Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

A $T-$coercivity approach to the nonlinear Stokes equations

Cristian Cárcamo[1], Patrick Ciarlet Jr.[2]

submitted: January 16, 2025

[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin, Germany
E-Mail: cristian.carcamosanchez@wias-berlin.de

[2] POEMS, CNRS, Inria, ENSTA Paris
Institut Polytechnique de Paris
828 Boulevard des Maréchaux, 91120 Palaiseau, France
E-mail: patrick.ciarlet@ensta-paris.fr

No. 3167
Berlin 2025

2020 *Mathematics Subject Classification.* 65N30, 76S05, 74F10, 65N15.

Key words and phrases. Nonlinear Stokes, quasi-newtonian flow, T-coercivity .

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

A $T-$coercivity approach to the nonlinear Stokes equations

Cristian Cárcamo, Patrick Ciarlet Jr.

Abstract

We address the nonlinear Stokes problem with Dirichlet boundary conditions, introducing additional variables into the standard formulation to accommodate solutions with reduced regularity requirements. To ground this analysis, we first review relevant preliminary results, emphasizing the significance of achieving $T-$coercivity in the context of nonlinear Stokes flows. We then introduce a specially designed operator T, proving its bijectivity and showing that it induces coercivity when applied to the test function space. This result provides a rigorous foundation for solving the quasi-Newtonian Stokes problem with minimal regularity constraint and also sets up the $T-$coercivity as an alternative to the well-posedness of the nonlinear Stokes problems.

1 Introduction

Nonlinear Stokes flow is common in industry and nature, being of special interest in areas such as geology and glaciology, commonly called the $p-$Stokes problem due to the viscosity depends on the gradient of the velocity or the strain rate tensor with an exponent $p \in (1, \infty)$, which is the parameter that determines the non-linearity of the material [1, 11]. This problem has been widely studied from differents perspectives. Some of them are by using Newton linearization [12], and clasic mixed formulation [13], among others. On the other hand, the $T-$coercivity is a tool, that has been widely applied in the last years as a great alternative to prove the well-posedness of some class of problems, most of them with linear characteristics [5, 6]. In many cases, this strategy plays an important role or even is the key to proving the existence of solutions [3, 7], despite the challenge that implies setting up the operator T. Another important aspect to consider is the regularity of the solution, above all when a problem is studied from a numerical point of view where usually in practice the solutions have low regularity. In this regard, [9] introduces an alternative path to deal with quasi-Newtonian fluid by means of the introduction of new variables in the continuous problem, being this path on which we will base our main result. Therefore, our contribution aim to set up an operator T whereby we will prove the existence and uniqueness of the solution for a nonlinear Stokes problem in its weak form, showing that this problem satisfy T-coerciveness.

Let us consider a domain $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$, an open bounded domain with Lipschitz continuous boundary $\partial\Omega = \Gamma$. A Dirichlet boundary condition is imposed on Γ. We employ conventional notation for Lebesgue spaces $L^p(\mathcal{O})$, with norm $\|\cdot\|_{L^p(\mathcal{O})}$ and $\|\cdot\|_{0,\mathcal{O}}$ for $p = 2$ and inner product $(\cdot,\cdot)_{\mathcal{O}}$, and Sobolev spaces $H^m(\mathcal{O})$ with norm $\|\cdot\|_{H^m(\mathcal{O})}$ and seminorm $|\cdot|_{H^m(\mathcal{O})}$, where $\mathcal{O} \in \{\Omega, \Gamma\}$. We will interpret these same spaces as vector spaces whenever they are written in bold, and as tensor spaces when written in calligraphic font.

We are interested to study the nonlinear Stokes equations given by

$$\left\{ \begin{array}{rll} -\mathrm{div}\boldsymbol{\sigma} & = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\boldsymbol{u} & = 0 & \text{in } \Omega, \\ \boldsymbol{u} & = \boldsymbol{h} & \text{on } \Gamma \end{array} \right. \tag{1.1}$$

DOI 10.20347/WIAS.PREPRINT.3167

where $\boldsymbol{\sigma}$ is the nonlinear stress tensor defined as

$$\boldsymbol{\sigma} = \nu(|\nabla \boldsymbol{u}|)\nabla \boldsymbol{u} - p\mathbb{I},$$

$\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is the velocity field, $p : \Omega \to \mathbb{R}$ is the pressure field, $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ represents the nonlinear kinematic viscosity, $|\cdot|$ corresponds to the Euclidean norm in $\mathbb{R}^{d\times d}$, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is a given source term, and $\boldsymbol{h} \in \boldsymbol{H}^{1/2}(\Omega)$. Furthermore, for any $\boldsymbol{\tau} = (\tau_{ij})$, $\boldsymbol{\zeta} = (\zeta_{ij}) \in \mathbb{R}^{d\times d}$ we use the notations $\mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^d \tau_{ii}$, $\boldsymbol{\tau}^t = (\tau_{ij})$ and $(\boldsymbol{\tau}, \boldsymbol{\zeta})_{\boldsymbol{L}^2(\Omega)^{d\times d}} = \boldsymbol{\tau} : \boldsymbol{\zeta} = \mathrm{tr}(\boldsymbol{\tau}\,\boldsymbol{\zeta}^t)$.

We define the mapping $\nu_{ij} : \mathbb{R}^{d\times d} \to \mathbb{R}$ as $\nu_{ij}(\boldsymbol{s}) := \nu(|\boldsymbol{s}|)\boldsymbol{s}_{ij}$ for all $\boldsymbol{s} := (s_{ij}) \in \mathbb{R}^{d\times d}$ and for all $i, j \in \{1, \ldots, d\}$. We then define the tensor $\boldsymbol{\nu} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ by $\boldsymbol{\nu}(\boldsymbol{s}) := (\nu_{ij}(\boldsymbol{s}))$ for every $\boldsymbol{s} \in \mathbb{R}^{d\times d}$. Throughout this paper, we assume that $\boldsymbol{\nu}$ is of class C^1 and that there exist constants $C_1, C_2 > 0$ such that, for all $\boldsymbol{s} := (s_{ij})$ and $\boldsymbol{r} := (r_{ij}) \in \mathbb{R}^{d\times d}$, the following holds:

$$\begin{aligned} &|\nu_{ij}(\boldsymbol{s})| \le C_1 \|\boldsymbol{s}\|_{\mathbb{R}^{d\times d}}, \quad \left|\frac{\partial}{\partial s_{kl}}\nu_{ij}(\boldsymbol{s})\right| \le C_1 \ \forall i,j,k,l \in \{1,\ldots,d\} \\ &\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial s_{kl}}\nu(\boldsymbol{s}) r_{ij} r_{kl} \ge C_2 \|\boldsymbol{r}\|^2_{\mathbb{R}^{2\times 2}}. \end{aligned} \tag{1.2}$$

Now, following the ideas developed in [9], introducing the variables $\boldsymbol{t} = \nabla \boldsymbol{u}$, and taking $\boldsymbol{\sigma}$ as a variable the problem (1.1) is reduced to

$$\left\{ \begin{aligned} \boldsymbol{t} - \nabla \boldsymbol{u} &= \boldsymbol{0} && \text{in } \Omega, \\ \boldsymbol{\sigma} - \boldsymbol{\nu}(\boldsymbol{t}) + p\mathbb{I} &= \boldsymbol{0} && \text{in } \Omega, \\ -\mathrm{div}\boldsymbol{\sigma} &= \boldsymbol{f} && \text{in } \Omega, \\ \mathrm{tr}(\boldsymbol{t}) &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{h} && \text{on } \Gamma \end{aligned} \right. \tag{1.3}$$

Let us start introducing the spaces

$$\begin{aligned} &\boldsymbol{X} := \mathbb{L}^2(\Omega), \quad \boldsymbol{Y} := \boldsymbol{H}(\mathbf{div};\Omega) \times L^2(\Omega), \quad \text{and} \quad \boldsymbol{Z} := \boldsymbol{L}^2(\Omega) \times \mathbb{R} \\ &\boldsymbol{H} = \boldsymbol{H}_0(\mathbf{div};\Omega) = \{\tau \in \boldsymbol{H}(\mathbf{div};\Omega) : \int_\Omega \mathrm{tr}(\tau) = 0\} \end{aligned}$$

As next, we define the next operators

$$\begin{aligned} \boldsymbol{A} : \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}, &\quad \boldsymbol{A}(\boldsymbol{r}, \boldsymbol{s}) := (\boldsymbol{\nu}(\boldsymbol{r}), \boldsymbol{s})_\Omega \\ \boldsymbol{B}_1 : \boldsymbol{X} \times \boldsymbol{Y} \to \mathbb{R}, &\quad \boldsymbol{B}_1(\boldsymbol{r}, (\boldsymbol{\tau}, q)) := -(\boldsymbol{\tau}, \boldsymbol{r})_\Omega - (q, \mathrm{tr}(\boldsymbol{r}))_\Omega \\ \boldsymbol{B}_2 : \boldsymbol{Y} \times \boldsymbol{Z} \to \mathbb{R}, &\quad \boldsymbol{B}_2((\boldsymbol{\tau}, q), (\boldsymbol{v}, \eta)) := -(\boldsymbol{v}, \mathbf{div}\,\boldsymbol{\tau})_\Omega + (\eta, \mathrm{tr}(\boldsymbol{\tau}))_\Omega \\ \boldsymbol{F} : \boldsymbol{Z} \to \mathbb{R}, &\quad \boldsymbol{F}(\boldsymbol{v}, \boldsymbol{\tau}) := (\boldsymbol{f}, \boldsymbol{v})_\Omega \\ \boldsymbol{G} : \boldsymbol{Y} \to \mathbb{R}, &\quad \boldsymbol{G}(\boldsymbol{\tau}, q) := -\langle \boldsymbol{h}, \boldsymbol{\tau}\boldsymbol{n}\rangle_{\Gamma_D}. \end{aligned}$$

The variational formulation of Problem (1.3) reads:
Find $\vec{\boldsymbol{t}} = (\boldsymbol{t}, (\boldsymbol{\sigma}, p), (\boldsymbol{u}, \xi)) \in \boldsymbol{X} \times \boldsymbol{Y} \times \boldsymbol{Z}$ *such that*

$$\langle \mathcal{A}(\vec{\boldsymbol{t}}), \vec{\boldsymbol{s}}\rangle = \langle \mathcal{F}, \vec{\boldsymbol{s}}\rangle, \tag{1.4}$$

for all $\vec{s} = (\boldsymbol{s}, (\boldsymbol{\tau}, q), (\boldsymbol{v}, \eta))$, with

$$\begin{aligned}\langle \mathcal{A}(\vec{t}), \vec{s}\rangle &:= \boldsymbol{A}(\boldsymbol{t}, \boldsymbol{s}) + \boldsymbol{B}_1(\boldsymbol{s}, (\boldsymbol{\sigma}, p)) + \boldsymbol{B}_1(\boldsymbol{t}, (\boldsymbol{\tau}, q)) \\ &\quad + \boldsymbol{B}_2((\boldsymbol{\tau}, q), (\boldsymbol{u}, \xi)) + \boldsymbol{B}_2((\boldsymbol{\sigma}, p), (\boldsymbol{v}, \eta)) \\ \langle \mathcal{F}, \vec{s}\rangle &:= \boldsymbol{F}(\boldsymbol{v}, \boldsymbol{\tau}) + \boldsymbol{G}(\boldsymbol{\tau}, q)\end{aligned}$$

Now, in order to prove that the continuous problem has one solution we are going to start showing the continuity of the operators $\mathcal{A}$ and $\mathcal{F}$. For this purpose, we introduce the norm

$$\|\vec{s}\| := \left\{\|\boldsymbol{s}\|_{\boldsymbol{X}}^2 + \|(\boldsymbol{\tau}, q)\|_{\boldsymbol{Y}}^2 + \|(\boldsymbol{v}, \eta)\|_{\boldsymbol{Z}}^2\right\}^{1/2}.$$

2 Preliminaries

We aim to prove that (1.4) by using $\boldsymbol{T}-$coercivity. Hence, we need to introduce the next defintion and lemma (details are explained in [6]).

Definition 1. *Let V and W be two Hilbert spaces and $A(\cdot,\cdot)$ be a continuous and linear on the second component form over $V \times W$. It is $\boldsymbol{T}-$coercive if*

$$\exists T \in \mathcal{L}(V, W),\ \textit{bijective},\ \exists\, \alpha > 0,\ \forall\, v \in V,\ |A(v, T(v)) \geq \alpha \|v\|_V^2. \tag{2.5}$$

Notice that if (2.5) is fulfilled, the injectivity of $\boldsymbol{T}$ follows.

Theorem 2. *Let $A(\cdot,\cdot)$ be a continuous form over $V \times W$. The problem*

$$A(u, v) = F(v)$$

is well-posed if, and only if, the form $A(\cdot,\cdot)$ is $T-$coercive in the sense of the Definition (1)*.*

The next result guarantees the surjectivity of the operator $\mathbf{div} : \boldsymbol{H} \to \boldsymbol{L}^2(\Omega)$, necessary to define the operator $\boldsymbol{T}$.

Lemma 3. *There exists $\hat{\beta} > 0$ such that s*

$$\sup_{\boldsymbol{0}\neq\boldsymbol{\tau}\in\boldsymbol{H}} \frac{(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{v})_\Omega}{\|\boldsymbol{\tau}\|_{\boldsymbol{H}(\mathbf{div},\Omega)}} \geq \hat{\beta}\|\boldsymbol{v}\|_{L^2(\Omega)^d}, \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega). \tag{2.6}$$

Proof. To prove (2.6) we refer to inequality (3.4) in [10, Lemma 3.6]. □

The next results plays an important role to prove the well-possednes of the continuous problem (1.4).

Lemma 4. *The following statements hold:*

a) The nonlinear operator $\boldsymbol{A}$ is strongly monotone and Lipschitz continuous. That is, there exists constants C_m, C_l such that for all $\boldsymbol{t}, \boldsymbol{r}, \boldsymbol{s} \in \boldsymbol{X}$ it holds

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r}, \boldsymbol{r}-\boldsymbol{s}) - \boldsymbol{A}(\boldsymbol{s}, \boldsymbol{r}-\boldsymbol{s}) &\geq C_m\|\boldsymbol{r}-\boldsymbol{s}\|_{\boldsymbol{X}}^2 \\ |\boldsymbol{A}(\boldsymbol{t}, \boldsymbol{r}) - \boldsymbol{A}(\boldsymbol{s}, \boldsymbol{r})| &\leq C_l\,\|\boldsymbol{t}-\boldsymbol{s}\|_{\boldsymbol{X}}\|\boldsymbol{r}\|_{\boldsymbol{X}}\end{aligned} \tag{2.7}$$

b) *Let us define the space*

$$\tilde{\boldsymbol{Y}} = \{(\boldsymbol{\tau}, q) \in \boldsymbol{Y} : \mathbf{div}(\boldsymbol{\tau}) = 0 \text{ in } \Omega \text{ and } \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) dx = 0\}.$$

For all $(\boldsymbol{\tau}, q) \in \tilde{\boldsymbol{Y}}$ *there exists a positive constant* β_1 *such that*

$$\sup_{\mathbf{0} \neq \boldsymbol{s} \in \boldsymbol{X}} \frac{\boldsymbol{B}_1(\boldsymbol{s}, (\boldsymbol{\tau}, q))}{\|\boldsymbol{s}\|_{\boldsymbol{X}}} \geq \beta_1 \|(\boldsymbol{\tau}, q)\|_{\boldsymbol{Y}} \tag{2.8}$$

c) *For all* $(\boldsymbol{v}, \eta) \in \boldsymbol{Z}$ *there exists a positive constant* β_2 *dependent of* Ω*, such that*

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}, q) \in \boldsymbol{Y}} \frac{\boldsymbol{B}_2((\boldsymbol{\tau}, q), (\boldsymbol{v}, \eta))}{\|(\boldsymbol{\tau}, q)\|_{\boldsymbol{Y}}} \geq \beta_2 \|(\boldsymbol{v}, \eta)\|_{\boldsymbol{Z}} \tag{2.9}$$

Proof. For further details we refer to [9, Theo. 2.4]. □

It is important to mention the relevance of the assumptions described in (1.2), because without them It would be impossible to prove the result (2.7).

3 Well-posedness of the problem

Lemma 5. *There exists a positive constant* $C_{\mathcal{A}_0}$*, such that*

$$\langle \mathcal{A}_0(\vec{\boldsymbol{t}}), \vec{\boldsymbol{s}} \rangle \leq C_{\mathcal{A}_0} \|\vec{\boldsymbol{t}}\| \|\vec{\boldsymbol{s}}\|, \quad \text{and} \quad \langle \mathcal{F}_0, \vec{\boldsymbol{s}} \rangle \leq C_{\mathcal{F}_0} \left[\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \right] \|\vec{\boldsymbol{s}}\|.$$

Proof. By applying triangle and Cauchy-Schwarz inequalites we get

$$\begin{aligned} |\langle \mathcal{A}_0(\vec{\boldsymbol{t}}), \vec{\boldsymbol{s}} \rangle| \leq & C_1 \|\boldsymbol{t}\|_{\boldsymbol{X}} \|\boldsymbol{s}\|_{\boldsymbol{X}} + \|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathbf{div}, \Omega)} \|\boldsymbol{s}\|_{\boldsymbol{X}} + \sqrt{d} \|p\|_{L^2(\Omega)} \|\boldsymbol{s}\|_{\boldsymbol{X}} \\ & + \|\boldsymbol{\tau}\|_{\boldsymbol{H}(\mathbf{div}, \Omega)} \|\boldsymbol{t}\|_{\boldsymbol{X}} + \sqrt{d} \|q\|_{L^2(\Omega)} \|\boldsymbol{t}\|_{\boldsymbol{X}} + \|\boldsymbol{u}\|_{L^2(\Omega)^d} \|\boldsymbol{\tau}\|_{H(\mathbf{div}, \Omega)} \\ & + \sqrt{d} |\xi| \|\boldsymbol{\tau}\|_{H(\mathbf{div}, \Omega)} + \|\boldsymbol{v}\|_{L^2(\Omega)^d} \|\boldsymbol{\sigma}\|_{H(\mathbf{div}, \Omega)} + \sqrt{d} |\eta| \|\boldsymbol{\sigma}\|_{H(\mathbf{div}, \Omega)} \\ \leq & C_{\mathcal{A}_0} \|\vec{\boldsymbol{t}}\| \|\vec{\boldsymbol{s}}\|, \end{aligned}$$

where $C_{\mathcal{A}_0} = \max\{1, C_1, \sqrt{d}\}$ Analogously, by applying triangle, trace inequality en $\boldsymbol{H}(\mathbf{div}; \Omega)$ (see e.g. [4, 8]), and Cauchy-Schwarz in $\mathbb{R}^2$ we get

$$\begin{aligned} \langle \mathcal{F}_0, \vec{\boldsymbol{s}} \rangle \leq & |(\boldsymbol{f}, \boldsymbol{v})_\Omega| + |\langle \boldsymbol{h}, \boldsymbol{\tau n} \rangle_{\Gamma_D}| \\ \leq & \|\boldsymbol{f}\|_{L^2(\Omega)^d} \|\boldsymbol{v}\|_{L^2(\Omega)^d} + \|\boldsymbol{h}\|_{H^{\frac{1}{2}}(\Gamma)} \|\boldsymbol{\tau n}\|_{H^{-\frac{1}{2}}(\Gamma)} \\ \leq & \sqrt{2} \left[\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|\boldsymbol{h}\|_{H^{\frac{1}{2}}(\Gamma)} \right] \left[\|\boldsymbol{v}\|_{L^2(\Omega)^d} + \|\boldsymbol{\tau}\|_{\boldsymbol{H}(\mathbf{div}; \Omega)} \right] \\ \leq & C_{\mathcal{F}_0} \left[\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|\boldsymbol{h}\|_{H^{\frac{1}{2}}(\Gamma)} \right] \|\vec{\boldsymbol{s}}\|, \end{aligned}$$

where $C_{\mathcal{F}_0} = \sqrt{2}$. □

Next, inspired by [6] and in virtue of the previous results, the next theorem is established.

Theorem 6. *The form $\langle \mathcal{A}_0(\cdot), \cdot\rangle$ is $\boldsymbol{T}-$coercive.*

Proof. Before to formulate the operator T we are going to call the Lemma (4). We know that $\boldsymbol{B}_1$ satisfy an inf-sup condition in the space $\tilde{\boldsymbol{Y}}$, and so, following the ideas developed in [2, Lemma. 3.1] and [9, Theo. 2.4], let us consider the pair $(\boldsymbol{\sigma}', p')$ such that

$$\overline{\boldsymbol{s}} = \begin{cases} -(\boldsymbol{\sigma}' - \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}')\mathbb{I}), & \text{if} \quad \|p'|_{L^2(\Omega)} \leq \|\boldsymbol{\sigma}'\|_{H(\mathbf{div},\Omega)} \\ -p'\mathbb{I} + \boldsymbol{\sigma}', & \text{if} \quad \|\boldsymbol{\sigma}'\|_{H(\mathbf{div},\Omega)} \leq \|p'\|_{L^2(\Omega)} \end{cases}$$

and in both cases it is posible to prove that

$$\boldsymbol{B}_1(\overline{\boldsymbol{s}}, (\boldsymbol{\sigma}', p')) \geq \beta_1 \|(\boldsymbol{\sigma}', p')\|^2_{\boldsymbol{Y}} \quad \text{and} \quad \|\overline{\boldsymbol{s}}\|_{\boldsymbol{X}} \leq \frac{1}{\beta_1}\|(\boldsymbol{\sigma}', p'\|)\|_{\boldsymbol{Y}}.$$

On the other hand, we know that each $\boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{div};\Omega)$, it can be decomposed uniquely as

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\mathbb{I}, \quad \text{with} \quad \boldsymbol{\tau}_0 \in \boldsymbol{H} \quad \text{and} \quad c := \frac{1}{d|\Omega|}\int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \in \mathbb{R}.$$

By Lemma (3) be know that given $\boldsymbol{u}' \in [L^2(\Omega)]^d$ there exists $\boldsymbol{\tau}_0 \in \boldsymbol{H}$ such that $\mathbf{div}\boldsymbol{\tau}_0 = \boldsymbol{u}'$. Let us also consider $\xi' \in \mathbb{R}$ and then we can build the function

$$\overline{\boldsymbol{\tau}} = -\boldsymbol{\tau}_0 - \frac{1}{d|\Omega|}\xi'\mathbb{I}$$

which clearly belongs to $\boldsymbol{H}(\mathbf{div};\Omega)$. Consequently, we have

$$\begin{aligned} \boldsymbol{B}_2((\overline{\boldsymbol{\tau}}, \overline{q}), (\boldsymbol{u}', \xi')) &= -(\boldsymbol{u}', \mathbf{div}(\overline{\boldsymbol{\tau}}))_\Omega - (\xi', \operatorname{tr}(\overline{\boldsymbol{\tau}}))_\Omega \\ &= (\boldsymbol{u}', \mathbf{div}(\boldsymbol{\tau}_0))_\Omega + (\xi', \operatorname{tr}(\frac{1}{d|\Omega|}\xi'\mathbb{I}))_\Omega \\ &= \|\boldsymbol{u}'\|^2_{[L^2(\Omega)]^d} + |\xi'|^2 \\ &= \|(\boldsymbol{u}', \xi')\|^2_{\boldsymbol{Z}}. \end{aligned}$$

Additionally, the inequality (2.9) implies particularly that

$$\beta_2 \|(\overline{\boldsymbol{\tau}}, \overline{q})\|_{\boldsymbol{Y}} \|(\boldsymbol{u}', \xi')\|_{\boldsymbol{Z}} \leq \boldsymbol{B}_2((\overline{\boldsymbol{\tau}}, \overline{q}), (\boldsymbol{u}', \xi')) = \|(\boldsymbol{u}', \xi')\|^2_{\boldsymbol{Z}},$$

and one obtains $\|(\overline{\boldsymbol{\tau}}, \overline{q})\|_{\boldsymbol{Y}} \leq \frac{1}{\beta_2}\|(\boldsymbol{u}', \xi')\|_{\boldsymbol{Z}}$.

Now, we are in position to build the operator T. Indeed, for $\vec{\boldsymbol{t}}' = (\boldsymbol{t}', (\boldsymbol{\sigma}', p'), (\boldsymbol{u}', \xi')) \in \boldsymbol{X} \times \boldsymbol{Y} \times \boldsymbol{Z}$ and $\zeta \in \mathbb{R}^+$, we define

$$\begin{aligned} \boldsymbol{T} : \boldsymbol{X} \times \boldsymbol{Y} \times \boldsymbol{Z} &\to \boldsymbol{X} \times \boldsymbol{Y} \times \boldsymbol{Z} \\ (\boldsymbol{t}', (\boldsymbol{\sigma}', p'), (\boldsymbol{u}', \xi')) &\mapsto \boldsymbol{T}(\boldsymbol{t}', (\boldsymbol{\sigma}', p'), (\boldsymbol{u}', \xi')) \\ &\qquad = (\zeta\boldsymbol{t}' + \overline{\boldsymbol{s}}, (-\zeta\boldsymbol{\sigma}' + \overline{\boldsymbol{\tau}}, -\zeta p' + \overline{q}), (\zeta\boldsymbol{u}', \zeta\xi')) \end{aligned}$$

Consequently, replacing the test vector function $\vec{s}$ by $\boldsymbol{T}(\vec{\boldsymbol{t}}')$ one gets

$$\begin{aligned} &\langle \mathcal{A}_0(\vec{\boldsymbol{t}}'), \boldsymbol{T}(\vec{\boldsymbol{t}}')\rangle \\ &= (\boldsymbol{\nu}(\boldsymbol{t}'), \zeta\boldsymbol{t}' + \overline{\boldsymbol{s}})_\Omega - (\boldsymbol{\sigma}', \zeta\boldsymbol{t}' + \overline{\boldsymbol{s}})_\Omega - (p', \operatorname{tr}(\zeta\boldsymbol{t}' + \overline{\boldsymbol{s}}))_\Omega - (-\zeta\boldsymbol{\sigma}' + \overline{\boldsymbol{\tau}}, \boldsymbol{t}')_\Omega \end{aligned}$$

$$
\begin{aligned}
&- (-\zeta p' + \overline{q}, \mathrm{tr}(\boldsymbol{t}'))_\Omega - (\boldsymbol{u}', \mathbf{div}(-\zeta\boldsymbol{\sigma}' + \overline{\boldsymbol{\tau}}))_\Omega + (\xi', \mathrm{tr}(-\zeta\boldsymbol{\sigma}' + \overline{\boldsymbol{\tau}}))_\Omega \\
&- (\zeta \boldsymbol{u}', \mathbf{div}(\boldsymbol{\sigma}'))_\Omega + (\zeta\xi', \mathrm{tr}(\boldsymbol{\sigma}'))_\Omega \\
= {}& \zeta(\boldsymbol{\nu}(\boldsymbol{t}'), \boldsymbol{t}')_\Omega + (\boldsymbol{\nu}(\boldsymbol{t}'), \boldsymbol{\sigma})_\Omega - (\boldsymbol{\sigma}', \overline{\boldsymbol{s}})_\Omega - (p', \mathrm{tr}(\overline{\boldsymbol{s}}))_\Omega - (\overline{\boldsymbol{\tau}}, \boldsymbol{t}')_\Omega - (\overline{q}, \mathrm{tr}(\boldsymbol{t}'))_\Omega \\
&- (\boldsymbol{u}', \mathbf{div}(\overline{\boldsymbol{\tau}}))_\Omega + (\xi', \mathrm{tr}(\overline{\boldsymbol{\tau}}))_\Omega \\
= {}& \zeta \boldsymbol{A}(\boldsymbol{t}', \boldsymbol{t}') - \boldsymbol{A}(\boldsymbol{t}', \overline{\boldsymbol{s}}) + \boldsymbol{B}_1(\overline{\boldsymbol{s}}, (\boldsymbol{\sigma}', p')) + \boldsymbol{B}_1(\boldsymbol{t}', (\overline{\boldsymbol{\tau}}, \overline{q})) + \boldsymbol{B}_2((\overline{\boldsymbol{\tau}}, \overline{q}), (\boldsymbol{u}', \xi')) \\
\geq {}& \zeta \boldsymbol{A}(\boldsymbol{t}', \boldsymbol{t}') - \boldsymbol{A}(\boldsymbol{t}', \overline{\boldsymbol{s}}) + \beta_1 \|(\boldsymbol{\sigma}', p')\|^2_{\boldsymbol{Y}} + \boldsymbol{B}_1(\boldsymbol{t}', (\overline{\boldsymbol{\tau}}, \overline{q})) + \|(\boldsymbol{u}', \xi')\|^2_{\boldsymbol{Z}}.
\end{aligned}
$$

Given that the inequalities given in (2.7) holds for all $\boldsymbol{r}, \boldsymbol{s} \in \boldsymbol{X}$, in particular if $\boldsymbol{s} = \boldsymbol{0}$ one obtains

$$
\boldsymbol{A}(\boldsymbol{r}, \boldsymbol{r}) \geq C_m \|\boldsymbol{r}\|^2_{\boldsymbol{X}},
$$

because $\boldsymbol{\nu}(\boldsymbol{0}) = \boldsymbol{0}$. In addition, it is clear that by Cauchy-Schwarz inequality and inequality (1.2) one proves

$$
|\boldsymbol{A}(\boldsymbol{r}, \boldsymbol{s})| \leq C_1 \|\boldsymbol{r}\|_{\boldsymbol{X}} \|\boldsymbol{s}\|_{\boldsymbol{X}}
$$

for all $\boldsymbol{r}, \boldsymbol{s} \in \boldsymbol{X}$. On the other hand, by applying again Cauchy-Schwarz inequality one gets

$$
|\boldsymbol{B}_1(\boldsymbol{s}, (\boldsymbol{\tau}, q))| \leq \|\boldsymbol{s}\|_{\boldsymbol{X}} \|(\boldsymbol{\tau}, q)\|_{\boldsymbol{Y}}
$$

for all $\boldsymbol{s} \in \boldsymbol{X}$ and $(\boldsymbol{\tau}, q) \in \boldsymbol{Y}$. Hence, by using Young inequality one arrives

$$
\begin{aligned}
\langle \mathcal{A}_0(\vec{\boldsymbol{t}}), \boldsymbol{T}(\vec{\boldsymbol{t}}) \rangle \geq {}& \zeta\, C_m \|\boldsymbol{t}'\|^2_{\boldsymbol{X}} - C_1 \|\boldsymbol{t}'\|_{\boldsymbol{X}} \|\overline{\boldsymbol{s}}\|_{\boldsymbol{X}} + \beta_1 \|(\boldsymbol{\sigma}', p')\|^2_{\boldsymbol{Y}} \\
&+ \|\boldsymbol{t}'\|_{\boldsymbol{X}} \|(\overline{\boldsymbol{\tau}}, \overline{q})\|_{\boldsymbol{Y}} + \|(\boldsymbol{u}', \xi')\|^2_{\boldsymbol{Z}} \\
\geq {}& \zeta\, C_m \|\boldsymbol{t}'\|^2_{\boldsymbol{X}} - C_1^2 \frac{\delta_1}{2} \|\boldsymbol{t}'\|^2_{\boldsymbol{X}} - \frac{1}{2\delta_1} \|\overline{\boldsymbol{s}}\|^2_{\boldsymbol{X}} + \beta_1 \|(\boldsymbol{\sigma}', p')\|^2_{\boldsymbol{Y}} \\
&- \frac{\delta_2}{2} \|\boldsymbol{t}'\|^2_{\boldsymbol{X}} - \frac{1}{2\delta_2} \|(\overline{\boldsymbol{\tau}}, \overline{q})\|^2_{\boldsymbol{Y}} + \|(\boldsymbol{u}', \xi')\|^2_{\boldsymbol{Z}} \\
\geq {}& \left(\zeta\, C_m - C_1^2 \frac{\delta_1}{2} - \frac{\delta_2}{2} \right) \|\boldsymbol{t}'\|^2_{\boldsymbol{X}} + \left(\beta_1 - \frac{1}{2\beta_1\delta_1} \right) \|(\boldsymbol{\sigma}', p')\|^2_{\boldsymbol{Y}} \\
&+ \left(1 - \frac{1}{2\beta_2\delta_2} \right) \|(\boldsymbol{u}', \xi')\|_{\boldsymbol{Z}}.
\end{aligned}
$$

Imposing the conditions $\delta_1 > \frac{1}{2\beta_1^2}$, $\delta_2 > \frac{1}{2\beta_2}$ and $2\,\zeta\, C_m > C_1^2 \delta_1 + \delta_2$ one can conclude that

$$
\langle \mathcal{A}(\vec{\boldsymbol{t}}), \boldsymbol{T}(\vec{\boldsymbol{t}}) \rangle \geq \alpha \left(\|\boldsymbol{t}'\|^2_{\boldsymbol{X}} + \|(\boldsymbol{\sigma}', p')\|^2_{\boldsymbol{Y}} + \|(\boldsymbol{u}', \xi')\|_{\boldsymbol{Z}} \right), \tag{3.10}
$$

where $\alpha = \min\left\{ \zeta\, C_m - C_1^2 \frac{\delta_1}{2} - \frac{\delta_2}{2}, \beta_1 - \frac{1}{2\beta_1\delta_1}, 1 - \frac{1}{2\beta_2\delta_2} \right\} > 0.$

Finally, note that if $\boldsymbol{T}(\vec{\boldsymbol{t}}) = \boldsymbol{0}$, replacing in (2.5) we get that $\vec{\boldsymbol{t}} = \boldsymbol{0}$. Thereby, the injectivity for operator $\boldsymbol{T}$ follows. Additionally, given $\vec{\boldsymbol{s}}* = (\boldsymbol{\tau}^*, (\boldsymbol{\tau}^*, q^*), (\boldsymbol{v}^*, \eta^*)) \in \boldsymbol{X} \times \boldsymbol{Y} \times \boldsymbol{Z}$, choosing

$$
(\boldsymbol{t}', (\boldsymbol{\sigma}', p'), (\boldsymbol{u}', \xi')) = \frac{1}{\zeta} (\boldsymbol{s}^* - \overline{\boldsymbol{s}}, -(\boldsymbol{\tau}^* - \overline{\boldsymbol{\tau}}, q^* - \overline{q}), (\boldsymbol{v}^*, \eta^*))
$$

yields $\boldsymbol{T}(\vec{\boldsymbol{t}'}) = \vec{\boldsymbol{s}}*$, and therefore the operator $\boldsymbol{T}$ is bijective.

□

Acknowledgments

The author thanks to the Prof. Paulo Zúñiga from Catholic University of Temuco for the discussions.

References

[1] Josefin Ahlkrona and Malte Braack. Equal-order stabilized finite element approximation of the p-Stokes equations on anisotropic cartesian meshes. *Computational Methods in Applied Mathematics*, 20(1):1–25, 2020.

[2] Douglas N. Arnold, Jim Douglas, and Chaitan P. Gupta. A family of higher order mixed finite element methods for plane elasticity. *Numerische Mathematik*, 45(1):1–22, 1984.

[3] Mathieu Barré, Céline Grandmont, and Philippe Moireau. Analysis of a linearized poromechanics model for incompressible and nearly incompressible materials. *Evolution Equations & Control Theory*, 12(3):1–61, 2023.

[4] Daniele Boffi, Franco Brezzi, and Michel Fortin. *Mixed Finite Element Methods and Applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Cham, 2013.

[5] Patrick Ciarlet Jr. T-coercivity: Application to the discretization of Helmholtz-like problems. *Computers & Mathematics with Applications*, 64:22–34, 2012.

[6] Patrick Ciarlet Jr and Erell Jamelot. Explicit T-coercivity for the Stokes problem: a coercive finite element discretization. *arXiv:2410.14444*, page 41, 2024.

[7] Cristian Cárcamo, Alfonso Caiazzo, Felipe Galarce, and Joaquín Mura. A stabilized total pressure-formulation of the Biot's poroelasticity equations in frequency domain: Numerical analysis and applications. *Computer Methods in Applied Mechanics and Engineering*, 432:117353, 2024.

[8] Gabriel N. Gatica. *A Simple Introduction to the Mixed Finite Element Method: Theory and Applications*. SpringerBriefs in Mathematics. Springer, Cham, 2014.

[9] Gabriel N. Gatica, María González, and Salim Meddahi. A low-order mixed finite element method for a class of quasi-newtonian Stokes flows. part i: a priori error analysis. *Computer Methods in Applied Mechanics and Engineering*, 193(9):881–892, 2004.

[10] Gabriel N. Gatica, Ricardo Oyarzúa, and Francisco-Javier Sayas. Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem. *Mathematics of Computation*, 80(276):1911–1948, 2011.

[11] Christian Helanow and Josefin Ahlkrona. Stabilized equal low-order finite elements in ice sheet modeling–accuracy and robustness. *Comput. Geosci.*, 22:951–974, 2018.

[12] Tobin Isaac, Georg Stadler, and Omar Ghattas. Solution of nonlinear stokes equations discretized by high-order finite elements on nonconforming and anisotropic meshes, with application to ice sheet dynamics. *SIAM Journal on Scientific Computing*, 37(6):B804–B833, 2015.

[13] Hassan Manouzi and Mohamed Farhloul. Mixed finite element analysis of a non-linear three-fields stokes model. *IMA Journal of Numerical Analysis*, 21(1):143–164, 2001.