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A very short proof of Sidorenko's inequality for counts of homomorphism between graphs

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Abstract

We provide a very elementary proof of a classical extremality result due to Sidorenko (Discrete Math. 131.1-3, 1994), which states that among all graphs G on k vertices, the $k-1$-edge star maximises the number of graph homomorphisms of G into any graph H.

Let $\sharp\hom(G,H)$ denote the number of graph homomorphisms from a graph $G=(V(G),H(G))$ into an *image graph* $H=(V(H),E(H))$, i.e. $\hom(G,H)=\{f:V(G)\to V(H)\mid(u,v)\in E(G)\Rightarrow(f(u),f(v))\in E(H)\}$. The following general inequality is due to Sidorenko [3].

Theorem 1. *Let G denote any connected graph on $k+1$ vertices and S_k the star graph with k edges. Then*

$$\sharp\hom(G,H)\le\sharp\hom(S_k,H) \text{ for any graph } H.$$

Three proofs of this result can be found in the literature. Sidorenko's original proof relies on the remarkable fact that the relation on graphs

$$\sharp\hom(G,H)\le\sharp\hom(G',H) \text{ for all } H$$

is in one-to-one correspondence to an ordering of integral functionals on measure spaces which can be associated with G and G'. Certain combinatorial operations on graphs map to $\log$-convex combinations of these functionals for which Hölder-type inequalities hold. The corresponding inequalities for the homomorphism counts are then used to establish the extremality of star graphs and further relations between the homomorphism counts of concrete examples of graphs.

Csikvári and Lin [1] provide another proof of Theorem 1 that is close in spirit to Sidorenko's work but uses the Wiener index (the sum of all distances between pairs of vertices in a graph) and more elementary combinatorial operations on graphs to conclude.

Finally, Levin and Peres [2] prove Theorem 1 by a brief and elegant probabilistic argument that connects the homomorphism count to the stationary distribution of the simple random walk on the target graph.

The aim of this note is to present a new short proof of Sidorenko's bound that relies solely on elementary counting arguments and Hölder's inequality on finite probability spaces.

Proof of Theorem 1. Fix an arbitrary image graph $H=(V(H),E(H))$. Observe that removing edges from G can only increase $\sharp\hom(G,H)$, hence it suffices to show

$$\sharp\hom(T,H)\le\sharp\hom(S_k,H), \tag{1}$$

whenever T is any k-edge tree. Let $\mathcal{T}(k,\ell)$ denote the set of k-edge trees with precisely $\ell\le k$ leaves. In particular, we have $\mathcal{T}(k,k)=\{S_k\}$. The bound (1) follows, if we can show that

$$\max_{T\in\mathcal{T}(k,\ell)}\sharp\hom(T,H)\le\max_{T\in\mathcal{T}(k,\ell+1)}\sharp\hom(T,H) \text{ for any } \ell<k. \tag{2}$$

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To this end, we demonstrate that for every non-star $T \in \mathcal{T}(k,\ell)$ there exits some k-edge tree T' with one more leaf that admits at least the same number of homomorphisms into H as T. Denote by $\mathrm{sk}(T)$, the *skeleton tree* of T, obtained by removing all leaves from T. Since $\ell < k$, $\mathrm{sk}(T)$ has at least 2 leaves. We designate two leaves b_1, b_2 of $\mathrm{sk}(T)$ and denote by $T(b_1,b_2)$ the graph obtained from T by removing all leaves adjacent to b_1 and b_2. We write $\vec{d_1}, \vec{d_2}$ for the number of leaves removed at b_1 and b_2, respectively. Calculating $\sharp\hom(T,H)$ by first counting all maps of $T(b_1,b_2)$ and then the possible choices for the images of the remaining leaves yields

$$\begin{aligned}\sharp\hom(T,H) &= \sum_{u,v\in V(H)} \sharp\{f\in\hom(T(b_1,b_2),H) : f(b_1)=u, f(b_2)=v\}\deg(u)^{\vec{d_1}}\deg(v)^{\vec{d_2}}\\ &= \sharp\hom(T(b_1,b_2),H)\sum_{u,v\in V(H)} p(u,v)\deg(u)^{\vec{d_1}}\deg(v)^{\vec{d_2}},\end{aligned}$$

where $p(u,v)$ denotes the probability that a uniformly chosen map in $\hom(T(b_1,b_2),H)$ maps b_1 to u and b_2 to v. Denoting the marginals of $p(\cdot,\cdot)$ by $p_1(\cdot), p_2(\cdot)$, we conclude with the help of Hölder's inequality

$$\begin{aligned}\sharp\hom(T,H) \leq &\sharp\hom(T(b_1,b_2),H)\\ &\times\left(\sum_{u\in V(H)} p_1(u)\deg(u)^{\vec{d_1}+\vec{d_2}}\right)^{\frac{\vec{d_1}}{\vec{d_1}+\vec{d_2}}}\left(\sum_{u\in V(H)} p_2(u)\deg(u)^{\vec{d_1}+\vec{d_2}}\right)^{\frac{\vec{d_2}}{\vec{d_1}+\vec{d_2}}}.\end{aligned} \tag{3}$$

We assume w.l.o.g. that

$$\sum_{u\in V(H)} p_1(u)\deg(u)^{\vec{d_1}+\vec{d_2}} \geq \sum_{u\in V(H)} p_2(u)\deg(u)^{\vec{d_1}+\vec{d_2}},$$

since if the opposite inequality holds, we may reverse the choice of b_1, b_2 at the beginning. Thus (3) leads to

$$\begin{aligned}\sharp\hom(T,H) &\leq \sharp\hom(T(b_1,b_2),H)\sum_{u\in V(H)} p_1(u)\deg(u)^{\vec{d_1}+\vec{d_2}}\\ &= \sum_{u\in V(H)} \sharp\{f\in\hom(T(b_1,b_2),H) : f(b_1)=u\}\deg(u)^{\vec{d_1}+\vec{d_2}}.\end{aligned}$$

The last expression equals $\sharp\hom(T',H)$, where $T'\in\mathcal{T}(k,\ell+1)$ is obtained from $T(b_1,b_2)$ by attaching $\vec{d_1}+\vec{d_2}$ leaves to b_1 and consequently has precisely one more leaf than T. □

In fact, the above line of reasoning also establishes a slightly stronger form of the statement that coincides with Sidorenko's original formulation of the result.

Corollary 2 ([3, Theorem 1.2])**.** *Let T denote any k-edge tree, S_k the star graph with k edges and $T_{k-1,1}$ any tree obtained by attaching a single leaf to a leaf of S_{k-1}. Then*

$$\sharp\hom(T,H) \leq \sharp\hom(T_{k-1,1},H) \leq \sharp\hom(S_k,H) \text{ for any graph } H.$$

Proof. All graphs in $\mathcal{T}(k,k-1)$ agree with $T_{k-1,1}$ up to relabelling, hence the statement follows from (2). □

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