mathbb{T}\times\Omega)} \right]}{\left(1 - \|\Phi\|_{\infty,\mathbb{T}} \left[C_S \|\sigma\|_{3,\Omega_R} + \frac{1}{8\pi R} \right] - \gamma \right)^{1/2}}.
\tag{4.24}
$$

By Poincaré inequality and the orthonormality conditions in $L^2(\mathbb{T}, H_R^{(M)})$, this implies

$$
\begin{aligned}
\|\alpha\|_{L^2(\mathbb{T})^M} & = \|\vartheta^{(M)}\|_{L^2(\mathbb{T};H_R)} \\
& \leq C_P(\Omega_R) \|\vartheta^{(M)}\|_{L^2(\mathbb{T};V_R)} = C_P(\Omega_R) \|\vartheta^{(M)}\|_{(\mathbb{T},R)} \\
& \leq \frac{C_P(\Omega_R) C(\Omega_R) \left[\|f\|_{L^2(\mathbb{T}\times\Omega)} + (1+|\zeta|) \|\tilde{h}\|_{H^1(\mathbb{T}\times\Omega)} + \|\tilde{h}\|^2_{H^1(\mathbb{T}\times\Omega)} \right]}{\left(1 - \|\Phi\|_{\infty,\mathbb{T}} \left[C_S \|\sigma\|_{3,\Omega_R} + \frac{1}{8\pi R} \right] - \gamma \right)^{1/2}},
\end{aligned}
$$

where $C_P(\Omega_R)$ is a Poincaré constant on Ω_R. Then, going back to (4.21), we conclude that $\alpha \in W^{1,4/3}(\mathbb{T})^M$ and α' is also bounded by the data in $L^{4/3}(\mathbb{T})^M$.

Now, we show that the mapping $\mathcal{M}$ is compact. Suppose that the sequence $\{\underline{\alpha}^{(k)}\}_{k\in\mathbb{N}} \subset W^{1,4/3}(\mathbb{T})^M$ is bounded. We have

$$
(\alpha^{(k)} - \alpha^{(\ell)})' + A(\alpha^{(k)} - \alpha^{(\ell)}) = G(\underline{\alpha}^{(k)}) - G(\underline{\alpha}^{(\ell)}) \text{ in } \mathbb{T}
\tag{4.25}
$$

where, for each $m \in \{1, ..., M\}$,

$$
\begin{aligned}
& G_m(\underline{\alpha}^{(k)}) - G_m(\underline{\alpha}^{(\ell)}) \\
= & \sum_{i=1}^{M} (\underline{\alpha}_i^{(\ell)} - \underline{\alpha}_i^{(k)}) \left[c(\Psi_i, \tilde{h}, \Psi_m) + c(\tilde{h}, \Psi_i, \Psi_m) \right] \\
& + \sum_{i,j=1}^{M} (\underline{\alpha}_i^{(\ell)} \underline{\alpha}_j^{(\ell)} - \underline{\alpha}_i^{(k)} \underline{\alpha}_j^{(k)}) c(\Psi_i, \Psi_j, \Psi_m) \\
= & \sum_{i=1}^{M} (\underline{\alpha}_i^{(\ell)} - \underline{\alpha}_i^{(k)}) \left[c(\Psi_i, \tilde{h}, \Psi_m) + c(\tilde{h}, \Psi_i, \Psi_m) \right] \\
& + \sum_{i,j=1}^{M} \left[\underline{\alpha}_j^{(\ell)} (\underline{\alpha}_i^{(\ell)} - \underline{\alpha}_i^{(k)}) + \underline{\alpha}_i^{(k)} (\underline{\alpha}_j^{(\ell)} - \underline{\alpha}_j^{(k)}) \right] c(\Psi_i, \Psi_j, \Psi_m).
\end{aligned}
\tag{4.26}
$$

The embedding $W^{1,4/3}(\mathbb{T})^M \hookrightarrow C(\mathbb{T})^M$ is compact, hence $\{\underline{\alpha}^{(k)}\}_{k\in\mathbb{N}}$ contains a subsequence $\{\underline{\alpha}^{(k')}\}_{k'\in\mathbb{N}}$ that converges in $C(\mathbb{T})^M$. Let

$$\underline{\vartheta}^{(M,k')}(t,x) := \sum_{i=1}^{M} \underline{\alpha}_i^{(k')}(t)\Psi_i(x), \quad \vartheta^{(M,k')}(t,x) := \sum_{i=1}^{M} \alpha_i^{(k')}(t)\Psi_i(x). \tag{4.27}$$

The sequence $\{\underline{\vartheta}^{(M,k')}\}_{k'\in\mathbb{N}}$ converges in $L^2(\mathbb{T};V_R^{(M)})$ and therefore it is a Cauchy sequence in $L^2(\mathbb{T};V_R)$. Taking the dot product of both sides of (4.25) with $\alpha^{(k)}-\alpha^{(\ell)}$ and recalling (4.26), we get

$$\begin{aligned}\|\vartheta^{(M,k')}-\vartheta^{(M,\ell')}\|^2_{L^2(\mathbb{T};V_R)} &\leq \int_{\mathbb{T}} a\big(\vartheta^{(M,k')}-\vartheta^{(M,\ell')},\vartheta^{(M,k')}-\vartheta^{(M,\ell')}\big)\mathrm{d}t\\ &\leq C(\Omega_R,M)\Big(\|\tilde h\|_{H^1(\mathbb{T}\times\Omega)}\|\vartheta^{(M,k')}-\vartheta^{(M,\ell')}\|_{L^2(\mathbb{T};V_R)}\|\underline{\vartheta}^{(M,k')}-\underline{\vartheta}^{(M,\ell')}\|_{L^2(\mathbb{T};V_R)}\\ &\qquad + \big(\|\underline{\vartheta}^{(M,k')}\|_{L^\infty(\mathbb{T};V_R)}+\|\underline{\vartheta}^{(M,\ell')}\|_{L^\infty(\mathbb{T};V_R)}\big)\\ &\qquad\quad \times \|\vartheta^{(M,k')}-\vartheta^{(M,\ell')}\|_{L^2(\mathbb{T};V_R)}\|\underline{\vartheta}^{(M,k')}-\underline{\vartheta}^{(M,\ell')}\|_{L^2(\mathbb{T};V_R)}\Big).\end{aligned}$$

As in previous estimates, by Poincaré inequality and the orthonormality conditions in $L^2(\mathbb{T},H_R^{(M)})$, we get

$$\begin{aligned}\|\alpha^{(k')}-\alpha^{(\ell')}\|_{L^2(\mathbb{T})^M} &= \|\vartheta^{(M,k')}-\vartheta^{(M,\ell')}\|_{L^2(\mathbb{T};H_R)}\\ &\leq C_P(\Omega_R)\|\vartheta^{(M,k')}-\vartheta^{(M,\ell')}\|_{L^2(\mathbb{T};V_R)},\end{aligned}$$

and now we use the fact that $\{\underline{\vartheta}^{(M,k')}\}_{k'\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{T};V_R)$ to conclude that $\{\alpha^{(k')}\}_{k'\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{T})^M$. From (4.25) and the previous estimates, we also get

$$\|(\alpha^{(k')})'-(\alpha^{(\ell')})'\|_{L^{4/3}(\mathbb{T})^M} \leq \|A(\alpha^{(k')}-\alpha^{(\ell')})\|_{L^{4/3}(\mathbb{T})^M}+\|G(\underline{\alpha}^{(k')})-G(\underline{\alpha}^{(\ell')})\|_{L^{4/3}(\mathbb{T})^M}.$$

Using the strong convergence of $\{\underline{\alpha}^{(k')}\}_{k'\in\mathbb{N}}$ in $C(\mathbb{T})^M$, we conclude that $\mathcal{M}$ maps bounded sequences into relatively compact ones. In conclusion, the Leray–Schauder Theorem shows that the mapping $\mathcal{M}$ has a fixed point.

We thus solved (4.16) with M fixed and obtained an approximate solution (4.15) which satisfies (4.24). Now, we derive additional estimates for the sequence $\{\vartheta^{(M)}\}_{M\in\mathbb{N}}$. Actually, $\vartheta^{(M)}\in C(\mathbb{T};V_R)$, and from the estimate (4.24) and the mean value theorem for continuous functions, we conclude the existence of $\bar t\in(0,\mathcal{T})$ such that

$$\begin{aligned}\|\vartheta^{(M)}(\bar t)\|^2_{(R)} &= \int_{\mathbb{T}}\|\vartheta^{(M)}\|^2_{(R)}\mathrm{d}t = \|\vartheta^{(M)}\|^2_{(\mathbb{T},R)}\\ &\leq \frac{C(\Omega_R)^2\Big[\|f\|_{L^2(\mathbb{T}\times\Omega)}+(1+|\zeta|)\|\tilde h\|_{H^1(\mathbb{T}\times\Omega)}+\|\tilde h\|^2_{H^1(\mathbb{T}\times\Omega)}\Big]^2}{1-\|\Phi\|_{\infty,\mathbb{T}}\big[C_S\|\sigma\|_{3,\Omega_R}+\frac{1}{8\pi R}\big]-\gamma}.\end{aligned} \tag{4.28}$$

From Poincaré inequality, we further get

$$\|\vartheta^{(M)}(\bar t)\|_{2,\Omega_R} \leq C_P(\Omega_R)\|\vartheta^{(M)}(\bar t)\|_{(R)}. \tag{4.29}$$

Now, on the time interval $[\bar t,\bar t+\mathcal{T}]$, we consider (4.22) (with $\lambda=1$). Taking into account (4.5), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\vartheta^{(M)}\|^2_{2,\Omega_R}+a(\vartheta^{(M)},\vartheta^{(M)})\\ &\quad = c(\vartheta^{(M)},\vartheta^{(M)},\tilde h)+\big(f-\partial_t\tilde h,\vartheta^{(M)}\big)_{\Omega_R}-a(\tilde h,\vartheta^{(M)})-c(\tilde h,\tilde h,\vartheta^{(M)}),\end{aligned} \tag{4.30}$$

where

$$a(\vartheta^{(M)},\vartheta^{(M)}) = \|\nabla\vartheta^{(M)}\|_{2,\Omega_R}^2 + \int_{\partial B_R}\left(\frac{1}{R}+\frac{|\zeta|}{2}\right)|\vartheta^{(M)}|^2 \mathrm{d}S \geq \|\vartheta^{(M)}\|_{(R)}^2,$$

and

$$\begin{aligned}
\big|(f-\partial_t\tilde{h},\vartheta^{(M)})_{\Omega_R}\big| &\leq \frac{1}{4}\|\vartheta^{(M)}\|_{2,\Omega_R}^2 + 2\|f\|_{2,\Omega_R}^2 + 2\|\partial_t\tilde{h}\|_{2,\Omega_R}^2,\\
|a(\tilde{h},\vartheta^{(M)})| &\leq C(\Omega_R)(1+|\zeta|)\|\tilde{h}\|_{1,2,\Omega_R}\|\vartheta^{(M)}\|_{(R)} + |\zeta|\|\tilde{h}\|_{1,2,\Omega_R}\|\vartheta^{(M)}\|_{2,\Omega_R}\\
&\leq \frac{1}{4}\|\vartheta^{(M)}\|_{(R)}^2 + \frac{1}{4}\|\vartheta^{(M)}\|_{2,\Omega_R}^2 + C(\Omega_R)(1+|\zeta|)^2\|\tilde{h}\|_{1,2,\Omega_R}^2,\\
|c(\tilde{h},\tilde{h},\vartheta^{(M)})| &\leq \frac{1}{4}\|\vartheta^{(M)}\|_{(R)}^2 + C(\Omega_R)\|\tilde{h}\|_{1,2,\Omega_R}^4.
\end{aligned}$$

We estimate the term $c(\vartheta^{(M)},\vartheta^{(M)},\tilde{h})$ using Lemma 4.6, apply the preceding estimates, and use assumption (4.11) to deduce

$$\begin{aligned}
\frac{d}{dt}\|\vartheta^{(M)}\|_{2,\Omega_R}^2 + &\left[1-2\|\Phi\|_{\infty,\mathbb{T}}\left(C_S\|\sigma\|_{3,\Omega_R}+\frac{1}{8\pi R}\right)-2\gamma\right]\|\vartheta^{(M)}\|_{(R)}^2\\
&\leq \|\vartheta^{(M)}\|_{2,\Omega_R}^2 + C_2(\Omega_R)(1+|\zeta|)^2\|\tilde{h}\|_{1,2,\Omega_R}^2 + C_3(\Omega_R)\|\tilde{h}\|_{1,2,\Omega_R}^4\\
&\quad + 4\|f\|_{2,\Omega_R}^2 + 4\|\partial_t\tilde{h}\|_{2,\Omega_R}^2,
\end{aligned}$$

which we combine with (4.29) and the estimate (4.28) for $\|\vartheta^{(M)}(\bar{t})\|_{(R)}$. The Grönwall Lemma and the time-periodicity of $\vartheta^{(M)}$ yield

$$\|\vartheta^{(M)}\|_{L^\infty(\mathbb{T};L^2(\Omega_R))} \leq C\left(\Omega_R,\mathcal{T},|\zeta|,\|f\|_{L^2(\mathbb{T};L^2(\Omega_R))},\|\tilde{h}\|_{H^1(\mathbb{T};H^1(\Omega))}\right). \tag{4.31}$$

An estimate for the time derivative of $\vartheta^{(M)}$ can be obtained as follows: for each $M\in\mathbb{N}$, let $\mathbb{P}_M$ be the orthogonal projector onto $\operatorname{span}\{\Psi_1,\dots,\Psi_M\}$ in V_R. Recall that, by Lemma 4.2, we have, for each $\Phi\in V_R$,

$$\|\mathbb{P}_M\Phi\|_{(R)} \leq \|\Phi\|_{(R)}, \qquad \mathbb{P}_M\Phi\to\Phi \text{ in } V_R \text{ as } M\to\infty.$$

Since $\{\Psi_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system in H_R, we have

$$\int_{\Omega_R}\partial_t\vartheta^{(M)}\cdot\Phi\,\mathrm{d}x = \int_{\Omega_R}\partial_t\vartheta^{(M)}\cdot(\mathbb{P}_M\Phi)\,\mathrm{d}x, \quad \forall\Phi\in H_R$$

and therefore

$$\begin{aligned}
&\big(\partial_t\vartheta^{(M)},\Phi\big)_{\Omega_R} + a(\vartheta^{(M)},\mathbb{P}_M\Phi) + c(\vartheta^{(M)},\tilde{h},\mathbb{P}_M\Phi)\\
&\quad + c(\tilde{h},\vartheta^{(M)},\mathbb{P}_M\Phi) + c(\vartheta^{(M)},\vartheta^{(M)},\mathbb{P}_M\Phi)\\
&\qquad = (f,\mathbb{P}_M\Phi)_{\Omega_R} - \big(\partial_t\tilde{h},\mathbb{P}_M\Phi\big)_{\Omega_R} - a(\tilde{h},\mathbb{P}_M\Phi) - c(\tilde{h},\tilde{h},\mathbb{P}_M\Phi), \quad \forall\Phi\in V_R,
\end{aligned}$$

which, by setting

$$\begin{aligned}
\langle\mathcal{A}v,w\rangle_{V_R',V_R} &:= a(v,w),\\
\langle\mathcal{C}(u,v),w\rangle_{V_R',V_R} &:= c(u,v,w),
\end{aligned}$$

can be written as

$$\begin{aligned}
\langle\partial_t\vartheta^{(M)},\Psi\rangle_{V_R',V_R} = -\langle\mathcal{A}v^{(M)} + \mathcal{C}(v^{(M)},v^{(M)}),\mathbb{P}_M\Psi\rangle_{V_R',V_R}&\\
+\langle f-\partial_t\tilde{h},\mathbb{P}_M\Psi\rangle_{V_R',V_R},& \quad \forall\Psi\in V_R, \text{ in } \mathbb{T},
\end{aligned}$$

where $v^{(M)} = \vartheta^{(M)} + \tilde{h}$. By interpolation, from (4.24) and (4.31) we deduce that $\{v^{(M)}\}_{M\in\mathbb{N}} \subset L^4(\mathbb{T}; L^3(\Omega_R))$ is uniformly bounded. Since

$$\begin{aligned} \|\mathcal{A}(v)\|_{V_R'} &\leq C(\Omega_R)(1+|\zeta|)\|v\|_{1,2,\Omega_R}, \qquad \forall v \in H^1(\Omega_R),\\ \|\mathcal{C}(v,v)\|_{V_R'} &\leq C(\Omega_R)\|v\|_{3;\Omega}\|v\|_{1,2;\Omega}, \qquad \forall v \in H^1(\Omega_R),\\ \|\partial_t\vartheta^{(M)}\|_{V_R'} &\leq \|\mathcal{A}(v^{(M)})\|_{V_R'} + \|\mathcal{C}(v^{(M)},v^{(M)})\|_{V_R'} + \|f\|_{2,\Omega_R} + \|\partial_t\tilde{h}\|_{2,\Omega_R}, \end{aligned} \tag{4.32}$$

we conclude that $\{\partial_t\vartheta^{(M)}\}_{M\in\mathbb{N}}$ remains in a bounded set of $L^{4/3}(\mathbb{T}; V_R')$. Here, we used that, by integration by parts and Sobolev embeddings, we have

$$c(u,v,w) = \frac{1}{2}\int_{\Omega_R} u\cdot\nabla v\cdot w\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_R} u\cdot\nabla w\cdot v\,\mathrm{d}x \leq C(\Omega_R)\|u\|_{3;\Omega}\|v\|_{1,2;\Omega}\|w\|_{1,2;\Omega}$$

if $u \in V_R$. A combination with the above uniform estimates (4.24) and (4.31) enable us to assert the existence of an element $\vartheta \in L^\infty(\mathbb{T}; H_R)\cap L^2(\mathbb{T}; V_R)$ with $\partial_t\vartheta \in L^{4/3}(\mathbb{T}; V_R')$, and a sub-sequence $\{\vartheta^{(M')}\}$ of $\{\vartheta^{(M)}\}_{M\in\mathbb{N}}$ such that

$$\begin{aligned} \nabla\vartheta^{(M')} &\to \nabla\vartheta && \text{in } L^2(\mathbb{T}; L^2(\Omega_R)) \text{ weakly},\\ \vartheta^{(M')} &\to \vartheta && \text{in } L^\infty(\mathbb{T}; H_R) \text{ weakly-*},\\ \partial_t\vartheta^{(M')} &\to \partial_t\vartheta && \text{in } L^{4/3}(\mathbb{T}; V_R') \text{ weakly},\\ \vartheta^{(M')} &\to \vartheta && \text{in } L^2(\mathbb{T}; L^{q_1}(\Omega)) \text{ strongly}, \quad 1\leq q_1 < 6,\\ \vartheta^{(M')}|_{\partial B_R} &\to \vartheta|_{\partial B_R} && \text{in } L^2(\mathbb{T}; L^{q_2}(\partial B_R)) \text{ strongly}, \quad 1\leq q_2 < 4, \end{aligned}$$

where the latter convergences follow from the Aubin–Lions Theorem and the embeddings (4.2). Passing to the limit $M' \to \infty$ in (4.16), with standard arguments, we find that $v := \vartheta + \tilde{h}$ satisfies

$$\begin{aligned} \int_{\mathbb{T}} (v(t), \psi'(t)\Psi)_{\Omega_R}\mathrm{d}t &= \int_{\mathbb{T}} a(v(t), \psi(t)\Psi)\mathrm{d}t\\ &+ \int_{\mathbb{T}} c(v(t), v(t), \psi(t)\Psi)\mathrm{d}t - \int_{\mathbb{T}} (f(t), \psi(t)\Psi)_{\Omega_R}\,\mathrm{d}t,\ \forall\Psi \in V_R,\ \forall\psi \in \mathcal{D}(\mathbb{T}). \end{aligned} \tag{4.33}$$

Since the function spaces

$$\left\{\Phi : \mathbb{T}\times\Omega_R \to \mathbb{R}^3 : \Phi(t,x) = \psi(t)\Psi(x),\ \psi \in \mathcal{D}(\mathbb{T}),\ \Psi \in W_R\right\},$$
$$\left\{\eta : \mathbb{T}\times\Omega_R \to \mathbb{R} : \eta(t,x) = \psi(t)\phi(x),\ \psi \in \mathcal{D}(\mathbb{T}),\ \phi \in L^2(\Omega_R)\right\}$$

are dense in $H^1(\mathbb{T}; W_R)$ and in $L^2(\mathbb{T}; L^2(\Omega_R))$, respectively, we obtain an equivalent definition of weak solution (in the velocity variable):

$$\begin{aligned} \int_{\mathbb{T}} (v(t), \Phi'(t))_{\Omega_R}\,\mathrm{d}t &= \int_{\mathbb{T}} a(v(t), \Phi(t))\,\mathrm{d}t\\ &+ \int_{\mathbb{T}} c(v(t), v(t), \Phi(t))\,\mathrm{d}t - \int_{\mathbb{T}} (f(t), \Phi(t))_{\Omega_R}\,\mathrm{d}t, \quad \forall\Phi \in H^1(\mathbb{T}; V_R). \end{aligned} \tag{4.34}$$

The energy inequality (4.13) for ϑ is obtained from (4.30) and integration over $\mathbb{T}$, as

$$\begin{aligned} \int_{\mathbb{T}} \|\vartheta^{(M')}\|^2_{(R,|\zeta|)}\,\mathrm{d}t = &\int_{\mathbb{T}} c(\vartheta^{(M')}, \vartheta^{(M')}, \tilde{h})\,\mathrm{d}t + \int_{\mathbb{T}} \left(f - \partial_t\tilde{h}, \vartheta^{(M')}\right)_{\Omega_R}\mathrm{d}t\\ &- \int_{\mathbb{T}} a(\tilde{h}, \vartheta^{(M')})\,\mathrm{d}t - \int_{\mathbb{T}} c(\tilde{h}, \tilde{h}, \vartheta^{(M')})\,\mathrm{d}t, \end{aligned}$$

where we use the above convergence results for the subsequence $\{\vartheta^{(M')}\}_{M'\in\mathbb{N}}$ to obtain, in particular,

$$\int_{\mathbb{T}} c(\vartheta^{(M')}, \vartheta^{(M')}, \tilde{h})\mathrm{d}t \to \int_{\mathbb{T}} c(\vartheta, \vartheta, \tilde{h})\mathrm{d}t,$$

and, by the lower semicontinuity of the norm,

$$\begin{aligned}
\int_{\mathbb{T}} \|\vartheta\|^2_{(R,|\zeta|)}\mathrm{d}t &\leq \liminf_{M'\to\infty} \int_{\mathbb{T}} \|\vartheta^{(M')}\|^2_{(R,|\zeta|)}\mathrm{d}t \\
&= \int_{\mathbb{T}} c(\vartheta, \vartheta, \tilde{h})\mathrm{d}t + \int_{\mathbb{T}} \big(f - \partial_t \tilde{h}, \vartheta\big)_{\Omega_R}\mathrm{d}t - \int_{\mathbb{T}} a(\tilde{h}, \vartheta)\mathrm{d}t - \int_{\mathbb{T}} c(\tilde{h}, \tilde{h}, \vartheta)\mathrm{d}t \\
&= - \int_{\mathbb{T}} c(\vartheta, \tilde{h}, \vartheta)\mathrm{d}t + \int_{\mathbb{T}} \big(f - \partial_t \tilde{h}, \vartheta\big)_{\Omega_R}\mathrm{d}t - \int_{\mathbb{T}} a(\tilde{h}, \vartheta)\mathrm{d}t - \int_{\mathbb{T}} c(\tilde{h}, \tilde{h}, \vartheta)\mathrm{d}t \\
&= - \int_{\mathbb{T}} c(v, \tilde{h}, \vartheta)\mathrm{d}t + \int_{\mathbb{T}} \big(f - \partial_t \tilde{h}, \vartheta\big)_{\Omega_R}\mathrm{d}t - \int_{\mathbb{T}} a(\tilde{h}, \vartheta)\mathrm{d}t,
\end{aligned}$$

which is (4.13).

We have solved problem (4.33) in $\mathscr{D}'(\mathbb{T}; V_R')$ for the velocity field. Our aim now is to recover the pressure. For this purpose, we follow the ideas of [16], and define $\mathscr{F} \in \mathscr{D}'(\mathbb{T}; W_R')$ as follows:

$$\begin{aligned}
\langle \mathscr{F}(\psi), \Psi\rangle_{\Omega_R} := - \int_{\mathbb{T}} (v(t), \Psi)_{\Omega_R}\, \psi'(t)\,\mathrm{d}t + \int_{\mathbb{T}} \langle \mathscr{A} v(t) + \mathscr{C}(v(t), v(t)), \Psi\rangle_{\Omega_R} \psi(t)\,\mathrm{d}t \\
- \int_{\mathbb{T}} (f(t), \Psi)_{\Omega_R}\, \psi(t)\,\mathrm{d}t \quad (\Psi \in W_R, \psi \in \mathscr{D}(\mathbb{T})),
\end{aligned}$$

where now $\langle \cdot, \cdot\rangle_{\Omega_R}$ represents the duality pairing between W_R' and W_R.

Let $\mathbb{P}_{V_R^\perp}$ be the projection operator from W_R onto $V_R^\perp$, when considering the decomposition $W_R = V_R \oplus V_R^\perp$, orthogonal with respect to the inner product of W_R. Then $\mathbb{P}^*_{V_R^\perp} : (V_R^\perp)' \to W_R'$ and its range is given by

$$\mathrm{Ran}(\mathbb{P}^*_{V_R^\perp}) = V_R^0 := \big\{F \in W_R' : \langle F, u\rangle_{\Omega_R} = 0,\ \forall u \in V_R\big\} \cong (V_R^\perp)'.$$

From the previous results for the velocity field, we have $\mathscr{F} \in \mathscr{D}'(\mathbb{T}; V_R^0)$. This means

$$\mathscr{F} = \mathbb{P}^*_{V_R^\perp}\mathscr{F},$$

and $\mathbb{P}^*_{V_R^\perp}\mathscr{F}$ is given by

$$\begin{aligned}
\langle \mathbb{P}^*_{V_R^\perp}\mathscr{F}(\psi), \Psi\rangle := &- \int_{\mathbb{T}} \langle \mathbb{P}^*_{V_R^\perp} v(t), \Psi\rangle_{\Omega_R} \psi'(t)\,\mathrm{d}t \\
&+ \int_{\mathbb{T}} \langle \mathbb{P}^*_{V_R^\perp}\mathscr{A} v(t) + \mathbb{P}^*_{V_R^\perp}\mathscr{C}(v(t), v(t)), \Psi\rangle_{\Omega_R} \psi(t)\,\mathrm{d}t \\
&- \int_{\mathbb{T}} \langle \mathbb{P}^*_{V_R^\perp} f(t), \Psi\rangle_{\Omega_R} \psi(t)\,\mathrm{d}t \quad (\Psi \in W_R, \psi \in \mathscr{D}(\mathbb{T})).
\end{aligned}$$

Consider the operator $B : V_R^\perp \to L^2(\Omega_R)$ defined by

$$\langle Bv, p\rangle = b(v, p) = - \int_{\Omega_R} (\nabla \cdot v) p\,\mathrm{d}x \qquad (p \in L^2(\Omega)),$$

which is an isomorphism. Then $B^* : L^2(\Omega_R) \to V_R^0 = \mathrm{Ran}(\mathbb{P}^*_{V_R^\perp})$,

$$\langle v, B^* p\rangle = b(v,p) = -\int_{\Omega_R} (\nabla \cdot v) p \, \mathrm{d}x \qquad (v \in V_R^\perp),$$

is also an isomorphism. Therefore, there exists $p_0 \in L^\infty(\mathbb{T}; L^2(\Omega))$, $p_1, p_3 \in L^2(\mathbb{T}; L^2(\Omega))$, $p_2 \in L^{4/3}(\mathbb{T}; L^2(\Omega))$ such that

$$\begin{aligned}
\int_{\mathbb{T}} \langle \mathbb{P}^*_{V_R^\perp} v(t), \Psi\rangle_{\Omega_R} \psi'(t)\mathrm{d}t &= -\int_{\mathbb{T}}\int_{\Omega_R} p_0(t,x)\psi'(t)(\nabla\cdot\Psi)(x)\,\mathrm{d}x\mathrm{d}t,\\
\int_{\mathbb{T}} \langle \mathbb{P}^*_{V_R^\perp} \mathscr{A} v(t), \Psi\rangle_{\Omega_R} \psi(t)\mathrm{d}t &= -\int_{\mathbb{T}}\int_{\Omega_R} p_1(t,x)\psi(t)(\nabla\cdot\Psi)(x)\,\mathrm{d}x\mathrm{d}t,\\
\int_{\mathbb{T}} \langle \mathbb{P}^*_{V_R^\perp} \mathcal{C}(v(t),v(t)), \Psi\rangle_{\Omega_R} \psi(t)\mathrm{d}t &= -\int_{\mathbb{T}}\int_{\Omega_R} p_2(t,x)\psi(t)(\nabla\cdot\Psi)(x)\,\mathrm{d}x\mathrm{d}t,\\
\int_{\mathbb{T}} \langle \mathbb{P}^*_{V_R^\perp} f(t), \Psi\rangle_{\Omega_R} \psi(t)\mathrm{d}t &= -\int_{\mathbb{T}}\int_{\Omega_R} p_3(t,x)\psi(t)(\nabla\cdot\Psi)(x)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Hence

$$\begin{aligned}
&-\int_{\mathbb{T}} (v(t),\Psi)_{\Omega_R}\,\psi'(t)\mathrm{d}t - \int_{\mathbb{T}} (p_0(t),\nabla\cdot\Psi)_{\Omega_R}\,\psi'(t)\,\mathrm{d}x\mathrm{d}t\\
&\quad+\int_{\mathbb{T}} a(v(t),\Psi)\psi(t)\mathrm{d}t + \int_{\mathbb{T}} c(v(t),v(t),\Psi)\psi(t)\mathrm{d}t\\
&\quad= \int_{\mathbb{T}} (f(t),\Psi)_{\Omega_R}\,\psi(t)\mathrm{d}t - \int_{\mathbb{T}} (p_1(t)+p_2(t)+p_3(t),\nabla\cdot\Psi)_{\Omega_R}\,\psi(t)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

for all $\Psi \in W_R$ and $\psi \in \mathscr{D}(\mathbb{T})$, which shows (4.12). This completes the existence proof.

Concerning uniqueness, let us suppose that, in addition to the weak solution (v,p) already constructed, there exists a more regular solution $(\tilde v, \tilde p)$ as formulated in the theorem. To derive an estimate of $\overline{v} := v - \tilde v$ in the norm $\|\cdot\|_{(R,|\zeta|)}$ defined in (4.10), we can argue as in the proof of Lemma 5.2 below, where we compare a weak solution with a strong solution in the exterior domain. Instead of using the strong formulation and integrating by parts in space, we here employ the weak formulation for $(\tilde v, \tilde p)$. Since $\mathscr{B}_{\mathrm{R}}(\tilde v, \tilde p) = 0$ on ∂B_R in a weak sense, several terms from the derivation of (5.2) do not appear, and we arrive at

$$\begin{aligned}
\int_{\mathbb{T}} \|\overline{v}\|^2_{(R,|\zeta|)}\,\mathrm{d}t &\le \int_{\mathbb{T}\times\Omega_R} \overline{v}\cdot\nabla\overline{v}\cdot\tilde v\,\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_{\mathbb{T}\times\partial B_R} \left(\overline{v}\cdot\frac{x}{R}\right)(\tilde v\cdot\overline{v})\,\mathrm{d}S\mathrm{d}t\\
&= -\int_{\mathbb{T}\times\Omega_R} \overline{v}\cdot\nabla\tilde v\cdot\overline{v}\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{\mathbb{T}\times\partial B_R} \left(\overline{v}\cdot\frac{x}{R}\right)(\tilde v\cdot\overline{v})\,\mathrm{d}S\mathrm{d}t.
\end{aligned}$$

Then

$$\int_{\mathbb{T}} \|\overline{v}\|^2_{(R,|\zeta|)}\,\mathrm{d}t \le C(\Omega_R)\big(\|\nabla\tilde v\|_{L^\infty(\mathbb{T};L^2(\Omega_R))} + \|\tilde v\|_{L^\infty(\mathbb{T};L^2(\partial B_R))}\big)\|\nabla\overline{v}\|^2_{L^2(\mathbb{T}\times\Omega_R)},$$

and $\|\nabla\overline{v}\|^2_{L^2(\mathbb{T}\times\Omega_R)} = \|\overline{v}\|^2_{L^2(\mathbb{T}\times\partial B_R)} = 0$ follows from the assumption (4.14) if $\delta > 0$ is sufficiently small. □

Remark 4.5. A similar uniqueness result can be established under the assumption

$$\|\tilde v\|_{L^\infty(\mathbb{T};L^3(\Omega_R))} + \|\tilde v\|_{L^\infty(\mathbb{T};L^2(\partial B_R))} \le \delta,$$

instead of (4.14), in which case the last estimate is replaced with

$$\int_{\mathbb{T}} \|\overline{v}\|^2_{(R,|\zeta|)}\,\mathrm{d}t \le C(\Omega_R)\left(\|\tilde v\|_{L^\infty(\mathbb{T};L^3(\Omega_R))} + \|\tilde v\|_{L^\infty(\mathbb{T};L^2(\partial B_R))}\right)\|\nabla\overline{v}\|^2_{L^2(\mathbb{T}\times\Omega_R)}.$$

5 Estimates of the truncation error

Consider the strong solution (u, p) to problem (1.1) in the exterior domain Ω and the weak solution $(u_R, p_R) = (v, \not p)$ to problem (1.2) in the truncated domain Ω_R, which were established in Theorem 3.1 and Theorem 4.4, respectively. We provide an estimate of the approximation error under the assumption that the total flux Φ through $\partial\Omega$, defined in (3.5), is constant in time.

Theorem 5.1. *Under the assumptions of Theorems 3.1 and 4.4, and if* $\frac{\mathrm{d}}{\mathrm{d}t}\Phi = 0$*, there exist positive constants* C_i*,* $i = 0, 1, 2$*, independent of* R*, such that if* $\varepsilon \leq {}^1/_{C_0}$ *then*

$$\|\nabla u - \nabla u_R\|_{L^2(\mathbb{T}\times\Omega_R)} + \|u - u_R\|_{L^2(\mathbb{T}\times\partial B_R)} \leq (C_1\varepsilon + C_2\varepsilon^2)\frac{1}{R^{1/2}}. \tag{5.1}$$

To prove Theorem 5.1, consider the error $(w, q) := (u, p) - (u_R, p_R)$ associated with the approximation of (u, p) by (u_R, p_R). We measure this error in terms of the following inequality.

Lemma 5.2. *The difference* $w := u - u_R$ *satisfies*

$$\begin{aligned}
&\|\nabla w\|^2_{L^2(\mathbb{T}\times\Omega_R)} + \frac{1}{R}\|w\|^2_{L^2(\mathbb{T}\times\partial B_R)} + \frac{|\zeta|}{2}\|w\|^2_{L^2(\mathbb{T}\times\partial B_R)} \\
&\leq \int_{\mathbb{T}\times\Omega_R} w\cdot\nabla w\cdot u\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}\times\partial B_R}\frac{1}{2}\Big(w\cdot\frac{x}{R}\Big)(u\cdot w)\,\mathrm{d}S(x)\mathrm{d}t \\
&\quad - \int_{\mathbb{T}\times\partial B_R}\frac{1}{2}\Big(u\cdot\frac{x}{R}\Big)(u\cdot w)\,\mathrm{d}S(x)\,\mathrm{d}t + \int_{\mathbb{T}\times\partial B_R}\frac{1}{R}\left(1+\jmath_\zeta(x)\right)(u\cdot w)\,\mathrm{d}S(x)\,\mathrm{d}t \\
&\quad + \int_{\mathbb{T}\times\partial B_R}\frac{x}{R}\cdot\nabla u\cdot w\,\mathrm{d}S(x)\,\mathrm{d}t - \int_{\mathbb{T}\times\partial B_R} p\Big(\frac{x}{R}\cdot w\Big)\,\mathrm{d}S(x)\,\mathrm{d}t.
\end{aligned} \tag{5.2}$$

Proof. In what follows, we again consider the inner product $(\cdot,\cdot)_{(R;|\xi|)}$ in $H^1(\Omega_R)^3$ defined in (4.10), and the multi-linear forms a and c defined in (4.8) and (4.4), respectively. Recall the notation $\vartheta := u_R - \tilde{h}$, and define $\mu := u - \tilde{h} \in H^1(\mathbb{T}; V_R)$, so that $w = u - u_R = \mu - \vartheta$. Then, we have

$$\int_{\mathbb{T}}\|w\|^2_{(R,|\zeta|)}\mathrm{d}t = \int_{\mathbb{T}}(\mu, w)_{(R,|\zeta|)}\mathrm{d}t - \int_{\mathbb{T}}(\vartheta,\mu)_{(R,|\zeta|)}\mathrm{d}t + \int_{\mathbb{T}}\|\vartheta\|^2_{(R,|\zeta|)}\mathrm{d}t$$

Integration by parts in Ω_R and the fact that (u, p) is a strong solution to (1.1) yield

$$\begin{aligned}
&\int_{\mathbb{T}}(\mu, w)_{(R,|\zeta|)}\mathrm{d}t \\
&= \int_{\mathbb{T}\times\partial B_R}\Big(\frac{1}{R}+\frac{|\zeta|}{2}\Big)(\mu\cdot w)\,\mathrm{d}S\mathrm{d}t - \int_{\mathbb{T}}(\partial_t u, w)_{\Omega_R}\,\mathrm{d}t + \int_{\mathbb{T}\times\Omega_R}\zeta\cdot\nabla u\cdot w\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{\mathbb{T}\times\Omega_R} u\cdot\nabla u\cdot w\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}}(f, w)_{\Omega_R}\mathrm{d}t - \int_{\mathbb{T}\times\Omega_R}\nabla\tilde{h}:\nabla w\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\mathbb{T}\times\partial B_R}\frac{x}{R}\cdot\nabla u\cdot w\,\mathrm{d}S(x)\mathrm{d}t - \int_{\mathbb{T}\times\partial B_R}\frac{x}{R}\cdot wp\,\mathrm{d}S(x)\mathrm{d}t.
\end{aligned}$$

We next take the test function $\Phi = \mu = u - \tilde{h}$ in the weak formulation (4.34), which is admissible since $\mu \in L^2(\mathbb{T}; V_R)$, $\tilde{h} \in H^1(\mathbb{T}; H^1(\Omega)^3)$ and $u \in H^1(\mathbb{T}; L^p(\Omega_R)^3)$ for any $p \in (1, \infty)$. Decomposing $\vartheta = u_R - \tilde{h}$, we get

$$\begin{aligned}
-\int_{\mathbb{T}}(\vartheta,\mu)_{(R,|\zeta|)}\mathrm{d}t = &-\int_{\mathbb{T}}(u_R, \partial_t\mu)\,\mathrm{d}t + \int_{\mathbb{T}} c(u_R, u_R, \mu)\,\mathrm{d}t - \int_{\mathbb{T}}(f,\mu)_{\Omega_R}\,\mathrm{d}t + \int_{\mathbb{T}} a(\tilde{h},\mu)\,\mathrm{d}t \\
&- \int_{\mathbb{T}}\int_{\Omega_R}\zeta\cdot\nabla\vartheta\cdot\mu\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}}\int_{\partial B_R}\frac{1}{R}\frac{(\zeta\cdot x)}{2}(\vartheta\cdot\mu)\,\mathrm{d}S(x)\mathrm{d}t.
\end{aligned}$$

Since $\vartheta = u_R - \tilde{h}$ satisfies the energy inequality (4.13), we further have

$$\int_{\mathbb{T}} \|\vartheta\|^2_{(R,|\zeta|)} \mathrm{d}t \leq \int_{\mathbb{T}} \big(f - \partial_t \tilde{h}, \vartheta\big)_{\Omega_R} \mathrm{d}t - \int_{\mathbb{T}} c(u_R, \tilde{h}, \vartheta)\, \mathrm{d}t - \int_{\mathbb{T}} a(\tilde{h}, \vartheta)\, \mathrm{d}t.$$

To combine the terms in the above expressions, we take into account the identities

$$(f, w)_{\Omega_R} - (f, \mu)_{\Omega_R} + (f, \vartheta)_{\Omega_R} = 0,$$

as well as

$$\begin{aligned}\int_{\mathbb{T}} \big[- (\partial_t u, w)_{\Omega_R} - (u_R, \partial_t \mu)_{\Omega_R} - (\partial_t \tilde{h}, \vartheta)_{\Omega_R} \big]\, \mathrm{d}t &= \int_{\mathbb{T}} \big[- (\partial_t u, u)_{\Omega_R} + (\tilde{h}, \partial_t \tilde{h})_{\Omega_R} \big]\, \mathrm{d}t \\ &= \int_{\mathbb{T}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_R} \big[-\frac{1}{2}|u|^2 + \frac{1}{2}|\tilde{h}|^2 \big]\, \mathrm{d}x\mathrm{d}t = 0.\end{aligned}$$

Due to the identity

$$\begin{aligned}&\int_{\Omega_R} \zeta \cdot \nabla u \cdot w \, \mathrm{d}x - \int_{\Omega_R} \nabla \tilde{h} : \nabla w \, \mathrm{d}x + a(\tilde{h}, \mu) - a(\tilde{h}, \vartheta) \\ &\qquad = \int_{\Omega_R} \zeta \cdot \nabla \mu \cdot w \, \mathrm{d}x + \int_{\partial B_R} \frac{1}{R}(1 + \jmath_\zeta(x))\tilde{h} \cdot w \, \mathrm{d}S(x),\end{aligned}$$

we can further collect the terms related with ζ as

$$\begin{aligned}&\int_{\partial B_R} \Big(\frac{1}{R} + \frac{|\zeta|}{2}\Big) (\mu \cdot w)\, \mathrm{d}S + \int_{\Omega_R} \zeta \cdot \nabla u \cdot w \, \mathrm{d}x - \int_{\Omega_R} \nabla \tilde{h} : \nabla w \, \mathrm{d}x + a(\tilde{h}, \mu) \\ &\qquad - \int_{\Omega_R} \zeta \cdot \nabla \vartheta \cdot \mu \, \mathrm{d}x + \int_{\partial B_R} \frac{1}{R} \frac{(\zeta \cdot x)}{2} (\vartheta \cdot \mu)\, \mathrm{d}S(x) - a(\tilde{h}, \vartheta) \\ &= \int_{\partial B_R} \frac{1}{R}(1 + \jmath_\zeta(x))(u \cdot w)\, \mathrm{d}S(x) - \int_{\partial B_R} \frac{1}{R} \frac{(\zeta \cdot x)}{2} (w \cdot \mu)\, \mathrm{d}S(x) + \int_{\Omega_R} \zeta \cdot \nabla \mu \cdot w \, \mathrm{d}x \\ &\qquad + \int_{\Omega_R} \zeta \cdot \nabla \mu \cdot \vartheta \, \mathrm{d}x - \int_{\partial B_R} \frac{1}{R} \frac{(\zeta \cdot x)}{2} (\vartheta \cdot \mu)\, \mathrm{d}S(x) \\ &= \int_{\partial B_R} \frac{1}{R}(1 + \jmath_\zeta(x))(u \cdot w)\, \mathrm{d}S(x),\end{aligned}$$

where we used integration by parts and that $\mu = w + \vartheta$. Recalling the property (4.5) of c, we further have

$$\begin{aligned}&c(u_R, u_R, \mu) - c(u_R, \tilde{h}, \vartheta) \\ &\quad = c(u_R, u_R, \mu) - c(u_R, u_R, \vartheta) + c(u_R, \vartheta, \vartheta) = c(u - w, u - w, w) \\ &\quad = c(u, u, w) - c(w, u, w) - c(u, w, w) + c(w, w, w) = c(u, u, w) + c(w, w, u).\end{aligned}$$

In this way, we arrive at

$$\begin{aligned}&\int_{\mathbb{T}} \|w\|^2_{(R,|\zeta|)} \mathrm{d}t \\ &\leq - \int_{\mathbb{T}\times\Omega_R} u \cdot \nabla u \cdot w \, \mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}\times\partial B_R} \frac{x}{R} \cdot \nabla u \cdot w \, \mathrm{d}S(x)\mathrm{d}t - \int_{\mathbb{T}\times\partial B_R} \frac{x}{R} \cdot wp \, \mathrm{d}S(x)\mathrm{d}t \\ &\quad + \int_{\mathbb{T}} c(u, u, w)\, \mathrm{d}t - \int_{\mathbb{T}} c(w, w, u)\, \mathrm{d}t + \int_{\mathbb{T}\times\partial B_R} \frac{1}{R}(1 + \jmath_\zeta(x))(u \cdot w)\, \mathrm{d}S(x)\mathrm{d}t.\end{aligned}$$

Invoking the definition of c, see (4.4), we conclude (5.2). □

With inequality (5.2) at hand, we now show that the velocity error w tends to zero in appropriate norms. It is useful to recall the properties of strong solutions in the exterior domain, as outlined in Remark 3.2.

Proof of Theorem 5.1. In order to estimate w, we use Lemma 5.2 and write (5.2) as

$$\begin{aligned}\int_{\mathbb{T}} \|w\|_{(R,|\zeta|)}^2 \,\mathrm{d}t \leq &\int_{\mathbb{T}\times\Omega_R} w\cdot\nabla w\cdot u \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}\times\partial B_R} \left[-\frac{1}{2}\left(w\cdot\frac{x}{R}\right)\right](u\cdot w)\,\mathrm{d}S(x)\,\mathrm{d}t \\ &+ \int_{\mathbb{T}\times\partial B_R} \left[-\frac{1}{2}\left(u\cdot\frac{x}{R}\right)\right](u\cdot w)\,\mathrm{d}S(x)\,\mathrm{d}t + \int_{\mathbb{T}\times\partial B_R} \frac{1}{R}\left(1+\mathcal{s}_\zeta(x)\right)(u\cdot w)\,\mathrm{d}S(x)\,\mathrm{d}t \\ &+ \int_{\mathbb{T}\times\partial B_R} \frac{x}{R}\cdot\nabla u\cdot w \,\mathrm{d}S(x)\,\mathrm{d}t + \int_{\mathbb{T}\times\partial B_R} \left[-p\left(\frac{x}{R}\cdot w\right)\right]\mathrm{d}S(x)\,\mathrm{d}t =: \sum_{i=1}^6 I_j,\end{aligned}$$

and we estimate $I_1, \ldots, I_6$ separately.

Take a fixed $S\in(0,\infty)$ with $\partial B_S\subset\Omega$. Let $R>S$. From Lemma 4.1 and estimate (3.2) in Theorem 3.1, we obtain

$$\begin{aligned}\left|\int_{\mathbb{T}\times B_R\setminus\overline{B_S}} w\cdot\nabla w\cdot u\,\mathrm{d}x\mathrm{d}t\right| &\leq \int_{\mathbb{T}\times B_R\setminus\overline{B_S}} \frac{|w(t,x)|}{|x|}|\nabla w(t,x)||x||u(t,x)|\,\mathrm{d}x\mathrm{d}t \\ &\leq C(S)\|w\|_{(\mathbb{T},R)}^2\|u\|_{\infty,\nu^1;\mathbb{T}\times B^S}\end{aligned}$$

and therefore, by Poincaré's and Hölder's inequalities,

$$\begin{aligned}I_1 &\leq \left|\int_{\mathbb{T}\times\Omega_S} w\cdot\nabla w\cdot u\,\mathrm{d}x\mathrm{d}t\right| + \left|\int_{\mathbb{T}\times B_R\setminus\overline{B_S}} w\cdot\nabla w\cdot u\,\mathrm{d}x\mathrm{d}t\right| \\ &\leq C(S,\partial\Omega)\|\nabla w\|_{L^2(\mathbb{T}\times\Omega_S)}^2\|u\|_{L^\infty(\mathbb{T};L^3(\Omega_S))} + C(S)\|u\|_{\infty,\nu^1;\mathbb{T}\times B^S}\|w\|_{(\mathbb{T},R)}^2 \\ &\leq C(S,\partial\Omega)\left[\|u\|_{L^\infty(\mathbb{T};L^3(\Omega_S))} + \|u\|_{\infty,\nu^1;\mathbb{T}\times B^S}\right]\|w\|_{(\mathbb{T},R)}^2 \\ &\leq C(S,\partial\Omega)\,\varepsilon\|w\|_{(\mathbb{T},R)}^2.\end{aligned}$$

From (3.2), we also get the following estimates for the integrals over $\mathbb{T}\times\partial B_R$ involving u:

$$\begin{aligned}I_2 &\leq \frac{1}{2}\int_{\mathbb{T}\times\partial B_R} \frac{1}{R}|w(t,x)|^2|x||u(t,x)|\,\mathrm{d}S(x)\mathrm{d}t \\ &\leq \frac{1}{2}\|u\|_{\infty,\nu^1;\mathbb{T}\times B^S}\frac{1}{R}\|w\|_{L^2(\mathbb{T}\times\partial B_R)}^2 \leq \varepsilon\|w\|_{(\mathbb{T},R)}^2,\end{aligned}$$

and analogously,

$$\begin{aligned}I_3 &\leq \frac{1}{2R^{3/2}}\int_{\mathbb{T}\times\partial B_R} |x|^2|u(t,x)|^2\frac{|w(t,x)|}{R^{1/2}}\,\mathrm{d}S(x)\mathrm{d}t \\ &\leq \frac{C}{R^{1/2}}\|u\|_{\infty,\nu^1;\mathbb{T}\times B^S}^2\|w\|_{(\mathbb{T},R)} \leq \frac{C}{R^{1/2}}\varepsilon^2\|w\|_{(\mathbb{T},R)}\end{aligned}$$

and

$$\begin{aligned}I_4 &\leq \frac{1}{R^{3/2}}\int_{\mathbb{T}\times\partial B_R} |x|\left(1+\mathcal{s}_\zeta(x)\right)|u(t,x)|\frac{|w(t,x)|}{R^{1/2}}\,\mathrm{d}S(x)\mathrm{d}t \\ &\leq \frac{C}{R^{1/2}}\|u\|_{\infty,\nu_1^1(\cdot;\zeta);\mathbb{T}\times B^S}\|w\|_{(\mathbb{T},R)} \leq \frac{C}{R^{1/2}}\varepsilon\|w\|_{(\mathbb{T},R)}.\end{aligned}$$

From (2.9), we obtain $\mathcal{g}_R(3,3)\leq CR^{-2}$. Combined with estimates (3.2), this yields

$$\begin{aligned}I_5 &\leq R^{1/2}\|\nabla u\|_{\infty,\nu_{3/2}^{3/2}(\cdot;\zeta);\mathbb{T}\times B^S}\mathcal{g}_R(3,3)^{1/2}\frac{\|w\|_{L^2(\mathbb{T}\times\partial B_R)}}{R^{1/2}} \\ &\leq C\frac{1}{R^{1/2}}\|\nabla u\|_{\infty,\nu_{3/2}^{3/2}(\cdot;\zeta);\mathbb{T}\times B^S}\|w\|_{(\mathbb{T},R)} \leq C\frac{1}{R^{1/2}}\varepsilon\|w\|_{(\mathbb{T},R)}.\end{aligned}$$

Finally, the term with the pressure p is estimated as

$$
\begin{aligned}
I_6 &\leq \frac{1}{R^{3/2}} \int_{\mathbb{T}\times\partial B_R} |x|^2 |p(t,x)| \frac{|w(t,x)|}{R^{1/2}} \,\mathrm{d}S(x)\mathrm{d}t \\
&\leq \frac{C}{R^{3/2}} \|p\|_{\infty,\nu^2;\mathbb{T}\times B^S} R \frac{\|w\|_{L^2(\mathbb{T}\times\partial B_R)}}{R^{1/2}} \leq \frac{C}{R^{1/2}} \varepsilon \|w\|_{(\mathbb{T},R)}
\end{aligned}
$$

by (3.4), which holds due to $\frac{\mathrm{d}}{\mathrm{d}t}\Phi = 0$.

In summary, we find

$$
(1 - C_0\varepsilon)\|w\|_{(\mathbb{T},R)}^2 + \frac{|\zeta|}{2}\|w\|_{L^2(\mathbb{T}\times\partial B_R)}^2 \leq \frac{1}{R^{1/2}}(C_1\varepsilon + C_2\varepsilon^2)\|w\|_{(\mathbb{T},R)}.
$$

If $(1 - C_0\varepsilon) > 0$, then (redefining the constants)

$$
\|w\|_{(\mathbb{T},R)} + \|w\|_{L^2(\mathbb{T}\times\partial B_R)} \leq C\left(\|w\|_{(\mathbb{T},R)} + \sqrt{\frac{|\zeta|}{2}}\|w\|_{L^2(\mathbb{T}\times\partial B_R)}\right) \leq \frac{1}{R^{1/2}}(C_1\varepsilon + C_2\varepsilon^2),
$$

which gives (5.1) and concludes the proof. □

Remark 5.3. For the convergence statement of Theorem 5.1, we had to assume $\frac{\mathrm{d}}{\mathrm{d}t}\Phi = 0$, that is, that the total flux through the boundary is constant in time. As shown in Theorem 3.1 this condition ensures that the decay rate of the pressure is $|x|^{-2}$, compare Remark 3.2. In the previous proof, this lead to a suitable estimate of the term I_6, which cannot be obtained form the weaker rate $|x|^{-1}$ that holds in the general case.

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