

Global well-posedness of the elastic-viscous-plastic sea-ice model with the inviscid Voigt-regularisation

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Abstract

In this paper, we initiate the rigorous mathematical analysis of the elastic-viscous-plastic (EVP) sea-ice model, which was introduced in [E. C. Hunke and J. K. Dukowicz, *J. Phys. Oceanogr.*, 27, 9 (1997), 1849–1867]. The EVP model is one of the standard and most commonly used dynamical sea-ice models. We study a regularised version of this model. In particular, we prove the global well-posedness of the EVP model with the inviscid Voigt-regularisation of the evolution equation for the stress tensor. Due to the elastic relaxation and the Voigt regularisation, we are able to handle the case of viscosity coefficients without cutoff, which has been a major issue and a setback in the computational study and analysis of the related Hibler sea-ice model, which was originally introduced in [W. D. Hibler, *J. Phys. Oceanogr.*, 9, 4 (1979), 815–846]. The EVP model shares some structural characteristics with the Oldroyd-B model and related models for viscoelastic non-Newtonian complex fluids.

1 Introduction

1.1 Formulation of the EVP model

Sea-ice dynamics has a strong influence on the global climate, for instance, through heat exchange with the atmosphere and the ocean around the Arctic area as well as the reflection of sunlight. Despite the development of sea-ice modeling over many decades, its complexity continues to pose significant challenges for numerical simulations of the sea-ice cover (cf. [25]). For these reasons, an enhanced theoretical understanding of the fundamental dynamical sea-ice models and their rigorous mathematical analysis is of great importance and substantial interest.

Over several decades there have been significant advances in the rigorous mathematical analysis of oceanic and atmospheric dynamics. In contrast, there have been relatively few works which obtain rigorous results in the mathematical study of sea-ice dynamics. It is the purpose of this paper to commence the rigorous mathematical analysis of the elastic-viscous-plastic (EVP) dynamical sea-ice model.

The EVP model was introduced in [35] as a numerical regularisation of the Hibler model from [32]. The latter model describes sea-ice dynamics by using a viscous-plastic rheology, and the viscosity becomes singular for small strain rates and degenerate for large strain rates.

We will denote the symmetric part of the gradient $D(u)$ by

$$D(u) := \frac{1}{2}[\nabla u + (\nabla u)^\top]. \quad (1.1)$$

The strain rate is given by

$$\bar{D} := \left(\frac{2}{\bar{e}^2} \left| D(u) - \frac{1}{2} [\text{Tr } D(u)] \mathbb{I}_2 \right|^2 + |\text{Tr } D(u)|^2 \right)^{1/2}. \quad (1.2)$$

Here $u : \mathbb{T}^2 \times [0, T] \rightarrow \mathbb{R}^2$ is the unknown velocity and $\bar{e} > 1$ is a constant, representing the ratio of the major axis to minor axis lengths of the elliptic yield curve. Note that throughout this paper we will use periodic boundary conditions and take our domain to be the two-dimensional flat unit torus \mathbb{T}^2 (i.e. $[0, 1]^2$).

Then, the momentum equations of the Hibler sea-ice model [32] can be written as

$$m \partial_t u + m u \cdot \nabla u = \nabla \cdot \sigma + \mathcal{T}_a + \mathcal{T}_w + m \Omega u^\perp - m g \nabla H_0, \quad (1.3a)$$

with the constitutive law for the stress tensor given by

$$\frac{\bar{e}^2 \bar{D}}{P} (\sigma - \frac{1}{2} \text{Tr } \sigma \mathbb{I}_2) + \frac{\bar{D}}{2P} \text{Tr } \sigma \mathbb{I}_2 + \frac{\bar{D}}{2} \mathbb{I}_2 = D(u). \quad (1.3b)$$

Here $\sigma : \mathbb{T}^2 \times [0, T] \rightarrow \mathbb{R}^{2 \times 2}$ is the stress tensor, and $P > 0$, $m > 0$, and Ω are given constants which denote the pressure, the mass per unit area and the coefficient of the Coriolis force, respectively. The functions \mathcal{T}_a and \mathcal{T}_w denote the wind and ocean stresses, and are defined in equations (1.3d) and (1.3e), respectively. The given function H_0 denotes the ocean surface topography. We also recall the notation $u^\perp = (-u_2, u_1)^\top$. In order to fix ideas, we have omitted the thermodynamical equations and purely focus on the dynamical part of the EVP model, and we therefore assume that P is a given constant. The analysis of the EVP model coupled with the thermodynamical equations is a subject of future work.

One can easily check from (1.3b) that σ becomes singular as the strain rate \bar{D} (see (1.2)) approaches zero, and becomes degenerate as \bar{D} approaches infinity. This special structure, in particular, causes significant difficulties for both the analysis as well as the numerical simulations. In fact, in Hibler's original paper [32], a cutoff of \bar{D} is introduced to avoid this problem.

In order to facilitate the usage of explicit numerical schemes and therefore also of parallel computing, it was proposed in [35] to modify the viscous-plastic sea-ice rheology (1.3b) by adding an artificial elastic component. That is, one replaces the diagnostic, constitutive equation (1.3b) by the prognostic, dynamically relaxed equation

$$\frac{1}{\mathcal{E}} \partial_t \sigma + \frac{\bar{e}^2 \bar{D}}{P} (\sigma - \frac{1}{2} \text{Tr } \sigma \mathbb{I}_2) + \frac{\bar{D}}{2P} \text{Tr } \sigma \mathbb{I}_2 + \frac{\bar{D}}{2} \mathbb{I}_2 = D(u), \quad (1.3c)$$

where $\mathcal{E} > 0$ is the (artificial) elastic modulus, which is constant. Formally, by sending $\mathcal{E} \rightarrow \infty$ in (1.3c), one can recover (1.3b). In addition, we will show in this paper, that a cutoff of the strain rate \bar{D} is not necessary in the dynamical case with equation (1.3c) for $\mathcal{E} > 0$. Note that in contrast to that of the Hibler model, the stress tensor of the EVP model has become an independent dynamical quantity, which requires its own initial data which need to be chosen adequately in numerical simulations. Moreover, the parabolic structure of the full Hibler model (1.3a)-(1.3b) is lost due to this modification, although it reemerges as a damping term in the EVP model.

The EVP model, which is given by equations (1.3a) and (1.3c), is one of the most important dynamical sea-ice models and is a fundamental component of many climate models [7, 25, 40], including the Los Alamos sea-ice model (CICE) [47], the Louvain-la-Neuve (LIM2) sea-ice model [7], the Massachusetts Institute of Technology general circulation model (MITgcm) [56], and the Finite-Volume sea ice-ocean

model (FESOM) [23] (an overview of several models can be found in [33]). Furthermore, the wind stress \mathcal{T}_a and the ocean stress \mathcal{T}_w are defined as follows

$$\mathcal{T}_a := c_a \rho_a |U_a| \left(U_a \cos \phi + U_a^\perp \sin \phi \right), \quad (1.3d)$$

$$\mathcal{T}_w := c_w \rho_w |U_w - u| \left[(U_w - u) \cos \theta + (U_w - u)^\perp \sin \theta \right], \quad (1.3e)$$

respectively. The other notation is introduced in Table 1. For the purposes of this paper, the geostrophic wind and ocean currents U_a and U_w can be arbitrary functions in $C([0, T]; H^4(\mathbb{T}^2))$, but we note that the ocean velocity U_w is typically chosen as follows, cf. e.g. [23, 34],

$$U_w = \begin{pmatrix} \frac{0.1(2y-L)}{L} \\ -\frac{0.1(2x-L)}{L} \end{pmatrix}, \quad (1.4)$$

where L is the length scale of the domain $[0, L]^2$. Note that this example is not periodic. Similarly, a typical wind velocity is given by (cf. [34])

$$U_a = \begin{pmatrix} 5 + \left[\sin\left(\frac{2\pi t}{T}\right) - 3 \right] \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \\ 5 + \left[\sin\left(\frac{2\pi t}{T}\right) - 3 \right] \sin\left(\frac{2\pi y}{L}\right) \sin\left(\frac{\pi x}{L}\right) \end{pmatrix}. \quad (1.5)$$

The goal of this paper is to establish the global well-posedness of a regularised version of the EVP system, i.e. equations (1.3a) and (1.3c)-(1.3e). To simplify our presentation, we will fix $\bar{e} \equiv 2$ in the rest of this paper, as is commonly done in the sea-ice modelling literature. However, we emphasise that the results of this paper do not depend on the choice of \bar{e} .

Symbol	Meaning	Typical value
c	ice compactness (area covered by ‘thick’ ice)	$0 \leq c \leq 1$
c_a	air drag coefficient	$1.2 \cdot 10^{-3}$
c_w	ocean drag coefficient	$5.5 \cdot 10^{-3}$
\bar{e}	ratio of major axis to minor axis lengths of the elliptic yield curve	2
\mathcal{E}	elastic modulus	0.25
g	gravitational constant	9.81 m s^{-1}
H_0	sea surface height	
H	height of ‘thick’ ice	
m	mass per unit area	
Ω	rotation parameter	$1.46 \cdot 10^{-4} \text{ s}^{-1}$
P	internal ice strength	
P_0	internal ice strength parameter	$27.5 \cdot 10^3 \text{ N/m}^3$
ϕ	air turning angle	25°
ρ_a	air density	1.3 kg/m^3
ρ_w	ocean water density	1026 kg/m^3
θ	water turning angle	25°
U_a	geostrophic wind	equation (1.5)
U_w	geostrophic ocean current	equation (1.4)

Table 1: Index of notation and typical values of several of the parameters, which are taken from [34, 35, 46, 57].

1.2 The linear ill-posedness of the unregularised EVP model

Before we state the main result of this paper, we emphasise here that a regularisation of the EVP model is necessary to establish the well-posedness in Sobolev spaces. Indeed, consider the following one-dimensional version of the EVP model

$$\partial_t u = \partial_x \sigma, \quad (1.6a)$$

$$\partial_t \sigma + \frac{5\sqrt{|\partial_x u|^2 + \epsilon^2}}{2P} \sigma + \frac{\sqrt{|\partial_x u|^2 + \epsilon^2}}{2} = \partial_x u, \quad (1.6b)$$

where u and σ are now scalar quantities, and we have regularised and simplified the strain rate $\overline{\mathcal{D}} = \sqrt{\frac{5}{4}|\partial_x u|}$ from (1.2) by $\sqrt{|\partial_x u|^2 + \epsilon^2}$ for some $\epsilon > 0$. Note that we have omitted the external forces as well as the advection term (cf. point (ii) of Remark 1.3). By linearising system (1.6) around a given solution $(\bar{u}, \bar{\sigma})$ to (1.6) we find the following equations

$$\partial_t u_l = \partial_x \sigma_l, \quad (1.7a)$$

$$\partial_t \sigma_l = -\frac{5}{2P} \sqrt{|\partial_x \bar{u}|^2 + \epsilon^2} \sigma_l + \left[1 - \frac{5}{2P} \frac{\bar{\sigma} \partial_x \bar{u}}{\sqrt{|\partial_x \bar{u}|^2 + \epsilon^2}} - \frac{1}{2} \frac{\partial_x \bar{u}}{\sqrt{|\partial_x \bar{u}|^2 + \epsilon^2}} \right] \partial_x u_l, \quad (1.7b)$$

where (u_l, σ_l) is the solution of the linearised system (1.7a)-(1.7b). Then, after taking a spatial derivative of equation (1.7b) and using equation (1.7a), one has

$$\begin{aligned} \partial_{tt} u_l = & -\frac{5}{2P} \sqrt{|\partial_x \bar{u}|^2 + \epsilon^2} \partial_t u_l + \left[1 - \frac{5}{2P} \frac{\bar{\sigma} \partial_x \bar{u}}{\sqrt{|\partial_x \bar{u}|^2 + \epsilon^2}} - \frac{1}{2} \frac{\partial_x \bar{u}}{\sqrt{|\partial_x \bar{u}|^2 + \epsilon^2}} \right] \partial_{xx} u_l \\ & - \frac{5}{2P} \frac{\partial_x \bar{u} \partial_{xx} \bar{u}}{\sqrt{|\partial_x \bar{u}|^2 + \epsilon^2}} \sigma_l - \frac{\partial}{\partial x} \left[\frac{5}{2P} \frac{\bar{\sigma} \partial_x \bar{u}}{\sqrt{|\partial_x \bar{u}|^2 + \epsilon^2}} + \frac{1}{2} \frac{\partial_x \bar{u}}{\sqrt{|\partial_x \bar{u}|^2 + \epsilon^2}} \right] \partial_x u_l. \end{aligned} \quad (1.8)$$

Therefore, if at a space-time point $(x_0, t_0) \in \mathbb{T} \times [0, T]$ we have

$$1 - \frac{5}{2P} \frac{\bar{\sigma} \partial_x \bar{u}}{\sqrt{|\partial_x \bar{u}|^2 + \epsilon^2}} - \frac{1}{2} \frac{\partial_x \bar{u}}{\sqrt{|\partial_x \bar{u}|^2 + \epsilon^2}} < 0, \quad (1.9)$$

then equation (1.8) becomes locally elliptic near the point (x_0, t_0) , which leads to an instability and therefore the ill-posedness in the sense of Hadamard [30]. In a forthcoming paper [8], we will address the linear ill-posedness of the (unregularised) EVP model in all Sobolev spaces in a rigorous manner using this new instability (ill-posedness) that we have discussed above.

Remark 1.1. We emphasise that the instability described above is fundamentally different than the (formal) ill-posedness/instability results that have been reported in the sea-ice modelling/physics literature [26–28, 61]. These results rely crucially on the absence of the viscosity cutoff (i.e. $\epsilon = 0$) and are found for mainly the Hibler model (with the exception of [52, 67], which consider the (revised) EVP model as well). The rigorous analysis in [53] demonstrates that the linear ill-posedness is not present in the case of the Hibler model with $\epsilon > 0$. The linear instability of the EVP model described in this section is present regardless of the value of ϵ and therefore pertains to a new kind of instability that has not been reported in the literature before, and in particular it does not apply to the Hibler model.

1.3 Main result of this paper

We consider the following regularisation of the EVP model (i.e. equations (1.3a) and (1.3c)):

$$\partial_t u = \nabla \cdot \sigma + \mathcal{T}_a + \mathcal{T}_w + \Omega u^\perp - g \nabla H_0, \quad (1.10a)$$

$$\frac{1}{\mathcal{E}} \partial_t (\sigma - \alpha^2 \Delta \sigma) + \frac{4\overline{\mathcal{D}}}{P} (\sigma - \frac{1}{2} \text{Tr} \sigma \mathbb{I}_2) + \frac{\overline{\mathcal{D}}}{2P} \text{Tr} \sigma \mathbb{I}_2 + \frac{\overline{\mathcal{D}}}{2} \mathbb{I}_2 = D(u), \quad (1.10b)$$

$$u|_{t=0} = u_0, \quad \sigma|_{t=0} = \sigma_0. \quad (1.10c)$$

In this model we have included the Voigt regularisation on the elastic part of the rheology in the EVP model [35] by means of the inclusion of the term $-\alpha^2 \partial_t \Delta \sigma$ for some fixed $\alpha > 0$ in equation (1.10b). The drag forces \mathcal{T}_a and \mathcal{T}_w are taken as in equations (1.3d) and (1.3e), respectively. Note that throughout this paper we will assume that $\sigma(x, t)$ is a symmetric matrix, although this is not necessary for the results that we will prove. Next, we will simplify the strain rate $\overline{\mathcal{D}}$ from equation (1.2) by

$$\mathcal{D} := |D(u)|, \quad (1.11)$$

without fundamentally changing the analysis presented below, see Section 5 for more discussion. The system we will therefore consider in this paper, which we call the Voigt-EVP system, is given by

$$\partial_t u = \nabla \cdot \sigma + \mathcal{T}_a + \mathcal{T}_w + \Omega u^\perp - g \nabla H_0, \quad (1.12a)$$

$$\frac{1}{\mathcal{E}} \partial_t (\sigma - \alpha^2 \Delta \sigma) + \frac{4\mathcal{D}}{P} (\sigma - \frac{1}{2} \text{Tr} \sigma \mathbb{I}_2) + \frac{\mathcal{D}}{2P} \text{Tr} \sigma \mathbb{I}_2 + \frac{\mathcal{D}}{2} \mathbb{I}_2 = D(u), \quad (1.12b)$$

$$u|_{t=0} = u_0, \quad \sigma|_{t=0} = \sigma_0. \quad (1.12c)$$

Now we are ready to state our main result, i.e., the global well-posedness of the Voigt-EVP system:

Theorem 1.2. *Let $u_0 \in H^2(\mathbb{T}^2)$ and $\sigma_0 \in H^3(\mathbb{T}^2)$ (which is symmetric) be given. Moreover, we assume that $P > 0$ is a given constant, $U_a, U_w, H_0 \in C([0, T]; H^4(\mathbb{T}^2))$ and we let $\theta \in [0, \frac{\pi}{4}]$. The Voigt-EVP system (1.12) has a unique global strong solution (u, σ) on the time interval $[0, T]$ for any $T > 0$. The solution (u, σ) has the following regularity*

$$u \in C([0, T]; H^2(\mathbb{T}^2)), \quad \sigma \in C([0, T]; H^3(\mathbb{T}^2)).$$

In addition, we have the estimate

$$\begin{aligned} & \|u\|_{C([0, T]; H^2(\mathbb{T}^2))} + \|\sigma\|_{C([0, T]; H^3(\mathbb{T}^2))} + \|\partial_t u\|_{L^\infty((0, T); H^2(\mathbb{T}^2))} + \|\partial_t \sigma\|_{L^\infty((0, T); H^3(\mathbb{T}^2))} \\ & \leq C \exp(\exp(\exp(\exp(\exp(CT))))) \end{aligned}$$

where $C > 0$ is a constant, depending on $\mathcal{E}, \alpha, U_a, U_w, g, H_0, u_0, \sigma_0, c_a, c_w, \rho_a, \rho_w, P, \phi$, and θ .

We note that the superexponential bound from Theorem 1.2 originates from the repeated use of (logarithmic) Grönwall inequalities for the a priori estimates at the $L^2(\mathbb{T}^2)$, $\dot{H}^1(\mathbb{T}^2)$ and $\dot{H}^2(\mathbb{T}^2)$ level.

Remark 1.3. (i) The Voigt regularisation in equation (1.12b) is in the same spirit as the original numerical regularisation of the Hibler model from [35]. Namely, we seek an inviscid regularisation of the EVP model, which is suitable for numerical computations. The advantage of the Voigt regularisation is that on the one hand, like the original EVP model, it shares the same steady state(s) with the Hibler model. On the other hand, the Voigt regularisation constitutes a relatively modest (nonviscous) modification of the (artificial) elastic component of the rheology in the EVP model.

- (ii) For the momentum equation of the Voigt-EVP model, we have omitted the advection term, in which we follow the approach in [35]. As it has been remarked in [35], the advection term is at least one order of magnitude smaller than the acceleration term and therefore it has not been included in the EVP model. In the evolution equation for the stress tensor we also do not include an advection term, as it would affect the steady states of the EVP model, which would then lose its purpose as an approximation of the Hibler model.
- (iii) We note that the assumption $\theta \in [0, \frac{\pi}{4}]$ on the water turning angle, lies within the standard range in sea-ice modelling (see for example [31, 49]).

In order to prove Theorem 1.2, we will consider an intermediate system given by

$$\partial_t u = \nabla \cdot \sigma + \mathcal{T}_a + \mathcal{T}_w + \Omega u^\perp - g \nabla H_0, \quad (1.13a)$$

$$\frac{1}{\mathcal{E}} \partial_t (\sigma - \alpha^2 \Delta \sigma) + \frac{4\mathcal{D}_\epsilon}{P} (\sigma - \frac{1}{2} \text{Tr} \sigma \mathbb{I}_2) + \frac{\mathcal{D}_\epsilon}{2P} \text{Tr} \sigma \mathbb{I}_2 + \frac{\mathcal{D}_\epsilon}{2} \mathbb{I}_2 = D(u), \quad (1.13b)$$

$$u|_{t=0} = u_0, \quad \sigma|_{t=0} = \sigma_0. \quad (1.13c)$$

In this model, the strain rate \mathcal{D} in system (1.12) has been regularised by

$$\mathcal{D}_\epsilon := \sqrt{|D(u)|^2 + \epsilon^2}, \quad (1.14)$$

for some $\epsilon > 0$, where we recall that $D(u)$ denotes the symmetric part of the gradient, see equation (1.1).

Then, by establishing uniform-in- ϵ estimates of the solutions to system (1.13) globally in time, we will be able to send $\epsilon \rightarrow 0$ and obtain the (unique) solution to system (1.12). As a byproduct of this proof, we also obtain the global well-posedness of the intermediate system (1.13a)-(1.13c) as stated below:

Corollary 1.4. *Under the same conditions on $u_0, \sigma_0, U_a, U_w, H_0$, and θ as in Theorem 1.2, for any $T > 0$ the intermediate system (1.13) is globally well-posed. The unique global solution (u, σ) has the same regularity as in Theorem 1.2.*

Remark 1.5. (i) The benefit of introducing \mathcal{D}_ϵ in the intermediate system (1.13) is that it regularises the strain rate \mathcal{D} and allows one to take its derivatives directly. We emphasise that such a regularisation serves as a cutoff of the bulk and shear viscosities and has an important role in sea-ice modelling as it describes the yield curve of sea-ice, as part of the viscous-plastic rheology introduced in [32].

Moreover, it is a natural numerical regularisation for the strain rate, which was first introduced in [41], cf. [57]. Note that in practice, ϵ should be smaller than 10^{-9} s^{-1} [7]. For these reasons the study of the intermediate system (1.13) has its value independent of its role as an approximate system for the Voigt-EVP model.

- (ii) The proof of Theorem 1.2 implies that the limit as $\epsilon \rightarrow 0$ of the intermediate system (1.13), i.e. removing the bulk and shear viscosity cutoff, is stable in the sense that the solutions remain close to each other for all small ϵ . Our results are therefore very much in agreement with the numerical simulations in [7], where it was found that the dynamics of the EVP model is not very much affected by the value of ϵ .
- (iii) In [53], the approximation (1.14) is also used for the Hibler model (1.3b) as a regularisation of the strain rate, and it is shown that for any fixed $\epsilon > 0$, the resulting system of equations is locally well-posed. However, as demonstrated by the numerical evidence in [7], the limit as $\epsilon \rightarrow 0$ for the Hibler model is very unstable. The analysis of the limit as $\epsilon \rightarrow 0$ remains open.

- (iv) Instead of (1.14), our analysis works for a larger class of regularisations for the strain rate \mathcal{D} . See Section 5 for more discussion.

1.4 Overview of the literature

The sea-ice modeling literature

Sea ice is a highly complex material and modelling its dynamics and rheology poses significant challenges. In fact, numerous sea-ice models have been developed over time [25]. The Arctic Ice Dynamics Joint Experiment suggested an elastic-plastic rheology for modelling sea ice [22]. The Hibler sea-ice model was introduced in [32], which uses a viscous-plastic rheology which has become one of the standard sea-ice rheologies.

The Hibler model has been a very successful model in reproducing observed sea-ice drift [54]. However, its use in practical computations turns out to be very expensive [7, 40], particularly so when using an explicit numerical method [38]. Instead, implicit methods are employed in order to compute solutions of the Hibler model, for example the line relaxation method [68] as well as the Jacobian-free Newton Krylov solver [46, 55, 62] to mention some.

The elastic-viscous-plastic (EVP) sea-ice model was introduced in [35] in order to improve the computational efficiency of sea-ice modelling. Specifically, the EVP model is a modification of the Hibler model, as an elastic (relaxation) term is introduced into the constitutive relation. The advantage of the EVP formulation over the Hibler model is that it is expected to be more amenable to the use of explicit numerical schemes for computational modelling, and hence facilitates the use of parallel computing [34], see also [36]. As a result, numerically solving the EVP model significantly reduces the computational cost compared to the use of implicit schemes for the Hibler model [7].

In addition, the EVP model (1.3a) and (1.3c) has the advantage that its computational performance is not affected by the value of the strain rate regularisation parameter ϵ , while the Hibler model requires a small value of ϵ in the viscosity regularisation to accurately model plastic behaviour as shown in [7], cf. Remark 1.5. Because of the aforementioned reasons the EVP model has become one of the standard and most commonly used sea-ice models in contemporary climate modelling [7, 25, 40].

Over time, several modifications of the EVP model were introduced which have better numerical performance in simulations. For example, in [34] the elastic parameter was redefined to depend on the strain rate. The reason for this modification was to prevent the computed stresses from lying outside the yield curve (cf. [32]) during the (numerical) subcycling process.

However, although the EVP model was originally introduced as a reformulation of the Hibler model which was more suitable for numerical purposes, it remains unclear whether it successfully approximates the Hibler model, i.e. by sending $\mathcal{E} \rightarrow \infty$ in equation (1.3c) to obtain (1.3b). In particular, it was found in [39, 46, 54, 56], see also [6], that the numerically computed solutions of the EVP and the Hibler models can differ substantially, even for very small subcycling time steps. Note that this does not necessarily imply that either sea-ice model is physically more accurate, but just that their rheologies are fundamentally different from a computational point of view [46].

Nevertheless, within the subcycling time steps, the effect of the artificial elasticity in the rheology is ‘damped’ over the larger timescale, although it was found in [46] that the damping effect is weaker in areas of rigid ice, i.e. the regions with high concentration of ice and small strain rates.

The mathematical analysis literature

The first results in the rigorous analysis of sea-ice dynamics only appeared very recently. The local well-posedness of the Hibler model was proved independently in [10, 53]. Both of these papers slightly modified the original viscosity cutoff from [32] by introducing a regularisation of the form (1.14), which is required to avoid the singularity when $D(u) = 0$.

To be more precise, in [10], additional artificial diffusion terms were considered in the thermodynamics, which make the system fully parabolic. These additional diffusion terms made it possible to prove global existence of strong solutions for small data in [10]. Meanwhile, the study in [53] retained the original mixed hyperbolic-parabolic character of the Hibler model. A different regularisation of the viscosities was studied in [17], namely the replacement of the Heaviside cutoff by a hyperbolic tangent, where the authors establish the linear well-posedness.

The interaction between sea-ice and a rigid body for the fully parabolic Hibler model was then studied in [5]. Time periodic solutions for the parabolic regularisation of the Hibler model, were constructed in [11] in the case of periodic wind forcing. See [4] for some results on the coupling with oceanic and atmospheric dynamics. The free-boundary problem of shallow (land) ice-sheets was studied in [60].

In addition, in [57], under the assumption of the existence of smooth solutions, a formal H^1 -estimate was derived for the Hibler model as well as for a modified version of the EVP model in the formulation introduced in [34]. This formulation of the EVP model from [34] has a very different structure to our setting. The gradient estimates in [57] strongly rely on the cutoff of the bulk and shear viscosities in the constitutive relation, and also on the linearisation of the ocean drag term. In the revised EVP case, the additional assumption that $\text{Tr } \sigma = 0$ is made in [57]. The a priori estimates in [57] do not seem sufficient to deduce the existence of solutions for the revised EVP model, but instead are used in [57] to check the consistency of a numerical discretisation scheme for these sea-ice models.

We also note that the EVP model (1.3a) and (1.3c) shares some structural similarities with several models for non-Newtonian flows, for example the Oldroyd-B model, see for example [18, 19, 21, 24, 29, 42, 50, 51] and references therein, and other viscoelastic flow models, which have been studied in [2, 14, 15] and references therein. Finally, we note that our treatment of the viscosity regularisation (1.14) is in some sense related to results for non-Newtonian fluid models with shear dependent viscosities, see for example [3] and references therein.

As has been mentioned in point (i) in Remark 1.3, in this work we have chosen to use the (inviscid) Voigt regularisation for the EVP model. To the best of our knowledge, the Voigt regularisation in the context of the Navier-Stokes equations has first been rigorously studied in [58, 59]. Much later, the existence of solutions for such Kelvin-Voigt-type systems with non-constant density has also been investigated in [1] and see references therein. Moreover, the Voigt regularisation has been investigated extensively in the context of turbulence modelling, for example in [43–45]. More recently, it has also been used as a regularisation mechanism to study steady states [20], see also [37]. We also remark that employing the Voigt regularisation typically leads to equations of pseudo-parabolic type, which have been studied in [63–65] (and see references therein).

The rest of this paper is organised as follows. In Section 2, we will recall some inequalities that we will use throughout this paper. In Section 3, we will establish the global well-posedness of the intermediate system (1.13). Following this, in Section 4, we finish the proof of Theorem 1.2, namely the global well-posedness of the Voigt-EVP model (1.12). We will provide some concluding remarks in Section 5.

2 Preliminaries

Throughout this paper, we will use the notation C to denote a generic constant, which might differ from line to line. For any quantities $A, B \geq 0$, we shorten the notation $A \leq CB$ for some positive but finite constant C as

$$A \lesssim B. \quad (2.1)$$

We recall a few results that will be used in this paper. We first recall the Brézis-Gallouët–Wainger inequality, which was originally proved in [12, 13]:

Lemma 2.1 (Brézis-Gallouët–Wainger inequality). *Let $v \in H^2(\mathbb{T}^2)$ (where v is dimensionless), then it satisfies the following inequality*

$$\|v\|_{L^\infty} \leq C \left(\|v\|_{H^1} \sqrt{\log \left(1 + C_L \|v\|_{H^2} \right)} + 1 \right), \quad (2.2)$$

where we note that the constant C_L depends on the length scale of the torus (which we take to be unit length in our case), while C is a dimensionless constant.

We will also use the Ladyzhenskaya inequality in two space dimensions

$$\|f\|_{L^4} \lesssim \|f\|_{L^2}^{1/2} \|\nabla f\|_{L^2}^{1/2} + \|f\|_{L^2}; \quad (2.3)$$

as well as the logarithmic Grönwall inequality, which in this form can be found in [16, Lemma 2.5]:

Lemma 2.2 (Logarithmic Grönwall inequality). *Let $T > 0$, and assume $A : [0, T] \rightarrow [0, \infty)$ is a non-negative absolutely continuous function. Further assume there exist nonnegative $L^1(0, T)$ functions f, g , and h such that*

$$\frac{d}{dt} A(t) \leq [g(t) + h(t) \log(A(t) + e)] A(t) + f(t). \quad (2.4)$$

Then we have the following estimate

$$A(t) \leq (2F(t) + 1)e^{F(t)}, \quad (2.5)$$

where we have introduced the following function

$$F(t) := e^{\int_0^t h(s) ds} \left(\log(A(0) + e) + \int_0^t [g(s) + f(s)] ds + t \right). \quad (2.6)$$

In order to prove the uniqueness of strong solutions in the subsequent sections, we will need to prove a bound on the difference of the strain rates.

Lemma 2.3. *Let $v_1, v_2 \in H^2(\mathbb{T}^2)$ and recall that the notation $D(v)$ was defined in equation (1.1), then we have*

$$\left\| \sqrt{|D(v_1)|^2 + \epsilon^2} - \sqrt{|D(v_2)|^2 + \epsilon^2} \right\|_{L^2} \leq \|\nabla v_1 - \nabla v_2\|_{L^2}, \quad (2.7)$$

for all $\epsilon \geq 0$.

Proof. One can directly compute

$$\begin{aligned} |\sqrt{|D(v_1)|^2 + \epsilon^2} - \sqrt{|D(v_2)|^2 + \epsilon^2}| &= \left| \frac{[D(v_1) + D(v_2)] : [D(v_1) - D(v_2)]}{\sqrt{|D(v_1)|^2 + \epsilon^2} + \sqrt{|D(v_2)|^2 + \epsilon^2}} \right| \\ &\leq |D(v_1) - D(v_2)|. \end{aligned}$$

This implies inequality (2.7) for smooth v_1 and v_2 . For $v_1, v_2 \in H^2(\mathbb{T}^2)$, one can proceed by choosing smooth approximating sequences of $H^2(\mathbb{T}^2)$ functions and conclude the proof. \square

Although it is not necessary for the construction, we show in the following lemma that σ remains symmetric under the evolution of the Voigt-EVP model if it is initially symmetric:

Lemma 2.4 (Invariance of the symmetry). *Consider a solution (u, σ) to either system (1.12) or (1.13), which has the regularity described in Theorem 1.2, with $\sigma_0 \in H^3(\mathbb{T}^2; \mathbb{R}^{2 \times 2})$ being symmetric. That is, $u \in C([0, T]; H^2(\mathbb{T}^2))$ and $\sigma \in C([0, T]; H^3(\mathbb{T}^2))$ for arbitrary $T > 0$. Then $\sigma(x, t) = \sigma(x, t)^\top$ for all $0 \leq t \leq T$ and all $x \in \mathbb{T}^2$.*

Proof. Denote the antisymmetric part of σ by

$$W(\sigma) := \frac{1}{2}(\sigma - \sigma^\top). \quad (2.8)$$

Then, from equation (1.12b) or equation (1.13b), one has

$$\frac{1}{\mathcal{E}} \partial_t (I - \alpha^2 \Delta) W(\sigma) + \frac{4H}{P} W(\sigma) = 0, \quad (2.9)$$

for $H = \mathcal{D}$ or $H = \mathcal{D}_\epsilon$. Note that this equation holds in $C([0, T]; H^1(\mathbb{T}^2))$. Subsequently, the standard L^2 -estimate for $W(\sigma)$ yields

$$\sup_{t \in [0, T]} [\|W(\sigma)(t)\|_{L^2}^2 + \alpha^2 \|\nabla W(\sigma)(t)\|_{L^2}^2] \leq \|W(\sigma_0)\|_{L^2}^2 + \alpha^2 \|\nabla W(\sigma_0)\|_{L^2}^2 = 0. \quad (2.10)$$

This finishes the proof. \square

3 Global solutions to the intermediate system (1.13)

Our strategy of establishing the global solvability of system (1.13) is the following: First, we will write down the Galerkin approximation scheme. At each level of the scheme, by directly applying the Picard theory we obtain a local-in-time solution of the Galerkin system. Second, we establish better regularity for the approximating solutions (as well as their global existence in time), and pass to the limit in the Galerkin scheme. Finally, we establish the uniqueness of the strong solution.

3.1 The Galerkin approximation scheme

We look for a solution of the type

$$u_N(x, t) = \sum_{k \in \mathbb{Z}^2, |k| \leq N} \hat{u}_k(t) e^{2\pi i k \cdot x}, \quad \sigma_N(x, t) = \sum_{k \in \mathbb{Z}^2, |k| \leq N} \hat{\sigma}_k(t) e^{2\pi i k \cdot x},$$

which satisfies the following Galerkin system (together with the usual condition that u_N and σ_N are real-valued):

$$\partial_t u_N = \nabla \cdot \sigma_N + P_N \mathcal{T}_a + P_N \mathcal{T}_w^N + \Omega u_N^\perp - g P_N \nabla H_0, \quad (3.1a)$$

$$\frac{1}{\mathcal{E}} \partial_t (\sigma_N - \alpha^2 \Delta \sigma_N) + P_N \left[\frac{4\mathcal{D}_\epsilon^N}{P} (\sigma_N - \frac{1}{2} \text{Tr} \sigma_N \mathbb{I}_2) \right] + P_N \left[\frac{\mathcal{D}_\epsilon^N}{2P} \text{Tr} \sigma_N \mathbb{I}_2 \right] \quad (3.1b)$$

$$+ \frac{P_N \mathcal{D}_\epsilon^N}{2} \mathbb{I}_2 = D(u_N), \quad (3.1c)$$

$$u|_{t=0} = P_N u_0, \quad \sigma|_{t=0} = P_N \sigma_0,$$

where P_N is the projection onto the Fourier modes up to order N , including the zeroth mode. In system (3.1), we have used the notation

$$\mathcal{T}_w^N = c_w \rho_w |U_w - u_N| \left[(U_w - u_N) \cos \theta + (U_w - u_N)^\perp \sin \theta \right] \quad (3.2)$$

and

$$\mathcal{D}_\epsilon^N = \sqrt{|D(u_N)|^2 + \epsilon^2}. \quad (3.3)$$

Notice that system (3.1) is a first-order system of ordinary differential equations for $\{\hat{u}_k, \hat{\sigma}_k\}_{|k| \leq N}$, with a locally Lipschitz continuous vector field with respect to the unknowns. Thus, the Picard theory implies that, for each initial datum $(P_N u_0, P_N \sigma_0)$, there exists a unique local-in-time solution.

3.2 Uniform-in- (N, ϵ) estimates

To establish the uniform-in- N estimates for the solutions of system (3.1), let

$$\tau_N := \sigma_N + \frac{P}{2} \mathbb{I}_2. \quad (3.4)$$

Recall that the pressure $P > 0$ is constant. We then rewrite equation (3.1b) as follows:

$$\frac{1}{\mathcal{E}} \partial_t \left(\tau_N - \alpha^2 \Delta \tau_N \right) + P_N \left[\frac{4\mathcal{D}_\epsilon^N}{P} \left(\tau_N - \frac{1}{2} \text{Tr} \tau_N \mathbb{I}_2 \right) \right] \quad (3.5)$$

$$+ P_N \left[\frac{\mathcal{D}_\epsilon^N}{2P} \text{Tr} \tau_N \mathbb{I}_2 \right] = D(u_N).$$

L^2 -estimate. We start by computing the $L^2(\mathbb{T}^2)$ estimates for the solution of system (3.1) on its maximal (positive) time interval of existence. Taking the L^2 -inner product of (3.1a) and (3.5) with u_N and τ_N , respectively, applying integration by parts, and summing the results yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|u_N\|_{L^2}^2 + \mathcal{E}^{-1} \|\tau_N\|_{L^2}^2 + \alpha^2 \mathcal{E}^{-1} \|\nabla \tau_N\|_{L^2}^2 \right] \\ & + \int_{\mathbb{T}^2} \left[\frac{4\mathcal{D}_\epsilon^N}{P} |\tau_N - \frac{1}{2} \text{Tr} \tau_N \mathbb{I}_2|^2 + \frac{\mathcal{D}_\epsilon^N}{2P} |\text{Tr} \tau_N|^2 \right] dx \quad (3.6) \\ & = \int_{\mathbb{T}^2} \left[\mathcal{T}_a + \mathcal{T}_w^N + \Omega u_N^\perp - g \nabla H_0 \right] \cdot u_N dx \leq C + C \|u_N\|_{L^2}^2, \end{aligned}$$

thanks to the following estimate:

$$\begin{aligned} \int_{\mathbb{T}^2} \mathcal{T}_w^N \cdot u_N \, dx &= c_w \rho_w \int_{\mathbb{T}^2} |U_w - u_N| \left[(U_w - u_N) \cos \theta + (U_w - u_N)^\perp \sin \theta \right] \cdot u_N \, dx \\ &= c_w \rho_w \int_{\mathbb{T}^2} |U_w - u_N| \left[U_w \cos \theta + U_w^\perp \sin \theta \right] \cdot u_N \, dx \\ &\quad - c_w \rho_w \int_{\mathbb{T}^2} |U_w - u_N| |u_N|^2 \cos \theta \, dx \lesssim 1 + \|u_N\|_{L^2}^2, \end{aligned}$$

where we use the fact that $\cos \theta \geq 0$ since $\theta \in [0, \frac{\pi}{4}]$. Notice that the constant $C \in (0, \infty)$ in (3.6) depends only on $\Omega, g, P, H_0, U_a, U_w, c_a, c_w, \phi, \theta, \rho_a$, and ρ_w , but not on ϵ and N . Moreover, we also have used the following cancellation, which is similar in structure to the Oldroyd-B model,

$$\int_{\mathbb{T}^2} \left[u_N \cdot (\nabla \cdot \sigma_N) + \tau_N : D(u_N) \right] dx = 0. \quad (3.7)$$

Therefore, applying the Grönwall inequality to estimate (3.6) on a time interval of existence $[0, T]$ of the approximating Galerkin solution leads to

$$\sup_{t \in [0, T]} \left[\|u_N(t)\|_{L^2}^2 + \mathcal{E}^{-1} \left\| \sigma_N(t) + \frac{P}{2} \mathbb{I}_2 \right\|_{L^2}^2 + \alpha^2 \mathcal{E}^{-1} \left\| \nabla \left[\sigma_N(t) + \frac{P}{2} \mathbb{I}_2 \right] \right\|_{L^2}^2 \right] \leq C e^{CT}, \quad (3.8)$$

uniformly in N . Since the right-hand side remains finite for any $T < \infty$, we conclude that the time interval of existence of the approximating Galerkin solutions is $[0, \infty)$ and estimate (3.8) therefore holds for any $T < \infty$.

H^1 -estimate. Now we turn to the $\dot{H}^1(\mathbb{T}^2)$ estimates. Denote by

$$V_w := U_w - u_N. \quad (3.9)$$

Recalling (3.2), one can write

$$\begin{aligned} - \int_{\mathbb{T}^2} P_N \mathcal{T}_w^N \cdot \Delta u_N \, dx &= \sum_{i=1}^2 \int_{\mathbb{T}^2} P_N \partial_i \mathcal{T}_w^N \cdot \partial_i u_N \, dx = \sum_{i=1}^2 \int_{\mathbb{T}^2} \partial_i \mathcal{T}_w^N \cdot \partial_i u_N \, dx \\ &= \sum_{i=1}^2 \int_{\mathbb{T}^2} \partial_i \mathcal{T}_w^N \cdot \partial_i U_w \, dx - \sum_{i=1}^2 \int_{\mathbb{T}^2} c_w \rho_w [\partial_i (|V_w| V_w) \cos \theta] \cdot \partial_i V_w \, dx \\ &\quad - \sum_{i=1}^2 \int_{\mathbb{T}^2} c_w \rho_w \partial_i (|V_w| V_w^\perp) \cdot \partial_i V_w \sin \theta \, dx =: I_1 + I_2 + I_3. \end{aligned} \quad (3.10)$$

For $i = 1, 2$, one can calculate

$$\partial_i (|V_w| V_w) = |V_w| \partial_i V_w + \frac{V_w \cdot \partial_i V_w}{|V_w|} V_w. \quad (3.11)$$

Then directly one can estimate (where the constants can depend on $\|\partial_i U_w\|_{L^\infty}$)

$$|I_1| \lesssim \|V_w\|_{L^2} \|\nabla V_w\|_{L^2} \lesssim 1 + \|u_N\|_{L^2}^2 + \|\nabla u_N\|_{L^2}^2, \quad (3.12)$$

$$|I_3| \leq c_w \rho_w |\sin \theta| \int_{\mathbb{T}^2} |V_w| |\nabla V_w|^2 \, dx. \quad (3.13)$$

Meanwhile, substituting (3.11) in I_2 , one can compute

$$I_2 = -c_w \rho_w \cos \theta \int_{\mathbb{T}^2} \left(|V_w| |\nabla V_w|^2 + \frac{|\nabla(|V_w|^2)|^2}{|V_w|} \right) dx. \quad (3.14)$$

Consequently, (3.10)–(3.14) leads to

$$- \int_{\mathbb{T}^2} P_N \mathcal{T}_w^N \cdot \Delta u_N dx = \int_{\mathbb{T}^2} \nabla \mathcal{T}_w^N : \nabla u_N \lesssim 1 + \|u_N\|_{L^2}^2 + \|\nabla u_N\|_{L^2}^2 \quad (3.15)$$

for θ satisfying

$$\cos \theta - |\sin \theta| \geq 0, \quad (3.16)$$

which holds for $\theta \in [0, \frac{\pi}{4}]$.

Taking the L^2 -inner product of (3.1a) and (3.1b) with $-\Delta u_N$ and $-\Delta \sigma_N$, respectively, applying integration by parts, and summing up the results leads to the following estimate

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|\nabla u_N\|_{L^2}^2 + \mathcal{E}^{-1} \|\nabla \sigma_N\|_{L^2}^2 + \alpha^2 \mathcal{E}^{-1} \|\Delta \sigma_N\|_{L^2}^2 \right] \\ &= \int_{\mathbb{T}^2} \left\{ \left[\frac{4\mathcal{D}_\epsilon^N}{P} (\sigma_N - \frac{1}{2} \text{Tr} \sigma_N \mathbb{I}_2) \right] : \Delta \sigma_N + \frac{\mathcal{D}_\epsilon^N}{2P} \text{Tr} \sigma_N \mathbb{I}_2 : \Delta \sigma_N + \frac{\mathcal{D}_\epsilon^N}{2} \text{Tr} \Delta \sigma_N \right\} dx \\ &+ \int_{\mathbb{T}^2} \nabla \left[\mathcal{T}_a + \Omega u_N^\perp - g \nabla H_0 \right] : \nabla u_N dx + \int_{\mathbb{T}^2} \nabla \mathcal{T}_w^N : \nabla u_N dx \\ &+ \sum_{i=1}^2 \int_{\mathbb{T}^2} \left[\partial_i u_N \cdot (\nabla \cdot \partial_i \sigma_N) + \partial_i \sigma_N : D(\partial_i u_N) \right] dx \\ &\lesssim \|\mathcal{D}_\epsilon^N\|_{L^2} \|\sigma_N\|_{L^\infty} \|\Delta \sigma_N\|_{L^2} + \|\Delta \sigma_N\|_{L^2}^2 + \|\nabla u_N\|_{L^2}^2 + \|u_N\|_{L^2}^2 + 1 \\ &\lesssim \left[\sqrt{\log(e + C_L R_1 \|\sigma_N\|_{H^2})} \|\sigma_N\|_{H^1} + 1 \right] \left[\|\nabla u_N\|_{L^2}^2 + \|\Delta \sigma_N\|_{L^2}^2 \right] \\ &+ \|\Delta \sigma_N\|_{L^2}^2 + \|\nabla u_N\|_{L^2}^2 + \|u_N\|_{L^2}^2 + 1, \end{aligned} \quad (3.17)$$

where we have used (3.15), the Brézis-Gallouët-Wainger inequality (2.2), as well as Young's inequality. Note that we have introduced a constant $R_1 > 0$ in order to make the quantity $R_1 \sigma_N$ dimensionless (so that we can apply the Brézis-Gallouët-Wainger inequality). One can choose $R_1 = \|\sigma_0\|_{L^\infty}^{-1}$ for example (for nonzero initial data).

Thanks to the L^2 -estimate in (3.8), we can apply the logarithmic Grönwall inequality from Lemma 2.2 to (3.17) which leads to the following triple-exponential bound

$$\sup_{t \in [0, T]} \left[\|\nabla u_N\|_{L^2}^2 + \mathcal{E}^{-1} \|\nabla \sigma_N\|_{L^2}^2 + \alpha^2 \mathcal{E}^{-1} \|\Delta \sigma_N\|_{L^2}^2 \right] \leq C \exp(\exp(\exp(CT))), \quad (3.18)$$

for some C independent of N , ϵ , and T , but depending on \mathcal{E} , α , and the initial data. Note that the triple-exponential nature of the bound comes from the application of the logarithmic Grönwall inequality to the single-exponential L^2 -estimate (3.8).

H^2 -estimate. One can compute directly that

$$\begin{aligned} \nabla \mathcal{D}_\epsilon^N &= \frac{D(u_N) : [\nabla D(u_N)]}{\mathcal{D}_\epsilon^N} = \frac{\frac{1}{2} \nabla |D(u_N)|^2}{\mathcal{D}_\epsilon^N}, \\ \|\nabla \mathcal{D}_\epsilon^N\|_{L^2} &\lesssim \|\Delta u_N\|_{L^2}, \quad \text{where the constant is independent of } \epsilon \text{ and } N. \end{aligned} \quad (3.19)$$

In addition, we recall the following computation from [53, p. 22-23], for $i, j = 1, 2$, with V_w given in (3.9),

$$\begin{aligned} \partial_{ij}^2(|V_w|V_w) &= |V_w|\partial_{ij}^2V_w + \frac{V_w \cdot \partial_i V_w}{|V_w|}\partial_j V_w + \frac{V_w \cdot \partial_j V_w}{|V_w|}\partial_i V_w \\ &+ \left(\frac{V_w \cdot \partial_{ij}^2V_w + \partial_i V_w \cdot \partial_j V_w}{|V_w|} - \frac{[V_w \cdot \partial_i V_w] \cdot [V_w \cdot \partial_j V_w]}{|V_w|^3} \right) V_w. \end{aligned} \quad (3.20)$$

Therefore, one has the following estimate

$$\begin{aligned} \|\nabla^2(|V_w|V_w)\|_{L^2} &\leq \|V_w\|_{L^\infty}\|V_w\|_{H^2} + \|V_w\|_{W^{1,4}}^2 \\ &\lesssim (\|V_w\|_{L^\infty} + \|V_w\|_{H^1})\|V_w\|_{H^2}, \end{aligned} \quad (3.21)$$

where we have applied the Ladyzhenskaya inequality (2.3).

Then, after taking the L^2 -inner product of (3.1a) and (3.1b) with $\Delta^2 u_N$ and $\Delta^2 \sigma_N$, respectively, using integration by parts, and summing up the results, one obtains

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\|\Delta u_N\|_{L^2}^2 + \mathcal{E}^{-1} \|\Delta \sigma_N\|_{L^2}^2 + \alpha^2 \mathcal{E}^{-1} \|\nabla \Delta \sigma_N\|_{L^2}^2 \right] \\ &= - \int_{\mathbb{T}^2} \left[\frac{4\mathcal{D}_\epsilon^N}{P} |\Delta(\sigma_N - \frac{1}{2} \text{Tr} \sigma_N \mathbb{I}_2)|^2 + \frac{\mathcal{D}_\epsilon^N}{2P} |\Delta \text{Tr} \sigma_N|^2 \right] dx \\ &\quad - \sum_{i=1,2} \int_{\mathbb{T}^2} \left[\frac{4\partial_i \mathcal{D}_\epsilon^N}{P} \partial_i(\sigma_N - \frac{1}{2} \text{Tr} \sigma_N \mathbb{I}_2) + \frac{\partial_i \mathcal{D}_\epsilon^N}{2P} \partial_i \text{Tr} \sigma_N \mathbb{I}_2 \right] : \Delta \sigma_N dx \\ &\quad + \sum_{i=1,2} \int_{\mathbb{T}^2} \left[\frac{4\partial_i \mathcal{D}_\epsilon^N}{P} (\sigma_N - \frac{1}{2} \text{Tr} \sigma_N \mathbb{I}_2) + \frac{\partial_i \mathcal{D}_\epsilon^N}{2P} \text{Tr} \sigma_N \mathbb{I}_2 \right] : \partial_i \Delta \sigma_N dx \\ &\quad + \int_{\mathbb{T}^2} \nabla \left(\frac{\mathcal{D}_\epsilon^N}{2} \mathbb{I}_2 \right) : \nabla \Delta \sigma_N dx + \int_{\mathbb{T}^2} \Delta \left[\mathcal{T}_a + \mathcal{T}_w^N + \Omega u_N^\perp - g \nabla H_0 \right] \cdot \Delta u_N dx \\ &\quad + \int_{\mathbb{T}^2} \left[\Delta u_N \cdot (\nabla \cdot \Delta \sigma_N) + \Delta \sigma_N : D(\Delta u_N) \right] dx \\ &\lesssim \|\nabla^2 u_N\|_{L^2} \|\nabla \sigma_N\|_{L^4} \|\Delta \sigma_N\|_{L^4} + [\|\nabla^2 u_N\|_{L^2} \|\sigma_N\|_{L^\infty} + \|\nabla^2 u_N\|_{L^2}] \|\nabla \Delta \sigma_N\|_{L^2} \\ &\quad + (1 + \|u_N\|_{L^\infty} + \|\nabla u_N\|_{L^2}) (1 + \|u_N\|_{H^2}) \|u_N\|_{H^2} + \|\Delta u_N\|_{L^2}^2 + 1 \\ &\lesssim \left[\|\sigma_N\|_{H^1} (1 + \sqrt{\log(e + C_L R_1 \|\sigma_N\|_{H^2})}) + \|u_N\|_{H^1} (1 + \sqrt{\log(e + C_L R_2 \|u_N\|_{H^2})}) \right. \\ &\quad \left. + \|\sigma_N\|_{H^2} + 1 \right] \cdot [\|u_N\|_{H^2}^2 + \|\sigma_N\|_{H^3}^2] + 1, \end{aligned} \quad (3.22)$$

where we have used (3.21), the Brézis-Gallouët-Wainger inequality (2.2), as well as Young's inequality. Note that as before, we have used parameters $R_1, R_2 > 0$ in order to make the quantities $R_1 \sigma_N$ and $R_2 u_N$ dimensionless, so that we can apply the Brézis-Gallouët-Wainger inequality. One can choose $R_2 = \|u_0\|_{L^\infty}^{-1}$ for example (in the case of nonzero initial data).

From (3.22), by using the L^2 -estimate (3.8), the H^1 -estimate (3.18), and the logarithmic Grönwall inequality in Lemma 2.2, we find the following bound:

$$\sup_{t \in [0, T]} \left[\|u_N(t)\|_{H^2}^2 + \|\sigma_N(t)\|_{H^3}^2 \right] \leq C \exp(\exp(\exp(\exp(\exp(CT))))), \quad (3.23)$$

for some constant C which is independent of N, ϵ , and T , but which depends on \mathcal{E}, α , and the initial data. Note that the quintuple-exponential growth in time of the bound originates from the single-exponential respectively triple-exponential growth in time of the L^2 and H^1 -estimates, and the subsequent application of the logarithmic Grönwall inequality.

Estimate on the time derivative and summary. Thanks to (3.23) and system (3.1), one can obtain directly that

$$\sup_{t \in [0, T]} \left[\|\partial_t u_N(t)\|_{H^2}^2 + \|\partial_t \sigma_N - \alpha^2 \partial_t \Delta \sigma_N\|_{H^1}^2 \right] \leq C \exp(\exp(\exp(\exp(\exp(CT)))). \quad (3.24)$$

Meanwhile, by directly applying integration by parts we deduce that

$$\|\partial_t \sigma_N - \alpha^2 \partial_t \Delta \sigma_N\|_{H^1}^2 = \|\partial_t \sigma_N\|_{H^1}^2 + \alpha^4 \|\partial_t \Delta \sigma_N\|_{H^1}^2 + 2\alpha^2 \|\partial_t \nabla \sigma_N\|_{H^1}^2.$$

Hence, equations (3.23) and (3.24) yield

$$\begin{aligned} \sup_{t \in [0, T]} \left[\|u_N(t)\|_{H^2} + \|\sigma_N(t)\|_{H^3} + \|\partial_t u_N(t)\|_{H^2} + \|\partial_t \sigma_N\|_{H^3} \right] \\ \leq C \exp(\exp(\exp(\exp(\exp(CT)))). \end{aligned} \quad (3.25)$$

again for some constant C , which is independent of N, ϵ , and T , but depends on \mathcal{E}, α , and the initial data.

3.3 Well-posedness of the intermediate system (1.13)

Passing the limit $N \rightarrow \infty$ and existence of the strong solution. For any fixed $T \in (0, \infty)$ and $\epsilon \in (0, \infty)$, from the bound (3.25) and by applying the Aubin-Lions compactness lemma (see [9, Theorem II.5.16]), one has the following convergence results: for a suitable subsequence of $N \rightarrow \infty$,

$$u_N \overset{*}{\rightharpoonup} u \quad \text{weakly-* in } L^\infty((0, T); H^2(\mathbb{T}^2)), \quad (3.26)$$

$$\partial_t u_N \overset{*}{\rightharpoonup} \partial_t u \quad \text{weakly-* in } L^\infty((0, T); H^2(\mathbb{T}^2)), \quad (3.27)$$

$$u_N \rightarrow u \quad \text{strongly in } C([0, T]; H^1(\mathbb{T}^2)), \quad (3.28)$$

$$\sigma_N \overset{*}{\rightharpoonup} \sigma \quad \text{weakly-* in } L^\infty((0, T); H^3(\mathbb{T}^2)), \quad (3.29)$$

$$\sigma_N \rightarrow \sigma \quad \text{strongly in } C([0, T]; H^2(\mathbb{T}^2)), \quad (3.30)$$

$$\partial_t \sigma_N \overset{*}{\rightharpoonup} \partial_t \sigma \quad \text{weakly-* in } L^\infty((0, T); H^3(\mathbb{T}^2)), \quad (3.31)$$

where $u \in L^\infty((0, T); H^2(\mathbb{T}^2))$, $\partial_t u \in L^\infty((0, T); H^2(\mathbb{T}^2))$, $\sigma \in L^\infty((0, T); H^3(\mathbb{T}^2))$, and $\partial_t \sigma \in L^\infty((0, T); H^3(\mathbb{T}^2))$ with the same estimates as in (3.25) with (u_N, σ_N) replaced by (u, σ) , i.e.,

$$\begin{aligned} \sup_{t \in [0, T]} \left[\|u(t)\|_{H^2} + \|\sigma(t)\|_{H^3} + \|\partial_t u(t)\|_{H^2} + \|\partial_t \sigma\|_{H^3} \right] \\ \leq C \exp(\exp(\exp(\exp(\exp(CT)))). \end{aligned} \quad (3.32)$$

for some constant C independent of ϵ and T , but depending on \mathcal{E}, α , and the initial data. We also have $u \in C([0, T]; H^2(\mathbb{T}^2))$ and $\sigma \in C([0, T]; H^3(\mathbb{T}^2))$ thanks to [66, Lemma 1.3.2, p. 196].

It is clear that one can pass to the limit in the linear terms of the Galerkin system (3.1). To show the convergence of the nonlinear terms, first we observe that $\mathcal{D}_\epsilon^N \rightarrow \mathcal{D}_\epsilon$ in $C([0, T]; L^2(\mathbb{T}^2))$. In fact, due to Lemma 2.3, we have by using (3.28)

$$\|\mathcal{D}_\epsilon^N - \mathcal{D}_\epsilon\|_{C([0, T]; L^2(\mathbb{T}^2))} \leq C \|u_N - u\|_{C([0, T]; H^1(\mathbb{T}^2))} \xrightarrow{N \rightarrow \infty} 0.$$

Notice that this bound holds pointwise in time, as will generally be the case for the estimates in this part of the paper. We will therefore omit the temporal part of the norm in this section. Next, by using (3.30) one can estimate that

$$\left\| \frac{4\mathcal{D}_\epsilon^N}{P}\sigma_N - \frac{4}{P}\mathcal{D}_\epsilon\sigma \right\|_{L^2} \lesssim \|\mathcal{D}_\epsilon^N - \mathcal{D}_\epsilon\|_{L^2}\|\sigma_N\|_{L^\infty} + \|\mathcal{D}_\epsilon\|_{L^2}\|\sigma_N - \sigma\|_{L^\infty} \xrightarrow{N \rightarrow \infty} 0,$$

by the convergence of both \mathcal{D}_ϵ^N as well as σ_N . Similarly, we know that the other nonlinear terms in the evolution equation for the stress tensor, i.e. $-\frac{3}{2P}\mathcal{D}_\epsilon^N \text{Tr} \sigma_N I$ and $\frac{\mathcal{D}_\epsilon^N}{2}I$, converge to $-\frac{3}{2P}\mathcal{D}_\epsilon \text{Tr} \sigma I$ and $\frac{\mathcal{D}_\epsilon}{2}I$ in $L^\infty((0, T); L^2(\mathbb{T}^2))$ respectively.

Finally, we need to show that the ocean stress term \mathcal{T}_w^N converges. We have

$$\begin{aligned} \|\mathcal{T}_w^N - \mathcal{T}_w\|_{L^2} &\lesssim \| |U_w - u| - |U_w - u_N| \|_{L^4} \left\| \left[(U_w - u_N) \cos \theta + (U_w - u_N)^\perp \sin \theta \right] \right\|_{L^4} \\ &\quad + \| |U_w - u| \|_{L^4} \left\| \left[(u - u_N) \cos \theta + (u - u_N)^\perp \sin \theta \right] \right\|_{L^4} \\ &\lesssim \|u - u_N\|_{L^4} \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

Therefore all the terms in the Galerkin system (3.1) converge, and (u, σ) solves the intermediate system (1.13). This concludes the proof of the existence of a strong solution to the intermediate model (1.13).

Stability and uniqueness.

Suppose that $(u_1, \sigma_1), (u_2, \sigma_2) \in C([0, T]; H^2(\mathbb{T}^2)) \times C([0, T]; H^3(\mathbb{T}^2))$ are two strong solutions to the intermediate model (1.13). Let

$$\delta u := u_1 - u_2, \quad \delta \sigma := \sigma_1 - \sigma_2.$$

The system satisfied by the difference $(\delta u, \delta \sigma)$ is then given by

$$\partial_t \delta u = \nabla \cdot \delta \sigma + \mathcal{T}_{w,1} - \mathcal{T}_{w,2} + \Omega(\delta u)^\perp, \quad (3.33a)$$

$$\frac{1}{\mathcal{E}} \partial_t (\delta \sigma - \alpha^2 \Delta \delta \sigma) + \frac{4\mathcal{D}_{\epsilon,1}}{P} (\sigma_1 - \frac{1}{2} \text{Tr} \sigma_1 \mathbb{I}_2) - \frac{4\mathcal{D}_{\epsilon,2}}{P} (\sigma_2 - \frac{1}{2} \text{Tr} \sigma_2 \mathbb{I}_2) \quad (3.33b)$$

$$+ \frac{\mathcal{D}_{\epsilon,1}}{2P} \text{Tr} \sigma_1 \mathbb{I}_2 - \frac{\mathcal{D}_{\epsilon,2}}{2P} \text{Tr} \sigma_2 \mathbb{I}_2 + \frac{\mathcal{D}_{\epsilon,1} - \mathcal{D}_{\epsilon,2}}{2} \mathbb{I}_2 = D(\delta u), \quad (3.33c)$$

where we have introduced the following notation for $i = 1, 2$

$$\mathcal{T}_{w,i} = c_w \rho_w |U_w - u_i| \left[(U_w - u_i) \cos \theta + (U_w - u_i)^\perp \sin \theta \right], \quad (3.33d)$$

$$\mathcal{D}_{\epsilon,i} = \sqrt{|D(u_i)|^2 + \epsilon^2}. \quad (3.33e)$$

We observe that $\partial_t(\delta u) \in L^\infty((0, T); H^2(\mathbb{T}^2))$ and $\partial_t(\delta \sigma) \in L^\infty((0, T); H^3(\mathbb{T}^2))$, and therefore we can take the L^2 -inner product of equations (3.33a) and (3.33b) with δu respectively $\delta \sigma$. We therefore compute the L^2 -estimates on the difference $(\delta u, \delta \sigma)$ as follows (using a cancellation similar to

(3.7))

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left[\|\delta u\|_{L^2}^2 + \mathcal{E}^{-1} \|\delta \sigma\|_{L^2}^2 + \mathcal{E}^{-1} \alpha^2 \|\nabla \delta \sigma\|_{L^2}^2 \right] &= - \int_{\mathbb{T}^2} \frac{\mathcal{D}_{\epsilon,1} - \mathcal{D}_{\epsilon,2}}{2} \operatorname{Tr} \delta \sigma \, dx \\
&- \int_{\mathbb{T}^2} \left[\frac{4\mathcal{D}_{\epsilon,1}}{P} (\sigma_1 - \frac{1}{2} \operatorname{Tr} \sigma_1 \mathbb{I}_2) - \frac{4\mathcal{D}_{\epsilon,2}}{P} (\sigma_2 - \frac{1}{2} \operatorname{Tr} \sigma_2 \mathbb{I}_2) \right] : \delta \sigma \, dx \\
&- \int_{\mathbb{T}^2} \left[\frac{\mathcal{D}_{\epsilon,1}}{2P} \operatorname{Tr} \sigma_1 \mathbb{I}_2 - \frac{\mathcal{D}_{\epsilon,2}}{2P} \operatorname{Tr} \sigma_2 \mathbb{I}_2 \right] : \delta \sigma \, dx \\
&+ \int_{\mathbb{T}^2} (\mathcal{T}_{w,1} - \mathcal{T}_{w,2}) \cdot \delta u \, dx =: J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.34}$$

Applying Hölder's inequality, inequality (2.7), and the Ladyzhenskaya inequality (2.3) yields

$$\begin{aligned}
|J_3| &\lesssim \int_{\mathbb{T}^2} |\mathcal{D}_{\epsilon,1} - \mathcal{D}_{\epsilon,2}| |\operatorname{Tr} \sigma_1| |\operatorname{Tr} \delta \sigma| + |\mathcal{D}_{\epsilon,2}| |\operatorname{Tr} \delta \sigma|^2 \, dx \\
&\lesssim \|\sigma_1\|_{L^\infty} \|\mathcal{D}_{\epsilon,1} - \mathcal{D}_{\epsilon,2}\|_{L^2} \|\delta \sigma\|_{L^2} + \|\mathcal{D}_{\epsilon,2}\|_{L^2} \|\delta \sigma\|_{L^4}^2 \\
&\lesssim \|\sigma_1\|_{L^\infty} [\|\nabla \delta u\|_{L^2}^2 + \|\delta \sigma\|_{L^2}^2] + \|\nabla u_2\|_{L^2} [\|\delta \sigma\|_{L^2}^2 + \|\nabla \delta \sigma\|_{L^2}^2].
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
|J_2| &\lesssim \|\sigma_1\|_{L^\infty} [\|\nabla \delta u\|_{L^2}^2 + \|\delta \sigma\|_{L^2}^2] + \|\nabla u_2\|_{L^2} [\|\delta \sigma\|_{L^2}^2 + \|\nabla \delta \sigma\|_{L^2}^2], \\
|J_1| &\lesssim \|\nabla \delta u\|_{L^2} \|\delta \sigma\|_{L^2},
\end{aligned}$$

and

$$|J_4| \lesssim (\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} + \|U_w\|_{L^\infty}) \|\delta u\|_{L^2}^2.$$

Combining these bounds we get that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left[\|\delta u\|_{L^2}^2 + \mathcal{E}^{-1} \|\delta \sigma\|_{L^2}^2 + \mathcal{E}^{-1} \alpha^2 \|\nabla \delta \sigma\|_{L^2}^2 \right] &\lesssim (\|\sigma_1\|_{L^\infty} + 1) [\|\nabla \delta u\|_{L^2}^2 + \|\delta \sigma\|_{L^2}^2] \\
&+ \|\nabla u_2\|_{L^2} [\|\delta \sigma\|_{L^2}^2 + \|\nabla \delta \sigma\|_{L^2}^2] + (\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} + \|U_w\|_{L^\infty}) \|\delta u\|_{L^2}^2.
\end{aligned} \tag{3.35}$$

Meanwhile, to calculate the H^1 estimate of the difference $(\delta u, \delta \sigma)$, similar to (3.17), we have (by using a cancellation similar to (3.7))

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left[\|\nabla \delta u\|_{L^2}^2 + \mathcal{E}^{-1} \|\nabla \delta \sigma\|_{L^2}^2 + \alpha^2 \mathcal{E}^{-1} \|\Delta \delta \sigma\|_{L^2}^2 \right] \\
&= \int_{\mathbb{T}^2} \frac{\mathcal{D}_{\epsilon,1} - \mathcal{D}_{\epsilon,2}}{2} \operatorname{Tr} \Delta \delta \sigma \, dx + \int_{\mathbb{T}^2} \left[\frac{4\mathcal{D}_{\epsilon,1}}{P} (\sigma_1 - \frac{1}{2} \operatorname{Tr} \sigma_1 \mathbb{I}_2) - \frac{4\mathcal{D}_{\epsilon,2}}{P} (\sigma_2 - \frac{1}{2} \operatorname{Tr} \sigma_2 \mathbb{I}_2) \right] : \Delta \delta \sigma \, dx \\
&+ \int_{\mathbb{T}^2} \left[\frac{\mathcal{D}_{\epsilon,1}}{2P} \operatorname{Tr} \sigma_1 \mathbb{I}_2 - \frac{\mathcal{D}_{\epsilon,2}}{2P} \operatorname{Tr} \sigma_2 \mathbb{I}_2 \right] : \Delta \delta \sigma \, dx + \int_{\mathbb{T}^2} \nabla (\mathcal{T}_{w,1} - \mathcal{T}_{w,2}) : \nabla \delta u \, dx \\
&=: J_5 + J_6 + J_7 + J_8.
\end{aligned} \tag{3.36}$$

Similarly to the L^2 -estimate, we establish (by using inequality (2.7))

$$\begin{aligned}
|J_5 + J_6 + J_7| &\lesssim \|\sigma_1\|_{L^\infty} \|\mathcal{D}_{\epsilon,1} - \mathcal{D}_{\epsilon,2}\|_{L^2} \|\Delta \delta \sigma\|_{L^2} + \|\mathcal{D}_{\epsilon,2}\|_{L^2} \|\delta \sigma\|_{L^\infty} \|\Delta \delta \sigma\|_{L^2} \\
&+ \|\mathcal{D}_{\epsilon,1} - \mathcal{D}_{\epsilon,2}\|_{L^2} \|\Delta \delta \sigma\|_{L^2} \\
&\lesssim (\|\sigma_1\|_{L^\infty} + \|\nabla u_2\|_{L^2} + 1) [\|\nabla \delta u\|_{L^2}^2 + \|\delta \sigma\|_{L^2}^2 + \|\Delta \delta \sigma\|_{L^2}^2].
\end{aligned}$$

Finally, we estimate J_8 . Recalling the notation (3.9), we denote by

$$V_{w,i} := U_w - u_i, \quad i = 1, 2, \quad (3.37)$$

and hence $V_{w,2} - V_{w,1} = \delta u$. Thus, by using identity (3.11), one can compute, for $i = 1, 2$,

$$\begin{aligned} \partial_i(|V_{w,1}|V_{w,1} - |V_{w,2}|V_{w,2}) &= -|V_{w,1}|\partial_i\delta u + \left[|V_{w,1}| - |V_{w,2}|\right]\partial_iV_{w,2} \\ &\quad - \frac{V_{w,1} \cdot \partial_iV_{w,1}}{|V_{w,1}|}\delta u - \frac{V_{w,1} \cdot \partial_i\delta u}{|V_{w,1}|}V_{w,2} - \frac{\delta u \cdot \partial_iV_{w,2}}{|V_{w,2}|}V_{w,2} \\ &\quad - \left[|V_{w,1}| - |V_{w,2}|\right]\frac{V_{w,1} \cdot \partial_iV_{w,2}}{|V_{w,1}||V_{w,2}|}V_{w,2}. \end{aligned}$$

Therefore, one has that, for $i = 1, 2$,

$$\|\partial_i(\mathcal{T}_{w,1} - \mathcal{T}_{w,2})\|_{L^2} \lesssim (\|V_{w,1}\|_{L^\infty} + \|V_{w,2}\|_{L^\infty})\|\partial_i\delta u\|_{L^2} + (\|\partial_iV_{w,1}\|_{L^4} + \|\partial_iV_{w,2}\|_{L^4})\|\delta u\|_{L^4}. \quad (3.38)$$

Hence one has by using (3.37)

$$\begin{aligned} |J_8| &\lesssim \|\nabla\delta u\|_{L^2} \left[(\|U_w - u_1\|_{L^\infty} + \|U_w - u_2\|_{L^\infty})\|\nabla\delta u\|_{L^2} \right. \\ &\quad \left. + (\|\nabla(U_w - u_1)\|_{L^4} + \|\nabla(U_w - u_2)\|_{L^4})\|\delta u\|_{L^4} \right] \\ &\lesssim (\|U_w - u_1\|_{H^2} + \|U_w - u_2\|_{H^2})\|\delta u\|_{H^1}^2. \end{aligned} \quad (3.39)$$

Thus, we have derived the following from estimate (3.36),

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\|\nabla\delta u\|_{L^2}^2 + \mathcal{E}^{-1}\|\nabla\delta\sigma\|_{L^2}^2 + \alpha^2\mathcal{E}^{-1}\|\Delta\delta\sigma\|_{L^2}^2 \right] \\ &\lesssim (1 + \|\sigma_1\|_{H^2} + \|\sigma_2\|_{H^2} + \|u_1\|_{H^2} + \|u_2\|_{H^2})(\|\delta u\|_{H^1}^2 + \|\delta\sigma\|_{H^2}^2). \end{aligned} \quad (3.40)$$

Consequently, combining estimates (3.35) and (3.40) leads to

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\|\delta u\|_{L^2}^2 + \|\nabla\delta u\|_{L^2}^2 + \mathcal{E}^{-1}\|\delta\sigma\|_{L^2}^2 + \mathcal{E}^{-1}(1 + \alpha^2)\|\nabla\delta\sigma\|_{L^2}^2 + \mathcal{E}^{-1}\alpha^2\|\Delta\delta\sigma\|_{L^2}^2 \right] \\ &\lesssim (1 + \|\sigma_1\|_{H^2} + \|\sigma_2\|_{H^2} + \|u_1\|_{H^2} + \|u_2\|_{H^2})(\|\delta u\|_{H^1}^2 + \|\delta\sigma\|_{H^2}^2), \end{aligned} \quad (3.41)$$

from which the stability (i.e. the continuous dependence on the initial data) and uniqueness of the strong solution follows. This concludes the proof of the global well-posedness of the intermediate model (1.13).

4 Proof of Theorem 1.2

Thanks to (3.32), where the estimate is independent of ϵ , one can pass to the limit $\epsilon \rightarrow 0$ for a subsequence of solutions to the intermediate system (1.13), denoted as $\{(u_\epsilon, \sigma_\epsilon)\}$, with the convergence in the same spaces as in equations (3.26)–(3.31) with $N \rightarrow \infty$ replaced by $\epsilon \rightarrow 0$. The limit is denoted as (u, σ) , and the limit satisfies estimate (3.32). It is left to verify that (u, σ) is a solution to the Voigt-EVP model (1.12).

First we prove that $\sqrt{|D(u_\epsilon)|^2 + \epsilon^2} \rightarrow |D(u)|$ in $L^\infty((0, T); L^2(\mathbb{T}^2))$. Indeed, note that for all $t \in [0, T]$

$$\begin{aligned} \left\| \sqrt{|D(u_\epsilon)|^2 + \epsilon^2} - |D(u)| \right\|_{L^2} &\leq \left\| \sqrt{|D(u_\epsilon)|^2 + \epsilon^2} - \sqrt{|D(u)|^2 + \epsilon^2} \right\|_{L^2} \\ &+ \left\| \sqrt{|D(u)|^2 + \epsilon^2} - |D(u)| \right\|_{L^2} \leq C \|u_\epsilon - u\|_{H^1} + \|\epsilon\|_{L^2} \xrightarrow{\epsilon \rightarrow 0} 0, \end{aligned}$$

where we have used Lemma 2.3, the convergence $u_\epsilon \rightarrow u$ in $C([0, T]; H^1(\mathbb{T}^2))$ and the elementary inequality $\sqrt{y+z} \leq \sqrt{y} + \sqrt{z}$ for $y, z \geq 0$. It is clear that the linear terms in equations (1.13a)-(1.13b) converge. We now show that the nonlinear terms in the equation for the stress tensor converge. In particular, for all $t \in [0, T]$ we have

$$\begin{aligned} \left\| \frac{4\sqrt{|D(u_\epsilon)|^2 + \epsilon^2}}{P} \sigma_\epsilon - \frac{4}{P} |D(u)| \sigma \right\|_{L^2} &\lesssim \left\| \sqrt{|D(u_\epsilon)|^2 + \epsilon^2} - |D(u)| \right\|_{L^2} \|\sigma_\epsilon\|_{L^\infty} \\ &+ \left\| |D(u)| \right\|_{L^2} \|\sigma_\epsilon - \sigma\|_{L^\infty} \xrightarrow{\epsilon \rightarrow 0} 0. \end{aligned}$$

In a similar fashion, one can show that the terms $-\frac{3}{2} \frac{\sqrt{|D(u_\epsilon)|^2 + \epsilon^2}}{P} \text{Tr} \sigma_\epsilon \mathbb{I}_2$ and $\frac{\sqrt{|D(u_\epsilon)|^2 + \epsilon^2}}{2} \mathbb{I}_2$ converge to $-\frac{3}{2} \frac{|D(u)|}{P} \text{Tr} \sigma \mathbb{I}_2$ and $\frac{|D(u)|}{2} \mathbb{I}_2$ in $L^\infty((0, T); L^2(\mathbb{T}^2))$. Finally, we show that the ocean stress term \mathcal{T}_w^ϵ converges. We find

$$\begin{aligned} \|\mathcal{T}_{w,\epsilon} - \mathcal{T}_w\|_{L^2} &\lesssim \left\| |U_w - u| - |U_w - u_\epsilon| \right\|_{L^4} \left\| \left[(U_w - u_\epsilon) \cos \theta + (U_w - u_\epsilon)^\perp \sin \theta \right] \right\|_{L^4} \\ &+ \left\| |U_w - u| \right\|_{L^4} \left\| \left[(u - u_\epsilon) \cos \theta + (u - u_\epsilon)^\perp \sin \theta \right] \right\|_{L^4} \\ &\lesssim \|u - u_\epsilon\|_{L^4} \xrightarrow{\epsilon \rightarrow 0} 0, \end{aligned}$$

where again the convergence is uniform in time for the fixed time interval $[0, T]$. Therefore we conclude that (u, σ) is a global strong solution of the Voigt-EVP model.

To show the (global) well-posedness, one can repeat similar arguments as in Section 3.3. Consequently, one can obtain the analogous version of estimate (3.41), and therefore conclude the stability and uniqueness of the Voigt-EVP model. This finishes the proof of Theorem 1.2.

Remark 4.1. We remark that the proof of the convergence of the solutions in the viscosity cutoff limit $\epsilon \rightarrow 0$, is in some sense related to the approach in [3], in which the local well-posedness of a model for non-Newtonian incompressible flows with power-law stress tensor was established uniformly in the cutoff parameter of the viscosity.

5 Discussion and conclusion

In this paper we commenced the rigorous analytical study of one of the fundamental models of sea-ice dynamics, namely the elastic-viscous-plastic (EVP) sea-ice model. We introduced a regularised version of the EVP model, by means of the Voigt regularisation, which is inspired by the original elastic regularisation of the Hibler model in [35] (as both regularisations formally do not impact the asymptotic dynamics). We have shown this Voigt-EVP model to be globally well-posed. One interesting question, which we leave to future work, is whether the Voigt-EVP model also has a suitable notion of weak solution.

Due to the (formal) linear ill-posedness of the EVP model as discussed in Section 1.2, it seems necessary to introduce a regularised version of the EVP model of the type (1.12a)-(1.12c), in order to obtain a well-posed sea-ice model. Finally, in order to conclude the proof of the global well-posedness we took the limit of the viscosity cutoff parameter ϵ going to zero (while keeping the Voigt regularisation), and obtain that the Voigt-EVP model is globally well-posed.

Moreover, we would like to mention that over time several related models to the EVP model have been developed in the sea-ice literature, see for example [7, 34]. The approach developed here will also have potential applications in the rigorous study of several of these related sea-ice models.

We will make several remarks on straightforward extensions of our results, which have been omitted for reasons of brevity but can be treated by using the approach developed in this paper.

About the advection term If one includes an advection term in the momentum equation of the intermediate system (1.13a)-(1.13c), by applying the same methods as in this paper, one can prove the local well-posedness of the intermediate system by using similar estimates as in this paper. Note however that the existence time will most likely depend on ϵ (unlike for Corollary 1.4). This therefore prohibits taking the limit $\epsilon \rightarrow 0$, as we have done in the proof of Theorem 1.2.

Using the original strain rate We remark that the replacement of $\overline{\mathcal{D}}$ by $|D(u)|$ in the Voigt-EVP model is done merely for the sake of convenience. In fact, by using the approach from this paper one can prove the global well-posedness of the model (1.13a)-(1.13c) in which \mathcal{D}_ϵ has been replaced by

$$\sqrt{\overline{\mathcal{D}}^2 + \epsilon^2} = \sqrt{\frac{2}{\epsilon^2} |D(u) - \frac{1}{2} [\text{Tr } D(u)] \mathbb{I}_2|^2 + |\text{Tr } D(u)|^2 + \epsilon^2}.$$

This can be done by using similar estimates as in Section 3, but by using the following inequalities

$$\left\| \sqrt{\overline{\mathcal{D}}^2 + \epsilon^2} \right\|_{L^2} \lesssim \|\nabla u\|_{L^2} + \epsilon, \quad \left\| \nabla \left(\sqrt{\overline{\mathcal{D}}^2 + \epsilon^2} \right) \right\|_{L^2} \lesssim \|\Delta u\|_{L^2}.$$

Subsequently, one can proceed as in the proof of Theorem 1.2 and send $\epsilon \rightarrow 0$ in order to obtain a solution of the Voigt-EVP model with the original strain rate.

Different regularisations of the strain rate We also note that, in addition to (1.14), different regularisations of the strain rate have been used in the sea-ice literature. For example, one can set maximal values for the shear and bulk viscosities, as is done in [32]. This means that, instead of \mathcal{D}_ϵ , one uses

$$\max\{\overline{\mathcal{D}}, \epsilon\} = \frac{\overline{\mathcal{D}} + \epsilon}{2} + \frac{1}{2} |\overline{\mathcal{D}} - \epsilon|. \quad (5.1)$$

One can choose to regularise this cutoff by the following function for some $\gamma \in (0, \epsilon)$

$$\overline{\mathcal{D}}_{\epsilon, \gamma} := \frac{\overline{\mathcal{D}} + \epsilon + \gamma}{2} + \frac{1}{2} \sqrt{|\overline{\mathcal{D}} - \epsilon|^2 + \gamma^2}. \quad (5.2)$$

Then one finds

$$\left\| \overline{\mathcal{D}}_{\epsilon, \gamma} \right\|_{L^2} \lesssim \|\nabla u\|_{L^2} + \gamma + \epsilon, \quad \left\| \nabla \overline{\mathcal{D}}_{\epsilon, \gamma} \right\|_{L^2} \lesssim \|\Delta u\|_{L^2}.$$

It is straightforward to repeat the proof of Theorem 1.2 to obtain a global strong solution of the Voigt-EVP model with the viscosity regularisation from equation (5.1) by taking the limit $\gamma \rightarrow 0$. Moreover,

one can also choose to regularise the viscosities by means of the hyperbolic tangent, as was introduced in [48], see also [17], but again one can treat this regularisation by similar methods as in this paper. Generally speaking, one can treat any C^2 approximation of $\overline{\mathcal{D}}$ by relying on the approach of this paper.

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