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Nils Berglund[1], Tom Klose[2] and Nikolas Tapia[3]

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[1] Institut Denis Poisson (IDP)
Université d'Orléans, Université de Tours, CNRS – UMR 7013
Bâtiment de Mathématiques, B.P. 6759
45067 Orléans Cedex 2, France
E-Mail: nils.berglund@univ-orleans.fr

[2] University of Oxford
Mathematical Institute
Woodstock Road
Oxford, OX2 6GG, United Kingdom
E-Mail: tom.klose@maths.ox.ac.uk

[3] Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin, Germany
and
Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12486 Berlin, Germany
E-Mail: nikolasesteban.tapiamunoz@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract. We consider the perturbative renormalisation of the Φ^4_d model from Euclidean Quantum Field Theory for any, possibly non-integer dimension $d < 4$. The so-called BPHZ renormalisation, named after Bogoliubov, Parasiuk, Hepp and Zimmermann, is usually encoded into extraction-contraction operations on Feynman diagrams, which have a complicated combinatorics. We show that the same procedure can be encoded in the much simpler algebra of polynomials in two unknowns X and Y, which represent the fourth and second Wick power of the field. In this setting, renormalisation takes the form of a "Wick map" which maps monomials into Bell polynomials. The construction makes use of recent results by Bruned and Hou on multiindices, which are algebraic objects of intermediate complexity between Feynman diagrams and polynomials.

1. Introduction

The Φ^4 model is a famous toy model in Euclidean Quantum Field theory, featuring a quartic perturbation to a quadratic Hamiltonian. We are interested here in the model defined on the d-dimensional torus, which is denoted Φ^4_d. In dimension $d = 1$, the Gibbs measure associated to the Hamiltonian is well-defined. In higher dimension, however, a renormalisation procedure is needed to make sense of the Gibbs measure. Such a procedure can be implemented by fixing an ultra-violet cut-off N, and first restricting the measure to functions whose Fourier modes have a wave number of size N at most. This truncated Gibbs measure becomes undefined as N is sent to infinity, but in some cases one can fix this problem by adding suitable N-dependent terms to the Hamiltonian, which diverge as N goes to infinity, but make the limiting measure well-defined.

The easiest case occurs in dimension $d = 2$. Then the Gibbs measure can be renormalised by replacing the quartic term of the Hamiltonian by its fourth Wick power, which is defined by a Hermite polynomial with a variance diverging like $\log(N)$. The variance is that of the truncated Gaussian free field describing the uncoupled system. In dimension $d = 3$, the situation is considerably more difficult. In addition to the Wick counterterm, which now diverges like N, an additional counterterm known as mass renormalisation and which diverges like $\log(N)$, has to be added to the quadratic part of the Hamiltonian. In dimensions $d \geq 4$, on the other hand, it has been shown that the model is trivial [1, 26], in the sense that any viable renormalised version will converge to a Gaussian model as $N \to \infty$.

Numerous approaches have been developed to deal with the complicated combinatorics of the Φ^4_3 model. The earliest works by Glimm and Jaffe and by Feldman tackled the problem via a detailed combinatorial analysis of Feynman diagrams [25, 27–29], entailing very long and technical proofs. The works [3, 4] introduced the idea of using a renormalisation group approach, consisting in a decomposition of the covariance of the underlying Gaussian reference measure into scales similar to Paley–Littlewood blocks, that can be iteratively integrated out. This method was further perfected in [12], using polymers to control error terms, an approach based on ideas from statistical physics [30]. Another approach, provided in [14, 15], yields bounds on correlation functions (also known as n-point functions), by using the Gibbs measure as a generating function. This involves the derivation of skeleton inequalities, which were obtained up to third order in [14], and later extended to all orders in [8]. A relatively compact derivation of bounds on the partition function based on the Boué–Dupuis formula was recently obtained in [2]. The latter work is part of a programme called "stochastic quantisation" which has received a lot of interest in the last ten years due to the advent of novel solution techniques for singular stochastic PDEs [20, 31, 34, 39]. It is far beyond the scope of this introduction to review those recent developments, so we only list a few representative works pertaining to the Φ^4_d model: [16, 32, 33, 36, 42, 45].

The Φ^4_d model can be extended to non-integer values of the dimension d, by modifying the power with which the Green function diverges near the origin; the corresponding SPDE was introduced in [13]. This makes it possible to study all models for $d \in [3, 4)$, and examine their behaviour as d approaches the critical value 4. As d increases, the combinatorics becomes increasingly difficult, as more and more counterterms have to be added to the Hamiltonian. From the viewpoint of stochastic quantisation, this procedure has been understood in a non-perturbative way in [10, 17, 21, 24].

In this work, we expand on an idea introduced in [6], based on BPHZ renormalisation and inspired by algebraic methods. BPHZ renormalisation, named after Bogoliubov, Parasiuk, Hepp and Zimmermann [7, 37, 47], allows to

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systematically analyse the divergence of Feynman diagrams obtained by a perturbative expansion of expectations of observables under the Gibbs measure. As realised by Connes and Kreimer [18, 19], there is a Hopf-algebraic structure underlying Feynman diagrams, which encodes the extraction-contraction operations occurring in BPHZ renormalisation. See [35] for a modern exposition of this procedure.

The main new idea in [6] is that the complicated algebra of extraction-contraction operations on Feynman diagrams can be encoded in a much simpler way by operations on polynomials in only two variables X and Y, representing, respectively, the quartic and quadratic parts of the Hamiltonian. This procedure is inspired by ideas in [22] on deformation of coproducts. Our main result, Theorem 2.5, states that one can indeed encode BPHZ renormalisation on the level of polynomials, by using a rather simple "generalised Wick map", which consists in replacing powers of X by multiples of Y, with coefficients given by mass renormalisation counterterms. Corollary 2.6 then provides a simple expression for an asymptotic expansion of the model's partition function.

The remainder of this paper is organised as follows. Section 2 contains a definition of the Φ^4_d model for general, not necessarily integer dimensions d, introduces perturbative BPHZ renormalisation, and states the main results of this work. In Section 3, we present the construction of the "Wick map", and discuss its connections to cumulant expansions and Bell polynomials. Section 4 contains the definition of multi-indices in the setting of [11], and an explicit expression for the coproduct of the multiindices representing powers of the quartic interaction term. Finally, Section 5 contains the proof of the main result, which amounts to showing that a certain diagram between polynomials and multi-indices is commutative.

2. Main result

2.1. **The Φ^4_d model from Euclidean Quantum Field Theory.** We start by defining the Φ^4_d measure for integer values of the dimension $d \geqslant 1$. Let $\mathbb{T}^d$ denote the d-dimensional torus. For a function $\phi : \mathbb{T}^d \to \mathbb{R}$ and parameters $m > 0, \alpha \geqslant 0$, define the energy

$$\mathcal{H}_{d,\alpha}(\phi) = \int_{\mathbb{T}^d} \left[\frac{1}{2}\|\nabla\phi(x)\|^2 + \frac{m^2}{2}\phi(x)^2 + \alpha\phi(x)^4\right] \mathrm{d}x \;. \tag{2.1}$$

The Φ^4_d measure is the Gibbs measure associated with $\mathcal{H}_{d,\alpha}$, defined formally as

$$\mu_{d,\alpha}(\mathrm{d}\phi) \sim \frac{1}{\mathcal{Z}_{d,\alpha}} \mathrm{e}^{-\mathcal{H}_{d,\alpha}(\phi)} \mathrm{d}\phi \;, \tag{2.2}$$

where the *partition function* $\mathcal{Z}_{d,\alpha}$ is the normalisation that renders $\mu_{d,\alpha}$ a probability measure.

As such, this definition does not make sense, since there is no Lebesgue measure $\mathrm{d}\phi$ on an infinite-dimensional function space. However, in the case $\alpha = 0$, one can define $\mu_{d,0}$ as the Gaussian measure with covariance $-\Delta + m^2$, also called (massive) *Gaussian free field* with this covariance. Taking this Gaussian free field as reference measure, instead of Lebesgue measure, one can define the expectation under $\mu_{d,\alpha}$ of a test function F as

$$\mathbb{E}^{\mu_{d,\alpha}}[F] = \frac{\mathcal{Z}_{d,0}}{\mathcal{Z}_{d,\alpha}} \mathbb{E}^{\mu_{d,0}}\left[F(\phi)\exp\left\{-\alpha\int_{\mathbb{T}^d}\phi(x)^4\,\mathrm{d}x\right\}\right] \;. \tag{2.3}$$

In particular, setting $F = 1$, we obtain

$$\frac{\mathcal{Z}_{d,\alpha}}{\mathcal{Z}_{d,0}} = \mathbb{E}^{\mu_{d,0}}\left[\exp\left\{-\alpha\int_{\mathbb{T}^d}\phi(x)^4\,\mathrm{d}x\right\}\right] \;. \tag{2.4}$$

One can show that in dimension $d = 1$, the ratio of partition functions is indeed well-defined. However, in dimensions $d \geqslant 2$, this is no longer the case, since the Gaussian free field has almost surely infinite variance. Therefore, a renormalisation procedure becomes necessary to have a chance of defining the Φ^4_d measure.

Given an integer $N \geqslant 0$, called *ultra-violet cutoff*, denote by $\mathcal{K}_N$ the space of functions $\phi : \mathbb{T}^d \to \mathbb{R}$ spanned by Fourier basis functions e_k with $|k| = \sum_{i=1}^d |k_i| \leqslant N$. When restricted to functions in $\mathcal{K}_N$, the ratio (2.4) is well-defined. However, for $d \geqslant 2$ the ratio does not admit a well-defined limit as $N \to \infty$.

In dimension $d = 2$, the solution consists in replacing $\phi(x)^4$ by its fourth Wick power, defined as

$$:\phi^4(x): = H_4(\phi(x), C_N) \;, \tag{2.5}$$

where C_N denotes the variance of the Gaussian free field, which diverges like $\log(N)$, and H_4 is the fourth Hermite polynomial with variance C_N, given by

$$H_4(\phi, C_N) = \phi^4 - 6C_N\phi^2 + 3C_N^2 . \tag{2.6}$$

Note that this amounts to modifying the mass m in the energy (2.1) by a quantity depending on the cutoff N, and to adding an N-dependent constant to the energy. The extra terms in the energy are therefore called *mass renormalisation* and *energy renormalisation* counterterms. One can then show that with these counterterms included in the definition of $\mathcal{H}_{2,\alpha}$, the ratio (2.4) does converge to a finite limit as $N \to \infty$.

In dimension $d = 3$, Wick renormalisation is no longer sufficient, and additional mass and energy renormalisation terms have to be added to the energy. In dimension $d = 4$, on the other hand, is is known that no renormalisation procedure of the above type will lead to a meaningful result. More precisely, the Φ^4_4 model is trivial in the sense that the associated renormalised Gibbs measure is Gaussian [1], as is Φ^4_d when $d > 4$ [26].

This raises the question whether the limit $d \to 4_-$ can be better understood, by allowing for non-integer dimensions. This is known as the $\Phi^4_{4-\varepsilon}$ model. Non-integer dimensions can be interpreted, for $3 < d < 4$, by keeping the three-dimensional torus $\mathbb{T}^3$ as domain of integration, but changing the behaviour of the *Green function*. The Green function associated with the covariance operator $-\Delta + m^2$ on $\mathbb{T}^d$ is defined as

$$G_d(x, y) = \mathbb{E}^{\mu_{d,0}}\bigl[\phi(x) - \phi(y)\bigr] = \sum_{k\in\mathbb{Z}^d} e_k(x)\bigl[-\Delta + m^2\bigr]^{-1} e_k(y) . \tag{2.7}$$

Note that translation invariance of the model implies that $G_d(x, y) = G_d(x - y, 0)$. In dimension $d = 3$, the Green function is known to be defined everywhere except on the diagonal $\{x = y\}$, and that it diverges like $\|x - y\|^{-1}$ when approaching the diagonal. For $3 < d < 4$, a natural choice for the Green function is thus

$$G_d(x, y) \sim \frac{1}{\|x - y\|^{d-2}} . \tag{2.8}$$

The general form of the renormalised energy for ultraviolet cutoff N is expected to be

$$\mathcal{H}_{d,\alpha,N}(\phi) = \int_{\mathbb{T}^d} \biggl[\frac{1}{2}\|\nabla\phi(x)\|^2 + \frac{m^2}{2}\phi(x)^2 + \alpha{:}\,\phi(x)^4{:} + \beta{:}\,\phi(x)^2{:} + \gamma\biggr]\,\mathrm{d}x , \tag{2.9}$$

for suitable mass renormalisation and energy renormalisation counterterms $\beta = \beta(d, \alpha, N)$ and $\gamma = \gamma(d, \alpha, N)$. Introducing the notations

$$X = \int_{\mathbb{T}^d} {:}\,\phi(x)^4{:}\,\mathrm{d}x , \qquad Y = \int_{\mathbb{T}^d} {:}\,\phi(x)^2{:}\,\mathrm{d}x , \tag{2.10}$$

the ratio (2.4) of partition functions can be written as

$$\frac{\mathcal{Z}_{d,\alpha}}{\mathcal{Z}_{d,0}} = \mathbb{E}^{\mu_{d,0}}\Bigl[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\Bigr] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\mu_{d,0}}\Bigl[\mathrm{e}^{-\alpha X - \beta Y}\Bigr] . \tag{2.11}$$

2.2. **Perturbative renormalisation and Feynman diagrams.** Perturbative renormalisation consists in expanding the exponential in (2.11), yielding the formal series

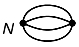
$$\frac{\mathcal{Z}_{d,\alpha}}{\mathcal{Z}_{d,0}} \asymp \mathrm{e}^{-\gamma} \sum_{n\geqslant 1} \frac{(-1)^n}{n!}\mathbb{E}^{\mu_{d,0}}\bigl[(\alpha X + \beta Y)^n\bigr] . \tag{2.12}$$

We emphasize that this series is known *not* to converge [38]. Instead, it is an asymptotic series of Gevrey-1 type, which is Borel summable (at least for $d \leqslant 3$, cf. [23, 41]).

By the Isserlis–Wick theorem, expectations of monomials in X and Y can be written in terms of integrals of products of Green functions. For instance,

$$\begin{aligned}
\mathbb{E}^{\mu_{d,0}}\bigl[X^2\bigr] &= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \mathbb{E}^{\mu_{d,0}}\bigl[{:}\,\phi(x)^4{:}\,{:}\,\phi(y)^4{:}\bigr]\,\mathrm{d}x\,\mathrm{d}y && (2.13)\\
&= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} 4!\,\mathbb{E}^{\mu_{d,0}}\bigl[\phi(x)\phi(y)\bigr]^4\,\mathrm{d}x\,\mathrm{d}y && (2.14)\\
&= 4!\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} G_{d,N}(x, y)^4\,\mathrm{d}x\,\mathrm{d}y . && (2.15)
\end{aligned}$$

Here the truncated Green function $G_{d,N}$ is defined as in (2.7), with the power -1 replaced by a suitable power $-s$ — power-counting arguments imply that $s = (7-d)/4$. In addition, the sum should be restricted to $k \in \mathbb{Z}^3$ with $|k| \leqslant N$. This can be conveniently represented by the graphical notation

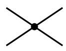
$$\mathbb{E}^{\mu_{d,0}}\Big[X^2\Big] = 4!\,\Pi_N$$  , (2.16)

where the vertices of the graph correspond to the two integration variables, the four edges correspond to the four copies of the Green function, and Π_N denotes the *valuation map*, which associates to the diagram a real number given by an integral of a product of Green functions. This is an example of *Feynman diagram* (in this case, a vacuum diagram, since there are no free legs).

For a general multigraph $\Gamma = (\mathcal{V}, \mathcal{E})$ (multiple edges are allowed), the valuation is defined by

$$\Pi_N(\Gamma) = \int_{(\mathbb{T}^3)^{|\mathcal{V}|}} \prod_{e\in\mathcal{E}} G_{d,N}(x_{e_+}, x_{e,-})\,\mathrm{d}x\;, \tag{2.17}$$

where x_{e_+} and x_{e_-} denote the endpoints of the edge e. We further represent the Wick powers X and Y by

$X =$  , $\qquad Y =$. (2.18)

With these notations, the expectation of a monomial is given by

$$\mathbb{E}^{\mu_{d,0}}\Big[X^n Y^m\Big] = \sum_p \Pi_N(\Gamma_p)\;, \tag{2.19}$$

where p runs over all *pairwise matchings* of the legs of X and Y.

Instead of expanding the exponential directly as in (2.12), it turns out to be advantageous to expand its logarithm, that is, to use a *cumulant expansion*. The *linked-cluster theorem* [9, 43, 44] states that

$$\log\frac{\mathcal{Z}_{d,\alpha}}{\mathcal{Z}_{d,0}} = \log\mathbb{E}^{\mu_{d,0}}\Big[\mathrm{e}^{-\alpha X-\beta Y-\gamma}\Big] \asymp -\gamma + \sum_{n\geqslant 0}\frac{(-1)^n}{n!}\Pi_N\mathcal{P}\Big((\alpha X+\beta Y)^n\Big)\;, \tag{2.20}$$

where $\mathcal{P}$ denotes the sum over *connected* pairwise matchings, i.e. connected vacuum diagrams.

2.3. **BPHZ renormalisation.** BPHZ renormalisation, a procedure named after Bogoliubov, Parasiuk, Hepp and Zimmermann [7, 37, 47], allows to determine expressions for the counterterms β and γ ensuring that the terms obtained by expanding (2.20) are uniformly bounded in the cut-off N.

The first step is to associate to a diagram $\Gamma = (\mathcal{V}, \mathcal{E})$ the *degree*

$$\deg(\Gamma) = d(|\mathcal{V}|-1) - (d-2)|\mathcal{E}|\;. \tag{2.21}$$

Diagrams of non-positive degree are called *divergent*. Simple examples suggest that $\Pi_N(\Gamma)$ diverges like $N^{-\deg(\Gamma)}$ if $\deg(\Gamma) < 0$, and like $\log(N)$ if $\deg(\Gamma) = 0$, while $\Pi_N(\Gamma)$ is uniformly bounded in N if $\deg(\Gamma) > 0$. This is not true in general, because a non-divergent diagram can contain subdiagrams that are divergent, making the valuation diverge. However, a deep result of the theory states that there exists a modification of Γ which behaves like its degree.

Theorem 2.1 (BPHZ renormalisation)**.** *There exists a linear map $\mathcal{A}$, acting on Feynman diagrams, such that*

$$\Pi_N(\mathcal{A}(\Gamma)) \asymp \begin{cases} N^{-\deg(\Gamma)} & \text{if } \deg(\Gamma) < 0\;,\\ \log(N)^\zeta & \text{if } \deg(\Gamma) = 0\;,\end{cases} \tag{2.22}$$

for a finite integer ζ, while $\Pi_N(\mathcal{A}(\Gamma))$ is bounded uniformly in N if $\deg(\Gamma) > 0$.

For a modern exposition, see [35], as well as [5]. The linear map $\mathcal{A}$ is in fact the antipode of the Connes–Kreimer Hopf algebra on Feynman diagrams. To define this Hopf algebra, we start with a collection $\mathfrak{G}$ of multigraphs. Let $\mathcal{G}$ be the real vector space freely generated by $\mathfrak{G}$. The vector space can be turned into a unitary associative algebra for the product $\cdot$ given by disjoint union. The neutral element for multiplication is the empty graph that we denote by $\mathbf{1}$. The *Connes–Kreimer extraction-contraction coproduct* is defined by

$$\Delta_{\mathrm{CK}}(\Gamma) = \Gamma\otimes\mathbf{1} + \mathbf{1}\otimes\Gamma + \sum_{\substack{\mathbf{1}\neq\overline{\Gamma}\subsetneq\Gamma\\ \deg\overline{\Gamma}\leqslant 0}} \overline{\Gamma}\otimes(\Gamma/\overline{\Gamma})\;, \tag{2.23}$$

where the sum ranges over all *divergent* subgraphs $\overline{\Gamma}$, and $\Gamma/\overline{\Gamma}$ denotes the graph obtained by replacing $\overline{\Gamma}$ by a single vertex. The subgraphs have to be *full*, in the sense that if an edge e belongs to $\overline{\Gamma}$, all edges connecting the same vertices also belong to $\overline{\Gamma}$. Note that the valuation Π_N is multiplicative, meaning that $\Pi_N(\Gamma_1 \cdot \Gamma_2) = \Pi_N(\Gamma_1)\Pi_N(\Gamma_2)$ for all $\Gamma_1, \Gamma_2 \in \mathcal{G}$.

Remark 2.2. We point out that if some divergent subgraphs have a degree smaller than -1, the expression (2.23) has to be modified, by adding decorated graphs on the right-hand side, as we will explain in Section 5.4 below. $\diamondsuit$

We endow the algebra $\mathcal{G}$ with two more linear maps. A *counit* $\mathbf{1}^\star : \mathcal{G} \to \mathbb{R}$, given by projection on the unit $\mathbf{1}$, and an *antipode* $\mathcal{A} : \mathcal{G} \to \mathcal{G}$, defined inductively by $\mathcal{A}(\mathbf{1}) = \mathbf{1}$ and

$$\mathcal{A}(\Gamma) = -\Gamma - \sum_{\substack{\mathbf{1}\neq\overline{\Gamma}\subsetneq\Gamma \\ \deg\overline{\Gamma}\leqslant 0}} \mathcal{A}(\overline{\Gamma}) \cdot (\Gamma/\overline{\Gamma}) \tag{2.24}$$

$$= -\Gamma - m(\mathcal{A} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma) . \tag{2.25}$$

Here $\mathring{\Delta}_{\mathrm{CK}} = \Delta_{\mathrm{CK}} - \Gamma \otimes \mathbf{1} - \mathbf{1} \otimes \Gamma$ denotes the *reduced coproduct*, and $m : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ is the multiplication map, defined by $m(\Gamma_1 \otimes \Gamma_2) = \Gamma_1 \cdot \Gamma_2$. The space $(\mathcal{G}, \cdot, \Delta, \mathbf{1}, \mathbf{1}^\star, \mathcal{A})$ constructed in this way is a Hopf algebra, called *Connes–Kreimer extraction-contraction Hopf algebra*.

To define BPHZ renormalisation, we first introduce the *twisted antipode*, defined as

$$\tilde{\mathcal{A}}(\Gamma) = \mathcal{A}(\Gamma)1_{\deg\Gamma\leqslant 0} . \tag{2.26}$$

Note that if $\deg(\Gamma) \leqslant 0$, then one has

$$\tilde{\mathcal{A}}(\Gamma) = -\Gamma - m(\tilde{\mathcal{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma) , \tag{2.27}$$

because $\mathring{\Delta}_{\mathrm{CK}}$ produces only divergent terms on the left of the tensor product.

Remark 2.3. The general definition of the twisted antipode given in [35, (2.28)] is rather that it should satisfy $m(\tilde{\mathcal{A}} \otimes \mathrm{id})\Delta_{\mathrm{CK}}(\Gamma) = 0$ whenever $\deg\Gamma \leqslant 0$, where m denotes the multiplication map, but this is equivalent – cf. [11, (4.3)], which uses the reduced coproduct. $\diamondsuit$

A *character* on $\mathcal{G}$ is a linear map $g : \mathcal{G} \to \mathbb{R}$ which is multiplicative in the sense that we have $g(\Gamma_1 \cdot \Gamma_2) = g(\Gamma_1)g(\Gamma_2)$ for all $\Gamma_1, \Gamma_2 \in \mathcal{G}$. With any character g, one can associate a linear map M^g defined by

$$M^g(\Gamma) = (g \otimes \mathrm{id})\Delta_{\mathrm{CK}}(\Gamma) , \tag{2.28}$$

and the set of these maps is known to form a group. The *BPHZ character* is the linear map $g^{\mathrm{BPHZ}} : \mathcal{G} \to \mathbb{R}$ given by

$$g^{\mathrm{BPHZ}}(\Gamma) = \Pi_N\tilde{\mathcal{A}}(\Gamma) . \tag{2.29}$$

The fact that g^{BPHZ} is indeed a character follows from multiplicativity of $\mathcal{A}$ and Π_N. The map $M^{g^{\mathrm{BPHZ}}}$ is called *BPHZ renormalisation map*. It defines a *renormalised valuation* given by

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma) = \Pi_N M^{g^{\mathrm{BPHZ}}}(\Gamma) = (g^{\mathrm{BPHZ}} \otimes \Pi_N)\Delta_{\mathrm{CK}}(\Gamma) = (\Pi_N\tilde{\mathcal{A}} \otimes \Pi_N)\Delta_{\mathrm{CK}}(\Gamma) . \tag{2.30}$$

The interest of this construction is the following result.

Lemma 2.4. *The BPHZ renormalised valuation satisfies*

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma) = \begin{cases} 0 & \text{if } \deg\Gamma \leqslant 0 , \\ -\Pi_N\mathcal{A}(\Gamma) & \text{if } \deg\Gamma > 0 . \end{cases} \tag{2.31}$$

PROOF: In the case $\deg\Gamma \leqslant 0$, using (2.27) we get

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma) = (\Pi_N \otimes \Pi_N)(\tilde{\mathcal{A}} \otimes \mathrm{id})\big[\Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \mathring{\Delta}_{\mathrm{CK}}(\Gamma)\big] \tag{2.32}$$

$$= (\Pi_N \otimes \Pi_N)\big[\tilde{\mathcal{A}}(\Gamma) \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + (\tilde{\mathcal{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma)\big] \tag{2.33}$$

$$= (\Pi_N \otimes \Pi_N)\big[-\Gamma \otimes \mathbf{1} - m(\tilde{\mathcal{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma) \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + (\tilde{\mathcal{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma)\big] , \tag{2.34}$$

which vanishes by multiplicativity of Π_N. In the case $\deg\Gamma > 0$, using $\tilde{\mathcal{A}}\Gamma = 0$ in the second line of the above computation, we obtain

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma) = (\Pi_N \otimes \Pi_N)\big[\mathbf{1} \otimes \Gamma + (\tilde{\mathcal{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma)\big] \tag{2.35}$$

$$= \Pi_N(\Gamma) + (\Pi_N\tilde{\mathcal{A}} \otimes \Pi_N)\mathring{\Delta}_{\mathrm{CK}}(\Gamma) \tag{2.36}$$

$$= \Pi_N(\Gamma) + \Pi_N m(\tilde{\mathcal{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma) \tag{2.37}$$

again by multiplicativity of Π_N. This is equal to $-\Pi_N\mathcal{A}(\Gamma)$ by (2.25). □

If follows from Theorem 2.1 that the renormalized valuation Π_N^{BPHZ} is bounded uniformly in the cut-off N for any $\Gamma \in \mathcal{G}$.

2.4. **Main result.** To formulate our main result, we introduce two sequences of critical dimensions, at which new energy or mass renormalisation terms appear. Note that any graph in $\mathcal{P}(X^n)$ has n vertices and $4n$ half-edges, which become $2n$ edges after pairing. Therefore,

$$\deg\mathcal{P}(X^n) = 4n - (n+1)d\ . \tag{2.38}$$

This implies that

$$\deg\mathcal{P}(X^n) \leqslant 0 \qquad \Leftrightarrow \qquad d \geqslant d_{\mathrm{e}}^*(n) := \frac{4n}{n+1} = 4 - \frac{4}{n+1}\ . \tag{2.39}$$

These thresholds control the appearance of new energy renormalisation counterterms. New mass renormalisation terms appear at the values

$$d_{\mathrm{m}}^*(n) = d_{\mathrm{e}}^*(2n-1) = 4 - \frac{2}{n} \tag{2.40}$$

of d. The two sequences are increasing and accumulate at $d = 4$. The first values are

$$\big(d_{\mathrm{e}}^*(n)\big)_{n\geqslant 1} = \left(2, \frac{8}{3}, 3, \frac{16}{5}, \frac{10}{3}, \frac{24}{7}, \frac{7}{2}, \frac{32}{9}, \dots\right) \qquad \text{and} \qquad \big(d_{\mathrm{m}}^*(n)\big)_{n\geqslant 1} = \left(2, 3, \frac{10}{3}, \frac{7}{2}, \dots\right)\ . \tag{2.41}$$

The inverse thresholds, expressing n in terms of d, are given by

$$n_{\mathrm{e}}^*(d) = \left\lfloor \frac{d}{4-d} \right\rfloor \qquad \text{and} \qquad n_{\mathrm{m}}^*(d) = \left\lfloor \frac{2}{4-d} \right\rfloor\ . \tag{2.42}$$

Our main result is the following.

Theorem 2.5 (Main result)**.** *For any dimension $d < 4$, there exists a linear map*

$$\mathrm{W} : \mathbb{R}[X] \to \mathbb{R}[X, Y]\,, \tag{2.43}$$

called Wick map*, such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{R}[X] & \xrightarrow{\ \mathcal{P}\ } & \mathcal{G} \\ {\scriptstyle \mathrm{W}}\big\downarrow & & \big\downarrow{\scriptstyle (\Pi_N\mathcal{A}\otimes\mathrm{id})\Delta_{\mathrm{CK}}+\Pi_N\Theta_{\mathrm{F}}} \\ \mathbb{R}[X,Y] & \xrightarrow{\ \mathcal{P}\ } & \mathcal{G} \end{array} \tag{2.44}$$

The Wick map W *satisfies*

$$\mathrm{W}(\mathrm{e}^{-\alpha X}) = \mathrm{e}^{-\alpha X - \beta Y}\ , \tag{2.45}$$

where the mass counterterm $\beta = \beta(d, \alpha, N)$ is given by

$$\beta(d, \alpha, N) = \sum_{n=2}^{n_{\mathrm{m}}^*(d)} \frac{(-\alpha)^n}{n!}\sigma_n(N)\ . \tag{2.46}$$

Here the $\sigma_n(N)$ can be expressed in terms of divergent Feynman diagrams, and diverge like $\sigma_n(N) \sim N^{2-(4-d)n}$, see Remark 2.7 below. In addition, for any $n \geqslant 2$,

$$\mathrm{W}(X^n) = B_n(X, -\sigma_2(N)Y, \dots, -\sigma_n(N)Y)\ , \tag{2.47}$$

where B_n is the nth complete Bell polynomial. Finally, the map Θ_{F}, which is associated with energy renormalisation, is such that

$$\gamma_0 := (\Pi_N\Theta_{\mathrm{F}} \circ \mathcal{P})(\mathrm{e}^{-\alpha X}) = -(\Pi_N\mathcal{A} \circ \mathcal{P})(\mathrm{e}^{-\alpha X})\ . \tag{2.48}$$

Together with the definition (2.30) of the BPHZ valuation, this implies that the following diagram commutes:

$$\begin{array}{ccccc} e^{-\alpha X} & \stackrel{\mathcal{P}}{\longmapsto} & \mathcal{P}(e^{-\alpha X}) & \xrightarrow{\Pi_N^{\text{BPHZ}}+\Pi_N\Theta_F} & \\ \Big\downarrow{\scriptstyle W} & & \Big\downarrow{\scriptstyle (\Pi_N\mathcal{A}\otimes\text{id})\Delta_{\text{CK}}+\Pi_N\Theta_F} & & \\ e^{-\alpha X-\beta Y} & \stackrel{\mathcal{P}}{\longmapsto} & \mathcal{P}(e^{-\alpha X-\beta Y}) & \stackrel{\Pi_N}{\longmapsto} & \mathbb{R} \end{array} \tag{2.49}$$

As a consequence, we have

$$\log\frac{\mathcal{Z}_{d,\alpha}}{\mathcal{Z}_{d,0}} = \Pi_N\mathcal{P}(e^{-\alpha X-\beta Y}) - \gamma = \Pi_N^{\text{BPHZ}}\mathcal{P}(e^{-\alpha X}) + \gamma_0 - \gamma \;. \tag{2.50}$$

Lemma 2.4 shows that $\Pi_N^{\text{BPHZ}}\mathcal{P}(X^n) = 0$ for $n \leqslant n_e^*(d)$, while it is bounded for $n > n_e^*(d)$. The energy renormalisation term should thus be chosen as the divergent part of γ_0, namely

$$\gamma = \gamma(d,\alpha,N) = -\sum_{n=2}^{n_e^*(d)} \frac{(-\alpha)^n}{n!}\Pi_N\mathcal{A}(\mathcal{P}(X^n)) \;. \tag{2.51}$$

Note that $\Pi_N\mathcal{A}(\mathcal{P}(X^n))$ diverges like $N^{(n+1)d-4n} = N^{(4-d)(n_e^*(d)-n)}$, with the tacit understanding that $N^0 = \log N$.

Theorem 2.1 then immediately implies the following result.

Corollary 2.6. *With the choice* (2.51) *of the energy renormalisation term, one has the asymptotic expansion*

$$\log\frac{\mathcal{Z}_{d,\alpha}}{\mathcal{Z}_{d,0}} \asymp -\sum_{n>n_e^*(d)} \frac{(-\alpha)^n}{n!}\Pi_N\mathcal{A}(\mathcal{P}(X^n)) \;. \tag{2.52}$$

All terms of this expansion are bounded uniformly in the cut-off N.

Remark 2.7. The counterterms σ_n occuring in the mass renormalisation β are linear combinations of valuations of divergent Feynman diagrams, the first of which are listed in Table 1. More precisely, they can be written

$$\sigma_n = -\Pi_N\mathcal{A}\mathcal{P}_{\text{M}}(\mathcal{Y}_n) \;, \tag{2.53}$$

where the $\mathcal{Y}_n$ are defined in (4.19) below, and the map $\mathcal{P}_{\text{M}}$ is defined in (4.8). They can also be computed by the relation

$$\sigma_n = -\Pi_{\text{M}}\mathcal{A}_{\text{M}}(\mathcal{Y}_n) \;, \tag{2.54}$$

where the maps Π_{M} and $\mathcal{A}_{\text{M}}$ are defined in [11, Corollary 4.5 and (4.3)]. ◊

2.5. **Structure of the proof.** We start, in Section 3, by explaining the construction of the Wick map W, and its (well known) relation to Bell polynomials. To prove the commutativity of the diagram (2.44), we will take advantage of recent results in [11], which show that instead of working with Feynman diagrams, one can work with somewhat simpler algebraic objects called *multiindices*. We introduce these objects in Section 4, where we also compute an explicit expression for the coproducts of the relevant Feynman diagrams, translated to the multiindex language. Finally, Section 5 contains the proof of the main result.

3. The Wick map

In this section, we give the construction of the Wick map W, and explain its connection to cumulants and Bell polynomials. The Wick map can be cast into a form closer to the map $(\Pi_N\mathcal{A}\otimes\text{id})\Delta_{\text{CK}}$ by working in the free commutative Hopf algebra $H = S(\mathbb{R}[X])$, as explained in Section 3.4. In Section 3.5, we explain how the construction extends to algebra-valued moments.

3.1. **Convolution algebra and power series.** Let $\mathbb{R}[X]$ denote the polynomial Hopf algebra on a single formal variable X. Its product and coproduct are given by

$$X^n\cdot X^m = X^{n+m} \;, \qquad \Delta X^n = \sum_{k=0}^{n}\binom{n}{k}X^k\otimes X^{n-k} \;. \tag{3.1}$$

The *convolution product* of two linear maps $\varphi,\psi\in\mathcal{L}(\mathbb{R}[X],\mathbb{R})$ is defined by

$$\varphi*\psi = m_{\mathbb{R}}(\varphi\otimes\psi)\Delta \;, \tag{3.2}$$

where $m_\mathbb{R}$ denotes the multiplication map, $m_\mathbb{R}(a \otimes b) = ab$. This means that

$$(\varphi * \psi)(X^n) = \sum_{k=0}^{n} \binom{n}{k} \varphi(X^k)\psi(X^{n-k}) . \tag{3.3}$$

The k-fold convolution product is denoted φ^{*k}.

We define two special subsets of $\mathcal{L}(\mathbb{R}[X], \mathbb{R})$, given by

$$\mathcal{L}_1 = \{\varphi \in \mathcal{L}(\mathbb{R}[X], \mathbb{R}) : \varphi(1) = 1\} , \tag{3.4}$$

$$\mathcal{L}_0 = \{\varphi \in \mathcal{L}(\mathbb{R}[X], \mathbb{R}) : \varphi(1) = 0\} . \tag{3.5}$$

Elements of $\mathcal{L}_1$ can be inverted, via the Neumann series

$$\varphi^{-1} = \sum_{k=0}^{\infty} (\varepsilon - \varphi)^{*k} , \tag{3.6}$$

where $\varepsilon : \mathbb{R}[X] \to \mathbb{R}$ is the counit, given by $\varepsilon(X^n) = \delta_{n0}$. One has explicitly

$$\varphi^{-1}(X^n) = \sum_{k=1}^{n} (-1)^k \sum_{\substack{n_1,\dots,n_k \geqslant 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1! \dots n_k!} \varphi(X^{n_1}) \dots \varphi(X^{n_k}) . \tag{3.7}$$

The exponential map $\exp_* : \mathcal{L}_0 \to \mathcal{L}_1$ and its inverse $\log_* : \mathcal{L}_1 \to \mathcal{L}_0$ are given by

$$\exp_*(\varphi) = \sum_{k\geqslant 0} \frac{1}{k!} \varphi^{*k} , \qquad \log_*(\varphi) = \sum_{k\geqslant 1} \frac{(-1)^k}{k} (\varphi - \varepsilon)^{*k} . \tag{3.8}$$

There is no issue of convergence, since the sums are always finite when evaluated on a basis element. In fact,

$$\exp_*(\varphi)(X^n) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geqslant 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1! \dots n_k!} \varphi(X^{n_1}) \dots \varphi(X^{n_k}) , \tag{3.9}$$

$$\log_*(\varphi)(X^n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1,\dots,n_k \geqslant 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1! \dots n_k!} \varphi(X^{n_1}) \dots \varphi(X^{n_k}) . \tag{3.10}$$

Let $\mathbb{R}[[t]]$ denote the algebra of (formal) power series in the variable t with real coefficients, endowed with pointwise multiplication. Then we can define a linear map

$$\Lambda : \mathcal{L}(\mathbb{R}[X], \mathbb{R}) \to \mathbb{R}[[t]] \tag{3.11}$$

$$\varphi \mapsto \sum_{n\geqslant 0} \varphi(X^n) \frac{t^n}{n!} . \tag{3.12}$$

Theorem 3.1. *Λ is an isomorphism between $\mathcal{L}(\mathbb{R}[X], \mathbb{R})$ and $\mathbb{R}[[t]]$.*

PROOF: Let $\varphi, \psi \in \mathcal{L}(\mathbb{R}[X], \mathbb{R})$. By the Cauchy product formula,

$$\Lambda(\varphi)(t)\Lambda(\psi)(t) = \left(\sum_{n\geqslant 0} \varphi(X^n) \frac{t^n}{n!}\right) \left(\sum_{n\geqslant 0} \psi(X^n) \frac{t^n}{n!}\right) \tag{3.13}$$

$$= \sum_{n\geqslant 0} \left(\sum_{k=0}^{n} \frac{\varphi(X^k)}{k!} \frac{\varphi(X^{n-k})}{(n-k)!}\right) t^n \tag{3.14}$$

$$= \sum_{n\geqslant 0} (\varphi * \psi)(X^n) \frac{t^n}{n!} \tag{3.15}$$

$$= \Lambda(\varphi * \psi)(t) , \tag{3.16}$$

showing that Λ is indeed a morphism. Bijectivity of Λ is straightforward to check. □

Corollary 3.2. *For any $\varphi \in \mathcal{L}_0$ and $\psi \in \mathcal{L}_1$, one has the relations*

$$\Lambda(\psi^{-1})(t) = \left[\Lambda(\psi(t))\right]^{-1} , \tag{3.17}$$

$$\Lambda(\exp_* \varphi)(t) = \exp(\Lambda(\varphi(t))) , \tag{3.18}$$

$$\Lambda(\log_* \psi)(t) = \log(\Lambda(\psi(t))) . \tag{3.19}$$

PROOF: We prove the second relation. Setting $\psi = \exp_* \varphi$, we have

$$\Lambda(\psi)(t) = \sum_{k\geqslant 0} \frac{1}{k!}\Lambda(\varphi^{*k})(t) = \sum_{k\geqslant 0} \frac{1}{k!}\Lambda(\varphi)(t)^k = \exp(\Lambda(\varphi)(t)) . \tag{3.20}$$

The other relations are proved in a similar way. □

3.2. **Moments, cumulants and Wick exponential.** Let $\mathcal{X}$ be a real-valued random variable having moments of all orders. We associate with it the linear map $\mu_{\mathcal{X}} : \mathbb{R}[X] \to \mathbb{R}$ given by

$$\mu_{\mathcal{X}}(X^n) = \mathbb{E}[\mathcal{X}^n] . \tag{3.21}$$

Note that $\mu_{\mathcal{X}} \in \mathcal{L}_1$, since $\mathbb{E}[1] = 1$. The associated power series

$$\Lambda(\mu_{\mathcal{X}})(t) = \sum_{n\geqslant 0} \frac{t^n}{n!}\mathbb{E}[\mathcal{X}^n] = \mathbb{E}[e^{t\mathcal{X}}] \tag{3.22}$$

is the *moment generating function* of $\mathcal{X}$. The *cumulant generating function* of $\mathcal{X}$ is defined as

$$K_{\mathcal{X}}(t) = \log \mathbb{E}[e^{t\mathcal{X}}] . \tag{3.23}$$

Corollary 3.2 implies

$$K_{\mathcal{X}}(t) = \Lambda(\log_* \mu_{\mathcal{X}})(t) = \Lambda(\kappa_{\mathcal{X}})(t) , \tag{3.24}$$

where

$$\kappa_{\mathcal{X}} = \log_* \mu_{\mathcal{X}} . \tag{3.25}$$

This is nothing but the classical moment-cumulant relation. In particular, (3.10) implies

$$\kappa_{\mathcal{X}}(X^n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1,\dots,n_k\geqslant 1\\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!}\mu_{\mathcal{X}}(X^{n_1})\dots\mu_{\mathcal{X}}(X^{n_k}) . \tag{3.26}$$

We now define the *Wick exponential* associated to $\mathcal{X}$ by

$$\mathrm{W} = (\mu_{\mathcal{X}}^{-1} \otimes \mathrm{id})\Delta = (\exp_*(-\kappa_{\mathcal{X}}) \otimes \mathrm{id})\Delta . \tag{3.27}$$

One easily checks that W is an element of $\mathcal{L}_1$, while (3.7) and (3.9) imply

$$\mathrm{W}(X^n) = \sum_{k=0}^{n}\sum_{j=1}^{k} \frac{(-1)^j}{j!} \sum_{\substack{k_1,\dots,k_j\geqslant 1\\ k_1+\dots+k_j=k}} \frac{n!}{(n-k)!k_1!\dots k_j!}\kappa_{\mathcal{X}}(X^{k_1})\dots\kappa_{\mathcal{X}}(X^{k_j})X^{n-k} . \tag{3.28}$$

The following result shows that $\mathrm{W}(t, X) := \Lambda(\mathrm{W})(t)$ can be interpreted as a generating function.

Proposition 3.3. *One has the relation*

$$\mathrm{W}(t, X) = \frac{e^{tX}}{\mathbb{E}[e^{t\mathcal{X}}]} = e^{tX-K_{\mathcal{X}}(t)} . \tag{3.29}$$

PROOF: Observe that

$$\mathrm{W}(X^n) = \sum_{k=0}^{n} \binom{n}{k}\mu_{\mathcal{X}}^{-1}(X^k)X^{n-k} = (\mu_{\mathcal{X}}^{-1} * \mathrm{id})(X^n) . \tag{3.30}$$

Therefore, Theorem 3.1 implies

$$\Lambda(\mathrm{W})(t) = \Lambda(\mu_{\mathcal{X}}^{-1} * \mathrm{id})(t) = \Lambda(\mu_{\mathcal{X}}^{-1})(t)\Lambda(\mathrm{id})(t) = \frac{\Lambda(\mathrm{id})(t)}{\Lambda(\mu_{\mathcal{X}})(t)} = \frac{e^{tX}}{\mathbb{E}[e^{t\mathcal{X}}]} , \tag{3.31}$$

where we have used Corollary 3.2 and (3.22). □

Note in particular that $\mathbb{E}[\mathrm{W}(t,\mathcal{X})] = 1$. This is a form of orthogonality relation, as it shows that $\mathbb{E}[\mathrm{W}(\mathcal{X}^n)] = 0$ for all $n \geqslant 1$.

Example 3.4. If $\mathcal{X}$ is centered Gaussian with variance σ^2, than $\kappa_{\mathcal{X}}(X^2) = \sigma^2$, while $\kappa_{\mathcal{X}}$ vanishes everywhere else. Therefore, $\mathrm{W}(t,X) = \mathrm{e}^{tX-\sigma^2t^2/2}$ is the generating function of Hermite polynomials, and (3.28) yields

$$\mathrm{W}(X^n) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!2^k} \sigma^{2k} X^{n-2k} , \tag{3.32}$$

which is the nth Hermite polynomial with variance σ^2. ♦

3.3. **Bell polynomials.** We consider now the case where the cumulants are given by

$$\kappa_{\mathcal{X}}(X^n) = \begin{cases} 0 & \text{if } n = 1 , \\ Y_n & \text{if } n \geqslant 2 , \end{cases} \tag{3.33}$$

where the Y_n are for now considered as real parameters. If we assume that only finitely many Y_n are different from zero, all sums will be well-defined.

Remark 3.5. Like in [22], we work with *formal* power series and do not deal with issues of convergence. In particular, the above definition of $\kappa_{\mathcal{X}}(X^n)$ is not in contradiction to Marcinkiewicz' theorem about the cumulants of real-valued random variables. Besides, we will only be interested in the case when the Y_n's are elements in an algebra, see Section 3.5 below. ♢

Then we have

$$\mathrm{W}(t,X) = \mathrm{e}^{tX-K_{\mathcal{X}}(t)} = \exp\biggl\{ tX - \sum_{n\geqslant 2} Y_n \frac{t^n}{n!} \biggr\} , \tag{3.34}$$

which is the generating function of the exponential Bell polynomials. Namely,

$$\mathrm{W}(t,X) = \sum_{n=0}^{\infty} B_n(X,-Y_2,\dots,-Y_n) \frac{t^n}{n!} , \tag{3.35}$$

where B_n is be definition the nth complete exponential Bell polynomial.

Lemma 3.6. *The nth complete exponential Bell polynomial can be written*

$$B_n(X,-Y_2,\dots,-Y_n) = n! \sum_{\substack{j_1,\dots,j_n\geqslant 0 \\ j_1+2j_2+\dots+nj_n=n}} \frac{1}{j_1!}\biggl(\frac{X}{1!}\biggr)^{j_1} \prod_{p=2}^{n} \frac{1}{j_p!}\biggl(\frac{-Y_p}{p!}\biggr)^{j_p} . \tag{3.36}$$

PROOF: By (3.28), and since $\kappa_{\mathcal{X}}(X) = 0$, we have

$$B_n(X,-Y_2,\dots,-Y_n) = \mathrm{W}(X^n) = n! \sum_{k=0}^{n} \sum_{j=1}^{k} \frac{(-1)^j}{j!} \sum_{\substack{k_1,\dots,k_j\geqslant 2 \\ k_1+\dots+k_j=k}} \frac{Y_{k_1}}{k_1!} \dots \frac{Y_{k_j}}{k_j!} \frac{X^{n-k}}{(n-k)!} . \tag{3.37}$$

We may rewrite this expression in a different way, by ordering terms according to the value of the k_i. For $p \geqslant 2$, let j_p denote the number of indices k_i equal to p (this number may be 0). For a given value of k and j, the sum of all j_p has to be equal to j, while the condition $k_1 + \dots + k_j = k$ translates into $2j_2 + 3j_3 + \dots + kj_k = k$. It is also important to note that in (3.37), permutations of the k_i are allowed, and count as different terms. As a result, we have

$$B_n(X,-Y_2,\dots,-Y_n) = n! \sum_{k=0}^{n} \sum_{j=1}^{k} \frac{(-1)^j}{j!} \sum_{\substack{j_2,\dots,j_k\geqslant 0 \\ j_2+j_3+\dots+j_k=j \\ 2j_2+3j_3+\dots+kj_k=k}} \binom{j}{j_2 \dots j_k} \biggl(\frac{Y_2}{2!}\biggr)^{j_2} \dots \biggl(\frac{Y_k}{k!}\biggr)^{j_k} \frac{X^{n-k}}{(n-k)!} , \tag{3.38}$$

where the multinomial coefficient accounts for the permutations of the k_i. Setting $j_1 = n-k$ and distributing the factor $(-1)^j$ over the Y_p, the sum over j and the condition on the sum of the j_p can be dropped, leading to (3.36). □

The complete Bell polynomial can be decomposed as a sum of incomplete Bell polynomials according to the sum of the j_p. Namely, one has

$$B_n(X, -Y_2, \dots, -Y_n) = \sum_{k=1}^{n} B_{n,k}(X, -Y_2, \dots, -Y_{n-k+1}) \;, \tag{3.39}$$

where

$$B_{n,k}(X, -Y_2, \dots, -Y_{n-k+1}) = n! \sum_{\substack{j_1,\dots,j_{n-k+1}\geqslant 0 \\ j_1+\dots+j_{n-k+1}=k \\ j_1+2j_2+\dots+(n-k+1)j_{n-k+1}=n}} \frac{1}{j_1!}\left(\frac{X}{1!}\right)^{j_1} \prod_{p=2}^{n-k+1} \frac{1}{j_p!}\left(\frac{-Y_p}{p!}\right)^{j_p} . \tag{3.40}$$

The Bell polynomials have a simple combinatorial interpretation. The coefficients of $B_{n,k}$ count the number of partitions of a set of cardinality n into k subsets, where the sizes of the subsets is encoded into the monomial. For instance,

$$B_{5,3}(X, Y_2, Y_3) = 15XY_2^2 + 10X^2Y_3 \tag{3.41}$$

means that there are 15 ways of partitioning a set of 5 elements into 3 subsets of sizes 1, 2 and 2, and 10 ways of partitioning it into 3 subsets of sizes 1, 1 and 3. Another interpretation is in terms of substitutions $X^2 \mapsto Y_2$, $X^3 \mapsto Y_3, \dots, X^n \mapsto Y_n$: then, $B_n(X, Y_2, \dots, Y_n)$ is obtained by applying these substitutions in all possible ways to the monomial X^n.

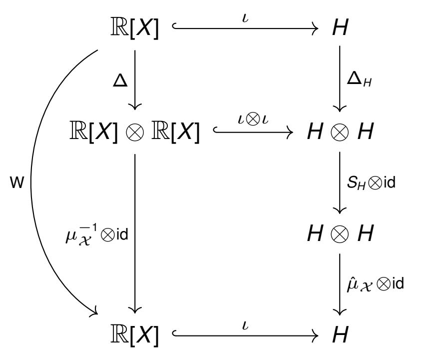
3.4. **Free algebra.** One drawback of the above definition of the Wick map W is that $\mu_{\mathcal{X}}$ is not a character (in general, $\mu_{\mathcal{X}}(X^nX^m) \neq \mu_{\mathcal{X}}(X^n)\mu_{\mathcal{X}}(X^m)$), and therefore $\mu_{\mathcal{X}}^{-1}$ cannot be expressed in terms of the antipode of the polynomial Hopf algebra $\mathbb{R}[X]$.

This problem can be fixed by lifting $\mu_{\mathcal{X}}$ to the symmetric Hopf algebra $H = S(\mathbb{R}[X])$, which is the free commutative algebra over $\mathbb{R}[X]$. We will use the symbol $\odot$ to denote the product in H. The algebra H being free means that $X^n \odot X^m$ is an element of H different from X^{n+m}. The map $\mu_{\mathcal{X}}$ lifts in a unique multiplicative way to a map $\hat\mu : H \to \mathbb{R}$.

We denote the antipode on H by $S_H : H \to H$. *Takeuchi's formula* [46] states that

$$S_H(X^n) = \sum_{k=1}^{n} (-1)^k \sum_{\substack{n_1,\dots,n_k\geqslant 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!} X^{n_1} \odot \dots \odot X^{n_k} \;. \tag{3.42}$$

The inverse of $\hat\mu_{\mathcal{X}}$ is then given by $\hat\mu_{\mathcal{X}}^{-1} = \hat\mu_{\mathcal{X}} \circ S_H$, which is compatible with (3.7). As a consequence, the following diagram commutes, where ι denotes the canonical injection and Δ_H denotes the coproduct on H:



(3.43)

We have explicitly

$$(S_H \otimes \mathrm{id})\Delta_H(X^n) = n! \sum_{k=0}^{n} \sum_{j=1}^{k} (-1)^j \sum_{\substack{k_1,\dots,k_j\geqslant 1 \\ k_1+\dots+k_j=k}} \frac{X^{k_1}}{k_1!} \odot \dots \odot \frac{X^{k_j}}{k_j!} \otimes \frac{X^{n-k}}{(n-k)!} \;. \tag{3.44}$$

As in the proof of Lemma 3.6, we can rewrite this expression in terms of the number j_p of indices k_i equal to p, and distributing the factor $(-1)^j$ over the powers of X. The result is

$$(S_H \otimes \mathrm{id})\Delta_H(X^n) = n! \sum_{k=0}^{n} \sum_{j=1}^{k} j! \sum_{\substack{j_1,\dots,j_k \geqslant 0 \\ j_1+j_2+\dots+j_k=j \\ j_1+2j_2+\dots+kj_k=k}} \bigodot_{p=1}^{j} \frac{1}{j_p!} \left(\frac{-X^p}{p!}\right)^{\odot j_p} \otimes \frac{X^{n-k}}{(n-k)!} \;. \tag{3.45}$$

Note the factor $j!$, which will disappear when applying $\hat{\mu}_{\mathcal{X}} \otimes \mathrm{id} = \exp_*(\hat{\kappa}_{\mathcal{X}}) \otimes \mathrm{id}$ to this expression to recover (3.36).

3.5. **Algebra-valued moments.** The above setting can be extended to the case where the map $\mu_{\mathcal{X}}$ takes values in an arbitrary commutative algebra A. This can be done by considering an *extension of scalars*: consider the space $A[X] = A \otimes \mathbb{R}[X]$, endowed with a left A-module structure given by $a \cdot (b \otimes P) = ab \otimes P$. It can be thought of as consisting of polynomials with coefficients in A.

For any algebra A, the space of linear maps $\mathcal{L}(\mathbb{R}[X], A)$ is an algebra for the convolution product

$$\varphi * \psi = m_A(\varphi \otimes \psi)\Delta \;, \tag{3.46}$$

where m_A is the multiplication map, defined by $m_A(a \otimes b) = ab$. Its unit element is $u_A \circ \varepsilon : \mathbb{R}[X] \to A$, where u_A denotes the projection on the unit of A.

All the properties of the previous sections hold in this setting as well. In particular, the Wick exponential (3.27) is now a map $\mathrm{W} : \mathbb{R}[X] \to A[X]$. It satisfies

$$m_A \circ \mu^A_{\mathcal{X}} \circ \mathrm{W} = u_A \circ \varepsilon \;, \tag{3.47}$$

where $\mu^A_{\mathcal{X}} = \mathrm{id} \otimes \mu_{\mathcal{X}} : A[X] \to A$ is the extension of $\mu_{\mathcal{X}}$.

We will work in the particular setting where $A = \mathbb{R}[Y]$ is the polynomial algebra in a single variable Y. Then one can identify $A[X] = \mathbb{R}[Y] \otimes \mathbb{R}[X]$ with $\mathbb{R}[X, Y]$, via the identification of $Y^m \otimes X^n$ with $X^n Y^m$. Note that we have

$$\mu^A_{\mathcal{X}}(Y^m \otimes X^n) = Y^m \mu_{\mathcal{X}}(X^n) \;. \tag{3.48}$$

4. Multi-indices

The combinatorics of Feynman diagrams becomes quite involved as their size increases. It can be simplified, however, by working with multiindices instead of graphs. A multiindices, introduced in [40], are monomials encoding information on the arity of a graph. A single multi-index corresponds in general to a linear combination of several different graphs, thereby reducing the complexity of the combinatorics, while keeping information that is essential for BPHZ renormalisation. We indroduce these objects by summarising material from [11]. The key result in this section is Proposition 4.3, which provides an explicit expression for the coproduct of the multi-indices corresponding to monomials X^n.

4.1. **Definition of multi-indices.** We introduce here some definitions and results from [11].

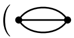
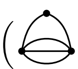
We fix a set of abstract variables $(z_k)_{k\in\mathbb{N}}$. Given a map $\beta : \mathbb{N} \to \mathbb{N}$ with finite support, meaning that the number of non-zero values of β is finite, the associated *multi-index* is the monomial

$$z^\beta = \prod_{k\in\mathbb{N}} z_k^{\beta(k)} \;. \tag{4.1}$$

Let $\Gamma = (\mathcal{V}, \mathcal{E}) \in \mathcal{G}$ be a Feynman diagram. We associate to it the multi-index

$$\Phi(\Gamma) = \prod_{v\in\mathcal{V}} z_{k(v)} \;, \tag{4.2}$$

where $k(v)$ denotes the *arity* of the vertex v, that is, the number of edges adjacent to v. The map Φ is called *counting map*. For instance, we have

$\Phi\Big($  $\Big) = z_3^2 \;, \qquad \Phi\Big($  $\Big) = z_2 z_4^2 \;.$ (4.3)

Note that in our situation, $\beta(k)$ can only differ from 0 for $k \in \{2, 3, 4\}$. The length and the degree of a multi-index are defined, respectively, by

$$|z^\beta| = \sum_{k\in\mathbb{N}} \beta(k) \;, \qquad \deg(z^\beta) = d\big(|z^\beta| - 1\big) - \frac{d-2}{2}\sum_{k\in\mathbb{N}} k\beta(k) \;. \tag{4.4}$$

The degree is preserved by the counting map, that is,

$$\deg\big(\Phi(\Gamma)\big) = \deg(\Gamma) \qquad \forall \Gamma \in \mathcal{G} \tag{4.5}$$

where the degree for a Feynman diagram Γ has been introduced in (2.21) above. In addition, a multi-index has a symmetry factor, defined by

$$S_{\mathrm{M}}(z^\beta) = \prod_{k\in\mathbb{N}} \beta(k)!(k!)^{\beta(k)} \;. \tag{4.6}$$

A Feynman diagram Γ also has a symmetry factor, defined as

$$S_{\mathrm{F}}(\Gamma) = |\mathrm{Aut}(\Gamma)| \;, \tag{4.7}$$

where $\mathrm{Aut}(\Gamma)$ denotes the *automorphism group* of Γ, that is, the number of permutations of vertices and edges of Γ that leave the diagram invariant (see [11, Definition 4.1] for details). This allows to define the inverse map of (4.2), given by

$$\mathcal{P}_{\mathrm{M}}(z^\beta) = \sum_{\Gamma\colon\, \Phi(\Gamma) = z^\beta} \frac{S_{\mathrm{M}}(z^\beta)}{S_{\mathrm{F}}(\Gamma)}\Gamma \;, \tag{4.8}$$

see [11, Definition 3.2 and Proposition 3.4].

We write **M** for the set of non-empty multi-indices (meaning that the $\beta(k)$ cannot all be zero), which also belong to the image $\Phi(\mathcal{G})$. It has a natural vector space structure, and can be equipped with the forest product $\cdot$, to form an algebra $\mathcal{M}$, whose neutral element we will denote $\mathbf{1}_{\mathrm{M}}$. The objects defined above can be extended to forests in $\mathcal{M}$. For instance,

$$\Phi(\Gamma_1 \cdot \ldots \cdot \Gamma_n) = \Phi(\Gamma_1) \cdot \ldots \cdot \Phi(\Gamma_n) \;, \tag{4.9}$$

$$|z^{\beta_1} \cdot \ldots \cdot z^{\beta_n}| = |\beta_1| + \cdots + |\beta_n| \;, \tag{4.10}$$

$$\deg\big(z^{\beta_1} \cdot \ldots \cdot z^{\beta_n}\big) = \deg\big(\beta_1\big) + \cdots + \deg\big(\beta_n\big) \;, \tag{4.11}$$

while the symmetry factor of a forest of multiindices is defined as

$$S_{\mathrm{M}}\big((z^{\beta_1})^{\cdot r_1} \cdot \ldots \cdot (z^{\beta_n})^{\cdot r_n}\big) = \prod_{i=1}^{n} r_i!\big(S_{\mathrm{M}}(z^{\beta_i})\big)^{r_i} \;, \tag{4.12}$$

where the β_i are all assumed to be different from each other.

In [11], the authors also construct a coproduct Δ_{M} and an antipode $\mathcal{A}_{\mathrm{M}}$ such that the following diagram commutes (see [11, Theorem 4.9]):

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\;\mathcal{P}_{\mathrm{M}}\;} & \mathcal{G} \\
\big\downarrow{\scriptstyle \Delta_{\mathrm{M}}} & & \big\downarrow{\scriptstyle \Delta_{\mathrm{CK}}} \\
\mathcal{M}_- \otimes \mathcal{M} & \xrightarrow{\;\mathcal{P}_{\mathrm{M}}\otimes\mathcal{P}_{\mathrm{M}}\;} & \mathcal{G}_- \otimes \mathcal{G} \\
\big\downarrow{\scriptstyle \Pi_{\mathrm{M}}(\mathcal{A}_{\mathrm{M}}\otimes\mathrm{id})} & & \big\downarrow{\scriptstyle \Pi_{N}(\mathcal{A}\otimes\mathrm{id})} \\
\mathcal{M} & \xrightarrow{\;\mathcal{P}_{\mathrm{M}}\;} & \mathcal{G}
\end{array} \tag{4.13}$$

Here $\mathcal{M}_-$ and $\mathcal{G}_-$ denote, respectively, forests of multi-indices and Feynman diagrams of non-positive degree, and Π_{M} is a valuation map on multi-indices. The structure of the renormalisation map on Feynman diagrams is thus exactly mirrored on the space of multi-indices.

Remark 4.1. In [11], the authors mention the *populatedness* condition for multi-indices without specifying it any further. The usual condition (see, for example, [40, eq. (6.3)]), however, is not satisfied for the multi-index z_4^n considered in [11, Sec. 5]. In their case, as in ours, the condition of populatedness is replaced by the requirement that we only consider multi-indices which are also in $\Phi(\mathcal{G})$. It is conceivable that one can restore populatedness by adding an auxiliary variable z_0 with the correct power; since this is not relevant for our considerations, we refrain from pursuing this line of thought any further. ♢

4.2. **Subdivergences.** Within the setting of multi-indices, it is easy to identify the structure of divergent subdiagrams, as we do in the following lemma.

Lemma 4.2. *Let $\Gamma \in \mathcal{G}$. If Γ is a divergent subdiagram, then we necessarily have*

$$\Phi(\Gamma) \in \Bigl\{z_2 z_4^{p-1}\Bigr\}_{p\geqslant 3} \bigcup \Bigl\{z_3^2 z_4^{p-2}\Bigr\}_{p\geqslant 2} . \tag{4.14}$$

In other words: The only potentially divergent subdiagrams are those having multi-index $z_2 z_4^{p-1}$ with $p \geqslant 3$ or $z_3^2 z_4^{p-2}$ with $p \geqslant 2$.

PROOF: Let β be such that $\beta(k) = 0$ for $k \notin \{2, 3, 4\}$. Then

$$\begin{aligned}\deg(z^\beta) &= -\frac{d-2}{2}\sum_{k=2}^{4} k\beta(k) + d\biggl(\sum_{k=2}^{4}\beta(k) - 1\biggr) &(4.15)\\ &= 2\beta(2) + \biggl(3 - \frac{d}{2}\biggr)\beta(3) + (4-d)\beta(4) - d\,. &(4.16)\end{aligned}$$

This is a decreasing function of d. In particular, in the limiting case $d = 4$, we get

$$\deg(z^\beta) = 2\beta(2) + \beta(3) - 4\,, \tag{4.17}$$

showing that z^β can only be divergent if $2\beta(2) + \beta(3) < 4$. In addition, the number of half-edges of the associated diagram should be even, which imposes $\beta(3) \in \{0, 2\}$. The only options are $\beta(3) = 0$ and $\beta(2) = 1$, leading to $z_2 z_4^{p-1}$, and $\beta(3) = 2$ and $\beta(2) = 0$, leading to $z_3^2 z_4^{p-2}$. □

Table 1 lists the first subdivergences appearing as the dimension d increases.

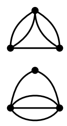
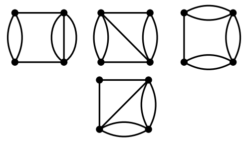
Graph	Multi-index	Degree	Critical d	Minimal n
	z_3^2	$6-2d$	$3 = d_{\mathrm{m}}^*(2)$	4
	$z_3^2 z_4$ $z_2 z_4^2$	$10-3d$	$\frac{10}{3} = d_{\mathrm{m}}^*(3)$	5
	$z_3^2 z_4^2$ $z_2 z_4^3$	$14-4d$	$\frac{7}{2} = d_{\mathrm{m}}^*(4)$	6

TABLE 1. List of the first subdivergent diagrams, with their multi-index, degree, value of d for which they become divergent, and minimal value of n such that they occur in $\mathcal{P}(X^n)$.

4.3. **Computation of $\boldsymbol{\Delta_{\mathrm{M}}(z_4^n)}$.** We will write $\mathring{\Delta}_{\mathrm{M}}$ for the *reduced coproduct*, which is such that

$$\Delta_{\mathrm{M}}(z^\beta) = \mathbf{1}_{\mathrm{M}} \otimes z^\beta + z^\beta \otimes \mathbf{1}_{\mathrm{M}} + \mathring{\Delta}_{\mathrm{M}}(z^\beta)\,. \tag{4.18}$$

(Note that in [11], the authors write Δ_{M} for the reduced coproduct, and Δ_{M}^- for the full coproduct). A simplification arises from the fact that in BPHZ renormalisation, only terms with non-positive degree on the left of the tensor product play a role. We can therefore restrict the coproduct to these terms, and we will use the same notation for that restricted form.

The following result establishes an explicit expression for the reduced coproduct of monomials z_4^n.

Proposition 4.3. *For any $p \geqslant 2$, define*

$$\mathcal{Y}_p = \begin{cases} 16 z_3^2 & \text{if } p = 2\,,\\ 6p z_2 z_4^{p-1} + 8p(p-1) z_3^2 z_4^{p-2} & \text{if } p \geqslant 3\,. \end{cases} \tag{4.19}$$

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Then for any $n \geqslant 4$, one has

$$\mathring{\Delta}_{\mathrm{M}}(z_4^n) = n! \sum_{k=2}^{n-1} \sum_{\substack{j_1,\dots,j_{n-2}\geqslant 0 \\ j_1+j_2+\dots+j_{n-2}=k \\ j_1+2j_2+\dots+(n-2)j_{n-2}=n}} \prod_{p=2}^{n-2} \frac{1}{j_p!}\left(\frac{\mathcal{Y}_p}{p!}\right)^{\cdot j_p} \otimes \frac{1}{j_1!} z_2^{k-j_1} z_4^{j_1} , \tag{4.20}$$

where the sum ranges over non-negative integers j_p.

We will apply [11, Proposition 3.7] to the particular case $z^\beta = z_4^n$. The general expression for the reduced coproduct is

$$\mathring{\Delta}_{\mathrm{M}}(z^\beta) = \sum_{z^{\beta_1}\cdot\ldots\cdot z^{\beta_m}\in\mathcal{M}} \sum_{z^\alpha\in\mathbf{M}} E(z^{\beta_1}\cdot\ldots\cdot z^{\beta_m}, z^\alpha, z^\beta)\left(z^{\beta_1}\cdot\ldots\cdot z^{\beta_m}\right)\otimes z^\alpha , \tag{4.21}$$

where the coefficients $E(\cdot,\cdot,\cdot)$ are given by

$$\begin{aligned} &E(z^{\beta_1}\cdot\ldots\cdot z^{\beta_m}, z^\alpha, z^\beta) \\ &= \sum_{k_1,\dots,k_m\in\mathbb{N}} \sum_{\hat\beta_1+\dots+\hat\beta_m+\hat\alpha=\beta} \frac{S_{\mathrm{M}}(z^\beta)}{S_{\mathrm{M}}(z^{\beta_1}\cdot\ldots\cdot z^{\beta_m})S_{\mathrm{M}}(z^\alpha)} \frac{\langle\prod_{i=1}^m \partial_{z_{k_i}} z^\alpha, z^{\hat\alpha}\rangle}{S_{\mathrm{M}}(z^{\hat\alpha})} \prod_{i=1}^m \frac{\langle D^{k_i} z^{\beta_i}, z^{\hat\beta_i}\rangle}{S_{\mathrm{M}}(z^{\hat\beta_i})} . \end{aligned} \tag{4.22}$$

Here the inner product for multi-indices is defined by

$$\langle z^\alpha, z^\beta\rangle = S_{\mathrm{M}}(z^\alpha)\delta^\alpha_\beta , \tag{4.23}$$

while D is the linear map

$$D = \sum_{k\in\mathbb{N}} z_{k+1}\partial_{z_k} . \tag{4.24}$$

In what follows, we write exponents of multi-indices in the form $\beta = [\beta(2),\beta(3),\beta(4)]$, since all other $\beta(k)$ vanish in our situation. We need to apply (4.21) to $\beta = [0,0,n]$. Since we restrict the coproduct to terms that are divergent on the left side of the tensor product, Lemma 4.2 implies that the β_i are of the form $[1,0,p-1]$ with $p \geqslant 3$ or $[0,2,p-2]$ with $p \geqslant 2$.

Lemma 4.4. *Denote by a_p the number of z^{β_i} equal to $z_2 z_4^{p-1}$, and by b_p the number of z^{β_i} equal to $z_3^2 z_4^{p-2}$, with the convention $a_2 = 0$ to avoid case distinctions between the a_p and b_p. Then the only non-vanishing coefficient in* (4.21) *for these β_i is*

$$E(z^{\beta_1}\cdot\ldots\cdot z^{\beta_m}, z_2^m z_4^{n-s}, z_4^n) = \frac{n!}{(n-s)!}\prod_{p=2}^{n-2}\frac{1}{a_p!}\left(\frac{4!}{4(p-1)!}\right)^{a_p}\frac{1}{b_p!}\left(\frac{4!}{3(p-2)!}\right)^{b_p} , \tag{4.25}$$

where

$$s = \sum_{p=2}^{n-2} p j_p , \qquad j_p = a_p + b_p . \tag{4.26}$$

PROOF: A direct computation shows that

$$D(z_2 z_4^{p-1}) = z_3 z_4^{p-1} + R_1 , \qquad D(z_3^2 z_4^{p-2}) = 2 z_3 z_4^{p-1} + R_3 , \tag{4.27}$$

$$D^2(z_2 z_4^{p-1}) = z_4^p + R_2 , \qquad D^2(z_3^2 z_4^{p-2}) = 2 z_4^p + R_4 , \tag{4.28}$$

where the R_i are residual terms that vanish when $z_k = 0$ for all $k \geqslant 5$. All higher derivatives also vanish when $z_k = 0$ for all $k \geqslant 5$.

We now observe that the condition $\hat\beta_1 + \dots + \hat\beta_m + \hat\alpha = \beta = [0,0,n]$ implies that all $\hat\beta_i$, as well as $\hat\alpha$, are of the form $[0,0,\star]$. It follows from (4.23) and (4.28) that $\langle D^{k_i} z^{\beta_i}, z^{\hat\beta_i}\rangle = 0$ unless $k_i = 2$ and $\hat\beta_i = [0,0,p]$, in which case

$$\frac{\langle D^2 z^{\beta_i}, z^{\hat\beta_i}\rangle}{S_{\mathrm{M}}(z^{\hat\beta_i})} = \begin{cases} 1 & \text{if } \beta_i = [1,0,p-1] , \\ 2 & \text{if } \beta_i = [0,2,p-2] . \end{cases} \tag{4.29}$$

Writing

$$r = \sum_{p=2}^{n-2} b_p , \qquad m - r = \sum_{p=2}^{n-2} a_p , \tag{4.30}$$

we obtain

$$\prod_{i=1}^{m} \frac{\langle D^2 z^{\beta_i}, z^{\hat{\beta}_i} \rangle}{S_{\mathrm{M}}(z^{\hat{\beta}_i})} = 2^r . \tag{4.31}$$

Furthermore, we have

$$\hat{\beta}_1 + \cdots + \hat{\beta}_m = [0, 0, \sum_{p=2}^{n-2} p j_p] = [0, 0, s] , \tag{4.32}$$

and therefore the condition $\hat{\beta}_1 + \cdots + \hat{\beta}_m + \hat{\alpha} = \beta = [0, 0, n]$ imposes

$$\hat{\alpha} = [0, 0, n - s] , \tag{4.33}$$

or, equivalently, $z^{\hat{\alpha}} = z_4^{n-s}$. Therefore, since all k_i are equal to 2, we obtain

$$\frac{\langle \prod_{i=1}^{m} \partial_{z_{k_i}} z^{\alpha}, z^{\hat{\alpha}} \rangle}{S_{\mathrm{M}}(z^{\hat{\alpha}})} = m! \mathbb{1}_{\alpha=[m,0,n-s]} \tag{4.34}$$

showing that we necessarily have $z^{\alpha} = z_2^m z_4^{n-s}$. It remains to compute some symmetry factors. First, we compute

$$\frac{S_{\mathrm{M}}(z_4^n)}{S_{\mathrm{M}}(z^{\alpha})} = \frac{n!(4!)^n}{m!(2!)^m (n-s)!(4!)^{n-s}} = \frac{n!(4!)^s}{m!(n-s)!(2!)^m} . \tag{4.35}$$

Next, since

$$S_{\mathrm{M}}(z_2 z_4^{p-1}) = 2!(p-1)!(4!)^{p-1} , \tag{4.36}$$

$$S_{\mathrm{M}}(z_3^2 z_4^{p-2}) = 2!(3!)^2 (p-2)!(4!)^{p-2} = 3(p-2)!(4!)^{p-1} , \tag{4.37}$$

we get

$$S_{\mathrm{M}}\left(z^{\beta_1} \cdot \ldots \cdot z^{\beta_m}\right) = \prod_{p=2}^{n-2} a_p! \left[2!(p-1)!(4!)^{p-1}\right]^{a_p} b_p! \left[3(p-2)!(4!)^{p-1}\right]^{b_p} . \tag{4.38}$$

Plugging (4.31), (4.34), (4.35) and (4.38) into (4.22), we finally obtain

$$E(z^{\beta_1} \cdot \ldots \cdot z^{\beta_m}, z_2^m z_4^{n-s}, z_4^n) = \frac{n!}{(n-s)!} \frac{(4!)^s 2^r}{(2!)^m} \prod_{p=2}^{n-2} \frac{1}{a_p! \left[2!(p-1)!(4!)^{p-1}\right]^{a_p} b_p! \left[3(p-2)!(4!)^{p-1}\right]^{b_p}} , \tag{4.39}$$

from which the result follows, upon expanding $(4!)^s = \prod_p 4!^{p(a_p+b_p)}$, $2^r = \prod_p 2^{b_p}$ and finally $(2!)^m = \prod_p (2!)^{a_p+b_p}$. □

PROOF OF PROPOSITION 4.3. Lemma 4.4 and (4.21) imply

$$\mathring{\Delta}_{\mathrm{M}}(z^{\beta}) = \sum_{s,m} F_{s,m}^{(n)} \otimes z_2^m z_4^{n-s} , \tag{4.40}$$

where

$$F_{s,m}^{(n)} = \frac{n!}{(n-s)!} \sum_{(a_p,b_p)} \prod_{p=2}^{n-2} \frac{1}{a_p!} \left(\frac{4! z_2 z_4^{p-1}}{4(p-1)!} \right)^{\cdot a_p} \frac{1}{b_p!} \left(\frac{4! z_3^2 z_4^{p-2}}{3(p-2)!} \right)^{\cdot b_p} , \tag{4.41}$$

with the sum running over all (a_p, b_p) satisfying

$$\sum_{p=2}^{n-2} j_p = m , \qquad \sum_{p=2}^{n-2} p j_p = s , \qquad j_p = a_p + b_p . \tag{4.42}$$

We first perform the sum over all (a_p, b_p) summing to j_p. The binomial formula yields

$$\sum_{a_p+b_p=j_p} \frac{1}{a_p!} \left(\frac{4! z_2 z_4^{p-1}}{4(p-1)!} \right)^{\cdot a_p} \frac{1}{b_p!} \left(\frac{4! z_3^2 z_4^{p-2}}{3(p-2)!} \right)^{\cdot b_p} = \frac{1}{j_p!} \left(\frac{\mathcal{Y}_p}{p!} \right)^{\cdot j_p} , \tag{4.43}$$

where the $\mathcal{Y}_p$ are given in (4.19). This implies

$$F_{s,m}^{(n)} = \frac{n!}{(n-s)!} \sum_{\substack{j_2+\cdots+j_{n-2}=m \\ 2j_2+\cdots+(n-2)j_{n-2}=s}} \prod_{p=2}^{n-2} \frac{1}{j_p!} \left(\frac{\mathcal{Y}_p}{p!} \right)^{\cdot j_p} . \tag{4.44}$$

The last step is to change the summation over variables (s, m) to that over variables (k, j_1), where $k = n + m - s$ and $j_1 = n - s$. In particular, the sum of the j_p starting from $p = 1$ is now k, while the sum of the $p j_p$ is n. □

Remark 4.5. The sum (4.20) can be extended to all k from 1 to $n-1$, because the conditions on the j_p cannot be both satisfied if $k = 1$ since $p \leqslant n-2$. ◊

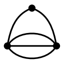
Remark 4.6. On the right-hand side of (4.20), the only k-dependence occurs in the term $z_2^{k-j_1}$. If the expression is evaluated in $z_2 = 1$, the sum over k and the condition $j_1 + j_2 + \dots + j_{n-2} = k$ can be dropped. ◊

Example 4.7. Consider the case $n = 4$. Then k can take the values 2 and 3. For $k = 2$, the only possible decomposition is $j_1 = 0$, $j_2 = 2$, while for $k = 3$, the only option is $j_1 = 2$, $j_2 = 1$. Since $\mathcal{Y}_2 = 16z_3^2$, we obtain

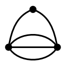
$$\mathring{\Delta}_{\mathrm{M}}(z_4^4) = 4!\biggl(\frac{1}{(2!)^3}\mathcal{Y}_2^{\cdot 2} \otimes z_2^2 + \frac{1}{(2!)^2}\mathcal{Y}_2 \otimes z_2 z_4^2\biggr) \tag{4.45}$$

$$= 4!\bigl(32(z_3^2)^{\cdot 2} \otimes z_2^2 + 4z_3^2 \otimes z_2 z_4^2\bigr) \;. \tag{4.46}$$

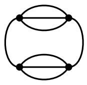
We can check commutativity of the upper part of diagram (4.13). Applying (4.8), we find

$\mathcal{P}_{\mathrm{M}}\bigl(z_2^2\bigr) = 2$  , $\quad \mathcal{P}_{\mathrm{M}}\bigl(z_3^2\bigr) = 6$  , $\quad \mathcal{P}_{\mathrm{M}}\bigl(z_2 z_4^2\bigr) = 2^6 \cdot 3$  , (4.47)

which implies

$(\mathcal{P}_{\mathrm{M}} \otimes \mathcal{P}_{\mathrm{M}})\mathring{\Delta}_{\mathrm{M}}(z_4^4) = 2^{11} \cdot 3^3 \Bigl($  $^{\cdot 2} \otimes$  $+ 2$  $\otimes$  $\Bigr)$. (4.48)

On the other hand, one finds

$\mathcal{P}_{\mathrm{M}}(z_4^4) = \mathcal{P}(X^4) = 2^{11} \cdot 3^3$  $+ \dots$, (4.49)

where the dots indicate diagrams without subdivergences. Applying $\mathring{\Delta}_{\mathrm{CK}}$ to this expression indeed yields (4.48), since we can extract one or two "sunset diagrams" .

For $n = 5$, the result is

$$\mathring{\Delta}_{\mathrm{M}}(z_4^5) = 5!\biggl(\frac{1}{2!3!}\mathcal{Y}_2 \cdot \mathcal{Y}_3 \otimes z_2^2 + \frac{1}{2!3!}\mathcal{Y}_3 \otimes z_2 z_4^2 + \frac{1}{(2!)^3}\mathcal{Y}_2^{\cdot 2} \otimes z_2^4 z_4 + \frac{1}{2!3!}\mathcal{Y}_2 \otimes z_2 z_4^3\biggr) \;, \tag{4.50}$$

where the coefficients $(10, 10, 15, 10)$ are as in the Bell polynomial

$$B_5(x, y_2, y_3, y_4, y_5) = y_5 + 5y_4 x + 10y_2 y_3 + 10y_3 x^2 + 15y_2^2 x + 10y_2 x^3 + x^5 \;, \tag{4.51}$$

up to boundary terms. ♦

4.4. **Extended coproduct.** Example 4.7 above shows some differences between the terms appearing in the reduced coproduct (such as (4.50)) and Bell polynomials (cf. (4.51)). These are partly due to the fact that the reduced coproduct is used, instead of the full coproduct. However, there is also a difference due to the condition $p \leqslant n-2$ on the subdivergences extracted from z_4^n. Therefore, the Bell polynomial (4.51) contains a term $5y_4 x$, while $\mathring{\Delta}_{\mathrm{M}}(z_4^5)$ has no term proportional to $\mathcal{Y}_4 \otimes X$.

Below, it will be convenient to artificially add this term to the reduced coproduct, that is, to consider its extended version

$$\mathring{\Delta}^+_{\mathrm{M}}(z_4^n) = n! \sum_{k=1}^{n-1} \sum_{\substack{j_1,\dots,j_{n-1} \geqslant 0 \\ j_1 + j_2 + \dots + j_{n-1} = k \\ j_1 + 2j_2 + \dots + (n-1)j_{n-1} = n}} \prod_{p=2}^{n-1} \frac{1}{j_p!}\biggl(\frac{\mathcal{Y}_p}{p!}\biggr)^{\cdot j_p} \otimes \frac{1}{j_1!} z_2^{k-j_1} z_4^{j_1} \;, \tag{4.52}$$

which differs from (4.20) by the condition on p within the product. The following lemma shows that this will not affect the end result.

Lemma 4.8. *For all $n \geqslant 4$, one has*

$$\mathcal{P}_{\mathrm{M}} \circ (\Pi_{\mathrm{M}} \tilde{\mathcal{A}}_{\mathrm{M}} \otimes \mathrm{id})\mathring{\Delta}^+_{\mathrm{M}}(z_4^n) = \mathcal{P}_{\mathrm{M}} \circ (\Pi_{\mathrm{M}} \tilde{\mathcal{A}}_{\mathrm{M}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{M}}(z_4^n) \;. \tag{4.53}$$

PROOF: The only difference between $\mathring{\Delta}^+_{\mathrm{M}}(z_4^n)$ and $\mathring{\Delta}_{\mathrm{M}}(z_4^n)$ is that the former has an additional term, corresponding to $j_{n-1} = 1$. The conditions on the j_p impose $j_1 = 1$ while all other j_p vanish, so that $k = 2$. One obtains

$$\mathring{\Delta}^+_{\mathrm{M}}(z_4^n) = \mathring{\Delta}_{\mathrm{M}}(z_4^n) + n\mathcal{Y}_{n-1} \otimes z_2 z_4 \;. \tag{4.54}$$

The result follows from the fact that $\mathcal{P}_{\mathrm{M}}(z_2 z_4) = 0$, because there is no admissible pairwise matching leaving no free legs. □

5. Commutative diagram

This section contains the proofs of the main results, Theorem 2.5 and Corollary 2.6.

5.1. **Identification of the counterterms.**

We assume now that the algebra-valued map $\hat{\kappa}_{\mathcal{X}} : H \to \mathbb{R}[Y]$ is given by

$$\hat{\kappa}_{\mathcal{X}}(X^p) = \begin{cases} 0 & \text{if } p = 1\,, \\ \sigma_p Y & \text{if } p \geqslant 2\,, \end{cases} \tag{5.1}$$

where the σ_p are real numbers. The Wick map $\hat{\mathsf{W}} = (\hat{\mu}_{\mathcal{X}} S_H \otimes \mathrm{id})\Delta_H : H \to \mathbb{R}[X,Y]$ can be written (cf. Lemma 3.6) in the form

$$\hat{\mathsf{W}}(X^n) = n! \sum_{k=0}^{n} \sum_{\substack{j_1,\dots,j_n \geqslant 0 \\ j_1+\dots+j_n=k \\ j_1+2j_2+\dots+nj_n=n}} \prod_{p=2}^{n} \frac{1}{j_p!} \biggl(\frac{-\sigma_p Y}{p!} \biggr)^{j_p} \frac{X^{j_1}}{j_1!}\,. \tag{5.2}$$

We now define a linear map $\eta : \mathbb{R}[X,Y] \to \mathcal{M}$ by

$$\eta(X) = z_4\,, \tag{5.3}$$

$$\eta(Y) = z_2\,, \tag{5.4}$$

extended multiplicatively. Since $Y^{j_2+\dots+j_n} = Y^{k-j_1}$, we have

$$(\eta \circ \hat{\mathsf{W}})(X^n) = n! \sum_{k=0}^{n} \sum_{\substack{j_1,\dots,j_n \geqslant 0 \\ j_1+\dots+j_n=k \\ j_1+2j_2+\dots+nj_n=n}} \prod_{p=2}^{n} \frac{1}{j_p!} \biggl(\frac{-\sigma_p}{p!} \biggr)^{j_p} z_2^{k-j_1} \frac{z_4^{j_1}}{j_1!}\,. \tag{5.5}$$

Proposition 5.1. *Define the σ_p by*

$$\sigma_p := -\Pi_{\mathsf{M}} \mathcal{A}_{\mathsf{M}}(\mathcal{Y}_p) = -\Pi_N \mathcal{A}\mathcal{P}_{\mathsf{M}}(\mathcal{Y}_p) \tag{5.6}$$

for all $p \geqslant 2$, where the second equality comes from the commutative diagram (4.13)*. Then*

$$(\eta \circ \hat{\mathsf{W}})(X^n) = (\Pi_{\mathsf{M}} \mathcal{A}_{\mathsf{M}} \otimes \mathrm{id})\mathring{\Delta}^+_{\mathsf{M}}(z_4^n) - \sigma_n z_2 + z_4^n \tag{5.7}$$

holds for all $n \geqslant 2$.

Proof: Proposition 4.3 and Lemma 4.8 imply

$$(\Pi_{\mathsf{M}} \mathcal{A}_{\mathsf{M}} \otimes \mathrm{id})\mathring{\Delta}^+_{\mathsf{M}}(z_4^n) = n! \sum_{k=2}^{n-1} \sum_{\substack{j_1,\dots,j_{n-1} \geqslant 0 \\ j_1+j_2+\dots+j_{n-1}=k \\ j_1+2j_2+\dots+(n-1)j_{n-1}=n}} \prod_{p=2}^{n-1} \frac{1}{j_p!} \biggl(\frac{\Pi_{\mathsf{M}} \mathcal{A}_{\mathsf{M}}(\mathcal{Y}_p)}{p!} \biggr)^{j_p} \frac{1}{j_1!} z_2^{k-j_1} z_4^{j_1}\,. \tag{5.8}$$

The only difference with $(\eta \circ \hat{\mathsf{W}})(X^n)$ is that the terms $k = 1$ and $k = n$ are missing. The term $k = 1$ allows only for $j_n = 1$ while all other j_i equal to 0, and accounts for $-\sigma_n z_2$. The term $k = n$ allows only for $j_1 = n$, while all other j_i equal to 0, and accounts for the term z_4^n. □

5.2. **Taking care of the boundary terms.**

It remains to add the boundary terms to the coproduct $\mathring{\Delta}_{\mathsf{M}}$. The result is as follows.

Proposition 5.2. *Define a linear map $\Theta_{\mathsf{M}} : \mathcal{M} \to \mathcal{M}$ by*

$$\Theta_{\mathsf{M}}(z_4^n) = \gamma_n := -\sigma_n z_2 - \Pi_{\mathsf{M}} \mathcal{A}_{\mathsf{M}}(z_4^n) \mathbf{1}_{\mathsf{M}}\,. \tag{5.9}$$

Then the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\ \eta\ } & \mathcal{M} \\ {\scriptstyle \hat{\mathsf{W}}}\Big\downarrow & & \Big\downarrow{\scriptstyle (\Pi_{\mathsf{M}}\mathcal{A}_{\mathsf{M}} \otimes \mathrm{id})\Delta^+_{\mathsf{M}} + \Theta_{\mathsf{M}}} \\ \mathbb{R}[X,Y] & \xrightarrow{\ \eta\ } & \mathcal{M} \end{array} \tag{5.10}$$

where Δ^+_{M} denotes the full *extended coproduct.*

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PROOF: This follows immediately by adding the relations

$$(\Pi_M \mathcal{A}_M \otimes \mathrm{id})(z_4^n \otimes \mathrm{id}) = \Pi_M \mathcal{A}_M(z_4^n)\mathbf{1}_M\ , \tag{5.11}$$

$$(\Pi_M \mathcal{A}_M \otimes \mathrm{id})(\mathrm{id} \otimes z_4^n) = z_4^n\ , \tag{5.12}$$

to (5.7), and incorporating the extra terms into Θ_M. □

5.3. **Proof of Theorem 2.5.** To lighten notations, we introduce two maps $\chi_M : \mathcal{M} \to \mathcal{M}$ and $\chi_F : \mathcal{G} \to \mathcal{G}$ given by

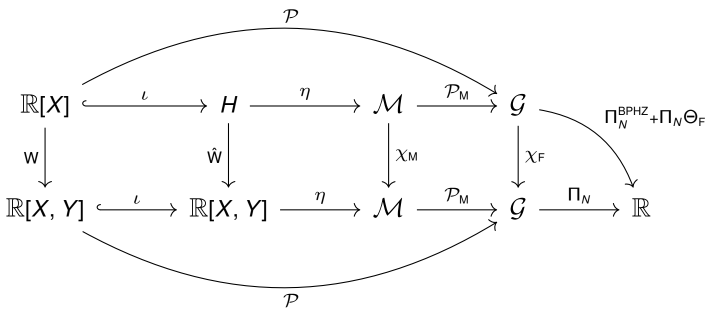
$$\chi_M = (\Pi_M \mathcal{A}_M \otimes \mathrm{id})\Delta_M^+ + \Theta_M\ , \tag{5.13}$$

$$\chi_F = (\Pi_N \mathcal{A} \otimes \mathrm{id})\Delta_{CK} + \Theta_F\ , \tag{5.14}$$

where Θ_F is defined by

$$\Theta_F \circ \mathcal{P}_M = \mathcal{P}_M \circ \Theta_M\ . \tag{5.15}$$

Then the results obtained so far show that the following diagram commutes:



(5.16)

Indeed, the left square is the commutative diagram (3.43), the middle square's commutativity is shown in Proposition 5.2, and the right square is commutative thanks to (4.13) and (5.15). The remaining commutativity relations follow from the definitions.

It follows directly from Proposition 3.3 that

$$W(e^{-\alpha X}) = \sum_{n\geqslant 0} \frac{(-\alpha)^n}{n!} W(X^n) = W(-\alpha, X) = e^{-\alpha X - K_{\mathcal{X}}(-\alpha)}\ , \tag{5.17}$$

where

$$K_{\mathcal{X}}(-\alpha) = \Lambda(\kappa_{\mathcal{X}})(-\alpha) = \sum_{n\geqslant 2} \frac{(-\alpha)^n}{n!}\sigma_n Y =: \beta Y\ . \tag{5.18}$$

This determines the value of the mass renormalisation term β. The fact that $W(X^n)$ is a Bell polynomial has been shown in Section 3.3.

It remains to derive an expression for the energy renormalisation term γ. This is determined by the fact that

$$\log \mathbb{E}[e^{-\alpha X - \beta Y}] = (\Pi_N \circ \mathcal{P})(e^{-\alpha X - \beta Y}) \tag{5.19}$$

$$= \Big(\big[\Pi_N^{BPHZ} + \Pi_N \Theta_F\big] \circ \mathcal{P}\Big)(e^{-\alpha X}) \tag{5.20}$$

$$= (\Pi_N^{BPHZ} \circ \mathcal{P})(e^{-\alpha X}) + \gamma_0\ , \tag{5.21}$$

where we have set

$$\gamma_0 = \Pi_N \Theta_F \circ \mathcal{P}(e^{-\alpha X})\ . \tag{5.22}$$

It follows that

$$\log \mathbb{E}[e^{-\alpha X - \beta Y - \gamma}] = \log \mathbb{E}[e^{-\alpha X - \beta Y}] - \gamma = \Pi_N^{BPHZ} \circ \mathcal{P}(e^{-\alpha X}) + \gamma_0 - \gamma\ . \tag{5.23}$$

Since $\Pi_N^{BPHZ} \circ \mathcal{P}(e^{-\alpha X})$ is convergent by Lemma 2.4, γ should be the divergent part of γ_0.

The constant γ_0 can be made explicit by noting that (5.15) implies

$$(\Theta_F \circ \mathcal{P}_M)(z_4^n) = \mathcal{P}_M(\gamma_n) \tag{5.24}$$

$$= -\sigma_n \mathcal{P}_M(z_2) - \Pi_M \mathcal{A}_M(z_4^n) \tag{5.25}$$

$$= -\sigma_n \mathcal{P}_M(z_2) - \Pi_N \mathcal{A}(\mathcal{P}(X^n))\ . \tag{5.26}$$

Therefore,

$$\begin{aligned}
\gamma_0 &= (\Pi_N \Theta_{\mathrm{F}} \circ \mathcal{P})(\mathrm{e}^{-\alpha X}) &(5.27)\\
&= (\Pi_N \Theta_{\mathrm{F}} \circ \mathcal{P}_{\mathrm{M}})(\mathrm{e}^{-\alpha z_4}) &(5.28)\\
&= \sum_{n\geqslant 2} \frac{(-\alpha)^n}{n!} (\Pi_N \Theta_{\mathrm{F}} \circ \mathcal{P}_{\mathrm{M}})(z_4^n) &(5.29)\\
&= -\sum_{n\geqslant 2} \frac{(-\alpha)^n}{n!} \sigma_n \Pi_N \mathcal{P}_{\mathrm{M}}(z_2) - \sum_{n\geqslant 2} \frac{(-\alpha)^n}{n!} (\Pi_N \mathcal{A} \circ \mathcal{P})(X^n) &(5.30)\\
&= -\sum_{n\geqslant 2} \frac{(-\alpha)^n}{n!} \sigma_n \Pi_N \mathcal{P}(Y) - (\Pi_N \mathcal{A} \circ \mathcal{P})(\mathrm{e}^{-\alpha X})\;. &(5.31)
\end{aligned}$$

Note however that $\Pi_N \mathcal{P}(Y) = 0$ since self-pairings are not allowed. Therefore, γ_0 reduces to $-(\Pi_N \mathcal{A} \circ \mathcal{P})(\mathrm{e}^{-\alpha X})$ as claimed. Accordingly, γ is given as in (2.51) since that is the divergent part of γ_0.

5.4. **The case of subdivergences of degree smaller than -1.** As mentioned in Remark 2.2, if a Feynman diagram Γ has subdivergences of degree below -1, the definition (2.23) of the Connes–Kreimer coproduct Δ_{CK} has to be modified for the BPHZ theorem, Theorem 2.1, to remain true. This is done by adding edge and node decorations to Feynman diagrams, where the node decorations represent additional monomials in the valuation, while edge decorations represent derivatives of the Green function. Denote by $\Gamma^{\mathfrak{n}}_{\mathfrak{e}}$ the graph $\Gamma = (\mathcal{V}, \mathcal{E})$ equipped with a node decoration $\mathfrak{n} : \mathcal{V} \to \mathbb{N}_0^3$ and an edge decoration $\mathfrak{e} : \mathcal{E} \to \mathbb{N}_0^3$. Its degree is defined by

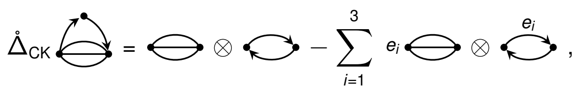
$$\deg(\Gamma^{\mathfrak{n}}_{\mathfrak{e}}) = d\bigl(|\mathcal{V}| - 1\bigr) + \sum_{v\in\mathcal{V}} |\mathfrak{n}(v)| + \sum_{e\in\mathcal{E}} \bigl[d - 2 - |\mathfrak{e}(e)|\bigr]\;, \tag{5.32}$$

where for any $\mathfrak{l} \in \mathbb{N}_0^3$, we set $|\mathfrak{l}| = \sum_{i=1}^3 |\mathfrak{l}_i|$. Given a distinguished vertex $v^\star \in \mathcal{V}$, the valuation of the decorated graph is

$$\Pi_N(\Gamma^{\mathfrak{n}}_{\mathfrak{e}}; v^\star) = \int_{(\mathbb{T}^d)^{\mathcal{V}\setminus v^\star}} \prod_{e\in\mathcal{E}} \partial^{\mathfrak{e}(e)} G_N(x_{e_+} - x_{e_-}) \prod_{w\in\mathcal{V}\setminus v^\star} (x_w - x_{v^\star})^{\mathfrak{n}(w)}\,\mathrm{d}x\;, \tag{5.33}$$

where $\partial^{\mathfrak{e}} = \prod_{i=1}^3 \partial^{\mathfrak{e}_i}$ and $x^{\mathfrak{n}} = \prod_{i=1}^3 x_i^{\mathfrak{n}_i}$. Since the derivative along an edge depends on its orientation, the graph Γ should actually be considered as a directed graph.

The Connes–Kreimer coproduct is then modified by adding to it terms with additional decorations. We will refrain from giving the general definition here, which can be found in [35, (2.19)], but give instead an illustrative example, when the graph is assumed to have a degree in $(-2, 1]$. Indicating the orientation of edges only where it matters, one has for the reduced coproduct



(5.34)

where the e_i denote the canonical basis vectors, and non-zero decorations have been placed near the node or edge they apply to.

In principle, the multiindex framework could be extended to decorated graphs, by adding suitable information to the multiindices. However, this is not needed in our situation. Indeed, the proof of Lemma 4.2 shows that for $d < 4$, all subdivergences have a degree larger than -2. This means that as in the above example, one can only extract graphs that have at most one node decoration of the form e_i. However, (5.33) shows that the valuation of such graphs vanishes by symmetry, since it involves the integral of an odd function. Therefore, decorated graphs can be ignored.

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