

**Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.**

Preprint

ISSN 2198-5855

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Lukas Lüchtrath¹, Benedikt Jahnel^{1,2}, Christian Mönch³

submitted: June 2, 2025

¹ Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: lukas.luechtrath@wias-berlin.de

² Technische Universität Braunschweig
Universitätsplatz 2
38106 Braunschweig
Germany
E-Mail: benedikt.jahnel@wias-berlin.de

³ Johannes Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz
Germany
E-Mail: cmoench@uni-mainz.de

No. 3199
Berlin 2025



2020 *Mathematics Subject Classification.* 60K35, 05C82, 91D30.

Key words and phrases. Local weak convergence, random network, SIS model, sparse random graph, locality of phase transition.

CM's research is funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) SPP 2265 443916008. BJ and LL acknowledge the financial support of the Leibniz Association within the Leibniz Junior Research Group on *Probabilistic Methods for Dynamic Communication Networks* as part of the Leibniz Competition as well as the Berlin Cluster of Excellence *MATH+* through the project *EF45-3* on *Data Transmission in Dynamical Random Networks*.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: <http://www.wias-berlin.de/>

Phase transitions for contact processes on sparse random graphs via metastability and local limits

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Abstract

We propose a new perspective on the asymptotic regimes of fast and slow extinction in the contact process on locally converging sequences of sparse finite graphs. We characterise the phase boundary by the existence of a metastable density, which makes the study of the phase transition particularly amenable to local-convergence techniques. We use this approach to derive general conditions for the coincidence of the critical threshold with the survival/extinction threshold in the local limit. We further argue that the correct time scale to separate fast extinction from slow extinction in sparse graphs is, in general, the exponential scale, by showing that fast extinction may occur on stretched exponential time scales in sparse scale-free spatial networks. Together with recent results by Nam, Nguyen and Sly (Trans. Am. Math. Soc. 375, 2022), our methods can be applied to deduce that the fast/slow threshold in sparse configuration models coincides with the survival/extinction threshold on the limiting Galton-Watson tree.

1 Introduction and main results

The contact process. Let $G = (V, E)$ denote a locally finite connected graph with a dedicated root $o \in V$ and we write (G, o) for the rooted graph. The *contact process* with infection rate $\lambda > 0$ on G is the family of set-valued continuous-time Markov processes $\{\xi^A = (\xi_t^A)_{t \geq 0} : A \subset V \text{ finite}\}$. The law of ξ^A is determined by setting $\xi_0^A = A$ and the transition dynamics, for every $v \in V$,

$$\begin{aligned} \xi_t^A &\rightarrow \xi_t^A \setminus \{v\} \text{ at rate } 1, \\ \xi_t^A &\rightarrow \xi_t^A \cup \{v\} \text{ at rate } \lambda \sum_{w: v \sim w} \mathbb{1}\{w \in \xi_t^A\}. \end{aligned} \tag{1}$$

Here and throughout we write $v \sim w$ for $\{v, w\} \in E$. For ease of notation, we set $\xi_t^v = \xi_t^{\{v\}}$ for $v \in V$ and analogously drop the set notation for singletons in similar instances. We write \mathbf{P}_G^λ for the law of the contact process based on the graph G with infection rate λ . Three fundamental and well-known properties of the contact process are

monotonicity: ξ_t^A is stochastically increasing in λ and G ,

attractivity: if $A \subset B$, then ξ^B stochastically dominates ξ^A , and

(self-)duality: $\mathbf{P}_G^\lambda(\xi_t^A \cap B = \emptyset) = \mathbf{P}_G^\lambda(\xi_t^B \cap A = \emptyset)$ for $A, B \subset V$ finite.

Duality immediately guarantees that ξ^A is well-defined for any $A \subset V$.

The *extinction time* $\tau_\emptyset(A)$ of ξ^A denotes the time at which $(\xi_t^A)_{t \geq 0}$ first gets absorbed into the state \emptyset . In this article, one central quantity of interest for the contact process on infinite graphs is the *survival/extinction threshold* of (G, o) , given by

$$\lambda_1(G) = \sup \{ \lambda > 0 : \mathbf{P}_G^\lambda(\tau_\emptyset(o) = \infty) = 0 \},$$

which is guaranteed to be well-defined by monotonicity. Note that the definition of the critical rate is independent of the choice of the root, since we assume G to be connected. For finite graphs, the critical rate is 0 and contains no information about the dynamics of ξ , and we discuss more appropriate thresholds for this setting below. Invoking monotonicity again, one sees that, for any infinite connected graph G , the value $\lambda_1(G)$ is at most the extinction/survival threshold for the contact process on \mathbb{N} with nearest neighbour edges. The latter is well-known to be finite [Har] and thus $\lambda_1(G) < \infty$. The survival/extinction threshold is also known as the *lower critical value* for the contact process on G , the *upper critical value* being the threshold at which the infection returns infinitely often to the root.

The contact process and related interacting particle systems were first systematically studied in this form in the 1970s, see e.g. [Har, Gric]. Classical choices for G are homogeneous lattices (usually the hypercubic lattice \mathbb{Z}^d) and their finite subgraphs [Gria, Grib, DG, CGOV, DL, DS]. Using the fact that the contact process is a form of oriented percolation, Bezuidenhout and Grimmett famously proved that the critical contact process on \mathbb{Z}^d dies out [BGa] and established exponential decay of the volume of the infected set in [BGb]. A comprehensive overview of the classical theory is provided in the monographs [Liga, Ligb].

The fast/slow threshold of the contact process on random graphs. With the advent of network science in the early 2000's, interest in infection models on finite sparse *random* graphs arose, as they serve as models for the spread of diseases or information in inhomogeneous populations. Since the contact process dies out on any finite graph, the question of extinction vs. survival becomes a question of fast extinction vs. slow extinction. Here, slow extinction means that the infection survives for a time that scales exponentially in the size of the graph. Let $(G_n)_{n \in \mathbb{N}}$ denote a sequence of finite graphs with $G_n = (V_n, E_n)$. Assume that $|V_n| \rightarrow \infty$, then the *fast/slow threshold* for $(G_n)_{n \in \mathbb{N}}$ is given by

$$\lambda_+((G_n)_{n \in \mathbb{N}}) = \sup \{ \lambda > 0 : \lim_{n \rightarrow \infty} \mathbf{P}_{G_n}^\lambda(\tau_\emptyset(V_n) > e^{c|V_n|}) = 0 \text{ for all } c > 0 \}.$$

The main purpose of this article is to investigate the question when $\lambda_+((G_n)_{n \in \mathbb{N}})$ and $\lambda_1(G)$ coincide, provided that the finite graphs $(G_n)_{n \in \mathbb{N}}$ approximate (G, o) in a suitable sense. To formalise this, we use the framework of [AL07, vdH23, vdH24]. Let \mathcal{G}_* denote the space of equivalence classes of connected locally finite rooted graphs modulo rooted isomorphisms, equipped with the local metric d_* and let $\mathcal{P}(\mathcal{G}_*)$ denote the space of Borel probability measures on \mathcal{G}_* . We do not usually distinguish between rooted graphs and their equivalence classes in the same vein as one commonly speaks of 'a random variable $X \in L_1$ '. Let us note that any locally finite rooted graph can be viewed as an element of \mathcal{G}_* if it is identified with the connected component of its root. More background on \mathcal{G}_* as a metric space can be found in [vdH24] and the references therein. Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ denote a sequence of finite connected random

graphs $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ with $|\mathcal{V}_n| \xrightarrow{n \rightarrow \infty} \infty$ in probability. We say $(\mathcal{G}_n)_{n \in \mathbb{N}}$ *converges locally in probability* to some random rooted graph (\mathcal{G}, o) with distribution $Q \in \mathcal{P}(\mathcal{G}_*)$ if, for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\mathfrak{d}_{\mathcal{P}}(Q_n, Q) > \varepsilon) = 0, \quad (2)$$

where

$$Q_n(\cdot) = |\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{(\mathcal{G}_n, v) \in \cdot\} \in \mathcal{P}(\mathcal{G}_*)$$

denotes the empirical distribution on \mathcal{G}_* associated with \mathcal{G}_n if a root is chosen uniformly at random, and $\mathfrak{d}_{\mathcal{P}}$ denotes the Lévy–Prokhorov metric on $\mathcal{P}(\mathcal{G}_*)$. We write

$$\mathcal{G}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} (\mathcal{G}, o)$$

for local convergence in probability. Note that, if $o_n \in \mathcal{V}_n$ is chosen uniformly at random and the distribution of (\mathcal{G}_n, o_n) is denoted by Q_n , then (2) implies that $Q_n \rightarrow Q$ weakly¹. This corresponds to *local weak convergence* of $(\mathcal{G}_n)_{n \in \mathbb{N}}$ to (\mathcal{G}, o) with distribution Q . In particular, the above definitions apply to sequences $(G_n)_{n \in \mathbb{N}}$ of *deterministic* finite graphs. It is well known, that every distribution $Q \in \mathcal{P}(\mathcal{G}_*)$ that arises as a local limit is *unimodular*, i.e., satisfies a certain mass-transport principle [AL07]. If a unimodular measure $Q \in \mathcal{P}(\mathcal{G}_*)$ does not admit any non-trivial representation as a convex combination of other unimodular measures, then we call Q *extremal*. Extremal distributions on rooted graphs are characterised by the property that

$$Q(A) \in \{0, 1\}, \text{ for all } A \in \mathcal{I},$$

where the σ -field \mathcal{I} consists of all Borel-events over \mathbb{G}_* that do not depend on the root of the involved graphs.

Remark 1.1. In general, if a sequence $(\mathcal{G}_n)_{n \in \mathbb{N}}$ of (not necessarily connected) random graphs converges locally in probability to a limit graph (\mathcal{G}, o) and $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is supercritical in the sense that, with high probability, a unique macroscopic *giant component* is formed, then the sequence of the corresponding giant components converges locally in probability to (\mathcal{G}, o) conditioned on $|\mathcal{V}| = \infty$, see [vdH23]. Examples where the limiting distribution is concentrated on trees include sparse Erdős–Rényi graphs, inhomogeneous random graphs, random regular graphs, preferential attachment graphs, and the configuration model. In the context of spatial random graphs, such as e.g. lattice bond and site percolation models, spatial scale-free networks, continuum percolation models and spatial random intersection graphs, the limits are usually not trees.

The classical example for coincidence of the critical values λ_+ and λ_1 is the case in which $(G_n)_{n \in \mathbb{N}}$ are boxes in \mathbb{Z}^d and the limit graph is the full lattice. In this case, it is also known that fast extinction actually occurs at a time scale that is logarithmic in the size of V_n [DL, DS]. A similar result is known to hold for random regular graphs converging to the d -ary tree [LS17]. The

¹Let us mention that the convergence in (2) still makes sense if Q is random, see [vdH24, Remarks 2.12 and 2.13]. In this case, the induced weak limit is the expectation of Q . However, since we are mostly interested in a law-of-large-numbers-type convergence of the contact process observables, we will always assume Q to be deterministic. Our results also hold for random Q in an almost-sure sense, but we do not present them in this way to spare the reader another layer of randomness that has no bearing on the technical core of our results.

situation is quite different if (\mathcal{G}, o) is not only random but also admits unbounded degrees. In two seminal works, Nam, Ngyuen and Sly [NNS] and Nguyen and Sly [NS25] recently showed that, in configuration graphs converging to a unimodular Bienaymé–Galton–Watson tree, the extinction time in the subcritical phase is polynomial in the size of the graph and determine explicit bounds on the polynomial power. Consequently, one defines

$$\lambda_-((\mathcal{G}_n)_{n \in \mathbb{N}}) = \sup \left\{ \lambda > 0 : \lim_{n \rightarrow \infty} \mathbf{P}_{\mathcal{G}_n}^\lambda (\tau_\emptyset(\mathcal{Y}_n) > |\mathcal{Y}_n|^c) = 0 \text{ for all } c > 0 \right\}.$$

The starting point of our investigation is the following conjecture of Nam, Ngyuen and Sly:

Conjecture 1.2 ([NNS, Conjecture 4]). *Let μ denote a probability distribution on the non-negative integers satisfying*

$$\sum_{k \geq 0} k(k-2)\mu(k) > 0$$

and let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ denote configuration graphs on n vertices derived from μ . Then,

$$\lambda_-((\mathcal{C}_n)_{n \in \mathbb{N}}) = \lambda_+((\mathcal{C}_n)_{n \in \mathbb{N}}) = \lambda_1(\mathcal{T}),$$

where for each n , \mathcal{C}_n is the maximal component of \mathcal{G}_n , and \mathcal{T} is the local limit of $(\mathcal{C}_n)_{n \in \mathbb{N}}$, i.e., the unimodular Bienaymé–Galton–Watson tree associated with μ conditioned on non-extinction.

For the moment, we remark that $\lambda_1(\mathcal{T}) > 0$ if and only if μ has an exponential tail [HD20, BNNS] and that the partial result $\lambda_+((\mathcal{G}_n)_{n \in \mathbb{N}}) \leq \lambda_1(\mathcal{T})$ was already established in [NNS].

Overview of main results. Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ denote a sequence of finite connected random graphs converging locally in probability to (\mathcal{G}, o) . Let $(s(m))_{m \in \mathbb{N}}$ be some diverging sequence. We say that the contact process on $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is *metastable at time scale $s(m)$* if, for all sequences $(t(n))_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \mathbb{P}(t(n) \leq s(|\mathcal{Y}_n|)) = 1$, there exists some $\eta > 0$, which may depend on $(\mathcal{G}_n)_{n \in \mathbb{N}}$, such that

$$\limsup_{n \rightarrow \infty} \mathbf{P}_{\mathcal{G}_n}^\lambda \left(|\mathcal{Y}_n|^{-1} |\xi_{t(n)}^{\mathcal{Y}_n}| > \eta \right) > 0.$$

Our goal is to quantify the infection density $|\xi^{\mathcal{Y}_n}|/|\mathcal{Y}_n|$, and thereby the occurrence of a metastable phase, through the *survival probability* of the contact process on the limit graph (\mathcal{G}, o) ,

$$\eta_\lambda(\mathbb{Q}) = \mathbb{E}[\mathbf{P}_{\mathcal{G}}^\lambda(\tau_\emptyset(o) = \infty)] = \mathbb{P}^\lambda(\tau_\emptyset(o) = \infty).$$

Observe that η_λ depends only on the *distribution* of the limit graph, while metastability as well as λ_+ and λ_- are, in principle, dependent on the realisation of $(\mathcal{G}_n)_{n \in \mathbb{N}}$. However, if (\mathcal{G}, o) is the infinite cluster of some percolation process, then its distribution \mathbb{Q} is usually extremal, such as in the setting of Conjecture 1.2, and then the limiting objects do not depend on the realisation of the graph sequence, cf Remark 1.1.

Our first main result is that the metastable density of the infection cannot exceed the survival probability in the limit.

Theorem 1.3. *Assume that $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is a sequence of connected locally finite graphs that converges locally in probability to some rooted locally finite random graph (\mathcal{G}, o) with extremal distribution \mathbb{Q} . Then, for any diverging sequence $(t(n))_{n \in \mathbb{N}}$ of times and any $\varepsilon > 0$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}^\lambda \left(|\mathcal{Y}_n|^{-1} |\xi_{t(n)}^{\mathcal{Y}_n}| \geq \eta_\lambda(\mathbb{Q}) + \varepsilon \right) = 0.$$

In particular, if $\lambda < \lambda_1(\mathcal{G}) \equiv \lambda_1(\mathbb{Q})$, then

$$|\mathcal{V}_n|^{-1} |\xi_{t(n)}^{\mathcal{V}_n}| \xrightarrow[n \rightarrow \infty]{\mathbb{P}^\lambda} 0.$$

Note that the density of the infection is a global quantity, therefore the extinction probability in the limit has to coincide with the ‘annealed’ survival probability η_λ .

The absence of metastability for some time scale does not imply that the contact process dies out on that time scale, as is illustrated by our next result. To formulate it, we define a sequence of random graphs $(\mathcal{G}_n)_{n \in \mathbb{N}}$ to be *sparse* if the family $\{\deg_{\mathcal{G}_n}(o_n) : n \in \mathbb{N}\}$ is uniformly integrable under \mathbb{P}^λ . Sparsity is a natural assumption in the context of locality of the fast/slow transition in the contact process as it guarantees that $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is tight in \mathcal{G}_* , see [BLS15].

Theorem 1.4. *For every $\varepsilon > 0$, there exists a sequence of sparse graphs $(\mathcal{G}_n)_{n \in \mathbb{N}}$ that converges locally in probability to some (\mathcal{G}, o) with extremal distribution \mathbb{Q} satisfying $\lambda_1(\mathcal{G}) > 0$ and such that, for all $\lambda > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}^\lambda \left(\tau_\emptyset(\mathcal{V}_n) > \exp(|\mathcal{V}_n| / \log^{1+\varepsilon}(|\mathcal{V}_n|)) \right) = 1. \quad (3)$$

In particular, for these graph sequences we have that $0 = \lambda_-((\mathcal{G}_n)_{n \in \mathbb{N}}) < \lambda_1(\mathcal{G})$.

This result suggests that, if $((\mathcal{G}_n, o_n))_{n \in \mathbb{N}}$ is sparse, then the correct time scale to distinguish the fast extinction regime from the slow extinction regime is in general the exponential one, i.e., $s(m) = e^{cm}$, $m \in \mathbb{N}$, for some $c > 0$. It is not difficult to see, cf. Lemma 2.5 below, that in this case, absence of metastability on the exponential time scale also implies extinction. On the other hand, it is straightforward that, in sparse graphs, there cannot be survival at super-exponential scales. We use the connection between metastability and survival on the exponential time scale to prove a general inequality for the critical values.

Theorem 1.5. *Suppose $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is a sequence of connected sparse graphs converging locally in probability to a graph (\mathcal{G}, o) with extremal distribution \mathbb{Q} . Then,*

$$\lambda_+((\mathcal{G}_n)_{n \in \mathbb{N}}) \geq \lambda_1(\mathcal{G}) \equiv \lambda_1(\mathbb{Q}).$$

In particular, our theorem implies the second equality in Conjecture 1.2.

Corollary 1.6. *Let μ denote a probability distribution on the non-negative integers and let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ denote configuration graphs on n vertices derived from μ . Then,*

$$\lambda_+((\mathcal{C}_n)_{n \in \mathbb{N}}) = \lambda_1(\mathcal{T}),$$

where, for each n , \mathcal{C}_n denotes the largest component in \mathcal{G}_n and \mathcal{T} is the local limit of $(\mathcal{C}_n)_{n \in \mathbb{N}}$.

Proof. It is well known that the configuration model converges locally in probability to the unimodular Bienaymé–Galton–Watson tree \mathcal{T} and that the corresponding giant components converge to the limit tree conditioned on non-extinction, see [vdH23]. Hence, Theorem 1.5 applies. The converse inequality is provided in [NNS, Theorem 5]. \square

Our final result concerns lower bounds in probability for the metastable density. For this, consider the condition

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}^\lambda \left(\xi_{t(n)}^{o_n} = \emptyset, \tau_R^{(n)}(o_n) < t(n) \right) = 0, \quad (4)$$

where $\tau_R^{(n)}(o_n)$ denotes the first times that a vertex at distance R from the root is infected in \mathcal{G}_n . The next statement asserts that (4) is necessary and sufficient for lower bounding the metastable density via the limit's survival probability.

Proposition 1.7. *Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be a sequence of connected random graphs with $\mathcal{G}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} (\mathcal{G}, o)$, where the limit is distributed according to some extremal distribution \mathbb{Q} . Let $(t(n))_{n \in \mathbb{N}}$ denote a sequence of diverging times. Then, (4) is equivalent to*

$$\lim_{n \rightarrow \infty} \mathbb{P}^\lambda \left(|\mathcal{V}_n|^{-1} |\xi_{t(n)}^{\mathcal{V}_n}| \leq \eta_\lambda(\mathbb{Q}) - \varepsilon \right) = 0, \quad \text{for all } \varepsilon > 0.$$

This, together with our Theorem 1.3 now implies that (4) is equivalent to convergence in probability of the metastable density to the limit's survival probability.

Corollary 1.8. *Under the assumptions of Proposition 1.7, the condition (4) is equivalent to*

$$|\mathcal{V}_n|^{-1} |\xi_{t(n)}^{\mathcal{V}_n}| \xrightarrow[n \rightarrow \infty]{\mathbb{P}^\lambda} \eta_\lambda(\mathbb{Q}).$$

Proof. This is a direct consequence of Theorem 1.3 and Proposition 1.7. □

Naturally, establishing (4) for a given time-scale is in general hard and the main challenge in proving metastability for a concrete graph sequence.

Further discussion and related work. Another useful observation pertaining to the locality of metastability on $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is that $\lambda_1(\mathcal{G}, o)$ can be characterised by tightness of the extinction times.

Lemma 1.9. *Suppose that $(\mathcal{G}_n)_{n \in \mathbb{N}}$ converges to (\mathcal{G}, o) locally weakly. For any $\lambda > 0$, $\{\tau_\emptyset^{(n)}(o_n), n \in \mathbb{N}\}$ is tight if and only if*

$$\mathbb{P}^\lambda(\tau_\emptyset(o) < \infty) = 1.$$

Lemma 1.9 provides yet another perspective on the fast/slow transition. The law of large numbers in Corollary 1.8 suggests the following definition for the critical value

$$\lambda_\rho((\mathcal{G}_n)_{n \in \mathbb{N}}) = \sup \left\{ \lambda > 0 : \lim_{n \rightarrow \infty} \mathbf{P}_{\mathcal{G}_n}^\lambda(\rho(e^{c|\mathcal{V}_n|}) > \varepsilon) = 0 \text{ for all } \varepsilon, c > 0 \right\},$$

where $\rho(t) = \rho_n(t) = |\xi_t^{\mathcal{V}_n}| / |\mathcal{V}_n|$ denotes the density process associated with $\xi^{\mathcal{V}_n}$ on \mathcal{G}_n . Denoting

$$\lambda_\rho^-((\mathcal{G}_n)_{n \in \mathbb{N}}) = \sup \left\{ \lambda > 0 : \lim_{n \rightarrow \infty} \mathbf{P}_{\mathcal{G}_n}^\lambda(\rho(t(|\mathcal{V}_n|)) > \varepsilon) = 0 \right. \\ \left. \text{for all } \varepsilon > 0 \text{ and } (t(n))_{n \in \mathbb{N}} \text{ with } t(n) \rightarrow \infty \right\},$$

we have that $\lambda_\rho^- \leq \lambda_\rho$. However, it is also elementary to see that

$$\lim_{n \rightarrow \infty} \mathbf{P}_{\mathcal{G}_n}^\lambda (\rho(t(|\mathcal{V}_n|)) > \varepsilon) = 0 \quad \text{for all } \varepsilon > 0, t(n) \rightarrow \infty$$

if and only if $\{\tau_\emptyset^{(n)}(o_n), n \in \mathbb{N}\}$ is tight. Consequently, $\lambda_1 = \lambda_\rho^- \leq \lambda_\rho$, in the case where the underlying graphs converge locally in probability. Note that $\lambda_1 \leq \lambda_\rho$ also follows from Theorem 1.3. For the configuration model, the bound $\lambda_\rho \geq \lambda_1$ is essentially [NNS, Theorem 6]. Our results show that this inequality always holds if $\mathcal{G}_n \rightarrow (\mathcal{G}, o)$ locally in probability. Moreover, our proof of Theorem 1.5 below implies that

$$\text{if } \mathcal{G}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} (\mathcal{G}, o) \text{ and } (\mathcal{G}_n)_{n \in \mathbb{N}} \text{ is sparse, then } \lambda_\rho = \lambda_+.$$

Further evidence that the polynomial time scale is the natural one for *fast* extinction in sparse locally tree-like graphs is, for instance, given in [Dur24] and [NS25]. Note that, unlike in our Theorem 1.4, the time scale of extinction in the graphs discussed there is *not* determined solely by the presence of stars. Furthermore, Theorem 1.4 should be contrasted with [SV, Theorem 1.2], where it is shown that the *supercritical* extinction time is at least as large as the time scale given in Theorem 1.4 on *any* finite graph provided that $\lambda > \lambda_1(\mathbb{Z})$. Our proof also shows that the time scale $\exp(|\mathcal{V}_n| / \log(|\mathcal{V}_n|)^{1+o(1)})$ is not optimal, cf. Remark 2.2, but that $\exp(\Theta(|\mathcal{V}_n|))$ cannot be achieved. It is an interesting question to determine whether there are sparse graphs on which fast extinction occurs on time scale $\exp(c^-(|\mathcal{V}_n|))$ and slow extinction occurs on time scale $\exp(c^+(|\mathcal{V}_n|))$ with $c^- < c^+$, or even with $c^- = c^+$. Another recent work proving exponential extinction times with logarithmic correction for all infection rates is [BHSV25], see in particular Theorem 1.1.(ii) therein. However, the random graphs considered in that paper are *small worlds* and it is highly likely that their local limits do not display an extinction phase in the parameter regime of [BHSV25, Theorem 1.1.(ii)]. This contrasts with our example in Theorem 1.4, which is not a small world graph but displays distances comparable to the lattice [L24] and does have a phase transition in the local limit.

Let us further mention that, if sparsity is violated, survival can occur on super-exponential time scales as demonstrated in [Can].

Finally, we would like to point out, that our approach to metastability in the contact process on random graph was inspired by [LMSV, Theorem 1.4], a metastability result on a concrete random graph sequence, and, most prominently, by the phenomenology developed for the corresponding percolation problem [vdH23].

Overview of the proof section. The remainder of the paper is devoted to the derivation of our results. We first recall some facts about local convergence and prove Lemma 1.9, which is independent of our main results. Then we construct the example leading to Theorem 1.4, and finally we prove the LLN-type results Theorem 1.3 and Proposition 1.7 and apply them to derive Theorem 1.5.

2 Proofs

2.1 Preparatory and auxiliary results

We recall some fundamental properties of the space \mathcal{G}_* . Recall that the local metric d_* on \mathcal{G}_* is given by

$$d_*((G, o_G), (H, o_H)) = 2^{-\sup\{k: B_G(o_G, k) = B_H(o_H, k)\}},$$

where $B_G(o_G, k)$ denotes the subgraph in G induced by all vertices of graph distances at most k from o_G . Recall that statements like $B_G(o_G, k) = B_H(o_H, k)$ or $G = H$ for elements of \mathcal{G}_* always implicitly refer to equality of equivalence classes under rooted isomorphisms.

We begin by noting that our definition of local convergence in probability is slightly different to the one given in [vdH23, vdH24]. In fact, there are several equivalent formulations.

Lemma 2.1. *Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be a sequence of finite random graphs and let $(\mathcal{G}, o) \in \mathcal{G}_*$ be a random graph with distribution \mathbb{Q} . The following three assertions are equivalent:*

(a) $\mathcal{G}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} (\mathcal{G}, o)$

(b) For all bounded continuous functions $h: \mathcal{G}_* \rightarrow \mathbb{R}$,

$$\mathbb{E}[h(\mathcal{G}_n, o_n) | \mathcal{G}_n] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int h(G, o) d\mathbb{Q}(G, o),$$

where $o_n \in \mathcal{V}_n$ is chosen uniformly at random and $\xrightarrow[n \rightarrow \infty]{\mathbb{P}}$ denotes convergence in probability in \mathbb{R} .²

1 For all bounded continuous functions $h_1, h_2: \mathcal{G}_* \rightarrow \mathbb{R}$,

$$\mathbb{E}[h_1(\mathcal{G}_n, o_n) h_2(\mathcal{G}_n, \tilde{o}_n)] \xrightarrow[n \rightarrow \infty]{} \int h_1(G, o) d\mathbb{Q}(G, o) \int h_2(G, o) d\mathbb{Q}(G, o),$$

where $o_n, \tilde{o}_n \in \mathcal{V}_n$ represent two independently and uniformly chosen roots.

Proof. First note that, by [vdH24, Thm 2.15 b)], the characterisation (b) is equivalent to

$$|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{B_{\mathcal{G}_n}(v, k) = (H, o_H)\} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mathbb{Q}(B_{\mathcal{G}}(v, k) = (H, o_H)), \quad (5)$$

for all $k \in \mathbb{N}$ and $(H, o_H) \in \mathcal{G}_*$. Now, to see that (b) implies (a), note that

$$\begin{aligned} \mathbb{P}(d_{\mathcal{P}}(Q_n, \mathbb{Q}) > \varepsilon) &= \mathbb{P}(\exists (H, o_H) \in \mathcal{G}_*: Q_n(H, o_H) > \mathbb{Q}(\mathcal{H}_*^\varepsilon) + \varepsilon) \\ &\leq \sum_{(H, o_H) \in \tilde{\mathcal{G}}_*} \mathbb{P}(Q_n(H, o_H) - \mathbb{Q}(H, o_H) > \varepsilon/2), \end{aligned}$$

²Recall that local convergence in probability in particular implies that $\mathbb{E}[h(\mathcal{G}_n, o_n)] \rightarrow \int h(G, o) d\mathbb{Q}(G, o)$, as n to infinity, for all bounded continuous functions h . This characterises local weak convergence by the Portmanteau theorem. Further, since the limit on the right-hand side is deterministic, local convergence in probability is equivalent to convergence in distribution of all random variables $\mathbb{E}[h(\mathcal{G}_n, o_n) | \mathcal{G}_n]$ towards $\int h(G, o) d\mathbb{Q}(G, o)$. On the other hand, converge in probability on \mathcal{G}_* with respect to the local topology induced by d_* , is stronger than local convergence in probability.

where $\mathcal{H}_*^\varepsilon = \{(H', o_{H'}): d_*((H', o_{H'}), (H, o_H)) < \varepsilon\}$ and $\tilde{\mathcal{G}}_*$ denotes the countable and dense subset of \mathcal{G}_* consisting of all finite graphs. But, due to (5), the right-hand side tends to zero as n tends to infinity. On the other hand, for all $k \in \mathbb{N}$ and $(H, o_H) \in \mathcal{G}_*$,

$$\begin{aligned} \mathbb{P}\left(\left|\mathcal{V}_n\right|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{B_{\mathcal{G}_n}(v, k) = (H, o_H)\} - \mathbb{Q}(B_{\mathcal{G}}(v, k) = (H, o_H))\right| > \varepsilon\right) \\ \leq \mathbb{P}(d_{\mathcal{P}}(Q_n, \mathbb{Q}) > \varepsilon) \rightarrow 0 \end{aligned}$$

as n tends to infinity. The equivalence of (b) and (c) is presented, for example, in [LRW23, Lemma 2.8.]. \square

We next explain, how the contact process can be incorporated into the local convergence setup. The notion of *random networks* describing processes on random graphs that are invariant under graph isomorphisms was coined in [AL07]. A *network* is a rooted graph (G, o) together with two maps $\Xi_V: V \rightarrow \mathcal{S}$, $\Xi_E: E \rightarrow \mathcal{S}$ on edges and vertices. Here, \mathcal{S} is some metric mark space. We choose $\mathcal{S} = \mathcal{N}(\mathbb{R} \times \{1, -1\})$, the space of all locally finite point measures on \mathbb{R} marked by -1 or 1 . We turn the network into a random network by allowing the underlying graph \mathcal{G} to be random, and conditionally on $\mathcal{G} = G = (V, E)$, letting $(\Xi_E(e))_{e \in E}$ be an independent collection of $\text{Poisson}(2\lambda)$ -processes where each point carries mark 1 or -1 with probability $1/2$ and by letting $(\Xi_V(v))_{v \in V}$ be an independent collection of $\text{Poisson}(1)$ -processes where each point carries mark 1 . A realisation of the contact process on G with initial infections at $A \subset V$ is now obtained by considering infection paths induced by interpreting $(\Xi_E(e))_{e \in E}$ as infection events with orientations along edges and $(\Xi_V(v))_{v \in V}$ as recovery events on the vertices. This is the well-known *graphical representation* of the contact process. It is straightforward to see that the law \mathbf{P}_G^λ of the induced oriented percolation model is invariant under rooted isomorphisms of (G, o) . If (G, o) is selected by some random mechanism, we are therefore justified in interpreting the annealed law \mathbb{P}^λ as a distribution on random networks in the sense of [AL07], in particular we may write

$$\mathbb{P}^\lambda = \int \mathbf{P}_G^\lambda d\mathbb{Q}(G, o)$$

to designate the law of the contact process on a random graph with distribution $\mathbb{Q} \in \mathcal{P}(\mathcal{G}_*)$. A similar definition applies to sequences of random graphs and henceforth we will assume, without loss of generality, that graph sequence, limit graph, and all associated contact processes live on the same probability space and have distribution \mathbb{P}^λ . This is only for notational convenience, since the framework of local convergence in probability is flexible enough to deal with graphs sequences that do not converge on the same probability space, see [vdH24, Remark 2.12] for a discussion. We occasionally omit the parameter λ from the notation if we refer to distributional properties of the underlying graphs under \mathbb{P}^λ only.

We close this section with the proof of Lemma 1.9, which is independent of our main results.

Proof of Lemma 1.9. Let λ be such that ξ dies out almost surely. Fix $s \in [0, \infty)$ and let $\tau_R^{(n)}$ and τ_R denote the first times that a vertex at distance R from the root is infected in \mathcal{G}_n and \mathcal{G} , respectively. We have, for fixed R , that

$$\begin{aligned} & |\mathbb{P}^\lambda(\tau_\emptyset^{(n)}(o_n) > s) - \mathbb{P}^\lambda(\tau_\emptyset(o) > s)| \\ & \leq |\mathbb{P}^\lambda(\tau_\emptyset^{(n)}(o_n) > s, \tau_R^{(n)} = \infty) - \mathbb{P}^\lambda(\tau_\emptyset(o) > s, \tau_R = \infty)| \\ & \quad + |\mathbb{P}^\lambda(\tau_R < \infty) - \mathbb{P}^\lambda(\tau_R^{(n)} < \infty)|. \end{aligned} \tag{6}$$

By local weak convergence, the second term on the right hand side of (6) vanishes with n , since the event in question depends solely on the R -neighbourhood of the roots. Let us consider the first term on the right-hand side of (6). If we choose n sufficiently large, then \mathcal{G}_n and \mathcal{G} agree on an $(R + 1)$ -neighbourhood with high probability. On this event, the contact processes can be perfectly coupled up to extinction, if it does not leave the R -neighbourhood in the respective base graph. Combining both observations, we see that extinction times converge in distribution, which implies tightness.

Now assume that λ is such that ξ survives with positive probability ε . Choose a subsequence $(\mathcal{G}_{n(R)})_{R \in \mathbb{N}}$ by asking that $(\mathcal{G}_{n(R)}, o_{n(R)})$ and (\mathcal{G}, o) can be coupled up to depth R with probability exceeding $1 - \varepsilon/2$. Under the graph coupling, the infection dynamics can be perfectly coupled as well up to the first time a vertex outside the R -neighbourhoods of the roots are infected. Thus ξ still survives with probability at least ε to distance R from the root in $\mathcal{G}_{n(R)}$. In particular, the overall probability of the event $\bigcap_R \{\tau_R < \tau_\emptyset^{(n(R))}(o_{n(R)})\}$ is positive. This contradicts tightness, since $\tau_R \rightarrow \infty$ almost surely as $R \rightarrow \infty$ conditionally on survival. \square

2.2 Slow extinction of subcritical contact processes on scale-free spatial networks

In this section, we prove Theorem 1.4. To this end, we need to construct, for any given $\varepsilon > 0$, a sequence of graphs $(\mathcal{G}_n)_{n \in \mathbb{N}}$ that satisfies (3). We first introduce an auxiliary graph. Place n independent random variables uniformly on the one-dimensional torus $(-n/2, n/2]$ of volume n , order them from smallest value to largest (with respect to the interval) and denote by $X_{-n/2+1} < \dots < X_{n/2}$ the ordered sequence; we assume n to be even for notational convenience. We now assign each vertex X_i an independent radius R_i , drawn as an i.i.d. copy of a positive random variable R , and form the associated Gilbert graph by connecting X_i and X_j precisely if $d_n(X_i, X_j) < R_i + R_j$, where d_n denotes the torus metric. We denote the resulting graph as $\tilde{\mathcal{G}}_n$. The graph \mathcal{G}_n now has the discrete torus $\{-n/2 + 1, \dots, n/2\}$ as its vertex set and

$$\mathcal{E}_n := \{i \sim j : X_i \sim X_j \text{ in } \tilde{\mathcal{G}}_n\} \cup \{i \sim i+1 : i = -n/2+1, \dots, n/2-1\} \cup \{n/2 \sim 1-n/2\}$$

as its edge set, writing ' \sim ' to indicate neighbours. It is straightforward to deduce that the local limit in probability of this model is the augmented Boolean model \mathcal{G} , analysed in [JLM25], which follows the same construction but with the Gilbert graph on a Poisson process on the real line as its auxiliary graph, which is mapped on \mathbb{Z} . Particularly, [JLM25] establishes that $\lambda_1(\mathcal{G}) > 0$ for the limiting graph, whenever R has finite expectation.

Proof of Theorem 1.4. The proof is based on a comparison of the survival time on the graph \mathcal{G}_n with the time the process survives on the star graph \mathcal{S}_n^* , induced by the vertex of maximal degree. To this end, let us first recapitulate some known results on the survival time on star graphs. Let $\mathcal{S}^{(k)}$ be the star graph with k leaves and centre o . The first result about the survival time on $\mathcal{S}^{(k)}$ is [BBCS, Lemma 5.2], stating that the infection survives, with high probability, exponentially long in the number of leaves. We use a refined version of this statement. First, by [SV, Lemma 2.5], we have

$$\mathbf{E}_{\mathcal{S}^{(k)}}^\lambda [\tau_\emptyset(\mathcal{S}^{(k)})] \geq e^{2c_\lambda k},$$

where $c_\lambda > 0$ is a λ -dependent constant. Secondly, we have, due to [Val24, Lemma 2.13], that

$$\mathbf{P}_{\mathcal{S}^{(k)}}^\lambda(\tau_\emptyset(\mathcal{S}^{(k)}) \leq t) \leq \frac{t}{\mathbf{E}_{\mathcal{S}^{(k)}}^\lambda[\tau_\emptyset(\mathcal{S}^{(k)})]}.$$

Combining both results and choosing $t = e^{c_\lambda k}$, we thus infer

$$\mathbf{P}_{\mathcal{S}^{(k)}}^\lambda(\tau_\emptyset(\mathcal{S}^{(k)}) > e^{c_\lambda k}) \geq 1 - e^{-c_\lambda k}. \quad (7)$$

Now, fix $\varepsilon > 0$, choose $1 < p < 1 + \varepsilon$, and consider the augmented Gilbert graph \mathcal{G}_n , constructed above, with radius distribution

$$\mathbb{P}(R > x) = (x \log^p(x))^{-1}. \quad (8)$$

Note that R has finite first moment as $p > 1$. By independence of the radii, we infer for the largest radius $R_n^* := \max\{R_{1-n/2}, \dots, R_{n/2}\}$, any $K > 0$ and large enough n that

$$\mathbb{P}(R_n^* \leq \frac{Kn}{\log^{1+\varepsilon}(n)}) = \left(1 - \frac{\log^{1+\varepsilon}(n)}{Kn \log^p(Kn/\log^{1+\varepsilon}(n))}\right)^n \leq e^{-\log^{1+\varepsilon-p}(n)/K},$$

which tends to zero, as $p < 1 + \varepsilon$. Put differently, $R_n^* > Kn/\log^{1+\varepsilon}(n)$, for any K , with high probability. Moreover, by construction, the degree of X_n^* , the vertex associated with R_n^* , has degree lower bounded by a Binomial number with parameters n and $2R_n^*/n$. Writing $X_{n,p}$ for a binomial with parameters n and p , Chernoff's inequality thus yields, for sufficiently large n ,

$$\begin{aligned} \mathbb{P}(\deg(X_n^*) > \frac{Kn}{\log^{1+\varepsilon}(n)}) &= \mathbb{E}[\mathbb{P}(\deg(X_n^*) > \frac{Kn}{\log^{1+\varepsilon}(n)}) \mid R_n^*] \\ &\geq \mathbb{E}[\mathbb{P}(X_{n, 2R_n^*/n} > \frac{Kn}{\log^{1+\varepsilon}(n)}) \mathbb{1}\{R_n^* > \frac{Kn}{\log^{1+\varepsilon}(n)}\}] \\ &\geq (1 - e^{-Kn/(4 \log^{1+\varepsilon}(n))}) \mathbb{P}(R_n^* > \frac{Kn}{\log^{1+\varepsilon}(n)}) \\ &\geq \left(1 - e^{-Kn/(4 \log^{1+\varepsilon}(n))}\right) (1 - e^{-\log^{1+\varepsilon-p}(n)/K}). \end{aligned}$$

Combining this with (7), and writing \mathcal{S}_n^* for the star graph induced by X_n^* and its neighbours, we have for $K = 1/c_\lambda$,

$$\begin{aligned} \mathbb{P}^\lambda(\tau_\emptyset(\mathcal{Y}_n) > e^{n/\log^{1+\varepsilon}(n)}) &\geq \mathbb{P}^\lambda(\tau_\emptyset(\mathcal{S}_n^*) > e^{n/\log^{1+\varepsilon}(n)}) \\ &\geq \mathbb{E}[\mathbb{1}\{\deg(X^*) \geq \frac{n}{c_\lambda \log^{1+\varepsilon}(n)}\} \mathbf{P}_{\mathcal{S}_n^*}^\lambda(\tau_\emptyset(\mathcal{S}_n^*) \geq e^{n/\log^{1+\varepsilon}(n)})] \\ &\geq \left(1 - e^{-\frac{n}{4c_\lambda \log^{1+\varepsilon}(n)}}\right) (1 - e^{-c_\lambda \log^{1+\varepsilon-p}(n)}) (1 - e^{-n/\log^{1+\varepsilon}(n)}) \xrightarrow{n \rightarrow \infty} 1, \end{aligned}$$

as desired. In particular, this holds true for any $\lambda \in (0, \lambda_1(\mathcal{G}))$ and hence $\lambda_1(\mathcal{G}) > \lambda_-(\mathcal{G}_n)_{n \in \mathbb{N}} = 0$. \square

Remark 2.2. The choice of the distribution in (8) can be adapted to yield examples in which the survival time is even closer to $e^{\Theta(|\mathcal{Y}_n|)}$, for instance, by setting

$$\mathbb{P}(R > x) \sim (x \log(x) \log^2(\log(x)))^{-1}.$$

In particular, the resulting graph would satisfy the lower bound in Theorem 1.4 for all ε simultaneously. We have chosen the particular scaling in (8) to illustrate that we can match the best known *universal* lower bound on supercritical extinction times given in [SV] with subcritical extinction times on certain classes of graphs.

2.3 Upper bounds on the metastable density from local limit

Before we provide the proof of Theorem 1.3, we introduce some notation and auxiliary results. Throughout this section, let $((\mathcal{G}_n, o_n))_{n \in \mathbb{N}}$ and its local limit (\mathcal{G}, o) with distribution \mathbb{Q} be given. As usually, o_n denotes a uniformly chosen vertex here. Fix $\lambda > 0$, set $\eta = \eta_\lambda(\mathbb{Q})$ and let $\tau_R^{(n)}(v)$ denote the first time that the infection started in $v \in \mathcal{V}_n$ reaches a vertex of graph distance R from v in \mathcal{G}_n . Let further $\tau_R(v), v \in \mathcal{V}$, denote the corresponding quantities for the rooted limit graph (\mathcal{G}, o) . Now define

$$\eta_{\geq R} = \mathbb{E}[\mathbb{P}_{\mathcal{G}}^\lambda(\tau_R(o) < \infty)], \quad R \in \mathbb{N},$$

and

$$Z_{\geq R} = Z_{\geq R}(n) = \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\tau_R^{(n)}(v) < \infty\}, \quad n \in \mathbb{N}.$$

The proof of Theorem 1.3 relies on the following two auxiliary statements.

Proposition 2.3. *Assume that $\mathcal{G}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} (\mathcal{G}, o)$. Then,*

$$|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) > t(n)\} \xrightarrow[n \rightarrow \infty]{\mathbb{P}^\lambda} 0,$$

for any $R \in \mathbb{N}$ and any diverging sequence $(t(n))_{n \in \mathbb{N}}$.

Proposition 2.4. *Assume that $\mathcal{G}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} (\mathcal{G}, o)$ where (\mathcal{G}, o) is distributed according to some extremal measure \mathbb{Q} . Then we have, for any $R \in \mathbb{N}$, that*

$$|\mathcal{V}_n|^{-1} Z_{\geq R} \xrightarrow[n \rightarrow \infty]{\mathbb{P}^\lambda} \eta_{\geq R}.$$

Before we prove Propositions 2.3 and 2.4, we show how they imply Theorem 1.3.

Proof of Theorem 1.3. By additivity, we have that

$$\mathbb{P}^\lambda(|\xi_{t(n)}^{\mathcal{V}_n}| \in \cdot) = \mathbb{P}^\lambda\left(\sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} \in \cdot\right)$$

and hence the assertion of the theorem is equivalent to showing

$$\left(|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} - \eta\right) \vee 0 \xrightarrow[n \rightarrow \infty]{\mathbb{P}^\lambda} 0. \quad (9)$$

Observe that, on one hand,

$$\eta = \mathbb{P}^\lambda\left(\bigcap_{R \in \mathbb{N}} \{\tau_R(o) < \infty\}\right) = \lim_{R \rightarrow \infty} \eta_{\geq R},$$

and hence, for any given ε , we may choose R_ε so large that, for any $R > R_\varepsilon$,

$$|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} \leq \eta_{\geq R} + \varepsilon/2 \quad (10)$$

implies

$$|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} \leq \eta + \varepsilon.$$

On the other hand, the bound

$$\begin{aligned} |\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} &= |\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) \leq t(n)\} \\ &\quad + |\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) > t(n)\} \\ &\leq |\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\tau_R^{(n)}(v) < \infty\} \\ &\quad + |\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) > t(n)\}, \end{aligned}$$

together with Proposition 2.3 tells us that

$$\left(|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} - |\mathcal{V}_n|^{-1} Z_{\geq R} \right) \vee 0 \xrightarrow[n \rightarrow \infty]{\mathbb{P}^\lambda} 0.$$

for any $R \in \mathbb{N}$. Choosing $R > R_\varepsilon$ and applying Proposition 2.4 thus yields that (10) occurs with probability tending to 1 as $n \rightarrow \infty$, which in turn establishes (9) and concludes the proof. \square

It remains to prove the two supporting results.

Proof of Proposition 2.3. For all $\varepsilon > 0$, we have

$$\begin{aligned} \mathbb{P}^\lambda \left(|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) > t(n)\} > \varepsilon \right) \\ &\leq \frac{1}{\varepsilon} \mathbb{E}^\lambda \left[|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) > t(n)\} \right] \\ &\leq \frac{1}{\varepsilon} \mathbb{E}^\lambda \left[|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} f_{t(n)}(B_{\mathcal{G}_n}(v, R)) \right] \\ &= \frac{1}{\varepsilon} \sum_{(H, o_H) \in \mathcal{G}_*} f_{t(n)}(H, o_H) \mathbb{P}(B_{\mathcal{G}_n}(o_n, R) = (H, o_H)), \end{aligned}$$

where

$$f_{t(n)}(G, o) := \mathbf{P}_G^\lambda(\xi_s^o \neq \emptyset \text{ for all } s \leq t(n)), \quad (G, o) \in \mathcal{G}_*.$$

By the assumption of local convergence in probability, the distributions of $B_{\mathcal{G}_n}(o_n, R)$ converge to the distribution of $B_{\mathcal{G}}(o, R)$. Since the limiting graph is locally finite, $\mathbb{P}(B_{\mathcal{G}}(o, R) \in \cdot)$ is a probability measure on \mathcal{G}_* and hence, by Prokhorov's theorem, the distributions of $(B_{\mathcal{G}_n}(o_n, R))_{n \geq 1}$ are tight. Hence, for all $\delta > 0$, there exists a finite set $\mathcal{A} \subset \mathcal{G}_*$ such that

$$\sup_{n \geq 1} \sum_{(H, o_H) \notin \mathcal{A}} \mathbb{P}^\lambda(B_{\mathcal{G}_n}(o_n, R) = (H, o_H)) \leq \delta.$$

We thus conclude that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{P}^\lambda \left(|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) > t(n)\} > \varepsilon \right) \\ & \leq \limsup_{n \rightarrow \infty} \sum_{(H, o_H) \in \mathcal{A}} f_{t(n)}(H, o_H) \mathbb{P}(B_{\mathcal{G}_n}(o_n, R) = (H, o_H)) + \frac{\delta}{\varepsilon} = \frac{\delta}{\varepsilon}, \end{aligned}$$

since $\lim_{n \rightarrow \infty} \mathbf{P}_H^\lambda(\xi_s^{o_H} \neq \emptyset \text{ for all } \forall s \leq t(n)) = 0$ for all finite H . This gives the result. \square

Proof of Proposition 2.4. We have that

$$\lim_{n \rightarrow \infty} \mathbb{E}^\lambda [|\mathcal{V}_n|^{-1} Z_{\geq R} - \eta_R] = \lim_{n \rightarrow \infty} \mathbb{E}[\mathbf{P}_{\mathcal{G}_n}^\lambda(\tau_R^{(n)}(o_n) < \infty) | \mathcal{G}_n] - \eta_R = 0,$$

by Lemma 2.1(b), hence the first moments asymptotically agree under \mathbb{P}^λ . On the other hand,

$$\mathbb{E}^\lambda [|\mathcal{V}_n|^{-2} Z_{\geq R}^2 | \mathcal{G}_n] = \mathbb{E}[\mathbf{P}_{\mathcal{G}_n}^\lambda(\tau_R^{(n)}(o_n) < \infty, \tau_R^{(n)}(o'_n) < \infty) | \mathcal{G}_n],$$

where (o_n, o'_n) is uniformly chosen (with replacement) from $\mathcal{V}_n \times \mathcal{V}_n$. Since the events $\{\tau_R^{(n)}(o_n) < \infty\}$ and $\{\tau_R^{(n)}(o'_n) < \infty\}$ are measurable with respect to the marks Ξ_V, Ξ_E of the random network and the graph inside a radius of $R + 1$ around the respective root, the right-hand side converges to $\eta_{\geq R}^2$ as $n \rightarrow \infty$ by Lemma 2.1(c). Combining the first and second moment limits, yields that the conditional variance of $Z_{\geq R} / |\mathcal{V}_n|$, given \mathcal{G}_n , vanishes, implying that $Z_{\geq R} / |\mathcal{V}_n|$ converges to its expectation $\eta_{\geq R}$ in probability and in L_1 . \square

2.4 Lower bounds on the metastable density from local limits

In this section, we prove Proposition 1.7.

Proof of Proposition 1.7. Assume that $\mathcal{G}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} (\mathcal{G}, o)$ and let $(t(n))_{n \in \mathbb{N}}$ denote a sequence of diverging times. We begin by showing that

$$|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} \xrightarrow[n \rightarrow \infty]{\mathbb{P}^\lambda} \eta_\lambda(\mathbf{Q})$$

implies

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}^\lambda(\xi_{t(n)}^{o_n} = \emptyset, \tau_R^{(n)}(o_n) < t(n)) = 0.$$

To this end, observe that, for all $\varepsilon > 0$,

$$\begin{aligned} & \mathbb{E}^\lambda \left[|\mathcal{V}_n|^{-1} \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v = \emptyset, \tau_R^{(n)}(v) \leq t(n)\} \right] \\ & \leq \varepsilon + \mathbb{P}^\lambda \left(\sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v = \emptyset, \tau_R^{(n)}(v) \leq t(n)\} > |\mathcal{V}_n| \varepsilon \right). \end{aligned} \tag{11}$$

The random variable of the second term can be rewritten as

$$\begin{aligned} & \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v = \emptyset, \tau_R^{(n)}(v) \leq t(n)\} \\ & \leq \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\tau_R^{(n)} < \infty\} - \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) \leq t(n)\} \\ & = \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\tau_R^{(n)} < \infty\} - \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} + \sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) > t(n)\}. \end{aligned}$$

After dividing both sides by $|\mathcal{V}_n|$, the right-hand side converges in probability to $\eta_R - \eta(Q)$, as $n \rightarrow \infty$ by assumption and Propositions 2.3 and 2.4. As $\eta_R \downarrow \eta$, as $R \rightarrow \infty$, the expectation in (11) can be made arbitrarily small by choosing n and R large enough, proving the claimed implication.

It remains to prove the other implication, i.e. that (4) implies, for all $\varepsilon > 0$,

$$\mathbb{P}^\lambda(|\xi_{t(n)}^{\mathcal{V}_n}| < |\mathcal{V}_n|(\eta - \varepsilon)) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (12)$$

Using additivity, this is equivalent to

$$\mathbb{P}^\lambda\left(\sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} < |\mathcal{V}_n|(\eta - \varepsilon)\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

To obtain this statement, note that

$$\begin{aligned} & \mathbb{P}^\lambda\left(\sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset\} < |\mathcal{V}_n|(\eta - \varepsilon)\right) \\ & \leq \mathbb{P}^\lambda\left(\sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v \neq \emptyset, \tau_R^{(n)}(v) < \infty\} < |\mathcal{V}_n|(\eta - \varepsilon)\right) \\ & \leq \mathbb{P}^\lambda\left(\sum_{v \in \mathcal{V}_n} \mathbb{1}\{\tau_R^{(n)}(v) < \infty\} < |\mathcal{V}_n|(\eta - \varepsilon/2)\right) \\ & \quad + \mathbb{P}^\lambda\left(\sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v = \emptyset, \tau_R^{(n)}(v) < \infty\} > |\mathcal{V}_n|\varepsilon/2\right), \end{aligned}$$

where the first summand tends to zero by Proposition 2.4. For the second summand, we note that $\xi_{t(n)}^v = \emptyset$ and $\tau_R^{(n)}(v) < \infty$ together imply $\tau_R^{(n)}(v) < t(n)$. A application of Markov's inequality then yields

$$\begin{aligned} & \mathbf{P}_{\mathcal{G}_n}^\lambda\left(\sum_{v \in \mathcal{V}_n} \mathbb{1}\{\xi_{t(n)}^v = \emptyset, \tau_R^{(n)}(v) < t(n)\} > \varepsilon|\mathcal{V}_n|\right) \\ & \leq \frac{1}{\varepsilon|\mathcal{V}_n|} \mathbf{E}_{\mathcal{G}_n}^\lambda\left[\#\{v \in \mathcal{V}_n : \tau_R^{(n)}(v) \leq t(n)\}\right] \\ & \leq \frac{1}{\varepsilon} \mathbf{P}_{\mathcal{G}_n}^\lambda(\xi_{t(n)}^{o_n} = \emptyset, \tau_R^{(n)}(o_n) < t(n)). \end{aligned}$$

Taking expectations on both sides, we hence see that (4) is a sufficient criterion for (12). \square

2.5 Fast extinction by absence of metastability

In this section, we prove Theorem 1.5. We begin with an auxiliary result.

Lemma 2.5. *Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be sparse with $\mathcal{G}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} (\mathcal{G}, o)$. Then, for every $\delta > 0$, there exists $\varepsilon > 0$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\max_{I \subset \mathcal{V}_n: |I| < \varepsilon |\mathcal{V}_n|} \sum_{v \in I} \deg_{\mathcal{G}_n}(v) > \delta |\mathcal{V}_n| \right) = 0.$$

Proof. Fix $\delta > 0$ and denote $|\mathcal{V}_n| = N$. Let $D_1^{(n)} \leq D_2^{(n)} \leq \dots \leq D_N^{(n)}$ denote the vertex degrees in \mathcal{G}_n ordered by magnitude. By sparsity and the convergence assumption, it follows that, for any $\varepsilon \in [0, 1]$,

$$\infty > \mathbb{E}[\deg_{\mathcal{G}}(o)] - \lim_{n \rightarrow \infty} N^{-1} \sum_{i=[\varepsilon N]+1}^N D_i^{(n)} = \lim_{n \rightarrow \infty} N^{-1} \sum_{i=1}^{[\varepsilon N]} D_i^{(n)} =: \sigma_\varepsilon,$$

where the convergence is to be understood in probability. In particular, the deterministic term σ_ε on the right-hand side vanishes as $\varepsilon \rightarrow 0$. Writing $\Sigma_\varepsilon(n) := N^{-1} \sum_{i=1}^{[\varepsilon N]} D_i^{(n)}$, we obtain for $\varepsilon \in (0, 1)$ fixed

$$\lim_{n \rightarrow \infty} \mathbb{P}(\Sigma_\varepsilon(n) > 2\sigma_\varepsilon) = 0.$$

The desired result now follows upon choosing $\varepsilon = \varepsilon(\delta)$ such that $\sigma_\varepsilon < \delta/2$. \square

Proof of Theorem 1.5. As $\lambda_\rho \geq \lambda_+$ by Theorem 1.3, it suffices to show $\lambda_\rho = \lambda_+$. Clearly, $\lambda_\rho \geq \lambda_+$ by definition, and it hence remains to show $\lambda_\rho \leq \lambda_+$. To this end, pick $\lambda < \lambda_\rho$ and show that $\xi^{\mathcal{Y}_n}$ does not survive on the exponential scale. Let $c > 0$ be arbitrary, denote by $T = T_{n,c} = e^{c|\mathcal{Y}_n|}$ the relevant time scale and let

$$r(\varepsilon, T) = T^{-1} \int_0^T \mathbb{1}\{|\xi_t^{\mathcal{Y}_n}| \leq \varepsilon |\mathcal{Y}_n|\} dt$$

represent the proportion of time that the infected set spends in low density states. Let

$$\tau_1 = \inf \{t > 0: |\xi_t^{\mathcal{Y}_n}| \leq \varepsilon |\mathcal{Y}_n|\}$$

and denote

$$\tau_k = \inf \{t > \tau_{k-1} + 1: |\xi_t^{\mathcal{Y}_n}| \leq \varepsilon |\mathcal{Y}_n|\}, \quad k \geq 2.$$

Define further $K = \max\{k: \tau_k \leq T-1\}$. Fix an arbitrary $\delta > 0$ and let ε be as in Lemma 2.5. Then, at each time τ_k , $k \leq K$, the total size of the infected set is at most $\varepsilon |\mathcal{Y}_n|$ and, consequently, has total degree no larger than $\delta |\mathcal{Y}_n|$ with probability $1 - o(1)$. Hence, conditionally on the evolution of the process up to the stopping time τ_k , the probability of immediate extinction just after τ_k , meaning that a recovery occurs at every vertex in the time interval $[\tau_k, \tau_k + 1)$ but no infection, is at least

$$(1 - e^{-1})^{|\xi_{\tau_k}|} e^{-2\lambda \sum_{v \in \xi_{\tau_k}} \deg_{\mathcal{G}_n}(v)} \geq (1 - e^{-1})^{\varepsilon |\mathcal{Y}_n|} e^{-2\lambda \delta |\mathcal{Y}_n|},$$

where the inequality holds with probability exceeding $1 - o(1)$ uniformly for all $k \leq K$. More precisely, if E_n denotes the exceptional event in Lemma 2.5, then we obtain from the strong Markov property and a coupling of the contact process to a geometric experiment,

$$\begin{aligned} \mathbb{P}^\lambda(\{\tau_\emptyset^{(n)}(\mathcal{V}_n) > T\} \mid |\mathcal{V}_n|) &\leq \mathbb{P}(E_n \mid |\mathcal{V}_n|) + \mathbb{P}^\lambda(K \leq T/3 \mid |\mathcal{V}_n|) \\ &\quad + \mathbb{E}^\lambda[\mathbb{1}\{E_n^c\} \mathbb{1}\{\tau_\emptyset^{(n)}(\mathcal{V}_n) > T\} \mathbb{1}\{K > T/3\} \mid |\mathcal{V}_n|] \\ &\leq \mathbb{P}(E_n \mid |\mathcal{V}_n|) \\ &\quad + \mathbb{P}^\lambda(K \leq T/3 \mid |\mathcal{V}_n|) + 3T^{-1} e^{-\log(1-e^{-1})\varepsilon|\mathcal{V}_n|+2\lambda\delta|\mathcal{V}_n|}. \end{aligned}$$

Here, the bound on the last term follows from the fact that $\mathbb{P}(G > t) \leq (pt)^{-1}$ for a Geometric(p) random variable G . Taking expectations yields

$$\mathbb{P}^\lambda(\tau_\emptyset^{(n)}(\mathcal{V}_n) > T) \leq \mathbb{P}(E_n) + \mathbb{P}^\lambda(K \leq T/3) + \mathbb{E}[3T^{-1} e^{-\log(1-e^{-1})\varepsilon|\mathcal{V}_n|+2\lambda\delta|\mathcal{V}_n|}]. \quad (13)$$

Decreasing the values of ε and δ if needed, the last term of the right-hand side vanishes since $T = e^{c|\mathcal{V}_n|}$ with fixed c , as $n \rightarrow \infty$, while the first term vanishes by Lemma 2.5. Hence it remains to show that $\gamma_n = \mathbb{P}^\lambda(K \leq T/3)$ converges to 0 as $n \rightarrow \infty$.

Note that, on the event $\{K \leq T/3\}$, the term $1 - r(\varepsilon, T)$ is bounded from below by $2/3$. Let W be uniform on $[0, T]$ under \mathbb{P}^λ and independent of the graph sequence and contact process. Denote by

$$\rho(t) = |\mathcal{V}_n|^{-1} |\xi_t^{\mathcal{V}_n}|, t \geq 0,$$

the density process and by \mathcal{I}_n the σ -field generated by \mathcal{G}_n and the corresponding edge marks in the network construction of ξ . We have

$$\mathbb{P}^\lambda(\rho(W) > \varepsilon) \geq \mathbb{E}^\lambda[\mathbb{1}\{K \leq T/3\} \mathbb{P}(\rho(W) > \varepsilon \mid \mathcal{I}_n)] \geq \frac{2}{3} \mathbb{P}^\lambda(K \leq T/3) = \frac{2}{3} \gamma_n.$$

On the other hand,

$$\mathbb{P}^\lambda(\rho(W) > \varepsilon) \leq \frac{1}{\varepsilon} \mathbb{E}^\lambda[\rho(W)],$$

and by additivity of ξ we have

$$\mathbb{E}^\lambda[\rho(W)] \leq \mathbb{P}^\lambda(\tau_\emptyset^{(n)}(o_n) > W) \leq \mathbb{P}^\lambda(W \leq \sqrt{T}) + \mathbb{P}^\lambda(\tau_\emptyset^{(n)}(o_n) > \sqrt{T}).$$

The first term vanishes by choice of W . Note that duality implies that the second term equals

$$\mathbb{E}[\mathbf{P}_{\mathcal{G}_n}^\lambda(\xi_{\sqrt{T}}^{\mathcal{V}_n} \cap \{o_n\} \neq \emptyset)] = \mathbb{E}^\lambda[\rho(\sqrt{T})],$$

as o_n is chosen uniformly. Since $\lambda < \lambda_\rho$, we have $\rho(\sqrt{T}) \rightarrow 0$ in probability, and the expectation on the right-hand side thus vanishes by dominated convergence, as $\rho \leq 1$. Hence, $\lim_{n \rightarrow \infty} \gamma_n = 0$ and the result follows from (13). \square

3 References

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