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Spectral bounds for the operator pencil of an elliptic system in an angle

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Abstract

The model problem of a plane angle for a second-order elliptic system with Dirichlet, mixed, and Neumann boundary conditions is analyzed. The existence of solutions is, for each boundary condition, reduced to solving a matrix equation. Leveraging these matrix equations and focusing on Dirichlet and mixed boundary conditions, optimal bounds on these solutions are derived, employing tools from numerical range analysis and accretive operator theory. The developed framework is novel and recovers known bounds for Dirichlet boundary conditions. The results for mixed boundary conditions are new and represent the central contribution of this work. Immediate applications of these findings are new regularity results in linear elasticity.

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1 Introduction

Regularity theory for differential equations is concerned with the question of how regular a solution can be with respect to the input data, such as the source function and the boundary data. A classic example is the Laplace equation, $\Delta u = f$, on a domain Ω with smooth boundary $\partial\Omega$, subject to Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. In this case, it is established that $u \in W^{k+2,p}(\Omega)$ when $f \in W^{k,p}(\Omega)$, for any $k \in \mathbb{N}$ and $p > 1$ (§2.4 in [11]). This result does not hold if we consider domains Ω which include edges or vertices. However, the irregularities of solutions in polygonal domains exhibit a similar structure. Considering a cone $\mathcal{K} \subset \mathbb{R}^n$, a solution u to an elliptic equation near the vertex of $\mathcal{K}$ can asymptotically be described by terms of the form (neglecting factors of $\log r$, see [8], [11], [19]):

$$u(r,\omega) \sim r^{\lambda} v(\omega), \tag{1.1}$$

where (r,ω) are spherical coordinates with r being the distance to the vertex.

Here, v is a function on a subset of the sphere $\omega \in \mathbb{S}^{n-1}$ (the cone opening), and $\lambda \in \mathbb{C}$ represents the regularity parameter linking integrability and differentiability of u to $\operatorname{Re}\lambda$. The associated operator pencil $\mathcal{A}(\lambda)$ is a λ-dependent differential operator such that $\mathcal{A}(\lambda)v = 0$, i.e., the spectrum of $\mathcal{A}$ consists of the possible exponents λ in the asymptotic expansion (1.1).

By localization, the regularity of boundary value problems in general polyhedral domains can be reduced to model problems in cones and angles. This means the (ir)regularity of solutions to elliptic equations is characterized by solutions of the form (1.1), or equivalently, by the spectrum of the corresponding operator pencil $\mathcal{A}$. A well-established reference in this regard is [20], which provides estimates for $\operatorname{Re}\lambda$ for various model problems. These include the Lamé system (for general boundary conditions) or general elliptic systems (for Dirichlet and Neumann boundary conditions) in cones. The follow-up work [22] applies these results to specific problems in three-dimensional polyhedral domains, translating the estimates for $\operatorname{Re}\lambda$ into regularity results. These references address a broad range of model problems and include extensive discussions. As noted in the Introduction of [21] *"No general, even to some extent, methods of obtaining this information are known, since even for the simplest problems of mathematical physics these spectral problems have a rather complicated form"*. Despite originating in the last century, this observation is still valid today.

From the above considerations, it is evident that regularity results for elliptic equations are inherently linked to solutions of the form $r^{\lambda}v$, arising from model problems that reflect the geometry of the original problem. The primary motivation for this paper is to improve the known regularity estimates for solutions in linear elasticity. The underlying equations can be nuanced because elasticity involves systems of elliptic equations, as the displacement is a vector field, and often incorporates mixed boundary conditions. Existing literature on regularity theory for linear elasticity is sparse, with many results providing only relatively weak estimates or lacking full generality. The regularity of solutions to the Lamé system with mixed boundary conditions is discussed in [23] for three-dimensional polygonal domains, including edges and vertices. In [15], it is shown (without using the expansion $r^{\lambda}v$ at all) that $u \in W^{1,2+\varepsilon}$ for some $\varepsilon > 0$, which is only a small improvement with respect to $u \in W^{1,2}$ from general solution theory. In [14], one can find results on three-dimensional *scalar* elliptic model problems, including mixed boundary conditions, that yield $u \in W^{1,p}$ for some $p > 3$. A famous counter-example in [25] demonstrates an upper bound, showing that we cannot expect $u \in W^{1,p}$ for $p \geq 4$ for scalar equations in the two-dimensional half-space, and consequently not for elliptic systems.

This work develops a framework to study the model problem in a two-dimensional angle for a second-order elliptic system with real-valued coefficient matrices. Utilizing this framework, we recover well-known bounds for Dirichlet boundary conditions in a novel way. The main result of this work, however, focuses on the model problem with mixed boundary conditions (Theorem 7.2). For these, we prove (under mild ellipticity conditions) that any solution $r^\lambda v$ of the model problem satisfies the bounds $|\operatorname{Re}\lambda| \geq \frac{1}{2}$ for $\alpha \leq \pi$ and $|\operatorname{Re}\lambda| \geq \frac{1}{4}$ for $\alpha \leq 2\pi$, where α is the opening angle. These bounds are optimal, and they coincide with those for the Laplace equation. Unlike implicit solution strategies in other works, this framework allows for the explicit construction of solutions. Also, it might be possible to adapt the framework to further boundary conditions, three-dimensional cones or higher-order elliptic equations.

As mentioned before, our findings have applications in the regularity theory of linear elasticity. For brevity, we only sketch a simple scenario: Consider the linear elastic equation $\operatorname{div}\sigma(u) = 0$ in a domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, where Dirichlet and Neumann boundary conditions are separated by a finite number of smooth, nonintersecting closed curves. Here, $\sigma(u) = \mathbb{C}e(u)$, where $\mathbb{C}$ denotes the elasticity tensor and $e(u)$ the symmetrized gradient. Under suitable conditions, it can be shown that solutions u satisfy $u \in W^{1,4-\varepsilon}$ for any $\varepsilon > 0$. For details, we refer to Theorem 8.1.7 and §8.3.1 in [22]. Applications to elasticity and other geometries will be addressed in a subsequent paper.

While the given approach provides new insights and results, its limitations must be acknowledged. The present work focuses exclusively on the model problem in a plane angle. Hence, the results are only applicable to domains $\Omega \subset \mathbb{R}^3$, where non-smoothness of $\partial\Omega$ manifests as an edge. More complex geometric structures, such as vertices, are not covered.

Structure of the paper

In Section 3, following [22] and [20], the model problem is introduced for an elliptic system of the form

$$L_A(x,y) = A_{11}\partial_x^2 + 2A_{12}\partial_x\partial_y + A_{22}\partial_y^2,$$

where $A_\bullet$ are real-valued, symmetric matrices satisfying the strong ellipticity condition (weaker than the *formal positivity* condition in elasticity, see Lemma B.3). The domain is given by the two-dimensional angle

$$\mathcal{K}_\alpha = \{(r\cos(\varphi), r\sin(\varphi)) : r > 0,\ 0 \leq \varphi \leq \alpha\} \subset \mathbb{R}^2,$$

for some fixed $0 < \alpha \leq 2\pi$. We consider either Dirichlet or Neumann boundary conditions on the two sides of the angle, respectively. Finding for the model problem solutions of the form $r^\lambda v$, where $\lambda \in \mathbb{C}$, is reformulated as determining the eigenvalues of the so-called *operator pencil* by translating L_A to a λ-dependent differential operator $\mathcal{L}_A(\lambda)$.

In Section 4, it is shown that it suffices to consider elliptic systems with $A_{22} = \operatorname{Id}$, referred to as *monic elliptic systems*. Moreover, we study the algebraic structure of the polynomial $\mathbb{C} \ni \beta \mapsto A_{11} + 2A_{12}\beta + \operatorname{Id}\beta^2$ and show, using a result in [10], that one has the factorization

$$A_{11} + 2A_{12}\beta + \operatorname{Id}\beta^2 = (V^* - \operatorname{Id}\beta)(V - \operatorname{Id}\beta),$$

for V a complex-valued matrix with $\sigma(V) \subset \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. The matrix V is unique (Theorem 4.6), and is referred to as the *standard root* of the matrix polynomial. Using V, one can show that all solutions of the model problem without boundary conditions are given by (Prop. 4.7):

$$u_\lambda : \mathbb{R}^2 \setminus \{0\} \to \mathbb{C}^\ell,\ (x_1, x_2) \mapsto (x_1 \operatorname{Id}_\ell + x_2 V)^\lambda c_1 + (x_1 \operatorname{Id}_\ell + x_2 \overline{V})^\lambda c_2, \quad c_1, c_2 \in \mathbb{C}^\ell, \quad (1.2)$$

where $\ell \in \mathbb{N}$ is the dimension of the system. The exponentiation of matrices here is defined via the functional calculus, and the choice of complex exponentiation $\bullet \mapsto \bullet^\lambda$ required for smooth solutions is discussed.

In Section 5, Dirichlet, mixed, and Neumann boundary conditions for u_λ are implemented. It is shown that existence of a nontrivial solution $r^\lambda v$ for the model problem with angle α is for each boundary condition equivalent to the vanishing of the determinant of a matrix $M_{\lambda,\alpha}$ (Prop. 5.1, Prop. 5.2, Prop. 5.5). E.g., for Dirichlet boundary conditions, we get the equivalent condition

$$0 = \det(M_{\lambda,\alpha}) \quad \text{for } M_{\lambda,\alpha} = Z_\alpha^\lambda - \overline{Z_\alpha}^\lambda, \tag{1.3}$$

where Z_α is a complex symmetric matrix derived from V. The matrices $M_{\lambda,\alpha}$ for mixed and Neumann boundary conditions have a similar structure. Additionally, we introduce two ellipticity conditions, *Neumann well-posedness* and *contractive Neumann well-posedness* (Def. 5.3), related to spectral properties of V. The former is equivalent to the *complementing boundary condition* for Agmon-Douglis-Nirenberg (ADN)-elliptic systems [3] implementing Neumann boundary conditions (see Appendix B). The latter relates to path-connectedness of Neumann well-posed systems to the Laplace operator (Lemma 5.4).

In Section 6, utilizing the numerical range and results on fractional powers of accretive operators [13], we are able to provide bounds on the spectrum of $M_{\lambda,\alpha}$ for Dirichlet (Theorem 6.4) and mixed boundary conditions (Theorem 6.5, Theorem 6.6). These results are used to bound $|\operatorname{Re}\lambda|$ for $\lambda \in \mathbb{C}$ a solution to equations like (1.3).

In Section 7, we summarize our findings and give bounds on $|\operatorname{Re}\lambda|$ for solutions $r^\lambda v$ of the model problem for Dirichlet and mixed boundary conditions. In particular, Theorem 7.2 establishes for mixed boundary conditions the bounds $|\operatorname{Re}\lambda| \geq \frac{1}{2}$ for $\alpha \leq \pi$ and $|\operatorname{Re}\lambda| \geq \frac{1}{4}$ for $\alpha \leq 2\pi$, provided that the system is contractive Neumann well-posed. If the system is not contractive Neumann well-posed, then cases with $|\operatorname{Re}\lambda| < \frac{1}{2}$, resp. $|\operatorname{Re}\lambda| < \frac{1}{4}$, may occur, but only in the form $\operatorname{Re}\lambda = 0$.

In Section 8, the paper is summarized. Also, Neumann boundary conditions, optimality of the given bounds, and the scalar case $\ell = 1$ are briefly discussed.

The appendices provide supplementary material and detailed proofs that might disturb the flow of the paper. In Appendix A, the concrete form of the differential operator $\mathcal{L}_A$ is derived. In Appendix B, Neumann well-posedness is related to the complementing boundary condition for ADN-elliptic systems. Also, it is shown that formal positivity implies contractive Neumann well-posedness. In Appendix C, a factorization result for nonnegative matrix polynomials is given. In Appendix D, the functional calculus is summarized. In Appendix E, the numerical range is summarized and results on accretive operators are adapted to our setting.

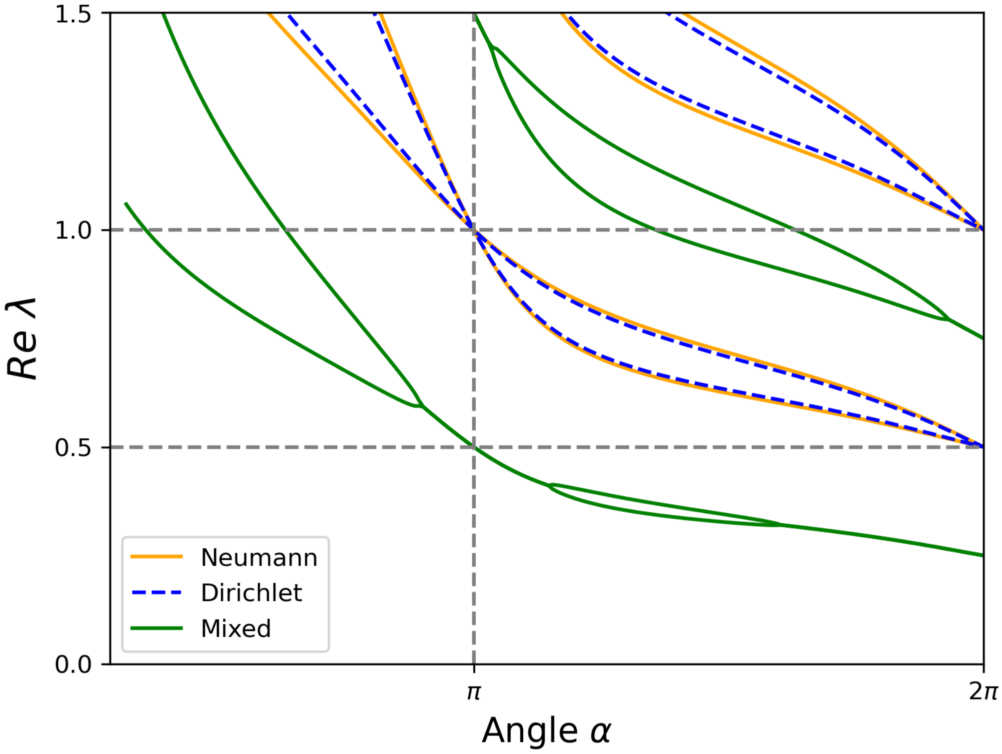

Figure 1: Relation between $\operatorname{Re}\lambda$ and $\alpha \in [1, 2\pi]$ for different boundary conditions. The elliptic tuple is defined by $A_{11} = \begin{pmatrix} 5 & 0.6 \\ 0.6 & 1.5 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 0.25 & -0.4 \\ -0.4 & -0.2 \end{pmatrix}$, $A_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The branches for Dirichlet and Neumann boundary conditions are very close to each other.

Contents

2 General prerequisites and notation

This section summarizes notation and standard results in linear algebra. The reference is [17]. For $r \in \mathbb{R} \setminus \{0\}$, let $\operatorname{sgn}(r) \in \{-1, +1\}$ denote the sign of r. $\mathbb{R}_{>0}$ denotes positive numbers, and $\mathbb{R}_{\geq 0} = \mathbb{R}_{>0} \cup \{0\}$. We use $(x_1, x_2) \in \mathbb{R}^2$ for Cartesian coordinates and $(r, \varphi) \in \mathbb{R}_{>0} \times [0, 2\pi)$ for polar coordinates. $\operatorname{Re} z$ and $\operatorname{Im} z$ will denote the real and imaginary part of a complex number $z \in \mathbb{C}$ (or matrix). Also, we write $\overline{z}$ for complex conjugation. For a set $A \subset \mathbb{C}$, we write $\operatorname{clos}(A)$ for its closure. LHS and RHS will be used as abbreviations for "left-hand sideänd "right-hand side", respectively.

Subsets of $\mathbb{C}$

Let us denote the (open) upper half-plane $\mathrm{UHP} \subset \mathbb{C}$ and the right half-plane $\mathrm{RHP} \subset \mathbb{C}$ by

$$\mathrm{UHP} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}, \quad \mathrm{RHP} := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}.$$

The lower half-plane and left half-plane are simply denoted by $-\mathrm{UHP}$ and $-\mathrm{RHP}$, where we use the following notation for set operations: For two sets $A, B \subset \mathbb{C}$ and $r \in \mathbb{C}$ we write

$$A + B := \{a + b : a \in A,\ b \in B\}, \quad rA := \{r \cdot a : a \in A\}, \quad \overline{A} := \{\overline{a} : a \in A\},$$
$$\operatorname{Re} A := \{\operatorname{Re}(a) : a \in A\}, \quad \operatorname{Im} A := \{\operatorname{Im}(a) : a \in A\}.$$

Vectors and Matrices

For the entire text, let us fix $\ell \in \mathbb{N}$ to denote the dimension of the vector spaces we consider. We denote by $\mathbb{R}^\ell$ and $\mathbb{C}^\ell$, respectively, the canonical real and complex vector spaces. By $\langle \bullet, \bullet \rangle$, we denote the scalar product on $\mathbb{C}^\ell$, and by $\| \bullet \|$ the vector norm $\|v\| = \sqrt{\langle v, v \rangle}$ for $v \in \mathbb{C}^\ell$. Write $\operatorname{Mat}_\ell(\mathbb{C})$ for matrices of size $\ell \times \ell$ with entries in $\mathbb{C}$, and let us assume $A \in \operatorname{Mat}_\ell(\mathbb{C})$ in what follows. We denote by $A^{i,j}$ the entry in the i-th row and j-th column. If the entries are only real-valued, we may also write $A \in \operatorname{Mat}_\ell(\mathbb{R})$ and often view this set as a subset of $\operatorname{Mat}_\ell(\mathbb{C})$. $\mathrm{Id}_\ell \in \operatorname{Mat}_\ell(\mathbb{C})$ will denote the identity matrix. We write A^T for the transpose and A^{-1} for the inverse matrix (if it exists). Further, we write $A^* = \overline{A}^T$ for the adjoint. We consider symmetric matrices $A^T = A$, Hermitian matrices $A^* = A$ and unitary matrices $A^* = A^{-1}$. The operator norm on $\operatorname{Mat}_\ell(\mathbb{C})$ is given by $\|A\| = \sup_{\|v\|=1} \|Av\|$. The commutator is denoted by $[A, B] = AB - BA$. Additionally, we rely on the following result on block matrices (see [1], Exercise 5.30).

Lemma 2.1. *Consider for $A, B, C, D \in \operatorname{Mat}_\ell(\mathbb{C})$ the block matrix*

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If A is invertible, then $\det(R) = \det(A) \cdot \det(D - CA^{-1}B)$.

Spectral Theory

If there exists a pair $(\lambda, v) \in \mathbb{C} \times \mathbb{C}^\ell$ such that $Av = \lambda v$, we call λ an eigenvalue of A and v the corresponding eigenvector. The set of all eigenvalues of A is called the spectrum of A, denoted by $\sigma(A)$. We say A is diagonalizable if there exist $Q, B \in \operatorname{Mat}_\ell(\mathbb{C})$, with Q invertible and B a diagonal matrix, such that

$$A = QBQ^{-1}. \tag{2.1}$$

In this case, $B^{i,i} \in \mathbb{C}$ for $1 \leq i \leq \ell$ is an eigenvalue of A, and the i-th row of Q is a corresponding eigenvector. We say A and $C \in \operatorname{Mat}_\ell(\mathbb{C})$ are similar if there is an invertible matrix $S \in \operatorname{Mat}_\ell(\mathbb{C})$

such that $A = SCS^{-1}$. In this case $\sigma(A) = \sigma(C)$. A key result in spectral theory ist that normal matrices ($A^*A = AA^*$) are always unitarily diagonalizable, meaning there exists a decomposition (2.1) with Q unitary. For Hermitian matrices, the spectrum satisfies $\sigma(A) \subset \mathbb{R}$, while for unitary matrices, $\sigma(A) \subset \mathbb{S}^1$, where $\mathbb{S}^1$ denotes the complex unit circle. Properties of the spectrum include the following: $\sigma(AB) = \sigma(BA)$ for A, B invertible, $0 \in \sigma(A) \iff \det(A) = 0$, and $\sigma(A^*) = \overline{\sigma(A)}$. The spectral radius of A is given by

$$\rho(A) = \sup_{\lambda \in \sigma(A)} |\lambda|.$$

It satisfies $\rho(A) \leq \|A\|$, and for Hermitian matrices, $\rho(A) = \|A\|$. The equality also holds for $A \in \mathrm{Mat}_\ell(\mathbb{R})$ skew-symmetric ($A^T = -A$), since iA is Hermitian.

Positive definite matrices

We call A positive definite, and write $A > 0$, if it is Hermitian and satisfies $\langle v, Av \rangle > 0$ for any vector $v \in \mathbb{C}^\ell \setminus \{0\}$. A Hermitian matrix A is positive definite if and only if $\sigma(A) \subset \mathbb{R}_{>0}$. If $\sigma(A) \subset \mathbb{R}_{\geq 0}$, A is called positive semi-definite, denoted $A \geq 0$. The product A^*A is always positive semi-definite, and it is positive definite if $0 \notin \sigma(A)$. Furthermore, $A^{-1} > 0$ if and only if $A > 0$. For matrices A and C, we have $A - C > 0$ if and only if $C^{-1} - A^{-1} > 0$. Additionally, if $A > 0$ and $C \geq 0$, then $A + C > 0$. Any positive definite matrix A has a unique positive definite square root $C \in \mathrm{Mat}_\ell(\mathbb{C})$, i.e., $C^2 = A$. We often write $C = A^{1/2}$.

3 The model problem

This section introduces the model problem as presented in §6 of [22]. The domain of interest is given, for $0 < \alpha \leq 2\pi$, by the two-dimensional angle

$$\mathcal{K}_\alpha := \{(r\cos(\varphi), r\sin(\varphi)) : r > 0,\ 0 \leq \varphi \leq \alpha\} \subset \mathbb{R}^2,$$

which has the boundary

$$\begin{aligned} &\partial\mathcal{K}_\alpha = \Gamma^- \cup \Gamma^+ \cup \{(0,0)\} \\ \text{for} \quad &\Gamma^- := \{(x,0) : x > 0\} \quad \text{and} \quad \Gamma^+ := \{(r\cos(\alpha), r\sin(\alpha)) : r > 0\}. \end{aligned}$$

For $A = (A_{11}, A_{12}, A_{22})$, where $A_\bullet \in \mathrm{Mat}_\ell(\mathbb{R})$ are symmetric matrices, the second-order differential operator L_A is given by

$$L_A(\partial_{x_1}, \partial_{x_2}) := \sum_{i,j=1}^{2} A_{ij}\partial_{x_i}\partial_{x_j} = A_{11}\partial_{x_1}^2 + 2A_{12}\partial_{x_1}\partial_{x_2} + A_{22}\partial_{x_2}^2, \tag{3.1}$$

where we set $A_{21} = A_{12}$ in the following. The conormal derivatives $N_A^\pm$ associated to L_A on $\Gamma^\pm$ are given by

$$N_A^-(\partial_{x_1}, \partial_{x_2}) = N_A(0, \partial_{x_1}, \partial_{x_2}) \quad \text{and} \quad N_A^+(\partial_{x_1}, \partial_{x_2}) = N_A(\alpha, \partial_{x_1}, \partial_{x_2}) \quad \text{for}$$

$$N_A(\varphi, \partial_{x_1}, \partial_{x_2}) := \sum_{i,j=1}^{2} A_{ij} n_i \partial_{x_j} = A_{11}n_1\partial_{x_1} + A_{12}(n_1\partial_{x_2} + n_2\partial_{x_1}) + A_{22}n_2\partial_{x_2}, \quad \varphi \in \{0, \alpha\},$$

where $n = (-\sin(\varphi), \cos(\varphi))$ for $\varphi \in \{0, \alpha\}$ is the normal vector perpendicular to $\Gamma^{\pm}$. We investigate complex-valued solutions $u : \mathcal{K}_\alpha \to \mathbb{C}^\ell$ to the equations

$$L_A(\partial_{x_1}, \partial_{x_2})u = 0 \quad \text{on } \mathcal{K}_\alpha, \quad B_A^{\pm}(\partial_{x_1}, \partial_{x_2})u = 0 \quad \text{on } \Gamma^{\pm}, \tag{3.2}$$

where u can be decomposed in the *radial form* $u(r, \varphi) = r^\lambda v(\varphi)$ for some $\lambda \in \mathbb{C}$ and $v : [0, \alpha] \to \mathbb{C}^\ell$ smooth. Here,

$$B_A^{\pm}(\partial_{x_1}, \partial_{x_2}) := (1 - d^{\pm})u + d^{\pm} N_A^{\pm}(\partial_{x_1}, \partial_{x_2})u \tag{3.3}$$

for $d^{\pm} \in \{0, 1\}$ such that $(d^+, d^-) = (0, 0)$ implements Dirichlet, $(d^+, d^-) = (1, 1)$ Neumann and $(d^+, d^-) = (0, 1)$ mixed boundary conditions. The problem of finding a solution u of radial form to the equations (3.2) is called the *model problem*. If we only consider

$$L_A(\partial_{x_1}, \partial_{x_2})u = 0 \quad \text{on } \mathcal{K}_\alpha, \tag{3.4}$$

then we call (3.4) the *model problem without boundary conditions*.

Remark. In the following, $\ell \in \mathbb{N}$ in $u : \mathcal{K}_\alpha \to \mathbb{C}^\ell$ always denotes the dimension of the codomain, and α in $0 < \alpha \leq 2\pi$ always the opening angle.

Our question is the following: For fixed $\alpha \in (0, 2\pi]$, for which $\lambda \in \mathbb{C}$ can we expect a solution of the form $r^\lambda v$ for the model problem (3.2)? More specifically, what is the smallest value for $|\operatorname{Re} \lambda|$ that we can expect for a solution? These results can be translated to regularity of solutions for strongly elliptic systems in polyhedral domains (see §2 and §6 in [22]).

3.1 Ellipticity

So far, we have not implemented ellipticity of L_A. One can find the next definition in a similar form in §1.1.2 of [22].

Definition 3.1. Consider the elliptic operator L_A in (3.1) where $A_\bullet \in \operatorname{Mat}_\ell(\mathbb{R})$ are symmetric. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we define the polynomial

$$L_A(\xi) := \sum_{i,j=1}^{2} A_{ij}\, \xi_i \xi_j.$$

We say

- L_A is *elliptic* if

$$\det L_A(\xi) \neq 0 \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}. \tag{3.5}$$

- L_A is *strongly elliptic* if there exists $\kappa > 0$ such that

$$\langle L_A(\xi)\eta, \eta \rangle \geq \kappa \|\eta\|^2 \|\xi\|^2 \quad \forall \eta \in \mathbb{C}^\ell,\ \xi \in \mathbb{R}^2. \tag{3.6}$$

Remark. $L_A(\xi) \in \operatorname{Mat}_\ell(\mathbb{R})$ is symmetric real-valued and thus Hermitian for any $\xi \in \mathbb{R}^2$. So the LHS in (3.6) is always real.

Obviously, strong ellipticity implies ellipticity. From now on, we always assume that A_{11} and A_{22} are positive definite. In this case, one can show that the two notions coincide, simplifying the analysis.

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Lemma 3.2. *Assume that $A_{11}, A_{12}, A_{22} \in \mathrm{Mat}_\ell(\mathbb{R})$ are symmetric matrices and A_{11}, A_{22} are positive definite. Then the following are equivalent:*

i) $\det(L_A(\xi)) \neq 0$ *for all* $\xi \in \mathbb{R}^2$ *of the form* $\xi = (1, \beta) \in \mathbb{R}^2$.

ii) L_A *is elliptic.*

iii) L_A *is strongly elliptic.*

In this case, in particular, $\det(L_A(\xi)) > 0$ *for any* $\xi \in \mathbb{R}^2 \setminus \{0\}$.

For the proof, see Appendix B. Throughout this work, we refer to A as an *elliptic system* or *elliptic tuple* if $A = (A_{11}, A_{12}, A_{22})$ are all symmetric and $A_{11} > 0$, $A_{22} > 0$. Correspondingly, we refer to the operator L_A as an *elliptic operator*.

Remark. The condition for strong ellipticity is sometimes referred to as *Legendre-Hadamard condition* (see §3.4.1 of [9]).

Remark. Thus far, we have only described the ellipticity of the operator L_A. However, to ensure the well-posedness of the elliptic problem, it is also necessary to specify a *complementing condition* for the boundary operators $B_A^{\pm}$. These conditions, introduced in the framework of elliptic systems by Agmon, Douglis, and Nirenberg in [3], establish compatibility between the boundary operators and the elliptic operator. The need for such a condition will arise naturally in our analysis, so we postpone this discussion to a later section. The ellipticity conditions are discussed in more detail in Appendix B.

3.2 Eigenvalues of the operator pencil

In this section, the model problem is translated to a parameter-dependent second-order ODE, and the operator pencil is introduced. The reference is §6.1.3 of [22].

Writing $u = r^\lambda v$ for $\lambda \in \mathbb{C}$, we define the λ-dependent differential operators $\mathcal{L}_A(\partial_\varphi, \lambda)$, $\mathcal{B}_A^{\pm}(\partial_\varphi, \lambda)$ and $\mathcal{N}_A(\partial_\varphi, \lambda)$ by:

$$\begin{aligned}
\mathcal{L}_A(\partial_\varphi, \lambda)v &:= r^{2-\lambda} L_A(\partial) r^\lambda v, \\
\mathcal{N}_A(\partial_\varphi, \lambda)v &:= r^{1-\lambda} N_A(\partial) r^\lambda v, \\
\mathcal{B}_A^{\pm}(\partial_\varphi, \lambda)v &:= (1 - d^{\pm})v + d^{\pm} \mathcal{N}_A(\partial_\varphi, \lambda)v.
\end{aligned}$$

A long but straightforward calculation (see Appendix A) shows that:

$$\begin{aligned}
\mathcal{L}_A(\partial_\varphi, \lambda) &= b_2(\varphi)\partial_\varphi^2 + (\lambda - 1)b_1(\varphi)\partial_\varphi + \lambda(\lambda - 1)b_0(\varphi) + \lambda b_2(\varphi), \\
\mathcal{N}_A(\partial_\varphi, \lambda) &= b_2(\varphi)\partial_\varphi + \frac{\lambda}{2} b_1(\varphi),
\end{aligned} \tag{3.7}$$

where the $b_\bullet$'s are the periodic functions:

$$\begin{aligned}
b_0(\varphi) &= A_{11}\cos(\varphi)^2 + A_{22}\sin(\varphi)^2 + 2A_{12}\sin(\varphi)\cos(\varphi), \\
b_1(\varphi) &= 2(A_{22} - A_{11})\sin(\varphi)\cos(\varphi) + 2A_{12}(\cos(\varphi)^2 - \sin(\varphi)^2), \\
b_2(\varphi) &= A_{11}\sin(\varphi)^2 + A_{22}\cos(\varphi)^2 - 2A_{12}\cos(\varphi)\sin(\varphi).
\end{aligned} \tag{3.8}$$

We define the λ-dependent mapping:

$$\mathcal{A}(\lambda) : W^{2,2}((0,\alpha),\mathbb{C}^\ell) \to L^2((0,\alpha),\mathbb{C}^\ell) \times \mathbb{C}^\ell \times \mathbb{C}^\ell \tag{3.9}$$

by

$$v \mapsto \left(\mathcal{L}_A(\lambda)v,\ \mathcal{B}_A^-(\lambda)v\big|_{\varphi=0},\ \mathcal{B}_A^+(\lambda)v\big|_{\varphi=\alpha}\right).$$

Here, $W^{2,2}$ and L^2 denote the usual Sobolev and Lebesgue space. In the literature, $\mathcal{A}(\lambda)$ is called the *operator pencil*. If there exist $\lambda \in \mathbb{C}$ and $v \neq 0$ such that $\mathcal{A}(\lambda)v = 0$, then λ is called an *eigenvalue* of $\mathcal{A}$ and v an *eigenvector* to λ. See §1 in [20] for an introduction to operator pencils and further references. With the above derivations, the model problem is reduced to a λ-dependent second-order ODE, and we have the following: *The model problem (3.2) has a solution of the form $r^\lambda v$ for $\lambda \in \mathbb{C}$ if and only if $\lambda \in \mathbb{C}$ is an eigenvalue of the corresponding operator pencil $\mathcal{A}(\lambda)$ in (3.9).*

Remark. Note that the leading coefficient $b_2(\varphi)$ in $\mathcal{L}_A$ is positive definite for any $\varphi \in [0, 2\pi)$. This can be seen by taking $\xi = (\sin(\varphi), -\cos(\varphi)) \in \mathbb{R}^2$ and observing that $b_2(\varphi) = L_A(\xi) > 0$ due to Lemma 3.2.

Remark (Solution theory for $\mathcal{L}_A(\partial_\varphi, \lambda)v = 0$). We refer to [16] for the following arguments. Fix $\lambda \in \mathbb{C}$, and observe that the system of ℓ second-order ODE's given by $\mathcal{L}_A(\partial_\varphi, \lambda)v = 0$ can be reduced, by a standard trick, to a system of first order ODE's of the form

$$\partial_\varphi y = M(\varphi, \lambda)y.$$

Here, the entries of $M(\varphi, \lambda) \in \mathrm{Mat}_{2\ell}(\mathbb{C})$ are analytic in φ. This reduction requires inverting $b_2(\varphi)$, which is possible due to the last remark. Using the results of §IV.10 in [16], this first order ODE has a fundamental matrix $Y(\varphi, \lambda)$ which is analytic in φ. This implies that one can choose 2ℓ linearly independent analytic solutions $v_{\lambda,1}(\lambda), \ldots, v_{\lambda,2\ell}(\lambda) : \mathbb{R} \to \mathbb{C}^\ell$ (depending on λ), and any solution $v(\lambda)$ of $\mathcal{L}_A(\lambda, \partial_\varphi)v(\lambda) = 0$ is given by a linear combination:

$$v(\lambda) := \sum_{l=1}^{2\ell} c_l v_{\lambda,l}(\lambda) \quad \text{for some } c_\bullet \in \mathbb{C}. \tag{3.10}$$

We do not discuss the analytic dependence of $v(\lambda)$ on $\lambda \in \mathbb{C}$ here, as it will be revealed at a later point (Section 4.2). Furthermore, by the preceding discussion, any solution u_λ to the model problem without boundary conditions can be expressed as $u_\lambda = r^\lambda v(\lambda)$, where $v(\lambda)$ is given in (3.10).

3.3 The case $\lambda = 0$

The subsequent derivation does not cover $\lambda \neq 0$, which is why we address this case now. A solution $u = r^\lambda v$ to the model problem (3.2) is called a *trivial solution* if $v = 0$.

Lemma 3.3. *The model problem (3.2) admits for $\lambda = 0$ and any angle $0 < \alpha \leq 2\pi$ only the trivial solution for Dirichlet and mixed boundary conditions, and only constant solutions for Neumann boundary conditions.*

This is not a new result but included for completeness.

Proof. Consider an elliptic tuple A, and assume $\mathcal{L}_A(\lambda)v = 0$ for $\lambda = 0$. By (3.7), this reduces to

$$\partial_\varphi(b_2\partial_\varphi v)(\varphi) = 0, \tag{3.11}$$

due to $\partial_\varphi b_2 = -b_1$. The 2ℓ linearly independent solutions are given by

$$v(\varphi) = c_1 + P(\varphi)c_2 \quad \text{for } c_\bullet \in \mathbb{C}^\ell,$$

where $P(\varphi) = \int_0^\varphi b_2^{-1}(s)ds$. Note that $P(\varphi) > 0$ for $\varphi > 0$, ensuring that $P(\varphi)$ is invertible. Dirichlet boundary conditions yield

$$c_1 = 0 \quad \text{and} \quad c_1 + P(\alpha)c_2 = 0 \implies c_1 = 0 = c_2,$$

so only the trivial solution exists. Neumann boundary conditions (check (3.7)) yield $c_2 = 0$ for $\varphi \in \{0, \alpha\}$ (b_2 is invertible), so $v(\varphi) = c_1$ is the most general solution. Finally, for mixed boundary conditions, both $c_2 = 0 = c_1$ are enforced, so only the trival solution exists. This completes the proof. □

3.4 Laplace equation

If we assume $A_{11} = \mathrm{Id}_\ell = A_{22}$ and $A_{12} = 0$, then the system reduces to decoupled Laplace equations in ℓ components. In this case, $\mathcal{L}_A(\lambda) = \mathrm{Id}_\ell(\partial_\varphi^2 + \lambda^2)$ and solutions to the λ-dependent ODE without boundary conditions are given by

$$v_\lambda(\varphi) = c_0 \sin(\lambda\varphi) + c_1 \cos(\lambda\varphi) \quad \text{for } c_0, c_1 \in \mathbb{C}^\ell.$$

Let us implement boundary conditions for $\varphi \in \{0, \alpha\}$. Note that $\mathcal{N}_A(\lambda) = \mathrm{Id}_\ell\,\partial_\varphi$, so nontrivial solutions are given by $c_1 = 0$ and $\lambda \in \frac{\pi}{\alpha} \cdot \mathbb{Z} \setminus \{0\}$ for Dirichlet, $c_0 = 0$ and $\lambda \in \frac{\pi}{\alpha} \cdot \mathbb{Z}$ for Neumann, and $c_1 = 0$ and $\lambda \in \frac{\pi}{2\alpha} \cdot \mathbb{Z} \setminus \{0\}$ for mixed boundary conditions (compare to §2.1 in [20]). This leads for $\alpha \leq \pi$ to the bounds $|\operatorname{Re}\lambda| \geq 1$ for Dirichlet and Neumann boundary conditions (ignoring constant solutions at $\lambda = 0$ for the latter) and $|\operatorname{Re}\lambda| \geq \frac{1}{2}$ for mixed boundary conditions. Although this is the simplest example of an elliptic system, we will derive similar lower bounds for more general elliptic systems.

4 Analysis of the model problem without boundary conditions

4.1 Algebraic properties of L_A

Reduction to monic matrix polynomials

The elliptic operator L_A is normalized in the sense that one only needs to consider the case of $A_{22} = \mathrm{Id}_\ell$.

Lemma 4.1. *Consider an elliptic tuple $A = (A_{11}, A_{12}, A_{22})$. The model problem*

$$L_A(\partial_{x_1}, \partial_{x_2})u = 0 \quad \text{on } \mathcal{K}_\alpha, \quad B_A^\pm(\partial_{x_1}, \partial_{x_2})u = 0 \quad \text{on } \Gamma^\pm$$

admits the solution $u = r^\lambda v$ if and only if the model problem

$$L_{\tilde{A}}(\partial_{x_1}, \partial_{x_2})u = 0 \quad \text{on } \mathcal{K}_\alpha, \quad B_{\tilde{A}}^\pm(\partial_{x_1}, \partial_{x_2})u = 0 \quad \text{on } \Gamma^\pm$$

for $\tilde{A} = (A_{22}^{-1/2}A_{11}A_{22}^{-1/2}, A_{22}^{-1/2}A_{12}A_{22}^{-1/2}, \mathrm{Id}_\ell)$ admits the solution $r^\lambda\tilde{v}$, where $\tilde{v} = A_{22}^{1/2}v$.

Here, we write $A_{22}^{-1/2} = \left(A_{22}^{1/2}\right)^{-1}$. Note that A is an elliptic tuple if and only if $\tilde{A}$ is an elliptic tuple: This can be shown by using Lemma 3.2 and the determinant product rule (observe $\tilde{A}_{11} > 0$ due to $A_{22}^{-1/2}$ being symmetric, invertible and Sylvester's law of inertia, see Theorem 4.5.8 in [17]). This ensures that $L_{\tilde{A}}$ is a well-defined elliptic operator.

Proof. Assume that $L_A u = 0$. It is clear that $\tilde{u} := A_{22}^{1/2} u$ will be a solution to $L_{\tilde{A}}\tilde{u} = 0$. Also, if u has the form $u = r^\lambda v$, then $\tilde{u}$ has the form $r^\lambda \tilde{v}$ for $\tilde{v} := A_{22}^{1/2} v$. Lastly, we show that boundary conditions are transformed accordingly: Assume $\varphi \in \{0, \alpha\}$. Observe for Dirichlet boundary conditions:

$$v(\varphi) = 0 \iff A^{1/2} v(\varphi) = 0, \tag{4.1}$$

due to invertibility of $A^{1/2}$. For Neumann boundary conditions, assume:

$$b_2(\varphi)(\partial_\varphi v)(\varphi) + \frac{\lambda}{2} b_1(\varphi) v(\varphi) = 0,$$

where $b_\bullet$ are defined as in (3.8) by A. Observe that $\tilde{b}_\bullet = A_{22}^{-1/2} b_\bullet A_{22}^{-1/2}$ are the corresponding coefficients for the tuple $\tilde{A}$ and that:

$$\tilde{b}_2(\varphi)(\partial_\varphi \tilde{v})(\varphi) + \frac{\lambda}{2} \tilde{b}_1(\varphi)\tilde{v}(\varphi) = A_{22}^{-1/2}\big(b_2(\varphi)(\partial_\varphi v)(\varphi) + \frac{\lambda}{2} b_1(\varphi) v(\varphi)\big) = 0.$$

Since $A_{22}^{1/2}$ is invertible, the converse implication is clear. □

In the following, we refer to elliptic tuples A with $A_{22} = \mathrm{Id}_\ell$ as *monic elliptic tuples*, and to L_A as a *monic elliptic operator*. Additionally, we call $\tilde{A}$, given in Lemma 4.1, the *monic reduction* of A.

Factorization of monic matrix polynomials

Consider a monic elliptic operator L_A. Taking $\xi = (1, \beta) \in \mathbb{R}^2$, we obtain the matrix polynomial

$$L_A(1, \beta) = L_A(\xi) = A_{11} + 2A_{12}\beta + \mathrm{Id}_\ell\,\beta^2,$$

which can be factorized into linear terms.

Lemma 4.2. 1 *Consider a monic elliptic tuple $A = (A_{11}, A_{12}, \mathrm{Id}_\ell)$. Then:*

i) There exists some $V \in \mathrm{Mat}_\ell(\mathbb{C})$ such that

$$A_{11} + 2A_{12}\beta + \mathrm{Id}_\ell\,\beta^2 = (V^* - \mathrm{Id}_\ell\,\beta)(V - \mathrm{Id}_\ell\,\beta) \quad \forall \beta \in \mathbb{C}, \tag{4.2}$$

and $\sigma(V) \subset \mathrm{UHP}$.

ii) Assume $V = C + iD$ with $C = \mathrm{Re}\, V$ and $D = \mathrm{Im}\, V$, then

$$A_{12} = -\frac{1}{2}(C + C^T), \qquad A_{11} = C^T C + D^T D, \tag{4.3}$$

$$D = D^T, \qquad C^T D = D^T C. \tag{4.4}$$

2 *On the other hand, assume $V = C + iD$ for $C, D \in \mathrm{Mat}_\ell(\mathbb{R})$ which satisfy the algebraic relations (4.4). Then:*

i) V *satisfies (4.2) for* $A_\bullet$ *as defined by (4.3).*

ii) If $\sigma(V) \cap \mathbb{R} = \emptyset$*, then the tuple defined by* $A_{22} = \mathrm{Id}_\ell$ *and (4.3) is monic elliptic.*

This result relies on a factorization property of nonnegative matrix polynomials as found in [10]. For the convenience of the reader, this result and all definitions necessary to understand it are summarized in Appendix C, which we refer to in the proof.

Proof. 1.i) Existence of such V with $\sigma(V) \subset \mathrm{clos}(\mathrm{UHP})$ follows from Theorem C.1 since the matrix polynomial $L_A(1, \beta)$ is monic, self-adjoint, and nonnegative due to Lemma 3.2. It remains to show $\sigma(V) \cap \mathbb{R} = \emptyset$. Assuming the contrary, we have $\beta \in \sigma(V) \cap \mathbb{R}$, implying

$$\det(V - \mathrm{Id}_\ell \beta) = 0 \overset{(4.2)}{\Longrightarrow} \det\bigl(A_{11} + 2A_{12}\beta + \mathrm{Id}_\ell \beta^2\bigr) = 0,$$

a contradiction to Lemma 3.2.
1.ii) Writing $V = C + iD$, $A_{11} = V^*V$, and $A_{12} = -\frac{1}{2}(V^* + V)$, we get:

$$\begin{aligned} A_{11} &= (C^T - iD^T)(C + iD) = C^T C + D^T D + i(-D^T C + C^T D), \\ A_{12} &= -\frac{1}{2}(C^T + C) + i(-D^T + D). \end{aligned}$$

Now, the imaginary part in both RHS's must vanish since A_{11} and A_{12} are real-valued. Thus, (4.3) and (4.4) follow.
2.i) Assume we define $V = C + iD$ for $C, D \in \mathrm{Mat}_\ell(\mathbb{R})$ satisfying (4.4). It is clear from the last calculation that we obtain (4.2) for $A_\bullet$ given in (4.3).
2.ii) We additionally assume $\sigma(V) \cap \mathbb{R} = \emptyset$ and prove that the corresponding operator L_A is elliptic. First, we need to show that $A_{11}, A_{12}, \mathrm{Id}_\ell$ are all symmetric and that $A_{11}, \mathrm{Id}_\ell > 0$. Symmetry is clear by the definition of A_{12} and A_{11} given in (4.3). For positive definiteness, note that $A_{11} = V^*V$, which is positive definite since $0 \notin \sigma(V)$. Lastly, using Lemma 3.2, it suffices to show that

$$\det(L_A(1, \beta)) \neq 0 \quad \forall \beta \in \mathbb{R}.$$

This follows from

$$\det\bigl((V^* - \mathrm{Id}_\ell \beta)(V - \mathrm{Id}_\ell \beta)\bigr) = \det(V^* - \mathrm{Id}_\ell \beta)\det(V - \mathrm{Id}_\ell \beta) \neq 0 \quad \forall \beta \in \mathbb{R},$$

since $\sigma(V) \cap \mathbb{R} = \emptyset$ and thus also $\sigma(V^*) \cap \mathbb{R} = \emptyset$. By definition, L_A is monic, which completes the proof. □

Definition 4.3. Assume that L_A is a monic elliptic operator. We call $V \in \mathrm{Mat}_\ell(\mathbb{C})$ fulfilling 1.i) in Lemma 4.2 a *standard root* of L_A.

Example 4.4. In the case of the Laplacian tuple $A = (\mathrm{Id}_\ell, 0, \mathrm{Id}_\ell)$, a standard root is given by $V = i\,\mathrm{Id}_\ell$.

By Lemma 4.2, it is clear that any monic elliptic operator admits at least one standard root. We derive some more interesting properties of standard roots.

Lemma 4.5. *Let* V *be a standard root of a monic elliptic differential operator* L_A*. Then* $D = \mathrm{Im}\, V$ *is positive definite.*

Proof. Let us write $V = C + iD$ for $C, D \in \mathrm{Mat}_\ell(\mathbb{R})$, where V is the standard root under consideration. Due to Lemma 4.2, we have $D = D^T$ such that all eigenvalues of D are real, and we need to show that they are all positive. First, we argue that $0 \notin \sigma(D)$. Assume the contrary, $0 \in \sigma(D)$, and derive a contradiction. In this case, there is $v \in \mathbb{R}^\ell \setminus \{0\}$ such that $Dv = 0$. Due to the commutativity relations in Lemma 4.2, we have $0 = C^T Dv = DCv$. From this, it follows that either $Cv = 0$ or $Cv \neq 0$ is also an eigenvector of D with eigenvalue 0. If $Cv = 0$, it would follow that $Vv = 0$, which is a contradiction since 0 cannot be an eigenvalue of V by Lemma 4.2. So, assuming the latter, $\mathrm{span}_\mathbb{C}(\{v, Cv\}) \subset \ker D$. But repeating the discussion with Cv instead of v, we conclude that $C^n v \neq 0$, for arbitrary $n \in \mathbb{N}$, must also be an eigenvector of D with eigenvalue 0. Define the cyclic (complex) subspace generated by v:

$$S_v := \mathrm{span}_\mathbb{C}(\{C^n v : n \in \mathbb{N}_0\}) .$$

By the above derivation, we have $D(S_v) = \{0\}$, and by its definition, it is clear that $C(S_v) \subset S_v$. This implies $C|_{S_v} = V|_{S_v}$. Thus, S_v is an eigenspace of V and moreover, V has a real-valued matrix representation (the same as C) for the subspace S_v. This implies that V has either a real eigenvalue or two different complex conjugated eigenvalues, which contradicts V being a standard root since $\sigma(V) \subset \mathrm{UHP}$. Thus, we have shown that $0 \notin \sigma(D)$.
Next, we argue why D cannot have negative eigenvalues. This follows by a simple scaling argument. Define for any $\rho \geq 1$

$$V_\rho := C + i \cdot \rho D.$$

Observe that $C_\rho := \mathrm{Re}\, V_\rho = C$ and $D_\rho := \mathrm{Im}\, V_\rho = \rho D$ still fulfill the algebraic relations (4.4). Moreover, the operator $L_{A,\rho}$ defined by V_ρ (second part of Lemma 4.2) is elliptic for any $\rho \geq 1$, since we can write the matrix polynomial as

$$L_{A,\rho}(1, \beta) = (\rho - 1)D^2 + A_{11} + 2A_{12}\beta + \mathrm{Id}_\ell\, \beta^2,$$

and $L_A(1, \beta) > 0$, for $\beta \in \mathbb{R}$, implies $L_{A,\rho}(1, \beta) > 0$ since $(\rho - 1)D^2$ is positive semi-definite. Now, let us assume that $r \in \sigma(D)$ for some $r < 0$ and derive a contradiction. Since all eigenvalues of D are nonzero, we get for the spectrum

$$\sigma(V_\rho) = \sigma(C + i\rho D) \xrightarrow{\rho \to \infty} i\rho \cdot \sigma(D)$$

in the appropriate sense. Since $0 > r \in \sigma(D)$, it means that for sufficiently large $\rho > 1$ there exists some $\beta_\rho \in \sigma(V_\rho)$ with $\mathrm{Im}\, \beta_\rho < 0$. Consequently, because the path

$$[1, \infty) \ni \rho \mapsto V_\rho \in \mathrm{Mat}_\ell(\mathbb{C})$$

is continuous and $\sigma(V) \subset \mathrm{UHP}$, there must be some intermediate $1 < \tilde{\rho} < \rho$ where

$$\sigma(V_{\tilde{\rho}}) \cap \mathbb{R} \neq \emptyset.$$

However, this contradicts the ellipticity of $L_{A,\tilde{\rho}}$ which we just have shown. □

The next statement summarizes the main result concerning standard roots.

Theorem 4.6. *Consider L_A, a monic elliptic operator, and V a standard root of L_A.*

i) The standard root of L_A is unique.

ii) V *satisfies* $A_{11} + 2A_{12}V + \mathrm{Id}_\ell V^2 = 0 = A_{11} + 2A_{12}\overline{V} + \mathrm{Id}_\ell \overline{V}^2$.

iii) V *can be written as* $V = (S + i\,\mathrm{Id}_\ell)D$*, where both* $S, D \in \mathrm{Mat}_\ell(\mathbb{R})$ *are symmetric and* $D > 0$.

Conversely, any $V = (S + i\,\mathrm{Id}_\ell)D$ *with* $S, D \in \mathrm{Mat}(\mathbb{R}, \ell)$ *symmetric and* $D > 0$ *is the standard root of the monic elliptic operator* L_A *for the tuple*

$$A = (A_{11} = V^*V, A_{12} = -\frac{1}{2}(V + V^*), A_{22} = \mathrm{Id}_\ell). \tag{4.5}$$

Proof. Consider L_A, a monic elliptic operator, and let $V \in \mathrm{Mat}_\ell(\mathbb{C})$ be a standard root.

i) Uniqueness of V follows from $L_A(1, \beta) = (V^* - \mathrm{Id}_\ell\,\beta)(V - \mathrm{Id}_\ell\,\beta)$, $\sigma(V) \subset \mathrm{UHP}$, $\sigma(V^*) \subset -\mathrm{UHP}$, and uniqueness of *monic* Γ*-spectral right divisors* for matrix polynomials. For the latter, we refer to §4.1 in [10], in particular Theorem 4.1 and the comment thereafter.

ii) The first equality follows by $A_{11} = V^*V$ and $A_{12} = -\frac{1}{2}(V + V^*)$. The second by complex conjugation and $A_\bullet \in \mathrm{Mat}_\ell(\mathbb{R})$.

iii) By Lemma 4.2 and Lemma 4.5, we can write $V = C + iD$ for $C, D \in \mathrm{Mat}_\ell(\mathbb{R})$, where $D > 0$ and (4.4) holds. In particular, we can invert D and thus $V = (S + i\,\mathrm{Id}_\ell)D$ for $S = CD^{-1}$. The symmetry of S follows from:

$$S^T = (D^{-1})^T C^T = D^{-1}C^T D D^{-1} \overset{(4.4)}{=} D^{-1}DCD^{-1} = CD^{-1} = S.$$

Lastly, consider $V = (S + i\,\mathrm{Id}_\ell)D$ with $S, D \in \mathrm{Mat}(\mathbb{R}, \ell)$ symmetric and $D > 0$. We show that V is the standard root of L_A for A given in (4.5). By Lemma 4.2, it suffices to verify the algebraic relations

$$D = D^T, \quad (SD)^T D = D(SD),$$

which are trivially fulfilled by symmetry of S and D, and to show $\sigma(V) \cap \mathbb{R} = \emptyset$. The latter statement follows, due to $D > 0$, from Lemma E.1 and $\sigma(V) = \sigma(D^{1/2}SD^{1/2} + iD)$. □

Remark. One may wonder if V with the properties given in Theorem 4.6 is diagonalizable. A counter example is given by:

$$V = \begin{pmatrix} 3i & -\frac{1}{\sqrt{3}} \\ -\sqrt{3} & i \end{pmatrix} = \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark. The discussion generalizes for nonmonic elliptic tuples $A = (A_{11}, A_{12}, A_{22})$ by defining a standard root $V \in \mathrm{Mat}_\ell(\mathbb{C})$ to satisfy $\sigma(V) \subset \mathrm{UHP}$ and

$$A_{11} + 2A_{12}\beta + A_{22}\beta^2 = (V^* - \mathrm{Id}_\ell\,\beta)A_{22}(V - \mathrm{Id}_\ell\,\beta).$$

One can show that V is related to the standard root $\tilde{V}$ of the monic reduction $\tilde{A}$ of A by $V = A_{22}^{-1/2}\tilde{V}A_{22}^{1/2}$. For our purposes, it suffices to work with the monic reduction.

4.2 Solutions for the model problem without boundary conditions

It is time to close the gap between algebra and analysis and justify the time spent on exploring the algebraic structure of L_A. In this section, solutions for the model problem without boundary conditions are derived. The idea is based on the solution basis given in §2.2 of [6]. For this, we first need to define the complex exponent $\lambda \in \mathbb{C}$ of a complex number $\sigma \in \mathbb{C}\setminus\{0\}$. We do so by using three branches of the complex logarithm, i.e. three different $\arg$-functions (see §8.2 in [4] for a reference). To distinguish between them, we introduce the set of symbols $a \in \{o, +, -\}$ and use these as decoration. Explicitly, we define

$$\sigma^{\lambda_a} := \exp(\lambda \log_a(\sigma)) \quad \text{for} \quad \log_a(\sigma) := \log(|\sigma|) + i\lambda \arg_a(\sigma). \tag{4.6}$$

Here, $\log(r)$ for $r \in \mathbb{R}$ is simply the standard logarithm for positive real numbers. The $\arg_a$-functions are uniquely determined by requiring that $\log_a$ inverts $\exp$ on the domain $\mathbb{C}\setminus\{0\}$ and by the following conditions:

$$\arg_+(\sigma) \in [0, 2\pi), \quad \arg_o(\sigma) \in (-\pi, \pi], \quad \arg_-(\sigma) \in (-2\pi, 0] \quad \forall \sigma \in \mathbb{C}\setminus\{0\}.$$

With this choice, $\log_o$ represents the principal logarithm, denoted by $\log$ in the following, and it exhibits a discontinuity along the branch cut of the negative real axis. The introduction of $\log_+$ and $\log_-$ serves to provide continuous extensions of the logarithm that avoid the discontinuity at $\arg(\sigma) = \pi$, shifting the branch cut instead to the positive real axis. Note that $\arg_o = \arg_+$ on UHP and $\arg_o = \arg_-$ on $-\mathrm{UHP}$.

Consider $0 < \alpha < 2\pi$ and the model problem for a monic elliptic tuple A with standard root V. We define, for $\lambda \in \mathbb{C}\setminus\{0\}$ and $c_1, c_2 \in \mathbb{C}^\ell$ arbitrary, the complex vector-valued functions:

$$u_\lambda : \mathbb{R}^2 \setminus \{0\} \to \mathbb{C}^\ell,\ (x_1, x_2) \mapsto (x_1 \operatorname{Id}_\ell + x_2 V)^{\lambda_+} c_1 + (x_1 \operatorname{Id}_\ell + x_2 \overline{V})^{\lambda_-} c_2. \tag{4.7}$$

Here, the exponentiation of matrices is defined via the functional calculus (see Appendix D). It is well-defined if $0 \notin \sigma(x_1 \operatorname{Id}_\ell + x_2 V) = x_1 + x_2\sigma(V)$ is ensured for any $(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$, which follows from $\sigma(V) \subset \mathrm{UHP}$. We will show that any solution to the model problem without boundary conditions (3.4) is of the form (4.7). For this, note that $u_\lambda = r^\lambda v_\lambda$, where

$$v_\lambda : [0, 2\pi) \to \mathbb{C}^\ell, \quad \varphi \mapsto (\cos(\varphi) \operatorname{Id}_\ell + \sin(\varphi)V)^{\lambda_+} c_1 + (\cos(\varphi) \operatorname{Id}_\ell + \sin(\varphi)\overline{V})^{\lambda_-} c_2. \tag{4.8}$$

Remark. Let us briefly discuss the choice of $\lambda_\pm$. Since $\sigma(V) \subset \mathrm{UHP}$, we have

$$\begin{aligned} \sigma(\cos(\varphi) \operatorname{Id}_\ell + \sin(\varphi)V) &\subset \mathrm{UHP} && \text{for } 0 < \varphi < \pi, \\ \sigma(\cos(\varphi) \operatorname{Id}_\ell + \sin(\varphi)V) &\subset -\mathrm{UHP} && \text{for } \pi < \varphi < 2\pi. \end{aligned}$$

Because the $\arg$-function in the principle logarithm has a discontinuity at $\varphi = \pi$, the function $\varphi \mapsto (\cos(\varphi) \operatorname{Id}_\ell + \sin(\varphi)V)^\lambda$ is not continuous. This issue is resolved by changing λ to λ_+. A similar reasoning applies to $\overline{V}$ and λ_-.

Also, note that u_λ satisfies $L_A u_\lambda = 0$. For this, observe that

$$\begin{aligned} &L_A(\partial_{x_1}, \partial_{x_2}) u_\lambda = \sum_{i,j=1}^{2} A_{ij} \partial_{x_i} \partial_{x_j} u_\lambda \\ =&(A_{11} + 2A_{12}V + V^2)(x_1 \operatorname{Id}_\ell + x_2 V)^{(\lambda-2)_+} c_1 + (A_{11} + 2A_{12}\overline{V} + \overline{V}^2)(x_1 \operatorname{Id}_\ell + x_2 \overline{V})^{(\lambda-2)_-} c_2 \end{aligned}$$

vanishes by Theorem 4.6. Here, the chain rule is applied, and it is important to note that the differentiation rules for $\bullet^{\lambda_a}$ are consistent regardless of the choice $a \in \{o, +, -\}$.

We have the following result.

Proposition 4.7. *Consider* $0 < \alpha < 2\pi$ *and a monic elliptic operator* L_A *with standard root* V*. For* $\lambda \in \mathbb{C} \setminus \{0\}$*, any solution* $u_\lambda = r^\lambda v$ *to* $L_A u_\lambda = 0$ *is of the form (4.7). Similarly, any solution to* $\mathcal{L}_A(\lambda) v_\lambda = 0$ *is of the form (4.8).*

Proof. By the preceding discussion and the final remark in Section 3.2, it suffices to show that there are no $(c_1, c_2) \in \mathbb{C}^{2\ell} \setminus \{0\}$ such that u_λ in (4.7) reduces to $u_\lambda = 0$. Let us assume the contrary and derive a contradiction. Then there are $c_1, c_2 \in \mathbb{C}^\ell$, at least one $c_\bullet \neq 0$, such that

$$(\cos(\varphi)\,\mathrm{Id}_\ell + \sin(\varphi)V)^{\lambda_+} c_1 + (\cos(\varphi)\,\mathrm{Id}_\ell + \sin(\varphi)\overline{V})^{\lambda_-} c_2 = 0 \quad \text{for } \varphi \in [0, \alpha). \tag{4.9}$$

In particular, for $\varphi = 0$, this leads to $c_1 = -c_2$. Differentiating (4.9) and evaluating at $\varphi = 0_+$, we get the condition

$$(V - \overline{V})c_1 = 0 \implies \det(\operatorname{Im} V) = 0,$$

which is a contradiction to Theorem 4.6. □

Remark. Note that we excluded $\lambda = 0$ because

$$u_0(x_1, x_2) = c_1 + c_2, \quad \text{for } c_1, c_2 \in \mathbb{C}^\ell,$$

does not yield 2ℓ linearly independent solutions. For this reason, the case $\lambda = 0$ was treated separately in Lemma 3.3.

Example 4.8. Assume the standard root V in the above discussion is diagonalizable. Then we can write $V = QBQ^{-1}$ for $Q, B \in \mathrm{Mat}_\ell(\mathbb{C})$ with $B = \mathrm{diag}(\beta_1, \ldots, \beta_\ell)$ the diagonal matrix of eigenvalues. A corresponding eigenvector q_k to β_k for $1 \leq k \leq \ell$ is given by the k-th row of Q. Using properties of the functional calculus (Appendix D), we derive:

$$(x_1\,\mathrm{Id}_\ell + x_2 V)^{\lambda_+} = (x_1\,\mathrm{Id}_\ell + x_2 QBQ^{-1})^{\lambda_+} = Q(x_1\,\mathrm{Id}_\ell + x_2 B)^{\lambda_+} Q^{-1}.$$

By a similar argument for $(x_1\,\mathrm{Id}_\ell + x_2\overline{V})^{\lambda_-}$, it is deduced that any solution $u_\lambda = r^\lambda v$ to $L_A u_\lambda = 0$ can be written as

$$u_\lambda(x_1, x_2) = \sum_{l=1}^{\ell} d_l q_l (x_1 + x_2\beta_l)^{\lambda_+} + d_{l+\ell}\overline{q}_l (x_1 + x_2\overline{\beta_l})^{\lambda_-} \quad \text{for } d_\bullet \in \mathbb{C}.$$

To see this, observe that $(x_1\,\mathrm{Id}_\ell + x_2 B)^{\lambda_+}$ is diagonal, with entries $(x_1 + x_2\beta_\bullet)^{\lambda_+}$, and that the vectors $d_\bullet q_\bullet$ are related to $c_\bullet$ in (4.7) through Q^{-1}.

Remark. The representation of solutions for the model problem without boundary conditions provided in (4.7) is new. However, an alternative representation involving complex contour integrals is known and can be found in [5] and [6]. In these references, the explicit solution basis discussed in Example 4.8 (with a slightly different representation) is introduced. In fact, this basis served as the foundational idea for the representation of solutions for general (non-diagonalizable) V as presented in this work.

Remark. The reason for avoiding $\alpha = 2\pi$ in the discussion and Prop. 4.7 is to prevent multivalued $\arg_a$-functions. Nonetheless, Prop. 4.7 can be canonically generalized to $\alpha = 2\pi$ by considering a continuous continuation of $\arg_a$ from the case $\alpha < 2\pi$.

5 Analysis of the model problem with boundary conditions

After deriving explicit formulas for solutions of the model problem without boundary conditions, we now investigate solutions of the model problem with boundary conditions. This will be equivalent to the vanishing of the determinant of some matrix $M_{\lambda,\alpha}$. For the remainder of this section, let us assume $0 < \alpha < 2\pi$, and that we consider a monic elliptic tuple $A = (A_{11}, A_{12}, \mathrm{Id}_\ell)$ with standard root V. We consistently write $V = (S + i\,\mathrm{Id}_\ell)D = C + iD$ for $C, D, S \in \mathrm{Mat}_\ell(\mathbb{R})$ given in Lemma 4.2 and Theorem 4.6. In this case, by Prop. 4.7, we have that $u = r^\lambda v$ for $\lambda \in \mathbb{C} \setminus \{0\}$ is a solution to $L_A(\partial_{x_1}, \partial_{x_2})u = 0$ if there exist $c_1, c_2 \in \mathbb{C}^\ell$ such that:

$$v(\varphi) = (\cos(\varphi)\,\mathrm{Id}_\ell + \sin(\varphi)V)^{\lambda_+} c_1 + (\cos(\varphi)\,\mathrm{Id}_\ell + \sin(\varphi)\overline{V})^{\lambda_-} c_2. \tag{5.1}$$

5.1 Dirichlet boundary conditions

Assume that $r^\lambda v$, for v in (5.1), satisfies Dirichlet boundary conditions on $\Gamma^\pm$. Consequently,

$$0 = v(0) = c_1 + c_2, \quad 0 = v(\alpha) = V_\alpha^{\lambda_+} c_1 + \overline{V_\alpha}^{\lambda_-} c_2,$$

where we denote

$$V_\alpha = \cos(\alpha)\,\mathrm{Id}_\ell + \sin(\alpha)V = \cos(\alpha)\,\mathrm{Id}_\ell + \sin(\alpha)SD + i\sin(\alpha)D. \tag{5.2}$$

Thus, finding $u = r^\lambda v \neq 0$ for the model problem with Dirichlet boundary conditions is equivalent to the vanishing of the determinant of $M_{\lambda,\alpha} \in \mathrm{Mat}_{2\ell}(\mathbb{C})$ which is given by

$$M_{\lambda,\alpha} := \begin{pmatrix} \mathrm{Id}_\ell & \mathrm{Id}_\ell \\ V_\alpha^{\lambda_+} & \overline{V_\alpha}^{\lambda_-} \end{pmatrix}.$$

To see this, apply the block vector $(c_1, c_2) \in \mathbb{C}^\ell$ from the right to $M_{\lambda,\alpha}$. Using Lemma 2.1, we derive:

$$0 = \det(M_{\lambda,\alpha}) \iff 0 = \det\Big(V_\alpha^{\lambda_+} - \overline{V_\alpha}^{\lambda_-}\Big). \tag{5.3}$$

We can further manipulate the RHS in (5.3):

$$0 = \det\Big(V_\alpha^{\lambda_+} - \overline{V_\alpha}^{\lambda_-}\Big) \iff 0 = \det\Big(D^{1/2}V_\alpha^{\lambda_+}D^{-1/2} - D^{1/2}\overline{V_\alpha}^{\lambda_+}D^{-1/2}\Big)$$

by the product rule for determinants since D is positive definite and thus $\det\big(D^{\pm 1/2}\big) \neq 0$. We set $Z_\alpha := D^{1/2}V_\alpha D^{-1/2}$ and note that

$$Z_\alpha = \cos(\alpha) + D^{1/2}SD^{1/2}\sin(\alpha) + iD\sin(\alpha) \tag{5.4}$$

is a symmetric matrix. By properties of the functional calculus like $D^{1/2}V_\alpha^{\lambda_+}D^{-1/2} = Z_\alpha^{\lambda_+}$ (Appendix D), we derive:

$$0 = \det(M_{\lambda,\alpha}) \iff 0 = \det\Big(Z_\alpha^{\lambda_+} - \overline{Z_\alpha}^{\lambda_-}\Big).$$

The discussion shows the following result.

Proposition 5.1. *Consider $0 < \alpha < 2\pi$ and a monic elliptic operator L_A with standard root $V = (S + i\,\mathrm{Id}_\ell)D$. The model problem with Dirichlet boundary conditions has for $\lambda \in \mathbb{C} \setminus \{0\}$ a solution $r^\lambda v \neq 0$ if and only if*

$$0 = \det\Big(Z_\alpha^{\lambda_+} - \overline{Z_\alpha}^{\lambda_-}\Big), \tag{5.5}$$

for Z_α given in (5.4).

5.2 Mixed boundary conditions

Before we continue, let us characterize the Neumann boundary condition $\mathcal{N}_A v(\varphi) = 0$ for $\varphi \in \{0, \alpha\}$. For this, we calculate (note that $n = (-\sin(\varphi), \cos(\varphi)) = \frac{1}{r}(-x_2, x_1)$ is the normal vector):

$$\sum_{i,j=1}^{2} A_{ij} n_i \partial_{x_j} (x_1 \operatorname{Id}_\ell + x_2 V)^{\lambda_+} = \frac{\lambda}{r}\big(-x_2(A_{11} + A_{12}V) + x_1(A_{12} + V)\big)(x_1 \operatorname{Id}_\ell + x_2 V)^{(\lambda-1)_+}.$$

Since V solves $A_{11} + A_{12}V = -A_{12}V - V^2$ (see Theorem 4.6), we obtain

$$\sum_{i,j=1}^{2} A_{ij} n_i \partial_{x_j} (x_1 \operatorname{Id}_\ell + x_2 V)^{\lambda_+} = \frac{2\lambda}{r}(A_{12} + V)(x_1 \operatorname{Id}_\ell + x_2 V)^{\lambda_+}.$$

Similarly, we derive $\sum_{i,j=1}^{2} A_{ij} n_i \partial_{x_j} (x_1 \operatorname{Id}_\ell + x_2 \overline{V})^{\lambda_-} = \frac{2\lambda}{r}(A_{12} + \overline{V})(x_1 \operatorname{Id}_\ell + x_2 \overline{V})^{\lambda_-}$. Now, assume that $r^\lambda v \neq 0$, for v in (5.1), satisfies Dirichlet boundary conditions on Γ^+ and Neumann boundary conditions on Γ^-. Thus, the coefficients $c_\bullet$ in (5.1) satisfy (recall that we assumed $\lambda \neq 0$):

$$0 = c_1 + c_2, \quad 0 = (A_{12} + V)V_\alpha^{\lambda_+} c_1 + (A_{12} + \overline{V})\overline{V_\alpha}^{\lambda_-} c_2,$$

for V_α in (5.2). This is equivalent to the condition $0 = \det(M_{\lambda,\alpha})$, where:

$$M_{\lambda,\alpha} := \begin{pmatrix} \operatorname{Id}_\ell & \operatorname{Id}_\ell \\ (A_{12} + V)V_\alpha^{\lambda_+} & (A_{12} + \overline{V})\overline{V_\alpha}^{\lambda_-} \end{pmatrix}.$$

Using Lemma 2.1, we derive:

$$0 = \det(M_{\lambda,\alpha}) \iff 0 = \det\Big((A_{12} + V)V_\alpha^{\lambda_+} - (A_{12} + \overline{V})\overline{V_\alpha}^{\lambda_-}\Big).$$

Write $A_{12} = -\frac{1}{2}(V + V^*)$ and $V = (S + i \operatorname{Id}_\ell)D$ (see Theorem 4.6) such that:

$$A_{12} + V = \frac{1}{2}[S, D] + iD, \quad A_{12} + \overline{V} = \frac{1}{2}[S, D] - iD, \tag{5.6}$$

and we conclude:

$$0 = \det(M_{\lambda,\alpha}) \iff 0 = \det\left(\left(\frac{1}{2}[S, D] + iD\right) V_\alpha^{\lambda_+} - \left(\frac{1}{2}[S, D] - iD\right)\overline{V_\alpha}^{\lambda_-}\right).$$

Now, again using $Z_\alpha = D^{1/2} V_\alpha D^{-1/2}$ and $\det\big(D^{\pm 1/2}\big) \neq 0$, we reformulate $0 = \det(M_{\lambda,\alpha})$ as:

$$\begin{aligned} 0 &= \det\left(D^{-1/2}\left(\frac{1}{2}[S, D] + iD\right) D^{-1/2} Z_\alpha^{\lambda_+} - D^{-1/2}\left(\frac{1}{2}[S, D] - iD\right) D^{-1/2}\overline{Z_\alpha}^{\lambda_-}\right) \\ &= \det\left(\frac{1}{2} D^{-1/2}[S, D]D^{-1/2}(Z_\alpha^{\lambda_+} - \overline{Z_\alpha}^{\lambda_-}) + i(Z_\alpha^{\lambda_+} + \overline{Z_\alpha}^{\lambda_-})\right). \end{aligned}$$

This leads to the following result.

Proposition 5.2. *Consider $0 < \alpha < 2\pi$ and a monic elliptic operator L_A with standard root $V = (S + i \operatorname{Id}_\ell)D$. The model problem with mixed boundary conditions has for $\lambda \in \mathbb{C} \setminus \{0\}$ a solution $r^\lambda v \neq 0$ if and only if*

$$0 = \det\left(\frac{1}{2}[D^{-1/2} S D^{-1/2}, D](Z_\alpha^{\lambda_+} - \overline{Z_\alpha}^{\lambda_-}) + i(Z_\alpha^{\lambda_+} + \overline{Z_\alpha}^{\lambda_-})\right),$$

for Z_α given in (5.4).

5.3 Additional ellipticity conditions for Neumann boundary

So far, we have only set ellipticity conditions for the operator L_A (Def. 3.1) but no conditions for $B_A^{\pm}$. For the model problem to be an elliptic system in the sense of Agmon-Douglis-Nirenberg, we additionally need that $B_A^{\pm}$ satisfies the so called *complementing boundary condition* (discussed in more detail in Appendix B). To motivate the following definitions, note that the matrix $M_{\lambda,\alpha}$ for Neumann boundary conditions will have the form:

$$M_{\lambda,\alpha} := \begin{pmatrix} (A_{12}+V)V_0^{\lambda_+} & (A_{12}+\overline{V})\overline{V_0}^{\lambda_-} \\ (A_{12}+V)V_\alpha^{\lambda_+} & (A_{12}+\overline{V})\overline{V_\alpha}^{\lambda_-} \end{pmatrix} = \begin{pmatrix} A_{12}+V & A_{12}+\overline{V} \\ (A_{12}+V)V_\alpha^{\lambda_+} & (A_{12}+\overline{V})\overline{V_\alpha}^{\lambda_-} \end{pmatrix}$$

This form is deduced by similar derivations as in the last section, and the last equation is due to $V_0 = \mathrm{Id}_\ell$. Now, using Lemma 2.1 is not immediate since it is not clear if $A_{12}+V$ is invertible. One can show that invertibility of $A_{12}+V$ is equivalent to the *complementing boundary condition* for $N_A^{\pm}$ (Appendix B).

Remark. A discussion of the complementing boundary condition in the case of Dirichlet boundary conditions for Γ^- was not necessary, as it is automatically satisfied for strongly elliptic systems. See Remark 3.2.7 in [7].

For the subsequent derivations, we introduce the following ellipticity conditions:

Definition 5.3. Consider an elliptic tuple A, where $V = C + iD$ for $C, D \in \mathrm{Mat}_\ell(\mathbb{R})$ is the standard root of its monic reduction. We say

- A is *Neumann well-posed* if $2i \notin \sigma([D^{-1}, C])$.
- A is *contractive Neumann well-posed* if $\rho([D^{-1}, C]) < 2$.

Clearly, contractive Neumann well-posedness implies Neumann well-posedness. How do these definitions relate to the considerations discussed above? From (5.6):

$$A_{12} + V = \frac{1}{2}[S, D] + iD = \frac{1}{2}D(D^{-1}SD - S + 2i\,\mathrm{Id}_\ell),$$

which is invertible if and only if $-2i$ is not an eigenvalue of:

$$D^{-1}SD - S = D^{-1}C - CD^{-1} = [D^{-1}, C]. \tag{5.7}$$

Note that, first, $\sigma([D^{-1}, C]) \subset i\mathbb{R}$, and second, $it \in \sigma([D^{-1}, C])$ if and only if $-it \in \sigma([D^{-1}, C])$ for any $t \in \mathbb{R}$. This follows from the computation:

$$\sigma([D^{-1}, C]) = \sigma(D^{1/2}[D^{-1}, C]D^{-1/2}) = \sigma([D^{-1/2}SD^{-1/2}, D]) \subset i\mathbb{R}. \tag{5.8}$$

The last inclusion holds since $D^{-1/2}SD^{-1/2}$ and D are symmetric, their commutator is skew-symmetric, and skew symmetric matrices have imaginary, complex conjugated eigenvalues. Thus, we have shown that invertibility of $A_{12}+V$ is equivalent to Neumann well-posedness.

Contractive Neumann well-posedness is related to path-connectedness to the Laplace operator via Neumann well-posed elliptic tuples. To understand this, consider a contractive Neumann well-posed tuple $A = (A_{11}, A_{12}, A_{22})$. We can find a continuous path

$$[0,1] \ni s \mapsto A(s) = (A_{11}(s), A_{12}(s), A_{22}(s)) \in (\mathrm{Mat}_\ell(\mathbb{R}))^3$$

such that $A(1) = (\mathrm{Id}_\ell, 0, \mathrm{Id}_\ell)$ and $A(0) = A$, with each $A(\bullet)$ being contractive Neumann well-posed. The path is constructed in three segments which can be glued together.
First Segment: Start with (A_{11}, A_{12}, A_{22}) and deform it as follows:

$$[0,1] \ni s \mapsto A(s) = (A_{22}^{-s/2} A_{11} A_{22}^{-s/2}, A_{22}^{-s/2} A_{12} A_{22}^{-s/2}, A_{22}^{1-s}).$$

Note that

$$A(0) = A, \quad A(1) = (A_{22}^{-1/2} A_{11} A_{22}^{-1/2}, A_{22}^{-1/2} A_{12} A_{22}^{-1/2}, \mathrm{Id}_\ell),$$

and that all $A(\bullet)$ are elliptic tuples and have the same monic reduction as $A(1)$. Thus, if $A(0)$ is contractive Neumann well-posed, then all $A(\bullet)$ are contractive Neumann well-posed.
Second Segment: Start with a contractive Neumann well-posed elliptic tuple of the form $(A_{11}, A_{12}, \mathrm{Id}_\ell)$. Let us write $V = C + iD$ for its standard root and define a path of matrices $V_s := (1-s)C + iD$ for $s \in [0,1]$. Using Theorem 4.6, these represent the standard roots of the elliptic tuples:

$$[0,1] \ni s \mapsto A(s) = \left(V_s^* V_s, -\frac{1}{2}(V_s^* + V_s), \mathrm{Id}_\ell \right).$$

Note that:

$$\rho([(\operatorname{Im} V_s)^{-1}, \operatorname{Re} V_s]) = (1-s)\rho([D^{-1}, C]) < 2(1-s) < 2$$

since $\rho([D^{-1}, C]) < 2$ due to $A(0)$ being contractive Neumann well-posed. Moreover, we have $A(1) = (D^2, 0, \mathrm{Id}_\ell)$ due to $-\frac{1}{2}(V_1^* + V_1) = 0$.
Third Segment: Start with an elliptic tuple of the form $(A_{11}, 0, \mathrm{Id}_\ell)$, set:

$$[0,1] \ni s \mapsto A(s) = (A_{11}^{-s/2} A_{11} A_{11}^{-s/2}, 0, \mathrm{Id}_\ell),$$

and note that $A(1) = (\mathrm{Id}_\ell, 0, \mathrm{Id}_\ell)$. Ellipticity along the path is clear by Lemma 3.2, as is contractive Neumann well-posedness since $\operatorname{Re} V = 0$ for any standard root V along the way.

On the other hand, any tuple not being contractive Neumann well-posed cannot be connected to the Laplace operator by a path of Neumann well-posed systems. To illustrate this, consider a continuous path of elliptic tuples

$$[0,1] \ni s \mapsto A(s) = (A_{11}(s), A_{12}(s), A_{22}(s)) \in (\mathrm{Mat}_\ell(\mathbb{R}))^3$$

such that $A(1) = (\mathrm{Id}_\ell, 0, \mathrm{Id}_\ell)$. Let $V_s = C_s + iD_s$ denote the standard root of the monic reduction of $A(s)$. If $A(0)$ is not contractive Neumann well-posed, then there is $t > 2$ such that $it \in \sigma([D_0^{-1}, C_0])$. Note that $\sigma([D_1^{-1}, C_1]) = \{0\}$, due to $C_1 = 0$ (see Example 4.4). So by continuity and $\sigma([D_s^{-1}, C_s]) \subset i\mathbb{R}$, there must be some intermediate value $s \in (0,1)$ such that $2i \in \sigma([D_s^{-1}, C_s])$ and Neumann well-posedness is violated. The discussion leads to the following result:

Lemma 5.4. *Consider A an elliptic touple. Then:*

i) If A is contractive Neumann well-posed, there exists a continuous path of contractive Neumann well-posed elliptic tuples connecting A to $(\mathrm{Id}_\ell, 0, \mathrm{Id}_\ell)$.

ii) If there exists a continuous path from A to $(\mathrm{Id}_\ell, 0, \mathrm{Id}_\ell)$ consisting of Neumann well-posed tuples, then A is contractive Neumann well-posed.

5.4 Neumann boundary conditions

We proceed with Neumann boundary conditions for the model problem. Assume that $r^\lambda v \neq 0$, for v in (5.1), satisfies Neumann boundary conditions on $\Gamma^\pm$. *Additionally, assume that the elliptic tuple A is Neumann well-posed.* As mentioned in the last section, we can derive the condition $\det(M_{\lambda,\alpha}) = 0$ for

$$M_{\lambda,\alpha} = \begin{pmatrix} A_{12} + V & A_{12} + \overline{V} \\ (A_{12} + V)V_\alpha^{\lambda_+} & (A_{12} + \overline{V})\overline{V_\alpha}^{\lambda_-} \end{pmatrix}.$$

Using Lemma 2.1 and determinant rules (here Neumann well-posedness is essential such that $A_{12} + V$ is invertible), we obtain:

$$0 = \det M_{\lambda,\alpha} \iff 0 = \det\Big((A_{12} + V)V_\alpha^{\lambda_+}(A_{12} + V)^{-1} - (A_{12} + \overline{V})\overline{V_\alpha}^{\lambda_-}(A_{12} + \overline{V})^{-1}\Big).$$

By substituting (5.6), we derive:

$$0 = \det\left(\left(\frac{1}{2}[S,D] + iD\right) V_\alpha^{\lambda_+} \left(\frac{1}{2}[S,D] + iD\right)^{-1} - \left(\frac{1}{2}[S,D] - iD\right) \overline{V_\alpha}^{\lambda_-} \left(\frac{1}{2}[S,D] - iD\right)^{-1}\right).$$

Rewriting $Z_\alpha = D^{1/2}V_\alpha D^{-1/2}$ and using similar arguments as before, we arrive at the following result.

Proposition 5.5. *Consider $0 < \alpha < 2\pi$ and a Neumann well-posed monic elliptic operator L_A with standard root $V = (S + i\,\mathrm{Id}_\ell)D$. The model problem with Neumann boundary conditions has for $\lambda \in \mathbb{C} \setminus \{0\}$ a solution $r^\lambda v \neq 0$ if and only if*

$$0 = \det\Big(E\, Z_\alpha^{\lambda_+} E^{-1} - \overline{E}\,\overline{Z_\alpha}^{\lambda_-}\overline{E}^{-1}\Big),$$

for Z_α given in (5.4) and $E = \frac{1}{2}D^{-1/2}[S,D]D^{-1/2} + i\,\mathrm{Id}_\ell$.

6 Matrix equations associated to the model problem

In the previous section, we derived matrix equations $0 = \det(M_{\lambda,\alpha})$ corresponding to solutions of the model problem with Dirichlet, mixed, or Neumann boundary conditions. This section provides bounds on $|\operatorname{Re}\lambda|$, where $0 = \det(M_{\lambda,\alpha})$ is solvable for the case of Dirichlet and mixed boundary conditions. We emphasize the following two points, which will be relevant throughout the section:

- Note that $0 = \det(A)$ for $A \in \mathrm{Mat}_\ell(\mathbb{C})$ if and only if $0 \in \sigma(A)$. Our strategy is to use the numerical range $W(A)$ and angular field $W'(A)$ to bound the eigenvalues of A away from zero. For details and the properties **N1**-**N8** of the numerical range, we refer to Appendix E.
- A matrix $Z \in \mathrm{Mat}_\ell(\mathbb{C})$ satisfying $Z^T = Z$ (i.e., both its real and imaginary part are symmetric) is called a (complex) symmetric matrix. Note that $Z^* = \overline{Z}$ for symmetric matrices.

Definition 6.1. Denote by $\mathrm{Mat}_\ell(\mathbb{C})_{+i}$ the subset of symmetric matrices $Z \in \mathrm{Mat}_\ell(\mathbb{C})$ satisfying $\mathrm{Im}(Z) > 0$. Similarly, denote by $\mathrm{Mat}_\ell(\mathbb{C})_{-i}$ the set of symmetric matrices satisfying $-\mathrm{Im}(Z) > 0$.

For $Z \in \mathrm{Mat}_\ell(\mathbb{C})$ with $Z = Z^T$, $\operatorname{Im} Z > 0$ is equivalent to $W(Z) \subset \pm\,\mathrm{UHP}$. This is derived by applying **N7** in Appendix E to $\mp iZ$.

We present two key results that will guide the subsequent proofs. The proofs of these results are given in Appendix E.

Lemma 6.2. *Consider $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{\pm i}$. For $\lambda \in [-1,1] \setminus \{0\}$ we have*

$$W(Z^\lambda) \subset \operatorname{sgn}(\lambda) \cdot \{z \in \mathbb{C} \setminus \{0\} : \pm \arg(z) \in (0, \lambda\pi)\},$$

and, in particular, $W'(Z^\lambda) \subset \pm \operatorname{sgn}(\lambda)\,\mathrm{UHP}$.

Lemma 6.3. *Consider $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Then the following holds:*

$$\begin{aligned}
&i) \quad \operatorname{sgn}(\lambda)\left(\mathrm{Id}_\ell - (Z^{i\lambda})^* Z^{i\lambda}\right) > 0,\\
&ii) \quad \rho((Z^{i\lambda})^* Z^{i\lambda}) \xrightarrow{\lambda\to\infty} 0,\\
&iii) \quad \min\{\beta : \beta \in \sigma((Z^{i\lambda})^* Z^{i\lambda})\} \xrightarrow{\lambda\to-\infty} \infty.
\end{aligned}$$

Finally, note that, in this section, complex exponentiation, as well as $\arg$- and $\log$-functions, will always refer to the principal branch.

6.1 Dirichlet boundary conditions

Theorem 6.4. *Consider $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$ and $\lambda \in \mathbb{C}$. The equation*

$$0 = \det\left(Z^\lambda - \overline{Z}^\lambda\right) \tag{6.1}$$

admits for $|\operatorname{Re}\lambda| \le 1$ only the trivial solution $\lambda = 0$.

Proof. Consider $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$ and $\lambda \in \mathbb{C}$. Decompose λ into real and imaginary part $\lambda = \lambda_1 + i\lambda_2$ and write:

$$\begin{aligned}
0 &= \det\Big(Z^\lambda - \overline{Z}^\lambda\Big) = \det\Big(Z^{i\lambda_2} Z^{\lambda_1} - \overline{Z}^{\lambda_1}\overline{Z}^{i\lambda_2}\Big) = \det\Big(Z^{i\lambda_2} Z^{\lambda_1} - \overline{Z}^{\lambda_1}(Z^{-i\lambda_2})^*\Big) \\
&= \det\Big(Z^{i\lambda_2} Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}\Big)\det\big((Z^{-i\lambda_2})^*\big) \iff 0 = \det\Big(Z^{i\lambda_2} Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}\Big).
\end{aligned} \tag{6.2}$$

Here, we used $Z = Z^T$ and that matrix exponentials of symmetric matrices are again symmetric, which allowed us to write $(Z^{-i\lambda_2})^* = \overline{Z^{-i\lambda_2}} = \overline{Z}^{i\lambda_2}$. We also relied on $\det\Big(\overline{Z}^{i\lambda_2}\Big) \neq 0$, $(Z^{-1})^\lambda = (Z^\lambda)^{-1}$, and $Z^{i\lambda_2} Z^{\lambda_1} = Z^{\lambda_1} Z^{i\lambda_2}$ (see Appendix D for all of these statements).

Case 1: $\operatorname{Re}\lambda \neq 0$.

Let us assume $\lambda_1 \in [-1,1] \setminus \{0\}$. We aim to show that the spectrum of $Z^{i\lambda_2} Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}$ is bounded away from 0, implying that $0 = \det\Big(Z^\lambda - \overline{Z}^\lambda\Big)$ is not possible. From properties of the numerical range:

$$\begin{aligned}
&\sigma(Z^{i\lambda_2} Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}) \subset W(Z^{i\lambda_2} Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}) \\
&\subset W(Z^{i\lambda_2} Z^{\lambda_1}(Z^{i\lambda_2})^*) - W(\overline{Z}^{\lambda_1}) \subset W'(Z^{i\lambda_2} Z^{\lambda_1}(Z^{i\lambda_2})^*) + \operatorname{sgn}(\lambda_1)\,\mathrm{UHP} \\
&\subset W'(Z^{\lambda_1}) + \operatorname{sgn}(\lambda_1)\,\mathrm{UHP} \subset \operatorname{sgn}(\lambda_1)\,\mathrm{UHP},
\end{aligned} \tag{6.3}$$

where we used properties **N3**, **N5**, **N4** given in Appendix E, and Lemma 6.2 (twice).

Case 2: $\operatorname{Re}\lambda = 0$.

Next, assume $\lambda_1 = 0$ and $\lambda_2 \in \mathbb{R}\backslash\{0\}$. By (6.2), the argument boils down to show $0 \neq \det\big(Z^{i\lambda_2}(Z^{i\lambda_2})^* - \mathrm{Id}_\ell\big)$. This follows directly from Lemma 6.3.

The above cases show that (6.1) for $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\operatorname{Re}\lambda| \leq 1$ does not have a solution. Finally, note that $\lambda = 0$ is always a solution for any $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$, since $Z^0 = \mathrm{Id}_\ell = \overline{Z}^0$. □

6.2 Mixed boundary conditions

Theorem 6.5. *Let $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$ and $A, B \in \mathrm{Mat}_\ell(\mathbb{R})$ be symmetric. Consider the equation:*

$$0 = \det\Big([A,B](Z^\lambda - \overline{Z}^\lambda) + i(Z^\lambda + \overline{Z}^\lambda)\Big). \tag{6.4}$$

Then:

i) Equation (6.4) has no solution $\lambda \in \mathbb{C}$ with $|\operatorname{Re}\lambda| \in (0, \frac{1}{2}]$.

ii) Equation (6.4) has a solution $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda = 0$ if and only if $\rho([A,B]) > 1$.

Note that $[A,B] \in \mathrm{Mat}_\ell(\mathbb{R})$ is skew-symmetric due to

$$[A,B]^T = (AB - BA)^T = B^T A^T - A^T B^T = BA - AB = -[A,B].$$

Thus, $\sigma([A,B]) \subset i\mathbb{R}$, and all eigenvalues come in conjugate pairs. Similarly, $i[A,B]$ is Hermitian, so $\sigma(i[A,B]) \subset \mathbb{R}$.

Proof. Consider Z,A,B as given in the statement, and $\lambda \in \mathbb{C}$ with $|\operatorname{Re}\lambda| \leq \frac{1}{2}$. Decompose λ into real and imaginary part $\lambda = \lambda_1 + i\lambda_2$. Rewrite (6.4) as:

$$\begin{aligned}
0 &= \det\Big(\big([A,B](Z^\lambda - \overline{Z}^\lambda) + i(Z^\lambda + \overline{Z}^\lambda)\big)\Big) \\
&= \det\Big([A,B](Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}) + i(Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* + \overline{Z}^{\lambda_1})\Big)\det\big((Z^{-i\lambda_2})^*\big) \\
\iff 0 &\in \sigma\Big([A,B](Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}) + i(Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* + \overline{Z}^{\lambda_1})\Big).
\end{aligned} \tag{6.5}$$

Here, we used the same arguments as in the derivation for (6.2). Again, we separate into two cases.

Case 1: $\lambda_1 \neq 0$.

In this case, $Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}$ is invertible as shown in (6.3). Thus, Equation (6.4) is due to (6.5) equivalent to:

$$0 \in \sigma\left(M_1^*[A,B]M_1 + iM_1^*M_2\right) \quad \text{for}$$
$$M_1 := Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1} \text{ and } M_2 := Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* + \overline{Z}^{\lambda_1}.$$

For Case 1, it suffices to prove the following claim.

Claim 1: $\sigma\left(M_1^*[A,B]M_1 + iM_1^*M_2\right) \subset \operatorname{sgn}(\lambda_1)\,\mathrm{RHP}$ for $\lambda_1 \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$.

Due to **N3** in Appendix E, it suffices to show $W(M_1^*[A,B]M_1 + iM_1M_2) \subset \operatorname{sgn}(\lambda_1)\,\mathrm{RHP}$. Note that:

$$W(M_1^*[A,B]M_1) \subset W'(M_1^*[A,B]M_1) \subset W'([A,B]) \subset -i \cdot W'(i[A,B]) \subset i\mathbb{R}.$$

Here, we used properties **N4**, **N2**, **N6** of the numerical range (Appendix E), as well as $i[A,B]$ being Hermitian. So by additivity of the numerical range **N5**, it suffices to show $W(iM_1^*M_2) \subset \operatorname{sgn}(\lambda_1)\,\mathrm{RHP}$ for the claim. This result follows from a more extensive derivation: Note that

$$\begin{aligned}\operatorname{sgn}(\lambda_1)\,\mathrm{RHP} \supset W(iM_1^*M_2) &= W\left(i\left(Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}\right)^*\left(Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* + \overline{Z}^{\lambda_1}\right)\right)\\ &= W\left(i\left(Z^{i\lambda_2}\overline{Z}^{\lambda_1}(Z^{i\lambda_2})^* - Z^{\lambda_1}\right)\left(Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* + \overline{Z}^{\lambda_1}\right)\right)\end{aligned}$$

is, due to **N7**, equivalent to:

$$\begin{aligned}0 <\ & i\operatorname{sgn}(\lambda_1)\left(Z^{i\lambda_2}\overline{Z}^{\lambda_1}(Z^{i\lambda_2})^* - Z^{\lambda_1}\right)\left(Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* + \overline{Z}^{\lambda_1}\right)\\ & - i\operatorname{sgn}(\lambda_1)\left(Z^{i\lambda_2}\overline{Z}^{\lambda_1}(Z^{i\lambda_2})^* + Z^{\lambda_1}\right)\left(Z^{i\lambda_2}Z^{\lambda_1}(Z^{i\lambda_2})^* - \overline{Z}^{\lambda_1}\right)\\ =\ & 2i\operatorname{sgn}(\lambda_1)\left(Z^{i\lambda_2}\overline{Z}^{\lambda_1}(Z^{i\lambda_2})^*\overline{Z}^{\lambda_1} - \left(Z^{i\lambda_2}\overline{Z}^{\lambda_1}(Z^{i\lambda_2})^*\overline{Z}^{\lambda_1}\right)^*\right)\\ =\ & 2\operatorname{sgn}(\lambda_1)\left(iZ^{i\lambda_2}\overline{Z}^{\lambda_1}(Z^{i\lambda_2})^*\overline{Z}^{\lambda_1} + \left(iZ^{i\lambda_2}\overline{Z}^{\lambda_1}(Z^{i\lambda_2})^*\overline{Z}^{\lambda_1}\right)^*\right),\end{aligned}$$

which is, again using **N7**, equivalent to:

$$\begin{aligned}&\operatorname{sgn}(\lambda_1)\,\mathrm{RHP} \supset W\left(iZ^{i\lambda_2}\overline{Z}^{\lambda_1}(Z^{i\lambda_2})^*\overline{Z}^{\lambda_1}\right) \overset{(*)}{=} W\left(iZ^{i\lambda_2}\overline{Z}^{2\lambda_1}(Z^{i\lambda_2})^*\right)\\ \iff\ & \operatorname{sgn}(\lambda_1)\,\mathrm{RHP} \supset W'\left(iZ^{i\lambda_2}\overline{Z}^{2\lambda_1}(Z^{i\lambda_2})^*\right) \overset{\textbf{N4}}{=} W'\left(i\overline{Z}^{2\lambda_1}\right)\\ \overset{(**)}{\iff}\ & W'\left(\overline{Z}^{2\lambda_1}\right) \subset -\operatorname{sgn}(\lambda_1)\,\mathrm{UHP} \overset{\textbf{N8}}{\iff} W'\left(Z^{2\lambda_1}\right) \subset \operatorname{sgn}(\lambda_1)\,\mathrm{UHP},\end{aligned}$$

where we used $\overline{Z}^{\lambda_1}(Z^{i\lambda_2})^* = (Z^{i\lambda_2})^*\overline{Z}^{\lambda_1}$ at $(*)$ and $\mathrm{UHP} = i\,\mathrm{RHP}$ at $(**)$. The last statement holds true by Lemma 6.2 and $0 < 2|\lambda_1| \leq 1$. This shows Claim 1 and closes Case 1.

Case 2: $\lambda_1 = 0$.

In this case, (6.5) can be rewritten as:

$$0 \in \sigma\left([A,B](Z^{it}(Z^{it})^* - \mathrm{Id}_\ell) + i(Z^{it}(Z^{it})^* + \mathrm{Id}_\ell)\right), \tag{6.6}$$

where we write $\lambda_2 = t \in \mathbb{R}$ in the following. For $t = 0$, this is equivalent to $0 \in \sigma\,(2i\,\mathrm{Id}_\ell)$, which is not possible. So assume $t \neq 0$ such that $-\operatorname{sgn}(t)(Z^{it}(Z^{it})^* - \mathrm{Id}_\ell) > 0$ due to Lemma 6.3. Define for $t \in \mathbb{R}\setminus\{0\}$:

$$K_t := \left(Z^{it}(Z^{it})^* - \mathrm{Id}_\ell\right)\left(Z^{it}(Z^{it})^* + \mathrm{Id}_\ell\right)^{-1}.$$

Alternatively, K_t can be given by the functional calculus $K_t = f(Z^{it}(Z^{it})^*)$ for

$$f : \mathbb{C}\setminus\{-1\} \to \mathbb{C}, \quad z \mapsto \frac{z-1}{z+1}. \tag{6.7}$$

Note that K_t is Hermitian, and by the spectral mapping theorem, along with Lemma 6.3, one has $-\operatorname{sgn}(t)K_t > 0$. Now (6.6) is equivalent to:

$$\begin{aligned}(6.6) &\iff 0 \in \sigma([A,B]K_t + i\,\mathrm{Id}_\ell) \iff -i \in \sigma([A,B]K_t) \\ &\iff -i\operatorname{sgn}(t) \in \sigma(-[A,B]\operatorname{sgn}(t)K_t) \iff -i\operatorname{sgn}(t) \in \sigma(|K_t|^{1/2}[A,B]|K_t|^{1/2}) \\ &\iff i \in \sigma(|K_t|^{1/2}[A,B]|K_t|^{1/2}),\end{aligned}$$

where we used $\det(Z^{it}(Z^{it})^* + \mathrm{Id}_\ell) \neq 0$ and the abbreviation $|K_t| = -\operatorname{sgn}(t)K_t > 0$. The sign flip in the last line is due to skew-symmetry (note $|K_t|^{1/2}[A,B]|K_t|^{1/2}$ is skew-symmetric since $|K_t|^{1/2}$ symmetric and $[A,B]$ skew-symmetric). Before we continue, let us show the following claim.

Claim 2: $\rho(|K_t|^{1/2}[A,B]|K_t|^{1/2}) < \rho([A,B])$ for any $t \in \mathbb{R}$.

By the spectral mapping theorem, (6.7), and $\sigma(Z^{it}(Z^{it})^*) \subset \mathbb{R}_{>0}$, we deduce that $\rho(K_t) = \|K_t\| < 1$, since K_t is Hermitian. Consequently, we have $\|[A,B]|K_t|\| < \|[A,B]\| = \rho([A,B])$. It follows that:

$$\rho(|K_t|^{1/2}[A,B]|K_t|^{1/2}) = \rho([A,B]|K_t|) \leq \|[A,B]|K_t|\| < \rho([A,B]),$$

which shows the claim.

It remains to show that Equation (6.4) has a solution $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda = 0$ if and only if $\rho[A,B]) > 1$. So far, we have shown that Equation (6.4) admits such a solution if and only if

$$\exists t \in \mathbb{R}\setminus\{0\} : i \in \sigma(|K_t|^{1/2}[A,B]|K_t|^{1/2}). \tag{6.8}$$

Case 2a: $\rho([A,B]) \leq 1$.

If $\rho([A,B]) \leq 1$, then Claim 2 implies $\rho(|K_t|^{1/2}[A,B]|K_t|^{1/2}) < 1$ for any $t \in \mathbb{R}\setminus\{0\}$. This shows that (6.8) cannot hold, and hence, no solution exists in this case.

Case 2b: $\rho([A,B]) > 1$.

In this case, there exists some $\beta \in \sigma(i[A,B])$ such that $\beta > 1$. To complete the proof, it suffices to consider from now on $t < 0$ such that $|K_t| = K_t$. Due to Lemma 6.3 and the definition of K_t, we have $K_t \xrightarrow{t\to 0-} 0$ and $K_t \xrightarrow{t\to-\infty} \mathrm{Id}_\ell$ such that:

$$iK_t^{1/2}[A,B]K_t^{1/2} \xrightarrow{t\to 0-} 0, \quad iK_t^{1/2}[A,B]K_t^{1/2} \xrightarrow{t\to-\infty} i[A,B].$$

Note that $iK_t^{1/2}[A,B]K_t^{1/2}$ is Hermitian and admits only real eigenvalues. By continuity of eigenvalues, and since $\beta > 1$, there exist some $t < 0$ such that $1 \in \sigma(iK_t^{1/2}[A,B]K_t^{1/2})$. Consequently, $\pm i \in \sigma(K_t^{1/2}[A,B]K_t^{1/2})$ which shows that (6.8) holds. This completes the proof. □

The next result is needed for mixed boundary conditions in the case where $\alpha \in \{\pi, 2\pi\}$.

Theorem 6.6. *Let $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$ and $A, B \in \mathrm{Mat}_\ell(\mathbb{R})$ be symmetric. Consider the equation:*

$$0 = \det\left([A,B](e^{i\lambda\pi} - e^{-i\lambda\pi}) + i(e^{i\lambda\pi} + e^{-i\lambda\pi})\,\mathrm{Id}_\ell\right). \tag{6.9}$$

Then:

i) All solutions $\lambda \in \mathbb{C}$ of (6.9) satisfy $\operatorname{Re}\lambda \in \frac{1}{2}\mathbb{Z}$.

ii) If $\rho([A,B]) \leq 1$, then all solutions $\lambda \in \mathbb{C}$ of (6.9) satisfy $\operatorname{Re}\lambda = \frac{1}{2} + \mathbb{Z}$.

iii) If $\rho([A,B]) < 1$, then for any $k \in \mathbb{Z}$ there exist ℓ solutions $\lambda \in \mathbb{C}$ (counted with multiplicity) satisfying $\operatorname{Re}\lambda = \frac{1}{2} + k$.

Proof. i) Equation (6.9) can be rewritten as:

$$0 = \det\left([A,B]\sin(\lambda\pi) + \cos(\lambda\pi)\,\mathrm{Id}_\ell\right). \tag{6.10}$$

First, note that $\lambda \in \mathbb{Z}$ cannot be a solution to (6.10) since then $\sin(\lambda\pi) = 0$ and $\cos(\lambda\pi) \neq 0$. Thus, we can assume $\sin(\lambda\pi) \neq 0$. Dividing by $\sin(\lambda\pi)$, the conditions becomes:

$$(6.9) \iff -\frac{1}{\tan(\lambda\pi)} \in \sigma([A,B]). \tag{6.11}$$

Now assume $\lambda \in \mathbb{C}\backslash\mathbb{Z}$ solves the RHS in (6.11). Since $\sigma([A,B]) \in i\mathbb{R}$, it follows that $\tan(\lambda\pi) \in i\mathbb{R}$. Using the following representation of complex tangent:

$$\tan(x+iy) = \frac{\sin(2x) + i\sinh(2y)}{\cosh(2y) + \cos(2x)} \quad \text{for } x, y \in \mathbb{R},$$

we deduce that (6.9) implies $0 = \sin(2\pi\operatorname{Re}\lambda)$. This is equivalent to $\operatorname{Re}\lambda \in \frac{1}{2}\mathbb{Z}$ which shows i).

ii) Assume $\rho([A,B]) \leq 1$. Write $\lambda = \frac{k}{2} + it$ for $k \in \mathbb{Z}$ and $t \in \mathbb{R}$. Using the tangent representation, we have:

$$\frac{1}{\tan(\lambda\pi)} = -i\frac{\cosh(2t\pi) + (-1)^k}{\sinh(2t\pi)}. \tag{6.12}$$

If $k \in 2\mathbb{Z}$, then $\left|\tan(\lambda\pi)^{-1}\right| > 1$. In this case, λ cannot solve (6.9) due to (6.11) and $\rho([A,B]) \leq 1$.

iii) Assume $\rho([A,B]) < 1$, so $\sigma([A,B]) \subset i(-1,1)$. If $k \in 1 + 2\mathbb{Z}$, then the RHS of (6.12) can be continuously extended to $t = 0$ (with the value 0) such that it defines a surjective function $f : \mathbb{R} \to i(-1,1)$. By relation (6.11) and $\sigma([A,B]) \subset i(-1,1)$, Equation (6.9) has ℓ solutions (counted with multiplicity) as t in (6.12) varies over $(-\infty,\infty)$. $\square$

7 Regularity results for the model problem

The main results of this work, bounds on $|\operatorname{Re}\lambda|$ for Dirichlet and mixed boundary conditions, are given. Neumann boundary conditions are briefly discussed in Section 8.1.

7.1 Dirichlet boundary conditions

Theorem 7.1. *Consider an elliptic tuple $A = (A_{11}, A_{12}, A_{22})$. Define*

$$\Lambda_\alpha := \{\lambda \in \mathbb{C} : \exists r^\lambda v \neq 0 \text{ solving (3.2) with angle } \alpha \text{ and Dirichlet b.c.}\}$$

Then, for all $\lambda \in \Lambda_\alpha$:

i) $|\operatorname{Re}\lambda| > 1$ *if* $0 < \alpha < \pi$.

ii) $\lambda \in \mathbb{Z} \setminus \{0\}$ *if* $\alpha = \pi$.

iii) $|\operatorname{Re}\lambda| > \frac{1}{2}$ *if* $\pi < \alpha < 2\pi$.

iv) $\lambda \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$ *if* $\alpha = 2\pi$.

This result is not new, compare with §8.6 and §11.3 in [20]. However, the proof is new and utilizes the methods derived in this work.

Proof. Due to Lemma 3.3, we can assume $\lambda \neq 0$ in the following. Furthermore, by Lemma 4.1, it suffices to prove the result for monic elliptic tuples. For such tuples, Theorem 4.6 guarantees the existence of a standard root $V = (S + i\operatorname{Id}_\ell)D \in \operatorname{Mat}_\ell(\mathbb{C})$, where $S, D \in \operatorname{Mat}_\ell(\mathbb{R})$ are symmetric, and $D > 0$. Due to Prop. 5.1, the corresponding model problem with Dirichlet boundary conditions admits a solution $r^\lambda v \neq 0$ for $\lambda \in \mathbb{C} \setminus \{0\}$ and $0 < \alpha < 2\pi$ if and only if

$$0 = \det\Big(Z_\alpha^{\lambda_+} - \overline{Z_\alpha}^{\lambda_-}\Big) \tag{7.1}$$

admits a solution. Here, $Z_\alpha = \cos(\alpha) + D^{1/2}SD^{1/2}\sin(\alpha) + iD\sin(\alpha)$ is a complex symmetric matrix. We now analyze the different cases for α:

i) $0 < \alpha < \pi$.

In this case, $Z_\alpha \in \operatorname{Mat}_\ell(\mathbb{C})_{+i}$ due to $D > 0$ and $\sin(\alpha) > 0$. By Lemma E.1, this implies $\sigma(Z_\alpha) \subset \mathrm{UHP}$ and $\sigma(\overline{Z_\alpha}) \subset -\mathrm{UHP}$. Consequently, we can replace λ_+ and λ_- with the principal branch $\lambda_o = \lambda$, and Equation (7.1) simplifies to:

$$0 = \det\Big(Z_\alpha^{\lambda} - \overline{Z_\alpha}^{\lambda}\Big).$$

The results follows by Theorem 6.4 and $\lambda \neq 0$.

ii) $\alpha = \pi$.

In this case, $Z_\pi = -\operatorname{Id}_\ell$, so (7.1) reads

$$0 = \det\big(((-1)^{\lambda_+} - (-1)^{\lambda_-})\operatorname{Id}_\ell\big) = \det\big((e^{i\pi\lambda} - e^{-i\pi\lambda})\operatorname{Id}_\ell\big) = 0 \iff \sin(\lambda\pi) = 0.$$

The last equation holds if and only if $\lambda \in \mathbb{Z}$. This shows $\Lambda_\pi = \mathbb{Z} \setminus \{0\}$ due to $\lambda \neq 0$.

iii) $\pi < \alpha < 2\pi$.

Before we can reduce $\lambda_\pm$ to λ as in i), we need a trick. Since $D > 0$ and $\sin(\alpha) < 0$, Lemma E.1 implies $W(Z_\alpha) \subset -\mathrm{UHP}$. By Lemma 6.2, this further implies $W(Z_\alpha^{1/2}) \subset -\mathrm{UHP}$. Define $Y_\alpha = -Z_\alpha^{1/2}$ and observe $W(Y_\alpha) \subset \mathrm{UHP}$ and $Y_\alpha^2 = Z_\alpha$. Since $\sigma(Y_\alpha) \subset \mathrm{UHP}$ by **N3**, it follows that $Y_\alpha^{2\lambda} = Y_\alpha^{(2\lambda)_+} = Z_\alpha^{\lambda_+}$. Similarly, we obtain $\overline{Y_\alpha}^{2\lambda} = \overline{Z_\alpha}^{\lambda_-}$. Thus, (7.1) reads:

$$0 = \det\left(Y_\alpha^{2\lambda} - \overline{Y_\alpha}^{2\lambda}\right).$$

Note that $Y_\alpha = -Z_\alpha^{1/2}$ is a complex symmetric matrix with $\operatorname{Im} Y_\alpha > 0$ (due to $W(Y_\alpha) \subset \mathrm{UHP}$ and **N7** applied to $-iY_\alpha$). The result follows by Theorem 6.4 and $\lambda \neq 0$.

iv) $\alpha = 2\pi$.

Here, $Z_{2\pi} = \mathrm{Id}_\ell$, and we cannot use the original definition of $\lambda_\pm$. However, as pointed out in the last remark of Section 4.2, the result can be obtained as a boundary case of $\alpha < 2\pi$. For this, define $Z_\varepsilon := (1 - i\varepsilon)\,\mathrm{Id}_\ell$ for $\varepsilon > 0$. Note $Z_\varepsilon \xrightarrow{\varepsilon \to 0_+} Z_{2\pi}$, as well as

$$(Z_\varepsilon)^{\lambda_+} \to e^{2\pi i\lambda} Id_\ell, \quad (\overline{Z}_\varepsilon)^{\lambda_-} \to e^{-2\pi i\lambda} Id_\ell.$$

Thus, taking the limit, (7.1) becomes:

$$\det\left((e^{2i\pi\lambda} - e^{-2i\pi\lambda})\,\mathrm{Id}_\ell\right) = 0 \iff \sin(2\lambda\pi) = 0.$$

This shows $\Lambda_{2\pi} = \frac{1}{2}\mathbb{Z} \setminus \{0\}$. □

7.2 Mixed boundary conditions

Theorem 7.2. *Consider an elliptic tuple $A = (A_{11}, A_{12}, A_{22})$. Define:*

$$\Lambda_\alpha := \{\lambda \in \mathbb{C} : \exists r^\lambda v \neq 0 \textit{ solving (3.2) with angle } \alpha \textit{ and mixed b.c.}\}$$

1 If A is contractive Neumann well-posed, then for all $\lambda \in \Lambda_\alpha$:

i) $|\operatorname{Re}\lambda| > \frac{1}{2}$ if $0 < \alpha < \pi$.

ii) $\operatorname{Re}\lambda \in \frac{1}{2} + \mathbb{Z}$ if $\alpha = \pi$.

iii) $|\operatorname{Re}\lambda| > \frac{1}{4}$ if $\pi < \alpha < 2\pi$.

iv) $\operatorname{Re}\lambda \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ if $\alpha = 2\pi$.

2 If the assumption on contractive Neumann well-posedness is dropped, the only additional solutions not satisfying the above conditions are of the form:

2.1 $\operatorname{Re}\lambda = 0$ for i) and iii).

2.2 $\operatorname{Re}\lambda \in \frac{1}{2}\mathbb{Z}$ for ii).

2.3 $\operatorname{Re}\lambda \in \frac{1}{4}\mathbb{Z}$ for iv).

This result is new and was the central motivation for developing the framework introduced in this work. Many arguments in the proof are similar to those for Dirichlet boundary conditions, so we omit details where appropriate.

Proof. By Lemma 3.3, we can assume $\lambda \neq 0$ in the following. Furthermore, by Lemma 4.1 (and the definition of contractive Neumann well-posedness), it suffices to prove the result for monic elliptic tuples (which are contractive Neumann well-posed). For such tuples, let $V = (S + i\,\mathrm{Id}_\ell)D \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$, where $S, D \in \mathrm{Mat}_\ell(\mathbb{R})$ are symmetric and $D > 0$, denote its standard root. Recall that contractive Neumann well-posedness means:

$$\rho([(\mathrm{Im}\,V)^{-1}, \mathrm{Re}\,V]) < 2 \overset{(5.8)}{\iff} \rho\left(\frac{1}{2}[D^{-1/2}SD^{-1/2}, D]\right) < 1. \tag{7.2}$$

Due to Prop. 5.2, the model problem with mixed boundary conditions admits a solution $r^\lambda v \neq 0$ for $\lambda \in \mathbb{C} \setminus \{0\}$ and $0 < \alpha < 2\pi$ if and only if

$$0 = \det\left(\frac{1}{2}[D^{-1/2}SD^{-1/2}, D](Z_\alpha^{\lambda_+} - \overline{Z_\alpha}^{\lambda_-}) + i(Z_\alpha^{\lambda_+} + \overline{Z_\alpha}^{\lambda_-})\right) \tag{7.3}$$

admits a solution for Z_α defined in (5.4).

i) $0 < \alpha < \pi$.

As in the case of Dirichlet boundary conditions, $\lambda_\pm = \lambda$, and Equation (7.3) becomes:

$$0 = \det\left(\frac{1}{2}[D^{-1/2}SD^{-1/2}, D](Z_\alpha^{\lambda} - \overline{Z_\alpha}^{\lambda}) + i(Z_\alpha^{\lambda} + \overline{Z_\alpha}^{\lambda})\right),$$

where Z_α is a complex symmetric matrix with $\mathrm{Im}\,Z_\alpha > 0$. The result for i) and (a) follows from Theorem 6.5 and (7.2).

ii) $\alpha = \pi$.

Similar to the case of Dirichlet boundary conditions, (7.3) becomes:

$$0 = \det\left(\frac{1}{2}[D^{-1/2}SD^{-1/2}, D](e^{i\lambda\pi} - e^{-i\lambda\pi}) + i(e^{i\lambda\pi} + e^{-i\lambda\pi})\,\mathrm{Id}_\ell\right).$$

The result for ii) and (b) follows from Theorem 6.6 and (7.2).

iii) $\pi < \alpha < 2\pi$.

By the same argument as in the Dirichlet case, we can rewrite (7.3) as

$$0 = \det\left(\frac{1}{2}[D^{-1/2}SD^{-1/2}, D](Y_\alpha^{2\lambda} - \overline{Y_\alpha}^{2\lambda}) + i(Y_\alpha^{2\lambda} + \overline{Y_\alpha}^{2\lambda})\right),$$

for some $Y_\alpha \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$. The result for iii) and (a) follows from Theorem 6.5 and (7.2).

iv) $\alpha = 2\pi$.

Using the same limiting argument as in the Dirichlet case, we conclude in this case:

$$0 = \det\left(\frac{1}{2}[D^{-1/2}SD^{-1/2}, D](e^{2i\lambda\pi} - e^{-2i\lambda\pi}) + i(e^{2i\lambda\pi} + e^{-2i\lambda\pi})\,\mathrm{Id}_\ell\right).$$

The result for iv) and (c) follows from Theorem 6.6 and (7.2). □

Remark. For 1.) in Theorem 7.2, we could also include elliptic tuples where the standard root V satisfies $\rho([(\mathrm{Im}\,V)^{-1}, \mathrm{Re}\,V]) = 2$. This follows because the main ingredients, Theorem 6.5 and Theorem 6.6, cover this case. Then, however, the system is not Neumann well-posed.

Remark. Neglecting the case mentioned in the previous remark, contractive Neumann well-posedness precisely distinguishes scenarios where purely imaginary solutions occur. Verifiying contractive Neumann well-posedness can be challenging in practice, however, Appendix B relates it to a stronger ellipticity condition that is commonly used in linear elasticity.

Remark. Theorem 7.1 and Theorem 7.2 are highly relevant in the context of regularity theory, as they provide the sharpest lower bounds on $|\operatorname{Re}\lambda|$ that can be expected for given $0 < \alpha \leq 2\pi$. This is also discussed in Section 8.4.

8 Summary and comments

In this work, we analyzed the model problem for an elliptic system in an angle $0 < \alpha \leq 2\pi$ under Dirichlet, mixed, and Neumann boundary conditions within a new framework. For all three boundary conditions, we derived a matrix equation of the form $\det(M_{\lambda,\alpha}) = 0$, which characterizes the pairs (α, λ) such that the model problem with angle α admits a solution of the form $r^\lambda v$. Equivalently, this equation determines eigenvalues of the corresponding operator pencil. For Dirichlet and mixed boundary conditions, we established lower bounds on $|\operatorname{Re}\lambda|$ for nontrivial solutions. For the former, these results align with those found in the literature. For the latter, our findings represent a new contribution.

8.1 Bounds on $|\operatorname{Re}\lambda|$ for Neumann boundary conditions

We did not discuss bounds for $|\operatorname{Re}\lambda|$ in the case of Neumann boundary conditions. Recall that the corresponding matrix equation in this case is:

$$0 = \det\Big(E\, Z_\alpha^{\lambda_+} E^{-1} - \overline{E}\, \overline{Z_\alpha}^{\lambda_-} \overline{E}^{-1}\Big),$$

where Z_α is given in (5.4) and $E = \frac{1}{2} D^{-1/2}[S, D] D^{-1/2} + i\,\mathrm{Id}_\ell$. For $\alpha = \pi$, we have $Z_\pi^{\lambda_\pm} = \mathrm{Id}_\ell\, e^{\pm\lambda\pi}$, so the equation reduces to $\sin(\lambda\pi) = 0$, as in the Dirichlet case (similarly for $\alpha = 2\pi$). For all other angles, the bounds on $\operatorname{Re}\lambda$ are less clear. As Figure 2 suggests, assuming contractive Neumann well-posedness does not guarantee $|\operatorname{Re}\lambda| > 1/2$ for $\pi < \alpha < 2\pi$. However, if the elliptic tuple A is *formal positive* (see Appendix B), the literature (see §12 in [20]) indicates that similar bounds to those for Dirichlet boundary conditions can be obtained (for $\lambda \neq 0$).

8.2 The scalar case

In the scalar case ($\ell = 1$), the matrices $A_\bullet$ reduce to real numbers, and the matrix equation $\det(M_{\lambda,\alpha}) = 0$ simplifies significantly. For Dirichlet and Neumann boundary conditions, the equations both reduce to $Z_\alpha^{\lambda_+} = \overline{Z_\alpha}^{\lambda_-}$. For mixed boundary conditions, the equation becomes $Z_\alpha^{\lambda_+} = -\overline{Z_\alpha}^{\lambda_-}$.

8.3 Location of zeros

Theorem 7.1 and Theorem 7.2 provide bounds on $|\operatorname{Re}\lambda|$, but they do not specify the existence and location of λ in the complex plane. In principle, the corresponding statements do not exclude the possibility of $\Lambda_\alpha = \emptyset$ for given α. However, as Figure 1 suggests, this is not the case. For the particular cases $\alpha \in \{\pi, 2\pi\}$, it is straightforward to show that $\lambda = 1$ (for $\alpha = \pi$) and $\lambda = 1/2$ (for

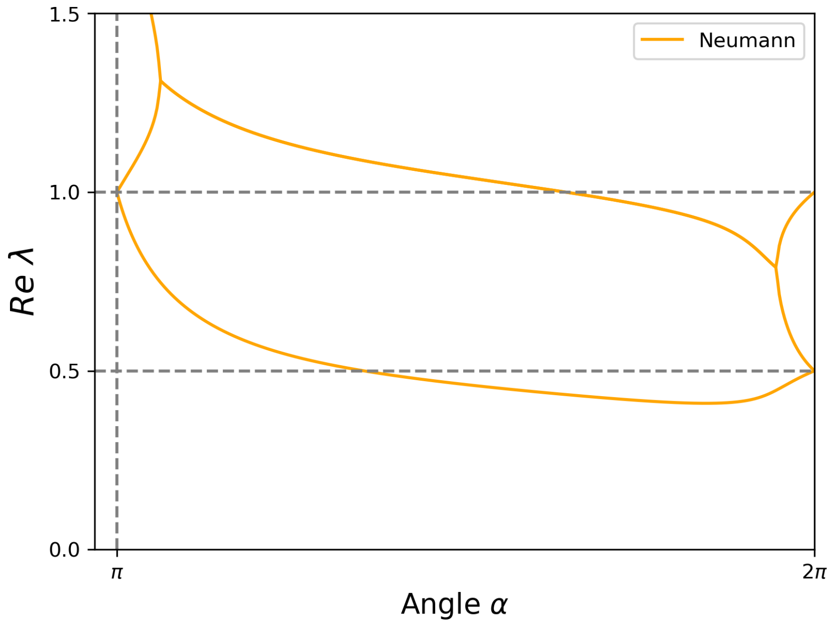

Figure 2: Relation between $\operatorname{Re}\lambda$ and $\alpha \in [\pi, 2\pi]$ for Neumann boundary conditions. The standard root of the monic elliptic tuple is given by $V = (S + i\,\mathrm{Id}_\ell)D$ for $S = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Although this tuple is contractive Neumann well-posed, it is not formal positive. The qualitative behavior differs from the Dirichlet (and mixed) case: Branch merging and $|\operatorname{Re}\lambda| < 1/2$ can be observed. Note that only selected branches are plotted.

$\alpha = 2\pi$) are solutions with multiplicity ℓ for Dirichlet and Neumann boundary conditions. This follows from $\det(\mathrm{Id}_\ell \sin(\lambda\pi)) = \sin(\lambda\pi)^\ell$. Similarly, by using iii) in Theorem 6.6, one can show for mixed boundary conditions that there exist ℓ solutions $r^\lambda v$ (counted with multiplicity) satisfying $\operatorname{Re}\lambda = \frac{1}{2}$ for $\alpha = \pi$ ($\operatorname{Re}\lambda = \frac{1}{4}$ for $\alpha = 2\pi$) if we assume contractive Neumann well-posedness. For angles $\alpha \notin \{\pi, 2\pi\}$, it is also possible to show the existence of solutions in certain strips of the complex plane. See the proof of Theorem 8.6.2 in [20] for details, where a generalization of Rouché's theorem is used.

8.4 Optimality of the bounds

The bounds on $|\operatorname{Re}\lambda|$ for different cases of $0 < \alpha \leq 2\pi$ in Theorem 7.1 and 1.) of Theorem 7.2 are sharp in the sense that one can construct a sequence of elliptic systems approaching these bounds. Note that the bounds for $\alpha \in \{\pi, 2\pi\}$ are sharp, as discussed in Section 8.3. To illustrate this for other angles, it suffices to consider the scalar case. For $k \in \mathbb{N}$, define $S_k := -k$ and $D_k := 1$. This yields the standard root $V_k = (S_k + i)D_k = -k + i$, and from (5.4) we derive:

$$Z_{\alpha,k} = \cos(\alpha) - k\sin(\alpha) + i\sin(\alpha).$$

For $0 < \alpha < \pi$ and Dirichlet boundary conditions, we obtain, as previously discussed, the condition $Z_{\alpha,k}^\lambda - \overline{Z_{\alpha,k}}^\lambda = 0$ (here $\lambda_\pm = \lambda$). This is solved by $\lambda_{\alpha,k} \in \mathbb{R}$ such that $\operatorname{Im}(Z_{\alpha,k}^{\lambda_{\alpha,k}}) = 0$. As $\arg(Z_{\alpha,k}) \xrightarrow{k\to\infty} \pi$, it follows that $\lambda_{k,\alpha} \xrightarrow{k\to\infty} 1$ for any $0 < \alpha < \pi$, which aligns with the bound given in Theorem 7.1. Similar arguments can be given for $\pi < \alpha < 2\pi$ and for mixed boundary conditions.

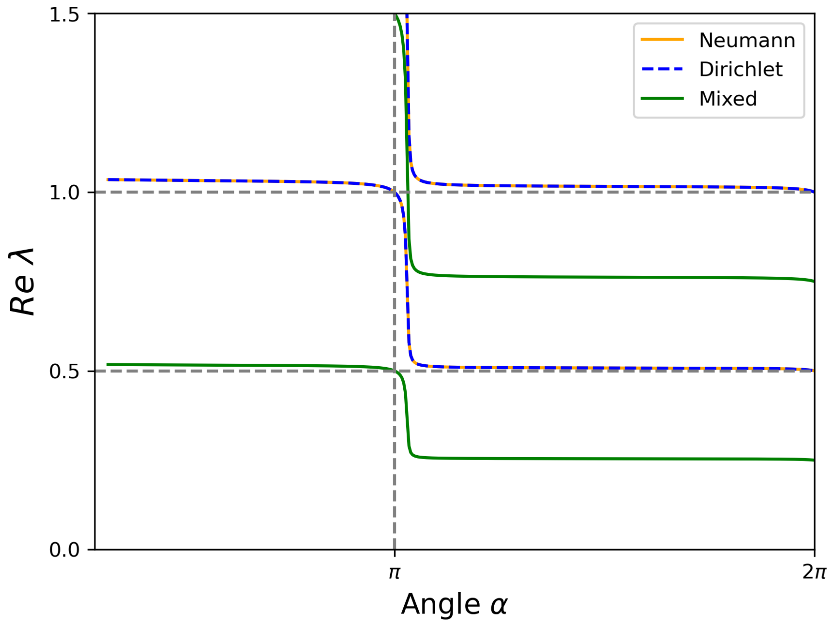

Figure 3: Relation between $\mathrm{Re}\,\lambda$ and $\alpha \in [1, 2\pi]$ for different boundary conditions. The (scalar) elliptic tuple is defined by the standard root $V = -10 + i$. In this case, the single branch for Neumann and Dirichlet boundary conditions coincides, and all branches closely approximate the bounds given in Theorem 7.1 and 1.) of Theorem 7.2.

Numerical implementation

Some parts of this work, such as the plots, can be found as a numerical implementation in the publicly available Jupyter Notebook: https://doi.org/10.5281/zenodo.14417259.

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A Derivation of $\mathcal{L}_A$ and $\mathcal{N}_A$

The explicit form of $\mathcal{L}_A$ and $\mathcal{N}_A$ in (3.7) is derived. A similar result is presented in [14] (see Def. 6), but without derivation. For this, we need to translate the differential operator

$$L_A(\partial_{x_1}, \partial_{x_2}) = A_{11}\partial_{x_1}^2 + 2A_{12}\partial_{x_1}\partial_{x_2} + A_{22}\partial_{x_2}^2$$

into radial coordinates. The relation between Cartesian and polar coordinates is $(x_1, x_2) = (r\cos(\varphi), r\sin(\varphi))$, so we have for the Jacobian and its inverse:

$$\frac{\partial(x_1,x_2)}{\partial(r,\varphi)} = \begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix}, \quad \frac{\partial(r,\varphi)}{\partial(x_1,x_2)} = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\frac{1}{r}\sin(\varphi) & \frac{1}{r}\cos(\varphi) \end{pmatrix}.$$

Using the chain rule, we compute (write ∂_i for ∂_{x_i}):

$$\begin{aligned} \partial_1 &= \frac{\partial r}{\partial x_1}\partial_r + \frac{\partial \varphi}{\partial x_1}\partial_\varphi = \cos(\varphi)\partial_r - \frac{1}{r}\sin(\varphi)\partial_\varphi, \\ \partial_2 &= \frac{\partial r}{\partial x_2}\partial_r + \frac{\partial \varphi}{\partial x_2}\partial_\varphi = \sin(\varphi)\partial_r + \frac{1}{r}\cos(\varphi)\partial_\varphi. \end{aligned}$$

The second-order derivatives are derived by using the product rule:

$$\begin{aligned} \partial_1\partial_1 =& \cos(\varphi)^2\partial_r^2 + \frac{2}{r^2}\cos(\varphi)\sin(\varphi)\partial_\varphi - \frac{2}{r}\cos(\varphi)\sin(\varphi)\partial_r\partial_\varphi + \frac{1}{r}\sin(\varphi)^2\partial_r + \frac{1}{r^2}\sin(\varphi)^2\partial_\varphi^2, \\ \partial_1\partial_2 =& \cos(\varphi)\sin(\varphi)\partial_r^2 + \frac{1}{r^2}(\sin(\varphi)^2 - \cos(\varphi)^2)\partial_\varphi + \frac{1}{r}(\cos(\varphi)^2 - \sin(\varphi)^2)\partial_r\partial_\varphi \\ & -\frac{1}{r}\cos(\varphi)\sin(\varphi)\partial_r - \frac{1}{r^2}\cos(\varphi)\sin(\varphi)\partial_\varphi^2 = \partial_2\partial_1, \\ \partial_2\partial_2 =& \sin(\varphi)^2\partial_r^2 - \frac{2}{r^2}\cos(\varphi)\sin(\varphi)\partial_\varphi + \frac{2}{r}\cos(\varphi)\sin(\varphi)\partial_r\partial_\varphi + \frac{1}{r}\cos(\varphi)^2\partial_r + \frac{1}{r^2}\cos(\varphi)^2\partial_\varphi^2. \end{aligned}$$

We calculate $\mathcal{L}_A$, where $L_A(\partial_{x_1}, \partial_{x_2})r^\lambda v = r^{\lambda-2}\mathcal{L}_A(\partial_\varphi, \lambda)v$, and obtain:

$$\mathcal{L}_A(\partial_\varphi, \lambda) = b_2(\varphi)\partial_\varphi^2 + (\lambda-1)b_1(\varphi)\partial_\varphi + \lambda(\lambda-1)b_0(\varphi) + \lambda b_2(\varphi),$$

where

$$\begin{aligned} b_0(\varphi) &= A_{11}\cos(\varphi)^2 + A_{22}\sin(\varphi)^2 + 2A_{12}\sin(\varphi)\cos(\varphi), \\ b_1(\varphi) &= 2(A_{22} - A_{11})\sin(\varphi)\cos(\varphi) + 2A_{12}(\cos(\varphi)^2 - \sin(\varphi)^2), \\ b_2(\varphi) &= A_{11}\sin(\varphi)^2 + A_{22}\cos(\varphi)^2 - 2A_{12}\cos(\varphi)\sin(\varphi). \end{aligned}$$

For the conormal derivative, we start with:

$$N_A(\varphi) = A_{11}n_1\partial_1 + A_{12}(n_1\partial_2 + n_2\partial_1) + A_{22}n_2\partial_2, \quad \text{where } n = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix}.$$

Exchanging Cartesian with polar coordinates yields:

$$\begin{aligned} N_A(\varphi) = & \frac{1}{r}\left(A_{11}\sin^2(\varphi) + A_{22}\cos^2(\varphi) - 2A_{12}\sin(\varphi)\cos(\varphi)\right)\partial_\varphi \\ & + \left((A_{22} - A_{11})\cos(\varphi)\sin(\varphi) + A_{12}(\cos^2(\varphi) - \sin^2(\varphi))\right)\partial_r, \end{aligned}$$

such that, using $N_A(\varphi)u = r^{\lambda-1}\mathcal{N}_A(\partial_\varphi, \lambda)v$, we derive:

$$\mathcal{N}_A(\partial_\varphi, \lambda) = b_2(\varphi)\partial_\varphi + \frac{\lambda}{2}b_1(\varphi).$$

B Ellipticity conditions

Proof of Lemma 3.2

We provide the proof of Lemma 3.2, restated here for completeness:

Lemma B.1. *Assume that $A_{11}, A_{12}, A_{22} \in \mathrm{Mat}_\ell(\mathbb{R})$ are symmetric matrices and that A_{11}, A_{22} are positive definite. Then the following are equivalent:*

i) $\det(L_A(\xi)) \neq 0$ *for all* $\xi \in \mathbb{R}^2$ *of the form* $\xi = (1, \beta) \in \mathbb{R}^2$.

ii) L_A *is elliptic.*

iii) L_A *is strongly elliptic.*

Moreover, in this case, $\det(L_A(\xi)) > 0$ *for any* $\xi \in \mathbb{R}^2 \setminus \{0\}$.

For clarity, let us write $L_A(1, \beta)$ instead of $L_A((1, \beta))$ in the following.

Proof. The implications iii) $\implies$ ii) $\implies$ i) are clear. Now, let us assume that

$$\det(L_A(1, \beta)) \neq 0 \quad \forall \beta \in \mathbb{R}, \tag{B.1}$$

and show that L_A is strongly elliptic.

Claim 1: $L_A(1, \beta) > 0$ for any $\beta \in \mathbb{R}$.

Take $\beta \in \mathbb{R}$. Symmetry of $L_A(1, \beta)$ follows from symmetry of the $A_\bullet$'s. Since $\det(L_A(1, \beta))$ is the product of eigenvalues of $L_A(1, \beta)$ (factors occurring with algebraic multiplicity), and because $L_A(1, 0) = A_{11} > 0$ has only positive eigenvalues, we conclude that $\sigma(L_A(1, \beta)) \subset \mathbb{R}_{>0}$ for all $\beta \in \mathbb{R}$. This follows from (B.1), continuity of $\beta \mapsto \det(L_A(1, \beta))$, and continuity of eigenvalues. This shows Claim 1.

Claim 2: $L_A(\beta, 1) > 0$ for any $\beta \in \mathbb{R}$.

Observe that $L_A(\beta, 1) = \beta^2 L_A(1, 1/\beta)$ for $\beta \neq 0$. Claim 2 then follows from Claim 1. For $\beta = 0$, the claim follows from $A_{22} > 0$.

To complete the proof, we need to show that there exists $\kappa > 0$ such that

$$\langle L_A(\xi)\eta, \eta\rangle \geq \kappa \|\eta\|^2 \|\xi\|^2 \quad \forall \eta \in \mathbb{C}^\ell,\ \xi \in \mathbb{R}^2. \tag{B.2}$$

Set $M_1 := \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \neq 0\}$ and $M_2 := \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 \neq 0\}$. We show uniform boundedness for each $M_\bullet$ separately.

Claim 3a: $\exists \kappa_1 > 0$ such that

$$\langle L_A(\xi)\eta, \eta\rangle \geq \kappa_1 \|\eta\|^2 \|\xi\|^2 \quad \forall \eta \in \mathbb{C}^\ell,\ \xi \in M_1. \tag{B.3}$$

Dividing both sides of (B.3) by $\xi_1^2 \neq 0$ yields the equivalent condition (note $\beta = \frac{\xi_2^2}{\xi_1^2}$):

$$\langle L_A(1, \beta))\eta, \eta\rangle \geq \kappa_1 \|\eta\|^2 (1 + \beta^2) \quad \forall \eta \in \mathbb{C}^\ell,\ \beta \in \mathbb{R}.$$

Assume this is false and derive a contradiction. Then there exist sequences $(\eta_n)_{n\in\mathbb{N}} \subset \mathbb{C}^\ell$ with $\|\eta_\bullet\| = 1$ and $(\beta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\frac{\langle L_A(1,\beta_n)\eta_n, \eta_n\rangle}{1+\beta_n^2} \le \frac{1}{n} \quad \forall n \in \mathbb{N}. \tag{B.4}$$

Note that $(\beta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ cannot be bounded. Otherwise it admits an accumulation point $\beta \in \mathbb{R}$ which implies $\langle L_A(1,\beta))\eta, \eta\rangle = 0$ for some $\eta \in \mathbb{C}^\ell \setminus \{0\}$, a contradiction to Claim 1 (not that the set of normalized vectors is compact). So without loss of generality, assume $|\beta_n| \xrightarrow{n\to\infty} \infty$. Rewriting (B.4) yields:

$$\frac{\beta_n^2}{1+\beta_n^2}\langle A_{22}\eta_n, \eta_n\rangle \le \frac{1}{n} - 2\frac{\beta_n}{1+\beta_n^2}\langle A_{12}\eta_n, \eta_n\rangle - \frac{1}{\beta_n^2}\langle A_{11}\eta_n, \eta_n\rangle \to 0.$$

As $|\beta_n| \to \infty$, and thus $\frac{\beta_n^2}{1+\beta_n^2} \to 1$, it follows that $\langle A_{22}\eta_n, \eta_n\rangle \to 0$. This contradicts the assumption that A_{22} is positive definite (again by compactness). Thus, Claim 3a is shown.

Claim 3b: $\exists \kappa_2 > 0$ such that

$$\langle L_A(\xi)\eta, \eta\rangle \ge \kappa_2 \|\eta\|^2 \|\xi\|^2 \quad \forall \eta \in \mathbb{C}^\ell,\ \xi \in M_2. \tag{B.5}$$

This follows by the same reasoning as for Claim 3a, swapping the roles of A_{11} with A_{22}, and using Claim 2 instead of Claim 1.

Since $M_1 \cup M_2 = \mathbb{R}^2 \setminus \{0\}$, we can define $\kappa = \min\{\kappa_1, \kappa_2\}$ to satisfy (B.2). For $\xi = 0$, (B.2) holds for any $\kappa > 0$. Thus, i) $\implies$ iii). Positivity of $\det(L_A(\xi))$ for $\xi \in \mathbb{R}^2 \setminus \{0\}$ is clear by ellipticity and continuity. □

Complementing b.c. and Neumann well-posedness

We aim to relate the ellipticity conditions presented in this work to those found in the literature, in particular to ADN-elliptic systems [3]. The reference for the next paragraphs is §1.1.2 of [22].

Recall the setup given in Section 3: Consider the domain $\mathcal{K}_\alpha$ with boundaries $\Gamma^\pm$, and let L_A denote the differential operator defined by $A = (A_{11}, A_{12}, A_{22})$ for $A_\bullet \in \mathrm{Mat}_\ell(\mathbb{R})$ symmetric and $A_{11}, A_{22} > 0$. On the boundary, we have two differential operators $B_A^\pm$, which we summarize as $B_A(x) = B_A^\pm(x)$ for $x \in \Gamma^\pm$. The system (L_A, B_A) is called *elliptic* (or ADN-elliptic) if the following two conditions are met:

1.) The operator L_A is *properly elliptic*.

2.) B_A satisfies the *complementing boundary condition* on $\partial\mathcal{K}_\alpha$.

For 1.), in the case of real-valued $A_\bullet$, proper ellipticity of L_A is equivalent to L_A being elliptic (3.5). For details, refer to §1 in [2].

To address 2.), we introduce some terminology. Consider $x_0 \in \partial\mathcal{K}_\alpha$ and ξ tangential to $\partial\mathcal{K}_\alpha$ at x_0. Denote by $\mathcal{M}(\xi)$ the subspace of solutions u to the ODE:

$$L_A(\xi - in\partial_t)u(t) = 0, \quad t > 0,$$

such that $u(t) \to 0$ for $t \to \infty$. Here, n denotes the unit vector orthogonal to ξ and pointing outward from $\mathcal{K}_\alpha$. Now, B_A is said to satisfy the complementing boundary condition if, for every $x_0 \in \partial\mathcal{K}_\alpha$, every ξ tangential to $\mathcal{K}_\alpha$ at x_0, and every $g \in \mathbb{C}^\ell$, there exists a unique $u \in \mathcal{M}(\xi)$ satisfying:

$$B_A(\xi - in\partial_t)u(t)|_{t=0} = g.$$

In our case, the tangential vectors ξ at $x_0 \in \partial\mathcal{K}_\alpha$ and the corresponding unit vectors n can be parameterized as:

$$\xi(r,\varphi) = r\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\end{pmatrix}, \quad n(\varphi) = \begin{pmatrix}-\sin(\varphi)\\ \cos(\varphi)\end{pmatrix},$$

for $r \neq 0$ and $\varphi \in \{0, \alpha\}$, depending on $x_0 \in \Gamma^\pm$. For $\varphi = 0$, the above n is actually pointing inward, but without loss of generality, we can reverse the sign for a simpler (but similar) argument. One can compute:

$$\begin{aligned} L_A(\xi(r,\varphi) - in(\varphi)\partial_t) &= \sum_{k,l=1}^{2} A_{kl}(\xi(r,\varphi) - in(\varphi)\partial_t)_k(\xi(r,\varphi) - in(\varphi)\partial_t)_l \\ &= r^2 b_0(\varphi) - irb_1(\varphi)\partial_t - b_2(\varphi)\partial_t^2, \end{aligned}$$

where the $b_\bullet$'s are given in Appendix A. Since the case of $\varphi = \alpha$ boils down to a rotated version of $\varphi = 0$, we restrict to discuss $\varphi = 0$. Then the above reduces to:

$$L_A(\xi(r,0) - in(0)\partial_t) = r^2 A_{11} - 2irA_{12}\partial_t - A_{22}\partial_t^2.$$

Let us assume A is monic, and let V be the standard root of A. Then for $r > 0$ any $u \in \mathcal{M}(\xi(r,0))$ is of the form (recall Theorem 4.6 ii):

$$u(t) = \exp(irtV)c \quad \text{for } c \in \mathbb{C}^\ell \text{ and } t \in [0,\infty).$$

Note that $\exp(irt\overline{V})$ does not occur, since $\|\exp(irt\overline{V})c\| \xrightarrow{t\to\infty} \infty$ for any $c \in \mathbb{C}^\ell \setminus \{0\}$ (deduced by the spectral mapping theorem). For $r < 0$, we simply swap V and $\overline{V}$. Continuing with $r > 0$, we now analyze the complementing boundary condition on Γ^- for $B_A^-(x_0) = N_A^-(x_0)$ (Neumann boundary conditions). In this case, we have:

$$B_A^-(\xi(r,0) - in(0)\partial_t) = \sum_{k,l=1}^{2} A_{kl}n(0)_k(\xi(r,0) - in(0)\partial_t)_l = rA_{12} - iA_{22}\partial_t.$$

So the complementing boundary condition reduces to the statement that, for any $g \in \mathbb{C}^\ell$, there exists a unique $c \in \mathbb{C}^\ell$ such that (recall $A_{22} = \mathrm{Id}_\ell$):

$$\sum_{k=1}^{\ell}(A_{12} + V)c = g,$$

which is equivalent to invertibility of $A_{12} + V$. Comparing to Secion 5.3, this shows that the complementing boundary condition at points $B_A(x_0) = N_A(x_0)$ is equivalent to Neumann well-posedness. The arguments can also be generalized to nonmonic tuples.

Formal positivity and contractive Neumann well-posedness

The tuple $A = (A_{11}, A_{12}, A_{22})$ is said to be *formal positive* (§3.2 in [7]) if there exists $\kappa > 0$ such that:

$$\sum_{i,j=1}^{2} \langle A_{ij} f^{(i)}, f^{(j)} \rangle \geq \kappa(\|f^{(1)}\|^2 + \|f^{(2)}\|^2) \quad \text{for all } f^{(1)}, f^{(2)} \in \mathbb{C}^\ell. \tag{B.6}$$

This condition is often found in the context of linear elasticity and is sometimes called *Legendre condition* (§3.1.4 in [9]). Note that the LHS of (B.6) can be rewritten in block matrix form as $f^T M_A f \geq \kappa\|f\|^2$, where

$$f := \begin{pmatrix} f^{(1)} & f^{(2)} \end{pmatrix} \in \mathbb{C}^{2\ell}, \quad M_A := \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \in \mathrm{Mat}_{2\ell}(\mathbb{R}).$$

Thus, formal positivity is equivalent to $M_A > 0$. Before continuing, let us state a result about block matrices, from Theorem 7.7.6 in [17].

Lemma B.2. *Consider $A, B, C \in \mathrm{Mat}_\ell(\mathbb{C})$ and the block matrix*

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

M is positive definite if and only if $A > 0$ and $C > B^* A^{-1} B$. Furthermore, this is equivalent to $\rho(B^* A^{-1} B C^{-1}) < 1$.

The next result relates formal positivity and contractive Neumann well-posedness.

Lemma B.3. *Consider an elliptic tuple A which is formal positive. Then it is contractive Neumann well-posed.*

Proof. Consider an elliptic tuple A. Note that A is formal positive if and only if its monic reduction is formal positive. To see this, one can apply $\mathrm{diag}(A_{22}^{-1/2}, A_{22}^{-1/2})$ from the left and right to M_A. So, without loss of generality, assume $A_{22} = \mathrm{Id}_\ell$. We prove the following claim.

Claim: Formal positivity of A implies that A is Neumann well-posed.

Before proving the claim, let us argue why the claim shows the statement. Due to $M_A > 0$ and Lemma B.2, we have:

$$M_A(s) := \begin{pmatrix} A_{11} & sA_{12} \\ sA_{12} & \mathrm{Id}_\ell \end{pmatrix} > 0 \quad \text{for all } s \in [0, 1].$$

Thus, using the claim, any $A(\bullet)$ on the path $s \mapsto A(s) = (A_{11}, sA_{12}, \mathrm{Id}_\ell)$ is Neumann well-posed. Using techniques from the proof of Lemma 5.4, we can further deform $(A_{11}, 0, \mathrm{Id}_\ell)$ to $(\mathrm{Id}_\ell, 0, \mathrm{Id}_\ell)$, showing that A is path-connected to $(\mathrm{Id}_\ell, 0, \mathrm{Id}_\ell)$ by a path of Neumann well-posed systems. From Lemma 5.4 it follows that A is contractive Neumann well-posed. It remains to prove the claim.

Proof of the claim: Using Theorem 4.6, we write for the standard root $V = (S + i\,\mathrm{Id}_\ell)D$ for $S, D \in \mathrm{Mat}_\ell(\mathbb{R})$ symmetric and $D > 0$, and M_A as:

$$M_A = \begin{pmatrix} D(S^2 + \mathrm{Id}_\ell)D & -\frac{1}{2}(SD + DS) \\ -\frac{1}{2}(SD + DS) & \mathrm{Id}_\ell \end{pmatrix}.$$

Now assume $M_A > 0$ and that A is not Neumann well-posed, and derive a contradiction. The latter implies $2i \in \sigma(D^{-1}SD - S)$ (recall (5.7)). $M_A > 0$ is by Lemma B.2 equivalent to:

$$\rho((SD + DS)(D(S^2 + \mathrm{Id}_\ell)D)^{-1}(SD + DS)) < 4 \iff \rho(N^*N) < 4 \iff \|N\| < 2, \tag{B.7}$$

where $N := (S + i\,\mathrm{Id}_\ell)^{-1}(D^{-1}SD + S)$. By assumption, there exists $y \in \mathbb{C}^\ell$ with $\|y\| = 1$ such that $(D^{-1}SD - S)y = 2iy$. This implies $(D^{-1}SD + S)y = 2(S + i\,\mathrm{Id}_\ell)y$, and thus $\|Ny\| = 2$, contradicting (B.7). This proves the claim. □

Remark. The statement that formal positivity implies Neumann well-posedness (a.k.a. the complementing boundary condition for N_A) is given, in a more general context, in Theorem 3.2.6. of [7].

Remark. Assuming that A is an elliptic tuple, the various (ellipticity) conditions discussed in this work are related as follows:

formal positive $\Longrightarrow$ contractive Neumann well-posed $\Longrightarrow$ Neumann well-posed
$\Longleftrightarrow$ complementing boundary condition for N_A.

C Factorization of nonnegative matrix polynomials

The reference is [10], in particular the Introduction, §1.4 and §12.5. A *matrix polynomial* L is a matrix-valued polynomial function $L(\xi) = \sum_{k=0}^{r} A_k \xi^k$, where $\xi \in \mathbb{C}$ and $A_\bullet \in \mathrm{Mat}_\ell(\mathbb{C})$. Here, $r \geq 0$ is called the *order* of L. L is called *monic* if $A_r = \mathrm{Id}_\ell$. If all $A_\bullet$ are Hermitian matrices, L is called *self-adjoint*. The spectrum of a matrix polynomial, denoted by $\sigma(L)$, generalizes the spectrum of matrices and is defined as:

$$\sigma(L) := \{\lambda \in \mathbb{C} : \exists v \in \mathbb{C}^\ell \setminus \{0\} \text{ with } L(\lambda)v = 0\}.$$

$L(\xi)$ is called *nonnegative* if one has $\langle L(\lambda)v, v\rangle \geq 0$ for all $\lambda \in \mathbb{R}$ and $v \in \mathbb{C}^\ell$. The main result is the following.

Theorem C.1. *For a monic, self-adjoint matrix polynomial $L(\lambda)$, the following statements are equivalent:*

i) $L(\lambda)$ is nonnegative.

ii) $L(\lambda)$ admits a representation of the form

$$L(\lambda) = M^*(\lambda)M(\lambda),$$

where $M(\lambda)$ is a monic matrix polynomial and $\sigma(M) \subset \mathrm{clos}(\mathrm{UHP})$.

For a matrix polynomial of order 2, this yields $M(\lambda)$ of the form $M(\lambda) = \mathrm{Id}_\ell\,\lambda - M_0$, for $M_0 \in \mathrm{Mat}_\ell(\mathbb{C})$ and $\sigma(M) = \sigma(M_0)$.

D Functional calculus

General concept

We briefly summarize the functional calculus adapted to finite-dimensional vector spaces. The reference is *Symbolic Calculus* in §10 of [24]. Let $A \in \mathrm{Mat}_\ell(\mathbb{C})$ be fixed for this subsection. For a polynomial $f(z) = c_0 + c_1 z + \cdots + c_n z^n$, where $c_\bullet \in \mathbb{C}$, we can canonically define $f(A)$ by setting:

$$f(A) := c_0 + c_1 A + \cdots + c_n A^n.$$

The functional calculus generalizes this idea, addressing whether a more general complex function $f : \mathbb{C} \to \mathbb{C}$ (with possible restrictions to the domain) can be meaningfully lifted to a function $f : \mathrm{Mat}_\ell(\mathbb{C}) \to \mathrm{Mat}_\ell(\mathbb{C})$, denoted by the same symbol. Remarkably, this is possible for a broad class of functions, including holomorphic ones. Using a generalization of the Cauchy integral formula, we define:

Definition D.1. Consider a holomorphic function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ on an open set Ω and let $A \in \mathrm{Mat}_\ell(\mathbb{C})$ with $\sigma(A) \subset \Omega$. We define

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z)(\mathrm{Id}_\ell \cdot z - A)^{-1} dz, \tag{D.1}$$

where Γ is a contour enclosing $\sigma(A)$.

Remark. i) The integral is understood componentwise in the entries of the integrand.
ii) The definition is independent of the choice of contour Γ, provided it encloses $\sigma(A)$. Then, $(\mathrm{Id}_\ell \cdot z - A)^{-1}$ and the integral are well-defined.

The functional calculus exhibits several key properties:

- The spectrum of $f(A)$ satisfies $\sigma(f(A)) = f(\sigma(A))$. This is called the *spectral mapping theorem*.
- For any invertible $Q \in \mathrm{Mat}_\ell(\mathbb{C})$, it holds that $f(QAQ^{-1}) = Qf(A)Q^{-1}$.
- If A is diagonalizable such that $A = QBQ^{-1}$ for $Q, B = \mathrm{diag}(\beta_1, \ldots, \beta_\ell) \in \mathrm{Mat}_\ell(\mathbb{C})$, the functional calculus simplifies to:

  $$f(A) := Qf(B)Q^{-1},$$

  where $f(B) = \mathrm{diag}(f(\beta_1), \ldots, f(\beta_\ell))$. This is consistent with (D.1), which boils down to Cauchy's integral formula for complex numbers in the diagonal components. This representation does not depend on the choice of Q, resp. eigenvectors.

Complex Exponentation

For $\lambda \in \mathbb{C}$ and $A \in \mathrm{Mat}_\ell(\mathbb{C})$ with $0 \notin \sigma(A)$, complex exponentiation was defined in Section 4.2 by $A^{\lambda_a} = \exp(\lambda \log_a(z))$. The following exponential rules hold for $\lambda, \mu \in \mathbb{C}$ and $a \in \{+, -, o\}$:

$$A^{(\lambda+\mu)_a} = A^{\lambda_a} A^{\mu_a}, \quad (A^{\lambda_a})^{\mu_a} = A^{(\lambda \cdot \mu)_a}.$$

These results can first be proven for diagonalizable A and then extended to general A using a density argument. Similarly, one can show $\overline{A^{\lambda_a}} = \overline{A}^{\overline{\lambda}_a}$. By the spectral mapping theorem, $0 \notin \sigma(A^{\lambda_a})$, implying that matrix exponentials are always invertible.

Remark. For $A_1, A_2 \in \mathrm{Mat}_\ell(\mathbb{C})$, it is generally not true that $(A_1 A_2)^{\lambda_a} = A_1^{\lambda_a} A_2^{\lambda_a}$.

Most of the time, we consider Z^{λ_a} for $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{\pm i}$ (see Def. 6.1). This is well-defined due to $0 \notin \sigma(Z)$ (see Lemma E.1). Also, $(Z^{\lambda_a})^T = Z^{\lambda_a}$. This follows from the Cauchy integral formula and the fact that $\mathrm{Id}_\ell\, z - Z$, and thus also its inverse, are complex symmetric matrices for any $z \in \mathbb{C}$.

E Numerical range and accretive operators

We summarize the relevant definitions and provide the proofs, which were used in Section 6 to bound the spectrum of $M_{\lambda,\alpha}$ away from zero.

Numerical range

The reference for this subsection is §1 in [18], where the numerical range is referred to as "field of values". Additional results can be found in [12]. For $A \in \mathrm{Mat}_\ell(\mathbb{C})$, the *numerical range of* A is defined as:

$$W(A) = \left\{ \frac{\langle x, Ax\rangle}{\langle x, x\rangle} : x \in \mathbb{C}^\ell \setminus \{0\} \right\},$$

and the *angular field of* A as:

$$W'(A) = \left\{ \langle x, Ax\rangle : x \in \mathbb{C}^\ell \setminus \{0\} \right\}.$$

Using the scaling $x \mapsto rx$ for $r > 0$, it follows that $W'(A)$ consists of rays connecting the origin $0 \in \mathbb{C}$ to points in $W(A)$. Also, it is clear that $W(A) \subset W'(A)$. For $A, B \in \mathrm{Mat}_\ell(\mathbb{C})$ and $\alpha, \beta \in \mathbb{C}$, the following hold:

N1: $W(A)$ is convex and compact.

N2: $W(\alpha A + \beta\,\mathrm{Id}_\ell) = \alpha W(A) + \{\beta\}$.

N3: The spectrum is bounded by the numerical range: $\sigma(A) \subset W(A)$.

N4: $W(A) = W(U^* A U)$ for any unitary $U \in \mathrm{Mat}_\ell(\mathbb{C})$ and $W'(A) = W'(C^* A C)$ for any $C \in \mathrm{Mat}_\ell(\mathbb{C})$.

N5: $W(A + B) \subset W(A) + W(B)$.

N6: $W(A)$ is a line segment $[\alpha, \beta]$ if and only if A is Hermitian. Then $\lambda_{\min}(A) = \alpha$ and $\lambda_{\max}(A) = \beta$.

N7: $W(A) \subset \mathrm{RHP}$ if and only if $A + A^* > 0$.

N8: $W(A^*) = \overline{W(A)}$.

Here, we used $\lambda_{\min}(A) := \min\{\sigma(A)\}$ and $\lambda_{\max}(A) := \max\{\sigma(A)\}$ for Hermitian A. Additionally, we have the following result.

Lemma E.1. *Consider $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{\pm i}$ (see Def. 6.1). Then $\sigma(Z), W(Z) \subset \pm\,\mathrm{UHP}$.*

Proof. This follows from $W(Z) \subset W(\mathrm{Re}\, Z) + iW(\mathrm{Im}\, Z) \subset \mathbb{R} \pm i\mathbb{R}_{>0}$, using **N2**, **N3**, **N5**, and **N6**. □

Accretive operators

Throughout this section, the $\arg$- and $\log$-function correspond to the principal branch. To bound the spectrum of $M_{\lambda,\alpha}$ away from zero, as discussed in Section 6, we used results on the numerical range of Z^λ for $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{\pm i}$ (recall Def. 6.1). These classes of matrices are naturally closely related to symmetric matrices $\mathrm{Mat}_\ell(\mathbb{C}) \ni A = A^T$ with $\mathrm{Re}\, A > 0$, which are examples of *accretive operators*. Bounds for fractional powers of accretive operators are known, and our task is to translate these to $\mathrm{Mat}_\ell(\mathbb{C})_{\pm i}$. The reference for all definitions and results in this section is the dissertation [13]. Some concepts have been simplified, in particular the notion of maximal accretive operators is not required for finite dimensions.

For $0 < \omega \leq \pi$, the open sector is defined as:

$$S_\omega := \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg(z)| < \omega\}.$$

The following definition, found on p.101 of [13], introduces the concept of ω-accretivity.

Definition E.2. Let $0 \leq \omega \leq \frac{\pi}{2}$. $A \in \mathrm{Mat}_\ell(\mathbb{C})$ is called ω-*accretive* if $W(A) \subset \mathrm{clos}(S_\omega)$. For $\omega = \frac{\pi}{2}$, i.e., $W(A) \subset \mathrm{clos}(\mathrm{RHS})$, A is simply called *accretive*.

Accretive operators often appear as generators of contraction semigroups.

Theorem E.3 (Theorem B.21 in [13])**.** *$A \in \mathrm{Mat}_\ell(\mathbb{C})$ is accretive if and only if $-A$ generates a strongly continuous contraction semigroup, i.e., $\|e^{-At}\| \leq 1$ for all $t \geq 0$.*

Remark. See Appendix A.7 in [13] for a summary on semigroups and generators. In our case, the semigroup can be expressed, using the functional calculus, as $T(t) = e^{-At}$.

The numerical range of fractional powers of accretive operators is well understood. The following results, all found in §*Fractional Powers of m-Accretive Operators and the Square Root Problem* of [13], will be used to extend these results to $\mathrm{Mat}_\ell(\mathbb{C})_{\pm i}$.

Proposition E.4. *Let $\delta > 0$ and $A - \delta\, \mathrm{Id}_\ell$ be accretive for $A \in \mathrm{Mat}_\ell(\mathbb{C})$. Then $A^\lambda - \delta^\lambda\, \mathrm{Id}_\ell$ is accretive for each $0 < \lambda \leq 1$.*

Proposition E.5. *Let $A \in \mathrm{Mat}_\ell(\mathbb{C})$ be accretive, and let $0 \leq \lambda \leq 1$. Then A^λ is $\frac{\lambda\pi}{2}$-accretive, i.e., $W(A^\lambda) \subset \mathrm{clos}(S_{\frac{\lambda\pi}{2}})$.*

Proposition E.6. *Let $A \in \mathrm{Mat}_\ell(\mathbb{C})$ be an injective ω-accretive operator for some $0 \leq \omega \leq \frac{\pi}{2}$. Then:*

$$W(\log(A)) \subset \{z \in \mathbb{C} : |\,\mathrm{Im}\, z| \leq \omega\}.$$

Now that we established the necessary tools, we can prove Lemma 6.2 and Lemma 6.3. For the reader's convenience, we restate them:

Lemma E.7. *Consider $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{\pm i}$. For $\lambda \in [-1, 1] \setminus \{0\}$ we have*

$$W(Z^\lambda) \subset \mathrm{sgn}(\lambda) \cdot \{z \in \mathbb{C} \setminus \{0\} : \pm \arg(z) \in (0, \lambda\pi)\},$$

and, in particular, $W'(Z^\lambda) \subset \pm\, \mathrm{sgn}(\lambda)\, \mathrm{UHP}$.

Proof. Consider $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{\pm i}$ and $0 < \lambda \leq 1$. Define $A_\pm = \mp iZ$ so $\operatorname{Re} A_\pm = \operatorname{Im} Z > 0$, implying $W(A_\pm) \subset \mathrm{RHP}$. Thus, A is $\frac{\pi-\varepsilon}{2}$-accretive for some $\varepsilon > 0$. Using this, we bound the numerical range of Z^λ as follows:

$$\begin{aligned} W(Z^\lambda) = (\pm i)^\lambda W(A_\pm^\lambda) = e^{\pm i\lambda\pi/2} W(A_\pm^\lambda) \subset e^{\pm i\lambda\pi/2} \operatorname{clos}\left(S_{\frac{\lambda(\pi-\varepsilon)}{2}}\right) \\ \subset \{z \in \mathbb{C} \setminus \{0\} : \pm \arg(z) \in (0, \lambda\pi)\} \cup \{0\}, \end{aligned} \tag{E.1}$$

where we applied **N2** and Prop. E.5. It remains to show $0 \notin W(Z^\lambda)$. For this, note that there exists $\delta > 0$ such that $Z \mp i\delta \operatorname{Id}_\ell \in \mathrm{Mat}_\ell(\mathbb{C})_{\pm i}$, implying $W(A_\pm - \delta \operatorname{Id}_\ell) \subset \mathrm{RHP}$. By Prop. E.4, $W(A_\pm^\lambda - \delta^\lambda \operatorname{Id}_\ell) \subset \operatorname{clos}(\mathrm{RHP})$, which implies $W(A_\pm^\lambda) \subset \mathrm{RHP}$. Thus, $0 \notin W(Z^\lambda)$ by the second equality in (E.1). This completes the proof for $\lambda \in (0, 1]$. For $\lambda \in [-1, 0)$, the proof follows from $(Z^{-1})^\lambda = Z^{-\lambda}$ and the next result. □

Lemma E.8. *We have* $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{\pm i} \iff Z^{-1} \in \mathrm{Mat}_\ell(\mathbb{C})_{\mp i}$.

Proof. Consider $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$ and show $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{-i}$. The reverse implication follows similarly. $(Z^{-1})^T = Z^{-1}$ is clear, and it remains to show $\operatorname{Im} Z^{-1} < 0$.

Claim: $Z^{-1} \in \mathrm{Mat}_\ell(\mathbb{C})_{-i}$ for $Z = \operatorname{Re} Z + i \operatorname{Id}_\ell$.

First, we assume that Z has the simple form $Z = \operatorname{Re} Z + i \operatorname{Id}_\ell$. Since $\operatorname{Re} Z$ is symmetric, there exists a diagonalization $\operatorname{Re} Z = QBQ^T$ for $Q, B \in \mathrm{Mat}_\ell(\mathbb{R})$ and Q orthogonal. Thus, we can write $Z = Q(B + i \operatorname{Id}_\ell)Q^T$. The inverse is given by $Z^{-1} = Q(B + i \operatorname{Id}_\ell)^{-1}Q^T$. Now observe that $\operatorname{Im} Z^{-1} = Q \operatorname{Im}[(B + i \operatorname{Id}_\ell)^{-1}]Q^T$, since Q is real-valued, as well as $\operatorname{Im}[(B + i \operatorname{Id}_\ell)^{-1}] < 0$. This shows the claim.

Now for general $Z = \operatorname{Re} Z + i \operatorname{Im} Z \in \mathrm{Mat}_n(\mathbb{C})_{+i}$ we have $Z^{-1} = (\operatorname{Im} Z)^{-1/2} \tilde{Z}^{-1} (\operatorname{Im} Z)^{-1/2}$ for $\tilde{Z} := (\operatorname{Im} Z)^{-1/2} \operatorname{Re} Z (\operatorname{Im} Z)^{-1/2} + i \operatorname{Id}_\ell$. Due to the claim, $\operatorname{Im} \tilde{Z}^{-1} < 0$, and by Sylvester's law of inertia, $\operatorname{Im} Z^{-1} = (\operatorname{Im} Z)^{-1/2} \operatorname{Im} \tilde{Z}^{-1} (\operatorname{Im} Z)^{-1/2} < 0$. This completes the proof. □

Lemma E.9. *Consider* $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$ *and* $\lambda \in \mathbb{R} \setminus \{0\}$. *Then the following holds:*

$$\begin{aligned} &i) \quad \operatorname{sgn}(\lambda)\left(\operatorname{Id}_\ell - (Z^{i\lambda})^* Z^{i\lambda}\right) > 0, \\ &ii) \quad \rho((Z^{i\lambda})^* Z^{i\lambda}) \xrightarrow{\lambda \to \infty} 0, \\ &iii) \quad \min\{\beta : \beta \in \sigma((Z^{i\lambda})^* Z^{i\lambda})\} \xrightarrow{\lambda \to -\infty} \infty. \end{aligned}$$

Proof. **Claim 1:** $\|Z^{i\lambda}\| < 1$ for $\lambda > 0$.

Consider $Z \in \mathrm{Mat}_\ell(\mathbb{C})_{+i}$ and $\lambda > 0$. We have $Z^{i\lambda} = \exp(i\lambda \log(Z))$, and due to Theorem E.3, it suffices to show that $-iW(\log(Z)) \subset \mathrm{RHP}$. Let $A = -iZ$, so $W(A) \subset \mathrm{RHP}$ and A is $\frac{\pi-\varepsilon}{2}$-accretive for some $\varepsilon > 0$. Then:

$$-i\log(Z) = -i\log(iA) = -i(\log(i)\operatorname{Id}_\ell + \log(A)) = \frac{\pi}{2}\operatorname{Id}_\ell - i\log(A),$$

where we used the functional calculus and properties of the $\log$-function. By **N2** and Prop. E.6, we obtain:

$$W(-i\log(iA)) \subset \frac{\pi}{2} - i \cdot \left\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \frac{\pi - \varepsilon}{2}\right\} \subset \mathrm{RHP}\,.$$

This establishes Claim 1.

Claim 2: $\rho(Z^{i\lambda}) < 1$ for $\lambda > 0$ and $\min\{|\beta| : \beta \in \sigma(Z^{i\lambda})\} > 1$ for $\lambda < 0$.

The first part follows from Claim 1, as $\rho(Z^{i\lambda}) \leq \|Z^{i\lambda}\|$. The second part follows since $(Z^{i\lambda})^{-1} = Z^{-i\lambda}$, and the spectral theorem implies:

$$\min\{|\beta| : \beta \in \sigma(Z^{i\lambda})\} = \max\{|\beta| : \beta \in \sigma(Z^{-i\lambda})\}.$$

This establishes Claim 2.

i) Note that for any $A \in \mathrm{Mat}_\ell(\mathbb{C})$ one has $\|AA^*\| = \|A^*A\| = \|A\|^2 = \rho(A^*A)$, so $\mathrm{Id}_\ell - A^*A > 0$ is equivalent to $\|A\| < 1$. Thus, the case $\mathrm{sgn}(\lambda) > 0$ follows from Claim 1. The case $\mathrm{sgn}(\lambda) < 0$ can be derived from the case $\mathrm{sgn}(\lambda) > 0$, the fact

$$A - B > 0 \iff B^{-1} - A^{-1} > 0$$

applied to $A = \mathrm{Id}_\ell$ and $B = (Z^{i\lambda})^* Z^{i\lambda}$, and $\sigma((Z^{-i\lambda})^* Z^{-i\lambda}) = \sigma(Z^{-i\lambda}(Z^{-i\lambda})^*)$.

ii) For any $N \in \mathbb{N}$, we have:

$$\rho((Z^{i\lambda})^* Z^{i\lambda}) = \|(Z^{i\lambda})^* Z^{i\lambda})\| = \|Z^{i\lambda}\|^2 = \|(Z^{i\lambda/N})^N\|^2 \leq \|Z^{i\lambda/N}\|^{2N},$$

where we used $\|A^N\| \leq \|A\|^N$. Claim 1 implies $\rho((Z^{i\lambda})^* Z^{i\lambda}) \xrightarrow{\lambda\to\infty} 0$.

iii) This follows from ii) using $(Z^{i\lambda})^{-1} = Z^{-i\lambda}$, $\sigma((Z^{-i\lambda})^* Z^{-i\lambda}) = \sigma(Z^{-i\lambda}(Z^{-i\lambda})^*)$, and the spectral theorem. □