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Solvability and optimal control of a multi-species Cahn–Hilliard–Keller–Segel tumor growth model

Pierluigi Colli[1], Gianni Gilardi[1], Andrea Signori[2], Jürgen Sprekels[3]

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[1] Dipartimento di Matematica "F. Casorati"
Università di Pavia
via Ferrata 5
27100 Pavia
Italy
E-Mail: pierluigi.colli@unipv.it
gianni.gilardi@unipv.it

[2] Dipartimento di Matematica
Politecnico di Milano
via E. Bonardi 9
20133 Milano
Italy
E-Mail: andrea.signori@polimi.it

[3] Department of Mathematics
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
and
Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: juergen.sprekels@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

This paper investigates an optimal control problem associated with a two-dimensional multi-species Cahn–Hilliard–Keller–Segel tumor growth model, which incorporates complex biological processes such as species diffusion, chemotaxis, angiogenesis, and nutrient consumption, resulting in a highly nonlinear system of nonlinear partial differential equations. The modeling derivation and corresponding analysis have been addressed in a previous contribution. Building on this foundation, the scope of this study involves investigating a distributed control problem with the goal of optimizing a tracking-type cost functional. This latter aims to minimize the deviation of tumor cell location from desired target configurations while penalizing the costs associated with implementing control measures, akin to introducing a suitable medication. Under appropriate mathematical assumptions, we demonstrate that sufficiently regular solutions exhibit continuous dependence on the control variable. Furthermore, we establish the existence of optimal controls and characterize the first-order necessary optimality conditions through a suitable variational inequality.

1 Introduction

This paper investigates an optimal control problem associated with a multi-species Cahn–Hilliard–Keller–Segel tumor growth model in a two-dimensional spatial domain $\Omega \subset \mathbb{R}^2$ over a given final time $T > 0$. The problem we aim to analyze consists of a distributed optimal control problem associated with an initial-boundary value problem:

$$\partial_t\varphi - \Delta\mu + \chi_\varphi \Delta n = -m\varphi + \mathbb{h}(\varphi) \quad \text{in } Q := \Omega \times (0,T), \tag{1.1}$$

$$-\Delta\varphi + F'(\varphi) = \mu \quad \text{in } Q, \tag{1.2}$$

$$\partial_t a - \Delta a + \chi_a \operatorname{div}(a\nabla\sigma) = a - a^2 + u \quad \text{in } Q, \tag{1.3}$$

$$\partial_t n - \Delta n - \chi_\varphi n = S_n \quad \text{in } Q, \tag{1.4}$$

$$\partial_t \sigma - \Delta\sigma - \chi_a a = S_\sigma \quad \text{in } Q, \tag{1.5}$$

$$\partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}} a = \partial_{\boldsymbol{n}} n = \partial_{\boldsymbol{n}}\sigma = 0 \quad \text{on } \Sigma := \partial\Omega \times (0,T), \tag{1.6}$$

$$\varphi(0) = \varphi_0, \quad a(0) = a_0, \quad n(0) = n_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \tag{1.7}$$

The primary variables in the system are φ, μ, a, n, and σ. These represent the density of tumor cells φ, the chemical potential μ, an angiogenetic phase composed of tumor-induced new vasculature with volume fraction a, a nutrient or signaling molecule n, and a concentration σ affecting tumor growth dynamics. The positive constants χ_φ and χ_a represent chemotaxis parameters quantifying the sensitivity of biological entities to chemical gradients. In the second equation (1.2), F' denotes the derivative of a configuration potential F characterized by a double-well shape. Prototypical choices

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for this latter include the so-called *classical regular potential* and the *logarithmic potential* defined as follows:

$$F_{reg}(r) := \frac{c_1}{4} r^2 (r-1)^2, \quad r \in \mathbb{R}, \tag{1.8}$$

$$F_{log}(r) := r \ln r + (1-r)\ln(1-r) + c_2\, r(1-r), \quad r \in (0,1), \tag{1.9}$$

where $c_1,\ c_2$ are two positive real coefficients. The mass of the tumor, represented by φ, is not conserved, as indicated by the presence of a source term $-m\varphi + \mathbb{h}(\varphi)$ on the right-hand side of the first equation, where $\mathbb{h}$ represents a smooth real function and m is a positive constant. Chemotaxis is modeled via a Keller–Segel type (cf., e.g., [20]) coupling, specifically through the nonlinear term $\chi_a \operatorname{div}(a\nabla\sigma)$ in the third equation. The logistic source term $a - a^2$ in (1.3) for the nutrient variable a is a common choice in Keller–Segel models to prevent solution blow-up in finite times, see, e.g., [12, 18, 19, 25, 29] and the references therein. Finally, φ_0, a_0, n_0, and σ_0 denote prescribed initial data for these variables, whereas S_n and S_σ stand for suitable source terms depending on the solution variables, details of which will be provided later on.

The model (1.1)–(1.7) originates from variational principles and was introduced in [2]. The postulated free energy of the system, which is defined as the internal energy minus the entropy, is given by

$$\begin{aligned}\mathcal{E}(\varphi, a, n, \sigma) = {} & \int_\Omega a(\ln a - 1) - \chi_\varphi \int_\Omega n\varphi - \chi_a \int_\Omega a\sigma \\ & + \frac{1}{2}\int_\Omega |\nabla n|^2 + \frac{1}{2}\int_\Omega |\nabla \sigma|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \int_\Omega F(\varphi).\end{aligned} \tag{1.10}$$

In [2], the modeling derivation and numerical simulations (cf. [1]) aim to optimize model parameters and support clinical decision-making, whereas the corresponding mathematical analysis is addressed in [3], where the existence of weak solutions was shown in two and three dimensions, and regularity results and uniqueness of regular enough solutions were proved in the two-dimensional setting. Here, we aim at considering an optimal control problem, where the distributed control u enters in the form of a source term in (1.3). The minimization problem we want to study consists in minimizing a suitable *cost functional* that we postulate to be of tracking type form, and expressed by

$$\mathcal{J}(\varphi, u) := \frac{b_1}{2}\int_Q |\varphi - \varphi_Q|^2 + \frac{b_2}{2}\int_\Omega |\varphi(T) - \varphi_\Omega|^2 + \frac{b_3}{2}\int_Q |u|^2\,, \tag{1.11}$$

where the coefficients b_i are given nonnegative numbers, with $b_3 > 0$, and φ_Q and φ_Ω are given functions on Q and Ω, respectively, representing clinical targets. Besides, we constrain the control variables to belong to the set of *admissible controls* defined by

$$\mathcal{U}_{\rm ad} := \left\{ u \in \mathcal{U} := L^\infty(Q) : 0 \le u \le u_{\max} \text{ a.e. in } Q \right\}, \tag{1.12}$$

where $u_{\max} \in L^\infty(Q)$ is a prescribed nonnegative function. Then the control problem under investigation can be formulated as follows:

$$\begin{aligned}&\text{Minimize } \mathcal{J}(\varphi, u) \text{ subject to } u \in \mathcal{U}_{\rm ad} \text{ and to the constraint that} \\ &(\varphi, \mu, a, n, \sigma) \text{ is the solution to the system (1.1)–(1.7).}\end{aligned} \tag{1.13}$$

Tumor growth models based on the phase field approach have gained significant popularity. While not exhaustive, we refer interested readers to [4, 10, 15, 16] and the references therein. Several studies within this framework consider the influence of velocity effects on the mixture dynamics, utilizing

Darcy's law and the Brinkman equation. For detailed discussions on these topics, see [7, 13, 15, 21]. The incorporation of chemotaxis, particularly through the Keller–Segel coupling, represents a relatively recent advancement in phase field models. This coupling has been explored in studies such as [2, 18, 25]. Finally, regarding the optimal control problem, we refer readers to [5, 8, 9, 14, 17, 27, 28] for comprehensive discussions and analyses.

The plan of the paper is as follows. In the next section, we state the problem in a precise form and present our results. The existence and the uniqueness of the solution to the state system, as well as proper stability and continuous dependence estimates, are proved in Sections 3 and 4. A technical result is given in Section 5, preparing the study of the control problem made in the last two sections, where we prove the existence of an optimal control and we establish first-order necessary conditions for optimality in terms of the solution to the adjoint problem.

2 Statement of the problem and results

Throughout the paper, Ω is a bounded open subset of $\mathbb{R}^2$ having a smooth boundary $\Gamma := \partial\Omega$. The symbols $|\Omega|$ and $\partial_{\boldsymbol{n}}$ denote the measure of Ω and the derivative in the direction of the outward unit normal vector $\boldsymbol{n}$ on Γ, respectively. With a prescribed final time $T > 0$, we set

$$Q := \Omega \times (0,T), \quad \Sigma := \Gamma \times (0,T). \tag{2.1}$$

Given a Banach space X, we denote by $\|\cdot\|_X$ both its norm and the norm in any power of X, with the exceptions of the space H introduced below and of the Lebesgue spaces $L^p(\Omega)$ $(1 \le p \le +\infty)$, for which we use the symbol $\|\cdot\|_p$. Sometimes, this symbol also denotes the norm in $L^p(Q)$. Moreover, in order to simplify the notation, we still write X (i.e., we avoid the exponent) when dealing with some power of X. Then, we introduce the shorthands

$$H := L^2(\Omega), \quad V := H^1(\Omega) \quad \text{and} \quad W := \{v \in H^2(\Omega) : \partial_{\boldsymbol{n}} v = 0 \text{ a.e. on } \Gamma\}, \tag{2.2}$$

and endow these spaces with their natural norms. For simplicity, we write $\|\cdot\|$ instead of $\|\cdot\|_H$. Moreover, we denote by V^* and $\langle\,\cdot\,,\,\cdot\,\rangle$ the dual space of V and the duality pairing between V^* and V, respectively, and we identify H with a subspace of V^* in the usual way, i.e., such that $\langle w, v\rangle = \int_\Omega wv$ for every $w \in H$ and $v \in V$. Hence, we have the continuous, dense, and compact embeddings

$$V \hookrightarrow H \hookrightarrow V^*,$$

yielding that (V, H, V^*) is a Hilbert triple.

At this point, we are ready to introduce our assumptions on the structure of the state system which involve, in particular, a specific choice of the functions S_n and S_σ that appear in equations (1.4) and (1.5). We assume that

$$m \in (0,+\infty), \quad \chi_\varphi, \chi_a \in (0,1), \quad c_\varphi, c_n, c_\sigma, c_0 \in \mathbb{R}, \tag{2.3}$$

$$\mathbb{h} : \mathbb{R} \to \mathbb{R} \text{ is such that } \mathbb{h} \in W^{2,\infty}(\mathbb{R}), \tag{2.4}$$

$$S_n := c_\varphi \varphi + c_n n + c_\sigma \sigma + c_0, \quad S_\sigma := 1 - \sigma - a\,\sigma. \tag{2.5}$$

As for the potential F, we confine ourselves to some conditions that generalize the cases of the classical and logarithmic potentials (1.8) and (1.9). In particular, we ignore the possibility of extending the latter to $[0,1]$ by continuity and prescribe suitable regularities on F in the open interval that is taken as the domain $D(F)$ in any case and is actually the effective domain of the derivative F'. We assume:

$$\begin{aligned}&\text{Either } D(F)=\mathbb{R} \text{ or } D(F)=(0,1), \text{ and it holds that}\\ &\quad F=F_1+F_2 \text{ with functions } F_1\,,F_2: D(F)\to\mathbb{R} \text{ of class } C^4,\\ &\quad \text{where } F_1 \text{ is convex and } F_2' \text{ is Lipschitz continuous.}\end{aligned} \tag{2.6}$$

$$\text{If } D(F)=\mathbb{R}, \text{ then } F \text{ satisfies } \lim_{|r|\to+\infty} r^{-2}F(r)=+\infty\,; \tag{2.7}$$

$$\begin{aligned}&\text{if } D(F)=(0,1), \text{ then } \lim_{r\searrow 0} F'(r)=-\infty, \quad \lim_{r\nearrow 1} F'(r)=+\infty, \quad \text{and there is}\\ &\quad \text{a constant } C_F \text{ such that } |F_1''(r)|\le e^{C_F(|F_1'(r)|+1)} \text{ for all } r\in(0,1).\end{aligned} \tag{2.8}$$

We notice that these assumptions imply that F is bounded from below and also ensure the existence of some $r_0\in\mathbb{R}$ satisfying

$$r_0\in D(F) \quad \text{and} \quad F_1'(r_0)=0\,. \tag{2.9}$$

As the reader can directly check, it turns out that the growth condition in (2.8) is satisfied by the convex part of the logarithmic potential in (1.9).

For the control variable u, we assume that

$$u\in L^\infty(Q) \quad \text{satisfies} \quad 0\le u\le u_{\max} \quad \text{a.e. in } Q\,, \tag{2.10}$$

where

$$u_{\max}\in L^\infty(Q) \quad \text{is nonnegative.} \tag{2.11}$$

To introduce our assumptions on the initial data, we use the following general notation for the mean value: we set

$$\overline{v}:=\frac{1}{|\Omega|}\int_\Omega v \quad \text{for } v\in L^1(\Omega)\,. \tag{2.12}$$

The same symbol will be used in the sequel even for time-dependent functions. Then, denoting by $(\,\cdot\,)^\pm$ the positive and negative parts, we assume that

$$\begin{aligned}&\varphi_0\in W \text{ with range in } D(F); \text{ moreover, } \varphi_0\in H^3(\Omega) \text{ if } D(F)=\mathbb{R},\\ &\quad \varphi_0\in H^4(\Omega) \text{ and } \mu_0:=-\Delta\varphi_0+F'(\varphi_0)\in W \text{ if } D(F)=(0,1).\end{aligned} \tag{2.13}$$

$$\begin{aligned}&r_0-(\overline{\varphi_0}-r_0)^- -R \text{ and } r_0+(\overline{\varphi_0}-r_0)^+ +R \text{ belong to } D(F),\\ &\quad \text{where } R:=\frac{1}{m}\sup_{r\in\mathbb{R}}|\mathbb{h}(r)-mr_0|\,.\end{aligned} \tag{2.14}$$

$$a_0\in V \quad \text{and} \quad a_0>0 \quad \text{a.e. in } \Omega\,. \tag{2.15}$$

$$n_0,\sigma_0\in W \quad \text{and} \quad 0\le\sigma_0\le 1 \quad \text{in } \Omega\,. \tag{2.16}$$

Of course, the assumption in (2.14) yields a restriction only in the case when $D(F)$ is bounded.

Finally, we are in a position to introduce our formulation of the state system. Even though some of the equations could be written in the strong form used in the Introduction, we prefer to present the whole problem in a variational framework. We look for a quintuple $(\varphi, \mu, a, n, \sigma)$ with the properties

$$\varphi \in \mathcal{Y}_1 := H^1(0,T;V) \cap L^\infty(0,T;W \cap H^3(\Omega)) \cap C^0(\overline{Q})$$
$$\text{and } \ \varphi \in D(F) \ \text{ a.e. in } Q\,, \tag{2.17}$$

$$\mu \in \mathcal{Y}_2 := L^\infty(0,T;V) \cap L^2(0,T;W \cap H^3(\Omega))\,, \tag{2.18}$$

$$a \in \mathcal{Y}_3 := H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$$
$$\text{and } \quad a > 0 \quad \text{a.e. in } Q\,, \tag{2.19}$$

$$n \in \mathcal{Y}_4 := H^1(0,T;V) \cap L^\infty(0,T;W) \cap L^2(0,T;H^3(\Omega)) \cap \mathcal{Z}$$
$$\text{where } \ \mathcal{Z} := W^{1,4}(0,T;L^4(\Omega)) \cap L^4(0,T;W^{2,4}(\Omega))\,, \tag{2.20}$$

$$\sigma \in \mathcal{Y}_4 \quad \text{and} \quad 0 \le \sigma \le 1 \quad \text{a.e. in } Q\,, \tag{2.21}$$

that solves the variational equations

$$\int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\mu \cdot \nabla v - \chi_\varphi \int_\Omega \nabla n \cdot \nabla v = -m \int_\Omega \varphi v + \int_\Omega \mathbb{h}(\varphi)\, v\,, \tag{2.22}$$

$$\int_\Omega \nabla\varphi \cdot \nabla v + \int_\Omega F'(\varphi)\, v = \int_\Omega \mu v\,, \tag{2.23}$$

$$\int_\Omega \partial_t a\, v + \int_\Omega \nabla a \cdot \nabla v - \chi_a \int_\Omega a\nabla\sigma \cdot \nabla v = \int_\Omega \bigl(a - a^2 + u\bigr) v\,, \tag{2.24}$$

$$\int_\Omega \partial_t n\, v + \int_\Omega \nabla n \cdot \nabla v - \chi_\varphi \int_\Omega \varphi v = \int_\Omega S_n\, v$$
$$\text{where} \quad S_n = c_\varphi \varphi + c_n n + c_\sigma \sigma + c_0\,, \tag{2.25}$$

$$\int_\Omega \partial_t\sigma\, v + \int_\Omega \nabla\sigma \cdot \nabla v = \int_\Omega \bigl((1-\sigma) + a(\chi_a - \sigma)\bigr) v\,, \tag{2.26}$$

for every $v \in V$ and a.e. in $(0,T)$, and satisfies the initial condition

$$(\varphi, a, n, \sigma)(0) = (\varphi_0, a_0, n_0, \sigma_0) \quad \text{a.e. in } \Omega\,. \tag{2.27}$$

Remark 2.1. We notice that the regularity properties (2.19) and (2.21) imply that both a and $\nabla\sigma$ are L^4 functions (see the forthcoming (2.36)), so that all of the terms occurring in (2.24) are meaningful. We also point out that by virtue of (2.17)–(2.21) all of the above equations may be written in their strong form. From (2.17) we also have that

$$\varphi \in L^\infty(0,T;H^3(\Omega))\,, \quad \text{whence} \quad \nabla\varphi \in L^\infty(Q)\,, \tag{2.28}$$

with the corresponding norms that are estimated by a constant like K_1 below.

For convenience, we set the state space as (cf. (2.17)–(2.21))

$$\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3 \times \mathcal{Y}_4 \times \mathcal{Y}_4. \tag{2.29}$$

Our first result regards well-posedness and stability of system (1.1)–(1.7).

Theorem 2.2. *Assume* (2.3)–(2.8) *on the structure, and* (2.10)–(2.11) *and* (2.13)–(2.16) *on the data. Then there exists a unique quintuple* $(\varphi, \mu, a, n, \sigma)$ *that satisfies* (2.17)–(2.21) *and solves problem* (2.22)–(2.27). *Moreover, the stability estimate and separation property*

$$\|(\varphi, \mu, a, n, \sigma)\|_{\mathcal{Y}} \le K_1\,, \tag{2.30}$$

$$r_- \le \varphi \le r_+ \quad \textit{a.e. in } Q\,, \tag{2.31}$$

hold true with constants $K_1 > 0$ and $r_\pm \in D(F)$ that depend only on Ω, T, the structure of the system, the initial data, and $u_{\max}$. In particular, they are independent of u.

Next, we have the following continuous dependence result.

Theorem 2.3. *Suppose the assumptions of Theorem 2.2 regarding the structure and the initial data are fulfilled, and let u_i, $i = 1, 2$, satisfy* (2.10) *and $(\varphi_i, \mu_i, a_i, n_i, \sigma_i) \in \mathcal{Y}$ be the corresponding solution. Then the inequality*

$$\begin{aligned} &\|\varphi_1 - \varphi_2\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\mu_1 - \mu_2\|_{L^2(0,T;V)} \\ &\quad + \|a_1 - a_2\|_{H^1(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;V)} + \|n_1 - n_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \\ &\quad + \|\sigma_1 - \sigma_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq K_2 \, \|u_1 - u_2\|_{L^2(0,T;H)} \end{aligned} \tag{2.32}$$

holds true with a constant $K_2 > 0$ that depends only on Ω, T, the structure of the system, the initial data, and $u_{\max}$.

Once well-posedness is established, we can deal with the control problem presented in the Introduction (see (1.11)–(1.13)). Let us refer to the last two sections for the precise statements. Here, we just mention that we first prove the existence of an optimal control, i.e., of an element $u^* \in \mathcal{U}_{\rm ad}$ that satisfies

$$\mathcal{J}(\varphi^*, u^*) \leq \mathcal{J}(\varphi, u) \quad \text{for every } u \in \mathcal{U}_{\rm ad} \tag{2.33}$$

where φ^* and φ are the first components of the solutions corresponding to u^* and u, respectively. Then, we derive a first-order necessary optimality condition for a given $u^* \in \mathcal{U}_{\rm ad}$ to be an optimal control in terms of a suitable variational inequality. Namely, u^* is an optimal control whether it fulfills

$$\int_Q (p_3 + b_3 u^*)(u - u^*) \geq 0 \quad \text{for every } u \in \mathcal{U}_{\rm ad} \,, \tag{2.34}$$

where p_3 is the third component of the solution to the adjoint problem introduced and discussed in Section 7.

In performing our proofs, we often make use of Hölder's inequality, as well as of Young's inequality

$$yz \leq \frac{\delta}{p}|y|^p + \frac{1}{p'}\delta^{-p'/p}\,|z|^{p'} \quad \text{for every } y, z \in \mathbb{R} \text{ and } \delta > 0 \,, \tag{2.35}$$

with $1 < p,\ p' < \infty$ conjugate exponents, i.e., $p + p' = p\,p'$. Moreover, we recall the two-dimensional embeddings

$$\begin{aligned} &V \hookrightarrow L^p(\Omega) \quad \text{for } p \in [1, +\infty), \quad W \hookrightarrow C^0(\overline{\Omega}), \\ &\text{and } \ L^\infty(0,T;H) \cap L^2(0,T;V) \hookrightarrow L^4(Q), \end{aligned} \tag{2.36}$$

and the corresponding inequalities

$$\begin{aligned} &\|v\|_p \leq C_{\Omega,p}\,\|v\|_V \quad \text{for every } v \in V, \quad \|v\|_\infty \leq C_\Omega\,\|v\|_W \quad \text{for every } v \in W \\ &\quad \text{and} \quad \|v\|_{L^4(Q)} \leq C_{\Omega,T}\,\|v\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\quad \text{for every } v \in L^\infty(0,T;H) \cap L^2(0,T;V), \end{aligned} \tag{2.37}$$

where C_Ω depend only on Ω and $C_{\Omega,p}$ and $C_{\Omega,T}$ depend on p and T, in addition. Furthermore, since the embeddings $V \subset H$ and $H \subset V^*$ are compact, we obtain from Ehrling's lemma the compactness inequality

$$\|v\| \le \delta \|\nabla v\| + C_\delta \|v\|_{V^*} \quad \text{for every } v \in V \text{ and } \delta > 0, \tag{2.38}$$

with some $C_\delta > 0$ that depends only on Ω and δ. We also account for the Poincaré–Wirtinger inequality, an inequality from the elliptic regularity theory, and the two-dimensional Ladyžhenskaya inequality. Namely, we sometimes owe to

$$\|v - \overline{v}\| \le C_\Omega \|\nabla v\| \quad \text{for every } v \in V, \tag{2.39}$$

$$\|\nabla v\| \le C_\Omega \left(\|v\| + \|\Delta v\|\right) \quad \text{for every } v \in W, \tag{2.40}$$

$$\|v\|_4^2 \le C_\Omega \|v\| \, \|v\|_V \quad \text{for every } v \in V,$$
$$\text{and } \|\nabla v\|_4^2 \le C_\Omega \|v\|_V \left(\|v\| + \|\Delta v\|\right) \quad \text{for every } v \in W, \tag{2.41}$$

with the same constant C_Ω as before, without loss of generality. We aim to point out that $v \mapsto \|v\| + \|\Delta v\|$ provides a norm in W which is equivalent to the standard norm in $H^2(\Omega)$.

We conclude this section by stating a convention that regards the constants appearing in the proofs of the forthcoming sections. The small-case symbol c denotes a generic constant that depends only on the structure of the system, Ω, T, the initial data, and $u_{\max}$ (see (2.11)). In particular, the values of c are independent of u. Notice that the meaning of c may vary from line to line and even within the same line. We use capital letters for precise constants we could refer to.

3 Existence and stability

This section is devoted to the existence of a solution $(\varphi, \mu, a, n, \sigma)$ to problem (2.22)–(2.27) that satisfies estimates (2.30) and (2.31). We mention that the well-posedness of a similar system can be compared with the analysis developed in [3]. The system studied there has a slightly more general structure, but it does not contain the control variable u, acting as a source term. The authors of [3] prove the existence of a (unique regular) solution, satisfying the restrictions on the values of a and σ given in our statement, by means of a proper argument based on regularization, truncation and discretization (see [3, Theorems 2.8–2.11]). On the other hand, here we construct our argumentation without giving the full detail of approximation and just perform formal a priori estimates on the solution $(\varphi, \mu, a, n, \sigma)$ to motivate the expected regularity and the stability estimates. In particular, we point out the treatment of the new terms involving the control u, which do not appear in the paper [3]. In agreement with the specific form of the energy $\mathcal{E}$ in (1.10), we assume in the following the component a to be positive.

Boundedness property. We first prove that σ attains its values in $[0, 1]$. To this end, we fix a monotone C^1 function $G : \mathbb{R} \to \mathbb{R}$ that grows linearly at infinity and satisfies $G(r) < 0$ for $r < 0$, $G(r) = 0$ for $r \in [0, 1]$, and $G(r) > 0$ for $r > 1$, and we test (2.26) by $G(\sigma)$. If $\widehat{G}$ is the antiderivative of G that vanishes at zero, we obtain that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \widehat{G}(\sigma) + \int_\Omega G'(\sigma) |\nabla \sigma|^2 = \int_\Omega \big((1 - \sigma) + a(\chi_a - \sigma)\big) G(\sigma) \quad \text{a.e. in } (0, T).$$

Since a is nonnegative and $\chi_a \in (0, 1)$, the right-hand side is nonpositive. We then integrate over time and observe that our assumptions on σ_0 (see (2.16)) imply that $\widehat{G}(\sigma_0) = 0$. Since both $\widehat{G}$ and

G' are nonnegative, we conclude that $\widehat{G}(\sigma)$ vanishes identically, entailing that

$$0 \le \sigma \le 1 \quad \text{a.e. in } Q. \tag{3.1}$$

Control of the mean value. The next estimate regards the mean value of φ. We test (2.22) by the constant $1/|\Omega|$ and obtain that

$$\frac{d}{dt}\,\overline{\varphi} + m\,\overline{\varphi} = \overline{g}\,, \quad \text{where} \quad g := \mathbb{h}(\varphi)\,. \tag{3.2}$$

In addition, recalling (2.9) we note that the constant function $v(t) = r_0$ fulfills the equation

$$\frac{d}{dt}\,v + m\,v = m\,r_0 \quad \text{in } (0,T) \tag{3.3}$$

and the initial condition $v(0) = r_0$. Hence, taking the difference between (3.2) and (3.3), and solving the resulting Cauchy problem for $\overline{\varphi} - v$, we easily find that

$$\overline{\varphi}(t) - r_0 = (\overline{\varphi_0} - r_0)\,e^{-mt} + \int_0^t e^{-m(t-s)}\,(\overline{g}(s) - mr_0)\,ds \quad \text{for every } t \in [0,T].$$

Since $|\overline{g}(s) - mr_0| \le \sup_{r\in\mathbb{R}} |\mathbb{h}(r) - mr_0| = mR$ for all $s \in [0,T]$ (see (2.14)), we easily conclude that

$$r_0 - (\overline{\varphi_0} - r_0)^- - R \le \overline{\varphi}(t) \le r_0 + (\overline{\varphi_0} - r_0)^+ + R \quad \text{for every } t \in [0,T]. \tag{3.4}$$

Finally, by recalling (2.14), we claim that there are positive constants δ_0 and C_0 such that

$$F_1'(r)(r - r') \ge \delta_0\,|F_1'(r)| - C_0$$
$$\text{for every } r \in D(F) \text{ and } r' \in [r_0 - (\overline{\varphi_0} - r_0)^- - R, r_0 + (\overline{\varphi_0} - r_0)^+ + R]. \tag{3.5}$$

This is a generalization of the inequality proved in [23, Appendix, Prop. A.1] in the case of a fixed r'. However, the proof also works in the present case with only minor changes since the values of r' we are considering belong to a compact subset of the open interval $D(F)$.

At this point, we start performing the estimates in the direction of the expected regularity of the solution. In each step, we test our equations at the time t by suitable test functions evaluated at the same time t. However, we do not write the symbol t for simplicity, and it is understood that the equalities we obtain hold a.e. in $(0,T)$. For the reader's convenience, we recall the definition of the energy $\mathcal{E}$ (see (1.10)) related to the system

$$\begin{aligned}\mathcal{E}(\varphi,a,n,\sigma) = {} & \int_\Omega a(\ln a - 1) - \chi_\varphi \int_\Omega n\varphi - \chi_a \int_\Omega a\sigma \\ & + \frac{1}{2}\int_\Omega \bigl(|\nabla\varphi|^2 + |\nabla n|^2 + |\nabla\sigma|^2\bigr) + \int_\Omega F(\varphi)\end{aligned} \tag{3.6}$$

and notice that its time derivative is given by

$$\begin{aligned}&\frac{d}{dt}\,\mathcal{E}(\varphi,a,n,\sigma) \\ &= \int_\Omega \partial_t a\,\ln a - \chi_\varphi \int_\Omega \partial_t n\,\varphi - \chi_\varphi \int_\Omega n\,\partial_t\varphi - \chi_a \int_\Omega \partial_t a\,\sigma - \chi_a \int_\Omega a\,\partial_t\sigma \\ &\quad + \frac{1}{2}\,\frac{d}{dt}\int_\Omega \bigl(|\nabla\varphi|^2 + |\nabla n|^2 + |\nabla\sigma|^2\bigr) + \frac{d}{dt}\int_\Omega F(\varphi)\,.\end{aligned} \tag{3.7}$$

First a priori estimate. We test (2.22) by μ and $-\chi_\varphi n$ to obtain the equalities

$$\int_\Omega \partial_t\varphi\,\mu + \int_\Omega |\nabla\mu|^2 - \chi_\varphi \int_\Omega \nabla n\cdot\nabla\mu = \int_\Omega \bigl(-m\varphi + \mathbb{h}(\varphi)\bigr)\mu,$$
$$-\chi_\varphi \int_\Omega \partial_t\varphi\, n - \chi_\varphi \int_\Omega \nabla\mu\cdot\nabla n + \chi_\varphi^2 \int_\Omega |\nabla n|^2 = \chi_\varphi\, m \int_\Omega \varphi n - \chi_\varphi \int_\Omega \mathbb{h}(\varphi)\, n\,.$$

Now, we recall the definition of R in (2.14) and set

$$M := \frac{1}{\delta_0}\,(mR+1)\,, \tag{3.8}$$

where δ_0 is the same as in (3.5). We notice at once that the value of M depends only on the structure of the original system, so that it can be absorbed in the notation c for the generic constants in performing estimates. Thus, we keep M explicitly only when it is needed. Then, we test (2.23) by $\partial_t\varphi$, $M(\varphi-\overline{\varphi})$, and $m\varphi - \mathbb{h}(\varphi)$ to infer that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi|^2 + \frac{d}{dt}\int_\Omega F(\varphi) = \int_\Omega \mu\,\partial_t\varphi\,,$$
$$M\int_\Omega |\nabla\varphi|^2 + M\int_\Omega F'(\varphi)(\varphi-\overline{\varphi}) = M\int_\Omega \mu(\varphi-\overline{\varphi})\,,$$
$$m\int_\Omega |\nabla\varphi|^2 - \int_\Omega \mathbb{h}'(\varphi)|\nabla\varphi|^2 + m\int_\Omega F'(\varphi)\,\varphi - \int_\Omega F'(\varphi)\,\mathbb{h}(\varphi) = \int_\Omega \mu\bigl(m\varphi - \mathbb{h}(\varphi)\bigr)\,.$$

Next, we test (2.24) by $\ln a - \chi_a\sigma$. By noting the identity $\nabla a - \chi_a a\nabla\sigma = a\nabla(\ln a - \chi_a\sigma)$, we have that

$$\int_\Omega \partial_t a\,\ln a - \chi_a \int_\Omega \partial_t a\,\sigma + \int_\Omega a|\nabla(\ln a - \chi_a\sigma)|^2 = \int_\Omega (a - a^2 + u)(\ln a - \chi_a\sigma)\,.$$

Finally, we test (2.25) and (2.26) by $\partial_t n$ and $\partial_t\sigma$, respectively, leading to

$$\int_\Omega |\partial_t n|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla n|^2 - \chi_\varphi \int_\Omega \varphi\partial_t n = \int_\Omega S_n\partial_t n\,,$$
$$\int_\Omega |\partial_t\sigma|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\sigma|^2 - \chi_a \int_\Omega a\partial_t\sigma = \int_\Omega \bigl((1-\sigma) - a\sigma\bigr)\partial_t\sigma\,.$$

At this point, we add all the above equalities to each other and to the sides of the resulting identity the equal terms $\frac{d}{dt}\int_\Omega |n|^2$ and $2\int_\Omega n\partial_t n$, respectively. Notice that four terms cancel out and that nine of the contributions on the left-hand side yield those of the time derivative (3.7) of the energy. By also rearranging terms, we conclude that

$$\frac{d}{dt}\,\mathcal{E}(\varphi,a,n,\sigma) + \int_\Omega |\nabla\mu|^2 + \chi_\varphi^2 \int_\Omega |\nabla n|^2$$
$$+ M\int_\Omega |\nabla\varphi|^2 + M\int_\Omega F_1'(\varphi)(\varphi-\overline{\varphi}) + m\int_\Omega |\nabla\varphi|^2$$
$$+ m\int_\Omega F_1'(\varphi)\,(\varphi - r_0) + \int_\Omega a|\nabla(\ln a - \chi_a\sigma)|^2$$
$$+ \int_\Omega |\partial_t n|^2 + \int_\Omega |\partial_t\sigma|^2 + \int_\Omega a^2(\ln a - \chi_a\sigma) + \frac{d}{dt}\int_\Omega |n|^2$$

$$
\begin{aligned}
&= 2\chi_\varphi \int_\Omega \nabla\mu\cdot\nabla n - \chi_\varphi \int_\Omega \bigl(\mathbb{h}(\varphi) - m\varphi\bigr) n \\
&\quad - M \int_\Omega F_2'(\varphi)(\varphi-\overline{\varphi}) + M \int_\Omega \mu(\varphi-\overline{\varphi}) \\
&\quad + \int_\Omega \mathbb{h}'(\varphi)|\nabla\varphi|^2 + \int_\Omega F_1'(\varphi)\bigl(-mr_0 + \mathbb{h}(\varphi)\bigr) + \int_\Omega F_2'(\varphi)\bigl(-m\varphi + \mathbb{h}(\varphi)\bigr) \\
&\quad + \int_\Omega (a+u)(\ln a - \chi_a\sigma) + \int_\Omega S_n \partial_t n \\
&\quad + \int_\Omega \bigl((1-\sigma) - a\sigma\bigr)\partial_t\sigma + 2\int_\Omega n\partial_t n \,. \qquad (3.9)
\end{aligned}
$$

We consider the terms on the left-hand side of (3.9) that involve F_1'. The second one is nonnegative, since F_1' is monotone and vanishes at r_0. As for the other, we recall (3.4) and apply (3.5) to obtain that

$$
M\int_\Omega F_1'(\varphi)(\varphi-\overline{\varphi}) \geq M\delta_0 \int_\Omega |F_1'(\varphi)| - c\,. \qquad (3.10)
$$

Since $\chi_a \in (0,1)$ and (3.1) holds, we have that

$$
\int_\Omega a^2(\ln a - \chi_a\sigma) \geq \int_\Omega a^2(\ln a - 1)\,,
$$

and we notice that the last integrand is bounded from below.

Let us come to the right-hand side of (3.9), where just some terms need an accurate treatment. Since $\varphi-\overline{\varphi}$ has zero mean value, by also applying the Poincaré–Wirtinger inequality and recalling (3.4), we derive that

$$
\begin{aligned}
&M\int_\Omega \mu(\varphi-\overline{\varphi}) = M\int_\Omega (\mu-\overline{\mu})(\varphi-\overline{\varphi}) \\
&\leq \frac14 \int_\Omega |\nabla\mu|^2 + c\int_\Omega |\varphi-\overline{\varphi}|^2 \leq \frac14\int_\Omega |\nabla\mu|^2 + c\int_\Omega |\varphi|^2 + c\,.
\end{aligned}
$$

As for the term involving F_1', we recall (2.14) and observe that

$$
\int_\Omega F_1'(\varphi)\bigl(-mr_0 + \mathbb{h}(\varphi)\bigr) \leq \sup_{r\in\mathbb{R}} |\mathbb{h}(r) - mr_0| \int_\Omega |F_1'(\varphi)| = mR\int_\Omega |F_1'(\varphi)|\,.
$$

Thus, due to the choice (3.8) of M, this term can be absorbed on the left-hand side. By (3.10), we have indeed

$$
\begin{aligned}
&M\int_\Omega F_1'(\varphi)(\varphi-\overline{\varphi}) - \int_\Omega F_1'(\varphi)\bigl(-mr_0 + \mathbb{h}(\varphi)\bigr) \\
&\geq (M\,\delta_0 - mR)\int_\Omega |F_1'(\varphi)| - c = \int_\Omega |F_1'(\varphi)| - c\,.
\end{aligned}
$$

The integral involving a and u is treated by recalling that a, σ, and u are nonnegative and that u is bounded by $u_{\max}$. Namely, we have that

$$
\int_\Omega (a+u)(\ln a - \chi_a\sigma) \leq \int_\Omega (a+u)\ln a \leq \frac14\int_\Omega a^2(\ln a - 1) + c\,.
$$

Finally, we observe that

$$\int_\Omega \big((1-\sigma)-a\sigma\big)\partial_t\sigma \le \int_\Omega (1+a)|\partial_t\sigma|$$
$$\le \frac{1}{2}\int_\Omega |\partial_t\sigma|^2 + \int_\Omega |a|^2 + c \le \frac{1}{2}\int_\Omega |\partial_t\sigma|^2 + \frac{1}{4}\int_\Omega a^2(\ln a - 1) + c\,.$$

By recalling the definition of S_n given in (2.25), that both F_2' and $\mathbb{h}$ are Lipschitz continuous, and that $\mathbb{h}$ is even bounded, the other terms on the right-hand side of (3.9) can easily be treated using Young's inequality. Hence, collecting all the above estimates and (3.9) itself, and ignoring some nonnegative terms on the left-hand side, we conclude that

$$\frac{d}{dt}\,\mathcal{E}(\varphi,a,n,\sigma) + \frac{1}{2}\int_\Omega |\nabla\mu|^2 + \int_\Omega |F_1'(\varphi)| + \int_\Omega a|\nabla(\ln a - \chi_a\sigma)|^2$$
$$+ \frac{1}{2}\int_\Omega |\partial_t n|^2 + \frac{1}{2}\int_\Omega |\partial_t\sigma|^2 + \frac{1}{2}\int_\Omega a^2(\ln a - 1) + \frac{1}{2}\,\frac{d}{dt}\int_\Omega |n|^2$$
$$\le c\int_\Omega |\nabla\varphi|^2 + c\int_\Omega |\varphi|^2 + c\int_\Omega |\nabla n|^2 + c\int_\Omega |n|^2 + c\,.$$

At this point, we integrate the resulting inequality over $(0,t)$, where $t \in (0,T)$ is arbitrary. The left-hand side contains two terms with no prescribed sign, specifically those of $\mathcal{E}$ involving the products $n\varphi$ and $a\sigma$. Consequently, we cannot directly apply the Gronwall lemma. We therefore move them to the right-hand side and estimate them. To this end, recall that $\chi_\varphi,\ \chi_a \in (0,1)$ and that $0 \le \sigma \le 1$. Moreover, we observe that

$$r^2 \le \frac{1}{2}\,F(r) + c \quad \text{for every } r \in D(F). \tag{3.11}$$

This readily follows from (2.7) in the case of regular potentials, and it is trivially satisfied in the case of potentials satisfying (2.8). Hence, we have that

$$\chi_\varphi \int_\Omega n(t)\varphi(t) + \chi_a \int_\Omega a(t)\sigma(t) \le \frac{1}{4}\int_\Omega |n(t)|^2 + \int_\Omega |\varphi(t)|^2 + \int_\Omega a(t)$$
$$\le \frac{1}{4}\int_\Omega |n(t)|^2 + \frac{1}{2}\int_\Omega F(\varphi(t)) + \frac{1}{2}\int_\Omega a(t)(\ln a(t) - 1) + c\,,$$

and this can be absorbed on the left-hand side. Moreover, there is one more term on the right-hand side to be treated, namely, the term arising from the time integration of $\int_\Omega |\varphi|^2$. This latter can be estimated by using (3.11) once more. Now recall that our assumptions on F imply that F is bounded from below. We therefore can apply Gronwall's lemma, owing also to the assumptions on the initial data, and conclude that

$$\begin{aligned}&\|\varphi\|_{L^\infty(0,T;V)} + \|F(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\mu\|_{L^2(0,T;H)}\\ &\quad + \|a(\ln a - 1)\|_{L^\infty(0,T;L^1(\Omega))} + \|a^2(\ln a - 1)\|_{L^1(Q)} + \|a^{1/2}\nabla(\ln a - \chi_a\sigma)\|_{L^2(Q)}\\ &\quad + \|n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c\,.\end{aligned} \tag{3.12}$$

The bound in (3.12) also implies that

$$\|a\|_{L^\infty(0,T;L^1(\Omega))\cap L^2(Q)} \le c\,. \tag{3.13}$$

Consequences. At this point, it is rather straightforward to infer that

$$\|\partial_t\varphi\|_{L^2(0,T;V^*)} + \|n\|_{L^2(0,T;W)} + \|\sigma\|_{L^2(0,T;W)} \le c\,. \tag{3.14}$$

Indeed, for the first of these estimates, one tests (2.22) by any $v \in L^2(0,T;V)$, integrates over $(0,T)$, and accounts for (3.12), to obtain that

$$\int_0^T \langle \partial_t\varphi(t), v(t)\rangle\,dt \le c\,\|v\|_{L^2(0,T;V)}\,.$$

By rewriting (2.25) and (2.26) as partial differential equations

$$\partial_t n - \Delta n = \chi_\varphi\varphi + c_\varphi\varphi + c_n n + c_\sigma\sigma + c_0 \quad \text{a.e. in } Q\,, \tag{3.15}$$
$$\partial_t\sigma - \Delta\sigma = (1-\sigma) + a(\chi_a - \sigma) \quad \text{a.e. in } Q\,, \tag{3.16}$$

and comparing the terms, we infer from (3.12), (3.13) and (3.1) that

$$\|\Delta n\|_{L^2(0,T;H)} + \|\Delta\sigma\|_{L^2(0,T;H)} \le c\,,$$

whence we obtain (3.14) with the help of the elliptic regularity theory.

Second a priori estimate. We test (2.23) by $\varphi - \overline{\varphi}$ and owe to (3.10) divided by M. By also accounting for the Poincaré–Wirtinger and Young inequalities and (3.4), we obtain that

$$\begin{aligned}
&\delta_0 \int_\Omega |F_1'(\varphi)| \le \int_\Omega F_1'(\varphi)(\varphi-\overline{\varphi}) + c \le \int_\Omega |\nabla\varphi|^2 + \int_\Omega F_1'(\varphi)(\varphi-\overline{\varphi}) + c \\
&= - \int_\Omega F_2'(\varphi)(\varphi-\overline{\varphi}) + \int_\Omega \mu(\varphi-\overline{\varphi}) + c \\
&= - \int_\Omega F_2'(\varphi)(\varphi-\overline{\varphi}) + \int_\Omega (\mu-\overline{\mu})(\varphi-\overline{\varphi}) + c \\
&\le c\int_\Omega |\varphi|^2 + c\,\|\nabla\mu\|\,\|\varphi-\overline{\varphi}\| + c\,.
\end{aligned} \tag{3.17}$$

Now we square, integrate over $(0,T)$, and apply (3.12). This yields that

$$\|F_1'(\varphi)\|_{L^2(0,T;L^1(\Omega))} \le c\,, \quad \text{whence} \quad \|\overline{F_1'(\varphi)}\|_{L^2(0,T)} \le c\,.$$

Therefore, by testing (2.23) by $1/|\Omega|$ and comparing, we infer that

$$\|\overline{\mu}\|_{L^2(0,T)} \le c\,.$$

By combining with (3.12), and using once more inequality (2.39), we conclude that

$$\|\mu\|_{L^2(0,T;V)} \le c\,. \tag{3.18}$$

Consequence. Now, we consider (2.23). By splitting F as $F = F_1 + F_2$, moving the term involving F_2' to the right-hand side and applying a usual argument based on the monotonicity of F_1' (i.e., one can test (2.23) by $F_1'(\varphi)$), we deduce that both $F_1'(\varphi)$ and $\Delta\varphi$ are estimated in H by the H norm of the right-hand side. Therefore, we conclude from elliptic regularity that

$$\|\varphi\|_{L^2(0,T;W)} + \|F_1'(\varphi)\|_{L^2(0,T;H)} \le c\,. \tag{3.19}$$

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Third a priori estimate. We take $v = a$ in (2.24) and, using the positivity of a and Young's inequality, we have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|a\|^2 + \int_\Omega |\nabla a|^2 + \int_\Omega |a|^3 \\ &\le \chi_a \int_\Omega a\nabla\sigma \cdot \nabla a + \frac{3}{2}\|a\|^2 + \frac{1}{2}\|u_{\max}\|_\infty^2 |\Omega|.\end{aligned} \tag{3.20}$$

Now, in order to deal with the first integral on the right-hand side, we invoke Hölder's inequality and the Ladyžhenskaya inequalities in (2.41), and find out that

$$\begin{aligned}\chi_a &\int_\Omega a\nabla\sigma \cdot \nabla a \le \|a\|_4 \|\nabla\sigma\|_4 \|\nabla a\|_2 \\ &\le c\|a\|^{1/2}\|a\|_V^{1/2}\|\sigma\|_V^{1/2}(\|\sigma\| + \|\Delta\sigma\|)^{1/2}\|a\|_V \\ &\le c\|a\|^{1/2}\|\sigma\|_{L^\infty(0,T;V)}^{1/2}\|\sigma\|_W^{1/2}\|a\|_V^{3/2} \\ &\le \frac{1}{2}\|a\|_V^2 + c\|\sigma\|_W^2\|a\|^2,\end{aligned} \tag{3.21}$$

where (3.12) and the Young inequality (2.35), with exponents $4/3$ and 4, have been exploited. Note that the function $t \mapsto \|\sigma(t)\|_W^2$ is known to be bounded in $L^1(0,T)$ by (3.14). Then, combining (3.20) and (3.21), we can integrate the resultant over $(0,t)$ with the help of the initial condition for a, see (2.27) and (2.15). Next, we apply the Gronwall lemma and deduce that

$$\|a\|_{L^\infty(0,T;H)\cap L^2(0,T;V)\cap L^3(Q)} \le c\,, \quad \text{whence also (cf. (2.37))} \quad \|a\|_{L^4(Q)} \le c\,. \tag{3.22}$$

Regularity for two variables. We set $g := (1-\sigma) + a(\chi_a - \sigma)$ for a while. Then, we have that $g \in L^4(Q)$ by (3.1) and (3.22). Moreover, since

$$\nabla g = -\nabla\sigma + \nabla a(\chi_a - \sigma) - a\nabla\sigma$$

and $\nabla\sigma \in L^4(Q)$ by (3.12), (3.14), and (2.36), we see that $\nabla g \in L^2(Q)$, so that g belongs to $L^2(0,T;V)$ as well. We also recall the property $\sigma_0 \in W$ from (2.16). Therefore, on the one hand, a comparison in (3.16) and maximal parabolic regularity (see [11, Thm. 2.1]) yield that

$$\|\sigma\|_{W^{1,4}(0,T;L^4(\Omega))\cap L^4(0,T;W^{2,4}(\Omega))} \le c\,. \tag{3.23}$$

On the other hand, by the abstract regularity theory for parabolic problems contained, e.g., in [22], we also infer that

$$\|\sigma\|_{H^1(0,T;V)\cap C^0([0,T];W)\cap L^2(0,T;H^3(\Omega))} \le c\,. \tag{3.24}$$

In view of (2.16) and (3.15), a similar conclusion holds for n since the right-hand side $(\chi_\varphi + c_\varphi)\varphi + c_n n + c_\sigma\sigma + c_0$ is bounded both in $L^4(Q)$ and in $L^2(0,T;V)$ due to (3.12). Thus, we have that

$$\|n\|_{W^{1,4}(0,T;L^4(\Omega))\cap L^4(0,T;W^{2,4}(\Omega))} + \|n\|_{H^1(0,T;V)\cap C^0([0,T];W)\cap L^2(0,T;H^3(\Omega))} \le c\,. \tag{3.25}$$

Fourth a priori estimate. From (2.24) and integration by parts, it follows that

$$\int_\Omega \partial_t a\, v + \int_\Omega \nabla a \cdot \nabla v = -\chi_a \int_\Omega (\nabla a \cdot \nabla\sigma + a\Delta\sigma)v + \int_\Omega \big(a - a^2 + u\big)v \tag{3.26}$$

for every $v \in V$, a.e. in $(0,T)$. Note that the first integral on the right-hand side makes sense since $\nabla\sigma$ is bounded in $L^4(0,T;L^\infty(\Omega))$ by (3.23) and the Sobolev embedding $W^{2,4}(\Omega) \subset L^\infty(\Omega)$, whereas a and $\Delta\sigma$ are bounded in $L^4(Q)$. We formally take $v = \partial_t a$ in (3.26) and, by the Hölder and Young inequalities, we easily obtain

$$\begin{aligned}
&\|\partial_t a\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla a\|^2 \\
&\leq \|\nabla a\|\|\nabla\sigma\|_\infty\|\partial_t a\| + \|a\|_4\|\Delta\sigma\|_4\|\partial_t a\| + \|a - a^2 + u\|_2\|\partial_t a\| \\
&\leq \frac{1}{2}\|\partial_t a\|^2 + c\|\sigma\|^2_{W^{2,4}(\Omega)}\|\nabla a\|^2 + c\|a\|^2_{L^4(\Omega)}\|\sigma\|^2_{W^{2,4}(\Omega)} + c\big(\|a\|^4_{L^4(\Omega)} + 1\big). \qquad (3.27)
\end{aligned}$$

Then, in view of (3.22) and (3.23), we are allowed to integrate over $(0,t)$ and apply the Gronwall lemma as $t \mapsto \|\sigma(t)\|^2_{W^{2,4}(\Omega)}$ is bounded in $L^2(0,T)$ to conclude that

$$\|a\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c\,. \qquad (3.28)$$

Then, going back to (3.26) and emphasizing that now the whole term $\nabla a\cdot\nabla\sigma + a\Delta\sigma$ is bounded in $L^2(0,T;H)$, by comparison and elliptic regularity we infer that

$$\|a\|_{L^2(0,T;W)} \leq c\,. \qquad (3.29)$$

Fifth a priori estimate. We proceed formally and take $v = \partial_t\mu$ in (2.22). Coincidently, we differentiate (2.23) and test the resulting equation by $\partial_t\varphi$. Then, we sum up, noting that a cancellation occurs, and integrate also by parts over $(0,t)$. We obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega|\nabla\mu(t)|^2 + \int_{Q_t}|\nabla\partial_t\varphi|^2 + \int_{Q_t}F_1''(\varphi)|\partial_t\varphi|^2 \\
&= \frac{1}{2}\int_\Omega|\nabla\mu(0)|^2 + \chi_\varphi\int_\Omega\nabla n(t)\cdot\nabla\mu(t) - \chi_\varphi\int_\Omega\nabla n_0\cdot\nabla\mu(0) \\
&\quad - \int_{Q_t}\nabla\partial_t n\cdot\nabla\mu + \int_\Omega(\mathbb{h}(\varphi) - m\varphi)(t)\mu(t) - \int_\Omega(\mathbb{h}(\varphi_0) - m\varphi_0)\mu(0) \\
&\quad - \int_{Q_t}(\mathbb{h}'(\varphi) - m)\partial_t\varphi\,\mu - \int_{Q_t}F_2''(\varphi)|\partial_t\varphi|^2, \qquad (3.30)
\end{aligned}$$

where we employed the notation $Q_t = \Omega\times(0,t)$. We point out that the third term on the left-hand side is nonnegative due to the monotonicity of F_1'. About the value $\mu(0)$, we recover it from (2.23) and realize from (2.13) that $\mu(0) = \mu_0$ is bounded in V. Then, recalling also (2.16) and (2.4), it is clear that the first, third and sixth terms on the right-hand side of (3.30) are under control. The second one can be easily treated as

$$\chi_\varphi\int_\Omega\nabla n(t)\cdot\nabla\mu(t) \leq \frac{1}{8}\int_\Omega|\nabla\mu(t)|^2 + c\|n\|^2_{L^\infty(0,T;V)} \leq \frac{1}{8}\int_\Omega|\nabla\mu(t)|^2 + c$$

by the Young inequality and (3.25). The fourth term on the right-hand side of (3.30) is already bounded due to (3.25) and (3.18). On the other hand, as the functions $\mathbb{h}'$ and F_2'' are bounded, by virtue of the Young inequality, the compactness inequality (2.38), and (3.14), we deduce that

$$\begin{aligned}
&-\int_{Q_t}(\mathbb{h}'(\varphi) - m)\partial_t\varphi\,\mu - \int_{Q_t}F_2''(\varphi)|\partial_t\varphi|^2 \leq c + c\int_{Q_t}|\partial_t\varphi|^2 \\
&\leq c + \frac{1}{2}\int_{Q_t}|\nabla\partial_t\varphi|^2 + \|\partial_t\varphi\|^2_{L^2(0,T;V^*)} \leq c + \frac{1}{2}\int_{Q_t}|\nabla\partial_t\varphi|^2.
\end{aligned}$$

Now, in (3.30) it remains to control one term, for which we use the boundedness of $\mathbb{h}$ and the estimate (3.12), along with the Poincaré–Wirtinger inequality (2.39) and, once more, the equation (2.23) with $v = 1/|\Omega|$. It follows that

$$\int_\Omega (\mathbb{h}(\varphi) - m\varphi)(t)\mu(t) \le c\|\mu(t)\| \le c\|\mu(t) - \overline{\mu}(t)\| + c\big|\overline{\mu}(t)\big|$$
$$\le c\|\nabla\mu(t)\| + c\int_\Omega |F_1'(\varphi(t))| + c\|\varphi(t)\| + c.$$

Now, we recall (3.17) and arrive at

$$\int_\Omega (\mathbb{h}(\varphi) - m\varphi)(t)\mu(t)$$
$$\le c\|\nabla\mu(t)\|\big(1 + \|\varphi(t) - \overline{\varphi}(t)\|\big) + c\|\varphi(t)\|^2 + c \le \frac{1}{8}\int_\Omega |\nabla\mu(t)|^2 + c.$$

Then, collecting the above computations in (3.30) and invoking (3.14) lead to the estimate

$$\|\nabla\mu\|_{L^\infty(0,T;H)} + \|\partial_t\varphi\|_{L^2(0,T;V)} \le c\,. \tag{3.31}$$

Hence, recalling (3.17) again, at this point we can infer that

$$\|F_1'(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} \le c\,,$$

whence, testing (2.23) by $1/|\Omega|$ and comparing, it holds that $\|\overline{\mu}\|_{L^\infty(0,T)} \le c$, and, consequently, using once more inequality (2.39), we conclude that

$$\|\mu\|_{L^\infty(0,T;V)} \le c\,. \tag{3.32}$$

Moreover, by (3.31) and a comparison of terms in the strong form of (2.22), i.e.,

$$\partial_t\varphi - \Delta(\mu - \chi_\varphi n) = -m\varphi + \mathbb{h}(\varphi) \quad \text{a.e. in } Q, \tag{3.33}$$

we find out that $\Delta(\mu - \chi_\varphi n)$ is bounded in $L^2(0,T;V)$, whence, by elliptic regularity, $\mu - \chi_\varphi n$ is bounded in $L^2(0,T;W \cap H^3(\Omega))$ and, consequently, in view of (3.25), it holds that

$$\|\mu\|_{L^2(0,T;W\cap H^3(\Omega))} \le c\,. \tag{3.34}$$

Now, we can argue in the same way as for (3.19) and find as well that

$$\|\varphi\|_{L^\infty(0,T;W)} + \|F_1'(\varphi)\|_{L^\infty(0,T;H)} \le c\,. \tag{3.35}$$

The information given by the estimates (3.31) and (3.35) is enough to conclude that $\varphi \in C^0(\overline{Q})$ so that the values assumed by φ range in a compact subset of $\mathbb{R}$, and in the case $D(F) = \mathbb{R}$ we have already proved the separation property (2.31).

Further estimate. From now on, we restrict ourselves to the case $D(F) = (0,1)$ and argue similarly as in the proof of [6, Proposition 2.6] in order to obtain the estimate

$$\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;W)} + \|\mu\|_{L^\infty(Q)} \le c\,. \tag{3.36}$$

In fact, the technique is as follows. From (2.22), considered at the initial time $t = 0$, and (2.13), (2.16) we recover that

$$\partial_t\varphi(0) = \Delta(\mu_0 - \chi_\varphi n_0) - m\varphi_0 + \mathbb{h}(\varphi_0)$$

is bounded in H. Next, we differentiate both (2.22) and (2.23) and test by $\partial_t\varphi$ and $-\Delta(\partial_t\varphi)$, respectively. Then, we sum up and integrate by parts and over $(0, t)$, paying attention to the cancellation of two terms. The integrals on the right-hand side

$$\chi_\varphi \int_{Q_t} \nabla\partial_t n \cdot \nabla\partial_t\varphi + \int_{Q_t} (\mathbb{h}'(\varphi) - m)|\partial_t\varphi|^2$$

are readily controlled thanks to (3.25) and (3.31). Then we arrive at some inequality similar to [6, formula (5.16) and following]. From this point we can proceed along the same line as in [6, pp. 2160-2162], by exploiting the inequality in (2.8), which gives a control for F_1'', along with the Trudinger inequality (see, e.g., [24])

$$\int_\Omega e^{|v|} \leq c_\Omega\, e^{c_\Omega\, \|v\|_V^2} \quad \text{for every } v \in V\,,$$

which holds in two dimensions. Let us omit the full proof here: this proof permits to arrive at the estimate

$$\|\partial_t\varphi\|_{L^\infty(0,T;H)} + \|\Delta(\partial_t\varphi)\|_{L^2(0,T;H)} \leq c,$$

from which by the elliptic regularity theory and (3.31), (3.35) we achieve the estimate (3.36) for the term involving φ. Next, a comparison argument in (3.33) reveals that $\Delta(\mu - \chi_\varphi n)$ is bounded in $L^\infty(0, T; H)$, whence by elliptic regularity and (3.25), μ is shown to be bounded in $L^\infty(0, T; W) \subset L^\infty(Q)$, which completes the proof of (3.36).

Separation property. Up to now, we have completely proved the stability estimate (2.30), which holds true with a constant K_1 satisfying the properties given in the statement, since the constants c we have introduced in the various steps enjoy these properties. We still have to check the separation property (2.31) in the case $D(F) = (0, 1)$. To this aim, we start from the bound for μ in $L^\infty(Q)$ in (3.36). Then, a Moser type procedure in (2.23) provides a bound for $F_1'(\varphi)$. Let us sketch this argument by proceeding formally and acknowledging that a truncation argument would suffice to obtain a rigorous proof. To simplify notation, we set $\psi := F_1'(\varphi)$ and $g := \mu - F_2'(\varphi)$. The argument just deals with the elliptic equation at the time t, which however is not written for simplicity. We take any $p > 2$ and test (2.23) by $|\psi|^{p-2}\psi$ to obtain that

$$(p-1)\int_\Omega |\psi|^{p-2}|\nabla\varphi|^2 + \int_\Omega |\psi|^p = \int_\Omega g|\psi|^{p-2}\psi\,.$$

By the Young inequality (2.35) we deduce that

$$\|\psi\|_p^p \leq \int_\Omega |g|\,|\psi|^{p-1} \leq \|g\|_p\, \||\psi|^{p-1}\|_{p'} = \|g\|_p\, \|\psi\|_p^{p/p'} \leq \frac{1}{p}\,\|g\|_p + \frac{1}{p'}\,\|\psi\|_p^p\,.$$

By rearranging, we infer that $\|\psi\|_p \leq \|g\|_p$. Then, letting p tend to infinity, we conclude that $\|\psi\|_\infty \leq \|g\|_\infty$, which entails $\|F_1'(\varphi(t))\|_\infty \leq \|\mu(t) - F_2'(\varphi(t))\|_\infty$ for a.a. $t \in (0, T)$, whence

$$\|F_1'(\varphi)\|_{L^\infty(Q)} \leq \|\mu - F_2'(\varphi)\|_{L^\infty(Q)} \leq c\,.$$

By accounting for assumption (2.8), we deduce that (2.31) holds true with some values $r_\pm$ as in the statement. □

4 Uniqueness and continuous dependence

In this section, we prove the uniqueness part of Theorem 2.2 and the continuous dependence estimate (2.32). We just prove the latter for arbitrary solutions corresponding to the control variables, so that uniqueness follows as a consequence of the case of the same control. In this direction, we fix $u_i \in \mathcal{U}_{\rm ad}, i = 1, 2$, and any two corresponding solutions $(\varphi_i, \mu_i, a_i, n_i, \sigma_i)$ with the regularity and the properties stated in Theorem 2.2, in particular, the separation property (2.31). We set for convenience

$$u := u_1 - u_2 \,, \quad \varphi = \varphi_1 - \varphi_2 \,, \quad \mu = \mu_1 - \mu_2 \,,$$
$$a = a_1 - a_2 \,, \quad n := n_1 - n_2 \,, \quad \sigma := \sigma_1 - \sigma_2 \,.$$

Then, we write all the equations (2.22)–(2.26) for both solutions and take the differences to obtain that

$$\int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\mu\cdot\nabla v - \chi_\varphi \int_\Omega \nabla n\cdot\nabla v = -m\int_\Omega \varphi v + \int_\Omega \big(\mathbb{h}(\varphi_1) - \mathbb{h}(\varphi_2)\big)\, v \,, \tag{4.1}$$

$$\int_\Omega \nabla\varphi\cdot\nabla v + \int_\Omega \big(F'(\varphi_1) - F'(\varphi_2)\big)\, v = \int_\Omega \mu v \,, \tag{4.2}$$

$$\int_\Omega \partial_t a\, v + \int_\Omega \nabla a\cdot\nabla v - \chi_a \int_\Omega \big(a\nabla\sigma_1 + a_2\nabla\sigma\big)\cdot\nabla v = \int_\Omega \big(a - (a_1 + a_2)a + u\big) v \,, \tag{4.3}$$

$$\int_\Omega \partial_t n\, v + \int_\Omega \nabla n\cdot\nabla v - \chi_\varphi \int_\Omega \varphi v = \int_\Omega \big(c_\varphi\varphi + c_n n + c_\sigma\sigma\big)\, v \,, \tag{4.4}$$

$$\int_\Omega \partial_t\sigma\, v + \int_\Omega \nabla\sigma\cdot\nabla v = \int_\Omega \big(-\sigma + \chi_a a - a\sigma_1 - a_2\sigma\big) v \,, \tag{4.5}$$

for every $v \in V$ and a.e. in $(0,T)$.

Remark 4.1. In our computations, we allow the values of the generic constants c to also depend on the solutions we are considering. However, at the end of the proof, since uniqueness follows as a consequence as already said before, one realizes that the solutions taken into account are exactly those provided by the already proved existence part of Theorem 2.2. This implies that the norms of the solutions considered in the present proof are bounded by the constant K_1 of the stability estimate (2.30), and thus they depend only on Ω, T, the structure of the system, the initial data, and $u_{\max}$.

First estimate. We test the above equations by φ, $-\Delta\varphi$, a, $\partial_t n$, and $\partial_t\sigma - \Delta\sigma$, respectively. More precisely, we test (4.1) and (4.3) as said, while we write (4.2), (4.4) and (4.5) in their strong form, multiply them by $-\Delta\varphi$, $\partial_t n$, and $\partial_t\sigma - \Delta\sigma$, and integrate over Ω. Notice that the regularity (2.17)–(2.21) for both solutions allows this procedure. After some rearrangement, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi|^2 + m\int_\Omega |\varphi|^2 = -\int_\Omega \nabla\mu\cdot\nabla\varphi + \chi_\varphi\int_\Omega \nabla n\cdot\nabla\varphi + \int_\Omega \big(\mathbb{h}(\varphi_1) - \mathbb{h}(\varphi_2)\big)\,\varphi \,,$$
$$\int_\Omega |\Delta\varphi|^2 = -\int_\Omega \big(F'(\varphi_1) - F'(\varphi_2)\big)\,(-\Delta\varphi) + \int_\Omega \nabla\mu\cdot\nabla\varphi \,,$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |a|^2 + \int_\Omega |\nabla a|^2 = \chi_a\int_\Omega \big(a\nabla\sigma_1 + a_2\nabla\sigma\big)\cdot\nabla a + \int_\Omega \big(a^2 - (a_1 + a_2)a^2 + ua\big) \,,$$
$$\int_\Omega |\partial_t n|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla n|^2 = \chi_\varphi\int_\Omega \varphi\partial_t n + \int_\Omega \big(c_\varphi\varphi + c_n n + c_\sigma\sigma\big)\,\partial_t n \,,$$
$$\int_\Omega |\partial_t\sigma|^2 + \frac{d}{dt}\int_\Omega |\nabla\sigma|^2 + \int_\Omega |\Delta\sigma|^2 = \int_\Omega \big(-\sigma + \chi_a a - a\sigma_1 - a_2\sigma\big)(\partial_t\sigma - \Delta\sigma) \,.$$

At this point, we take the sum of these identities. Moreover, we add

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |n|^2 \quad \text{and} \quad \frac{d}{dt}\int_\Omega |\sigma|^2$$

to the left-hand side of the resulting equality and the same terms, written in the form $\int_\Omega n\partial_t n$ and $2\int_\Omega \sigma\partial_t\sigma$, to its right-hand side. The new left-hand side then becomes

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\int_\Omega |\varphi|^2 + m\int_\Omega |\varphi|^2 + \int_\Omega |\Delta\varphi|^2 + \frac{1}{2}\,\frac{d}{dt}\int_\Omega |a|^2 + \int_\Omega |\nabla a|^2 \\
&\quad + \int_\Omega |\partial_t n|^2 + \frac{1}{2}\,\frac{d}{dt}\int_\Omega \bigl(|n|^2 + |\nabla n|^2\bigr) \\
&\quad + \int_\Omega |\partial_t \sigma|^2 + \frac{d}{dt}\int_\Omega \bigl(|\sigma|^2 + |\nabla\sigma|^2\bigr) + \int_\Omega |\Delta\sigma|^2 \,,
\end{aligned}$$

and we have to estimate the terms on the corresponding right-hand side. However, two of them cancel each other, and most of the others can be simply dealt with by means of Young's inequality, possibly on account of the Lipschitz continuity of both $\mathbb{h}$ and F' in the interval $[r_-, r_+]$, considering that we want to apply (after time integration) the Gronwall lemma. Hence, we only estimate the terms that need some treatment. On account of (2.40), we have that

$$\chi_\varphi \int_\Omega \nabla n \cdot \nabla\varphi \le \|\nabla\varphi\|\,\|\nabla n\| \le \frac{1}{4}\int_\Omega \bigl(|\varphi|^2 + |\Delta\varphi|^2\bigr) + c\int_\Omega |\nabla n|^2 \,.$$

As for the terms involving the product of three factors, we first employ (2.20) to find that

$$\begin{aligned}
&\chi_a \int_\Omega a\nabla\sigma_1 \cdot \nabla a \le \|a\|_4 \|\nabla\sigma_1\|_4\,\|\nabla a\|_2 \\
&\le \frac{1}{8}\int_\Omega |\nabla a|^2 + c\,\|\sigma_1\|^2_{L^\infty(0,T;W)}\,\|a\|\,(\|a\| + \|\nabla a\|) \le \frac{1}{4}\int_\Omega |\nabla a|^2 + c\,\|a\|^2 \,.
\end{aligned}$$

Moreover, using the Sobolev inequality in (2.37) and the second Ladyžhenskaya inequality in (2.41) after Young's inequality, we have that

$$\begin{aligned}
&\chi_a \int_\Omega a_2\nabla\sigma \cdot \nabla a \le \|a_2\|_4 \|\nabla\sigma\|_4\,\|\nabla a\|_2 \\
&\le \frac{1}{4}\int_\Omega |\nabla a|^2 + c\,\|a_2\|_4^2\,\|\sigma\|_V(\|\sigma\| + \|\Delta\sigma\|) \\
&\le \frac{1}{4}\int_\Omega |\nabla a|^2 + c\,\|a_2\|_4^2\,\|\sigma\|_V^2 + \frac{1}{4}\int_\Omega |\Delta\sigma|^2 + c\,\|a_2\|_4^4\,\|\sigma\|_V^2 \,,
\end{aligned}$$

and we observe that the function $t \mapsto \|a_2(t)\|_4$ belongs to $L^4(0,T)$ (cf. (2.19) and (2.37)). In addition, it is clear that

$$-\int_\Omega (a_1 + a_2)a^2 \le \|a_1 + a_2\|_4\,\|a\|_4\,\|a\|_2 \le \frac{1}{4}\,\|a\|_V^2 + c\,\|a_1 + a_2\|_4^2\,\|a\|_2^2 \,,$$

where the function $t \mapsto \|(a_1 + a_2)(t)\|_4$ belongs to $L^4(0,T)$. The terms involving $a\sigma_1$ and $a_2\sigma$ are treated in a similar way. Finally, from Young's inequality it follows that

$$\int_\Omega ua \le \frac{1}{2}\int_\Omega |a|^2 + \frac{1}{2}\int_\Omega |u|^2 \,.$$

Therefore, collecting all these inequalities and those we have omitted, we estimate the right-hand side we are considering in a form that is suitable for the application of the Gronwall lemma. After time integration, using the assumptions on the initial data as well as elliptic regularity theory, we conclude that

$$\begin{aligned}&\|\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;W)} + \|a\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\\ &\quad + \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c\,\|u\|_{L^2(0,T;H)}\,. \end{aligned} \tag{4.6}$$

Notice that this implies that

$$\|a\|_{L^4(Q)} + \|\nabla\sigma\|_{L^4(Q)} \le c\,\|u\|_{L^2(0,T;H)}\,, \tag{4.7}$$

thanks to the third inequality in (2.37).

Consequence. A comparison argument in equations (4.4) and (4.2), along with elliptic regularity theory, thus yields that

$$\|n\|_{L^2(0,T;W)} \le c\,\|u\|_{L^2(0,T;H)} \quad \text{and} \quad \|\mu\|_{L^2(0,T;H)} \le c\,\|u\|_{L^2(0,T;H)}\,. \tag{4.8}$$

Second estimate. We test (4.1) by μ and (4.2) by $\partial_t\varphi$ and add the resulting equalities. Noting a cancellation, we have that

$$\begin{aligned}&\int_\Omega |\nabla\mu|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi|^2 = \chi_\varphi\int_\Omega \nabla n\cdot\nabla\mu - m\int_\Omega \varphi\mu\\ &\quad + \int_\Omega \big(\mathbb{h}(\varphi_1) - \mathbb{h}(\varphi_2)\big)\mu - \int_\Omega \big(F'(\varphi_1) - F'(\varphi_2)\big)\partial_t\varphi\,, \end{aligned} \tag{4.9}$$

and just the last term needs some attention as the others can be easily controlled using Young's inequality. To deal with it, we recall the Lipschitz continuity of F in the interval $[r_-, r_+]$ and test (4.1) by $-(F'(\varphi_1) - F'(\varphi_2))$. We obtain that

$$\begin{aligned}&-\int_\Omega \big(F'(\varphi_1) - F'(\varphi_2)\big)\partial_t\varphi\\ &= \int_\Omega \nabla\mu\cdot\nabla\big(F'(\varphi_1) - F'(\varphi_2)\big) + \chi_\varphi\int_\Omega \Delta n\,\big(F'(\varphi_1) - F'(\varphi_2)\big)\\ &\quad + \int_\Omega \big(m\varphi - (\mathbb{h}(\varphi_1) - \mathbb{h}(\varphi_2))\big)\,\big(F'(\varphi_1) - F'(\varphi_2)\big)\,. \end{aligned} \tag{4.10}$$

In view of (4.6) and (4.8), and using Young's inequality and the Lipschitz continuity of both $\mathbb{h}$ and F' on $[r_-, r_+]$, we easily conclude that the sum of the second and third terms on the right-hand side is bounded by $c\|u\|^2_{L^2(0,T;H)}$. Regarding the first term on the right-hand side, we recall the regularity of φ_1 and φ_2 given by Theorem 2.2 and (2.28), to infer that

$$\begin{aligned}&\int_\Omega \nabla\mu\cdot\nabla\big(F'(\varphi_1) - F'(\varphi_2)\big) = \int_\Omega \nabla\mu\cdot\Big(\big(F''(\varphi_1) - F''(\varphi_2)\big)\nabla\varphi_1 + F''(\varphi_2)\nabla\varphi\Big)\\ &\le c\,\|\nabla\mu\|_2\,\big(c\,\|\varphi\|_2\,\|\nabla\varphi_1\|_\infty + \|F''(\varphi_2)\|_\infty\,\|\nabla\varphi\|_2\big) \le c\,\|\nabla\mu\|\,(\|\varphi\| + \|\nabla\varphi\|)\,. \end{aligned}$$

Now, by combining it with (4.9) and (4.10), then applying the Young inequality and integrating over $(0,t)$, we arrive at

$$\frac{1}{2}\int_{Q_t} |\nabla\mu|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi(t)|^2 \le c\int_{Q_t}\big(|\nabla n|^2 + |\Delta n|^2 + |\varphi|^2 + |\nabla\varphi|^2\big)\,,$$

so that, in view of (4.6) and (4.8), we conclude that

$$\|\nabla\mu\|_{L^2(0,T;H)} + \|\nabla\varphi\|_{L^\infty(0,T;H)} \le c\,\|u\|_{L^2(0,T;H)}\,.$$

From this, (4.6) and (4.8), we deduce that

$$\|\mu\|_{L^2(0,T;V)} + \|\varphi\|_{L^\infty(0,T;V)L^2(0,T;W)} \le c\,\|u\|_{L^2(0,T;H)}\,. \tag{4.11}$$

Consequence. A comparison argument in equation (4.1) then readily yields that

$$\|\partial_t\varphi\|_{L^2(0,T;V^*)} \le c\,\|u\|_{L^2(0,T;H)}\,. \tag{4.12}$$

Third estimate. Next, we take an arbitrary $v \in L^2(0,T;V)$, test (4.3) by v, and integrate over $(0,T)$. We have that

$$\int_Q \partial_t a\, v = -\int_Q \nabla a\cdot\nabla v \\ + \chi_a \int_Q \bigl(a\nabla\sigma_1 + a_2\nabla\sigma\bigr)\cdot\nabla v + \int_Q \bigl(a - (a_1+a_2)a + u\bigr)v\,.$$

Just some of the terms on the right-hand side need some treatment. The first one is the following:

$$\chi_a \int_Q a\nabla\sigma_1\cdot\nabla v \le \|a\|_{L^4(Q)}\,\|\nabla\sigma_1\|_{L^4(Q)}\,\|\nabla v\|_{L^2(Q)} \le c\,\|u\|_{L^2(0,T;H)}\,\|v\|_{L^2(0,T;V)}\,,$$

where the last inequality is due to the regularity of σ_1 and (4.7). For analogous reasons we have that

$$\chi_a \int_Q a_2\nabla\sigma\cdot\nabla v \le \|a_2\|_{L^4(Q)}\,\|\nabla\sigma\|_{L^4(Q)}\,\|\nabla v\|_{L^2(Q)} \le c\,\|u\|_{L^2(0,T;H)}\,\|v\|_{L^2(0,T;V)}\,,$$

as well as

$$-\int_Q (a_1+a_2)\,a\,v \le \|a_1+a_2\|_{L^2(0,T;L^4(\Omega))}\,\|a\|_{L^\infty(0,T;L^2(\Omega))}\,\|v\|_{L^2(0,T;L^4(\Omega))} \\ \le c\,\|a_1+a_2\|_{L^2(0,T;V)}\,\|a\|_{L^\infty(0,T;H)}\,\|v\|_{L^2(0,T;V)} \le c\,\|u\|_{L^2(0,T;H)}\,\|v\|_{L^2(0,T;V)}\,.$$

Since the other terms on the right-hand side can be estimated in a straightforward way, we conclude that

$$\int_Q \partial_t a\, v = \int_0^T \langle\partial_t a(t), v(t)\rangle\,dt \le c\,\|u\|_{L^2(0,T;H)}\,\|v\|_{L^2(0,T;V)}\,,$$

whence, due to the arbitrariness of v, this entails that

$$\|\partial_t a\|_{L^2(0,T;V^*)} \le c\,\|u\|_{L^2(0,T;H)}\,. \tag{4.13}$$

By recalling Remark 4.1, we see that the proof of the uniqueness part of Theorem 2.2 and of (2.32) is complete. □

5 An auxiliary result

This section is devoted to stating and proving an auxiliary result that will be used twice in the sequel. In the following sections, we will repeatedly analyze systems related to the state system (1.1)–(1.7). Since these share a very similar mathematical structure, we have decided to introduce an abstract result that encompasses both cases to be analyzed later. Theorem 5.1, proved below, will be used to show the well-posedness of the linearized system and the Fréchet differentiability of the solution operator. We fix some u^* satisfying (2.10) and the corresponding solution $(\varphi^*, \mu^*, a^*, n^*, \sigma^*)$ given by Theorem 2.2, and we recall at once that $\mathbb{h}'(\varphi^*)$ and $F''(\varphi^*)$ are bounded, since φ^* satisfies the separation property (2.31). Moreover, we fix

$$g_1\,,\,g_4\,,\,g_5 \in L^2(Q), \quad g_2 \in L^2(0,T;V) \quad \text{and} \quad \boldsymbol{g}_3 \in (L^2(Q))^2\,, \tag{5.1}$$

and we notice that $\boldsymbol{g}_3$ is a vector-valued function. Nevertheless, we often prefer to write g_3 (i.e., we do not use the boldface character) for uniformity. Then, we look for the solution $(\varphi, \mu, a, n, \sigma)$ to the problem stated below. We remark that the notation $(\varphi, \mu, a, n, \sigma)$ adopted in this section is unrelated to the original problem (2.22)–(2.27). We look for a quintuple $(\varphi, \mu, a, n, \sigma)$ with the regularity properties

$$\varphi \in \mathcal{X}_1 := H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\,, \tag{5.2}$$
$$\mu \in \mathcal{X}_2 := L^2(0,T;V)\,, \tag{5.3}$$
$$a \in \mathcal{X}_3 := H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V)\,, \tag{5.4}$$
$$n \in \mathcal{X}_4 := H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\,, \tag{5.5}$$
$$\sigma \in \mathcal{X}_4\,, \tag{5.6}$$

that solves the variational equations

$$\langle \partial_t \varphi, v\rangle + \int_\Omega \nabla\mu\cdot\nabla v - \chi_\varphi \int_\Omega \nabla n\cdot\nabla v$$
$$= -m\int_\Omega \varphi v + \int_\Omega \mathbb{h}'(\varphi^*)\varphi\, v + \int_\Omega g_1\, v\,, \tag{5.7}$$
$$\int_\Omega \nabla\varphi\cdot\nabla v = -\int_\Omega F''(\varphi^*)\varphi\, v + \int_\Omega \mu v + \int_\Omega g_2\, v\,, \tag{5.8}$$
$$\langle \partial_t a, v\rangle + \int_\Omega \nabla a\cdot\nabla v - \chi_a \int_\Omega (a\nabla\sigma^* + a^*\nabla\sigma)\cdot\nabla v$$
$$= -\int_\Omega \boldsymbol{g}_3\cdot\nabla v + \int_\Omega (a - 2a^* a)v + \int_\Omega g_4\, v\,, \tag{5.9}$$
$$\int_\Omega \partial_t n\, v + \int_\Omega \nabla n\cdot\nabla v - \chi_\varphi \int_\Omega \varphi v = \int_\Omega (c_\varphi\varphi + c_n n + c_\sigma\sigma)v\,, \tag{5.10}$$
$$\int_\Omega \partial_t \sigma\, v + \int_\Omega \nabla\sigma\cdot\nabla v = \int_\Omega (-\sigma + \chi_a a - a\sigma^* - a^*\sigma)v + \int_\Omega g_5\, v\,, \tag{5.11}$$

for every $v \in V$ and a.e. in $(0,T)$, and satisfies the initial condition

$$(\varphi, a, n, \sigma)(0) = (0,0,0,0) \quad \text{a.e. in } \Omega\,. \tag{5.12}$$

Let us introduce the space (cf. (5.2)–(5.6))

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \times \mathcal{X}_4 \times \mathcal{X}_4 \tag{5.13}$$

for the solutions to (5.7)–(5.12).

Theorem 5.1. *Let the assumptions of Theorem 2.2 and* (5.1) *be fulfilled, and assume that* u^* *satisfies* (2.10) *and* $(\varphi^*, \mu^*, a^*, n^*, \sigma^*)$ *is the corresponding solution. Then the problem* (5.7)–(5.12) *has a unique solution* $(\varphi, \mu, a, n, \sigma)$ *satisfying* (5.2)–(5.6)*, and the estimate*

$$\|(\varphi, \mu, a, n, \sigma)\|_{\mathcal{X}} \le K_3 \Big(\sum_{i=1}^{5} \|g_i\|_{L^2(Q)} + \|\nabla g_2\|_{L^2(Q)} \Big) \tag{5.14}$$

holds true with a constant $K_3 > 0$ *that depends only on* Ω*,* T*, the structure of the original system, the initial data, and* $u_{\max}$*.*

Proof. To establish existence, we just prove formal estimates for the solution, but those computations do suggest that the same estimates can be performed on the solution to the k-dimensional system obtained from the Faedo–Galerkin scheme constructed by using the first k eigenfunctions of the Laplace operator with homogeneous boundary conditions. These bounds can then be used to pass to the limit as k tends to infinity and to construct a solution to the problem satisfying (5.2)–(5.6) and (5.14).

First a priori estimate. We test the above equations (5.7)–(5.11), in the order, by φ, $-\Delta\varphi$, a, $\partial_t n$ and $\partial_t \sigma - \Delta\sigma$, respectively. We obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi|^2 + \int_\Omega \nabla\mu\cdot\nabla\varphi + m\int_\Omega |\varphi|^2 \\
&= \chi_\varphi \int_\Omega \nabla n\cdot\nabla\varphi + \int_\Omega \mathbb{h}'(\varphi^*)|\varphi|^2 + \int_\Omega g_1\,\varphi\,, \\
&\int_\Omega |\Delta\varphi|^2 = \int_\Omega F''(\varphi^*)\varphi\,\Delta\varphi + \int_\Omega \nabla\mu\cdot\nabla\varphi - \int_\Omega g_2\,\Delta\varphi\,, \\
&\frac{1}{2}\frac{d}{dt}\int_\Omega |a|^2 + \int_\Omega |\nabla a|^2 - \chi_a\int_\Omega (a\nabla\sigma^* + a^*\sigma)\cdot\nabla a \\
&= \int_\Omega |a|^2 - 2\int_\Omega a^*|a|^2 - \int_\Omega \boldsymbol{g}_3\cdot\nabla a + \int_\Omega g_4\,a\,, \\
&\int_\Omega |\partial_t n|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla n|^2 = \chi_\varphi\int_\Omega \varphi\,\partial_t n + \int_\Omega (c_\varphi\varphi + c_n n + c_\sigma\sigma)\,\partial_t n\,, \\
&\int_\Omega |\partial_t\sigma|^2 + \frac{d}{dt}\int_\Omega |\nabla\sigma|^2 + \int_\Omega |\Delta\sigma|^2 \\
&= \int_\Omega (-\sigma + \chi_a a - a\sigma^* - a^*\sigma)(\partial_t\sigma - \Delta\sigma) + \int_\Omega g_5\,(\partial_t\sigma - \Delta\sigma)\,.
\end{aligned}$$

Then, we take the sum of these identities and add $(1/2)d/dt\int_\Omega |n|^2$ and $d/dt\int_\Omega |\sigma|^2$ to the left-hand side of the resulting equality and the same terms, written in the form $\int_\Omega n\partial_t n$ and $2\int_\Omega \sigma\partial_t\sigma$, to the right-hand side. The left-hand side then becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi|^2 + m\int_\Omega |\varphi|^2 + \int_\Omega |\Delta\varphi|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |a|^2 + \int_\Omega |\nabla a|^2 \\
&\quad + \int_\Omega |\partial_t n|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega \big(|n|^2 + |\nabla n|^2\big) \\
&\quad + \int_\Omega |\partial_t\sigma|^2 + \frac{d}{dt}\int_\Omega \big(|\sigma|^2 + |\nabla\sigma|^2\big) + \int_\Omega |\Delta\sigma|^2\,,
\end{aligned}$$

and we have to estimate the terms on the corresponding right-hand side. However, two of these cancel each other, one is nonpositive, and most of the others can be easily dealt with using Young's inequality. So, we discuss only the most delicate terms. As usual, we intend to apply Gronwall's lemma after time integration. The first term we consider is the following:

$$\chi_a \int_\Omega a\nabla\sigma^* \cdot \nabla a \le \|a\|_2 \, \|\nabla\sigma^*\|_\infty \, \|\nabla a\|_2 \le \frac{1}{4}\int_\Omega |\nabla a|^2 + \|\nabla\sigma^*\|_\infty^2 \int_\Omega |a|^2 \,.$$

To deal with the next one, we owe to the second inequality in (2.41) to find that

$$\begin{aligned}
&\chi_a \int_\Omega a^*\nabla\sigma \cdot \nabla a \le \|a^*\|_4 \, \|\nabla\sigma\|_4 \, \|\nabla a\|_2 \le \frac{1}{4}\int_\Omega |\nabla a|^2 + \|a^*\|_4^2 \, \|\nabla\sigma\|_4^2 \\
&\le \frac{1}{4}\int_\Omega |\nabla a|^2 + \|a^*\|_4^2 \, \|\sigma\|_V \left(\|\sigma\| + \|\Delta\sigma\|\right) \\
&\le \frac{1}{4}\int_\Omega |\nabla a|^2 + \frac{1}{4}\int_\Omega |\Delta\sigma|^2 + c\left(\|a^*\|_4^2 + \|a^*\|_4^4\right) \|\sigma\|_V^2 \,.
\end{aligned}$$

We notice that the functions $t \mapsto \|\nabla\sigma^*(t)\|_\infty^2$ and $t \mapsto (\|a^*(t)\|_4^2 + \|a^*(t)\|_4^4)$ belong to $L^1(0,T)$. The last integral we consider is the following:

$$\begin{aligned}
&\int_\Omega a^*\sigma(\partial_t\sigma - \Delta\sigma) \le \|a^*\|_4 \, \|\sigma\|_4 \left(\|\partial_t\sigma\|_2 + \|\Delta\sigma\|_2\right) \\
&\le \frac{1}{8}\int_\Omega \left(|\partial_t\sigma|^2 + |\Delta\sigma|^2\right) + c\, \|a^*\|_4^2 \, \|\sigma\|_V^2 \,.
\end{aligned}$$

By collecting, rearranging, and applying the Gronwall lemma, we conclude that

$$\begin{aligned}
&\|\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;W)} + \|a\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \\
&\quad + \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c \sum_{i=1}^5 \|g_i\|_{L^2(Q)} \,.
\end{aligned} \tag{5.15}$$

Consequence. Notice that (5.15) also yields an estimate for a and $\nabla\sigma$ in $L^4(Q)$, thanks to (2.37). Moreover, by comparison, first in (5.10) and then in (5.8), and elliptic regularity, we derive estimates for n and μ. In conclusion, we have that

$$\|a\|_{L^4(Q)} + \|\nabla\sigma\|_{L^4(Q)} + \|n\|_{L^2(0,T;W)} + \|\mu\|_{L^2(0,T;H)} \le c \sum_{i=1}^5 \|g_i\|_{L^2(Q)} \,. \tag{5.16}$$

Second a priori estimate. We test (5.9) by an arbitrary $v \in L^2(0,T;V)$ and integrate over $(0,T)$ to obtain that

$$\begin{aligned}
&\int_0^T \langle \partial_t a(t), v(t)\rangle \, dt = -\int_Q \nabla a \cdot \nabla v + \chi_a \int_Q (a\nabla\sigma^* + a^*\nabla\sigma)\cdot\nabla v \\
&\quad - \int_Q \boldsymbol{g}_3 \cdot \nabla v + \int_Q (a + g_4)v - \int_Q 2a^* a v \,.
\end{aligned}$$

Just some of the terms on the right-hand side need a treatment. We have that

$$\begin{aligned}&\chi_a \int_Q (a\nabla\sigma^* + a^*\nabla\sigma)\cdot\nabla v\\&\le \|a\|_{L^4(Q)}\,\|\nabla\sigma^*\|_{L^4(Q)}\,\|\nabla v\|_{L^2(Q)} + \|a^*\|_{L^4(Q)}\,\|\nabla\sigma\|_{L^4(Q)}\,\|\nabla v\|_{L^2(Q)}\\&\le c\,\|v\|_{L^2(0,T;V)}\,\sum_{i=1}^5 \|g_i\|_{L^2(Q)}\,,\end{aligned}$$

and we similarly obtain that

$$-\int_Q 2a^* a v \le c\,\|v\|_{L^2(0,T;V)}\,\sum_{i=1}^5 \|g_i\|_{L^2(Q)}\,.$$

Hence, it turns out that

$$\int_0^T \langle \partial_t a(t), v(t)\rangle\,dt \le c\,\|v\|_{L^2(0,T;V)}\,\sum_{i=1}^5 \|g_i\|_{L^2(Q)}\,,$$

and, since v is arbitrary in $L^2(0,T;V)$, this means that

$$\|\partial_t a\|_{L^2(0,T;V^*)} \le c\sum_{i=1}^5 \|g_i\|_{L^2(Q)}\,. \tag{5.17}$$

Third a priori estimate. We test (5.7) by both μ and $g_2 - F''(\varphi^*)\varphi$. At the same time, we test (5.8) by $\partial_t\varphi$. We obtain that

$$\begin{aligned}&\int_\Omega \partial_t\varphi\,\mu + \int_\Omega |\nabla\mu|^2 = \chi_\varphi\int_\Omega \nabla n\cdot\nabla\mu - m\int_\Omega \varphi\mu + \int_\Omega \mathbb{h}'(\varphi^*)\,\varphi\mu + \int_\Omega g_1\mu\,,\\&\int_\Omega \partial_t\varphi\,\bigl(g_2 - F''(\varphi^*)\varphi\bigr) = \int_\Omega \bigl(-\nabla\mu + \chi_\varphi\nabla n\bigr)\cdot\nabla\bigl(g_2 - F''(\varphi^*)\varphi\bigr)\\&\quad + \int_\Omega \bigl(-m\varphi + \mathbb{h}'(\varphi^*)\varphi + g_1\bigr)\bigl(g_2 - F''(\varphi^*)\varphi\bigr)\,,\\&\frac12\,\frac{d}{dt}\int_\Omega |\nabla\varphi|^2 = \int_\Omega \mu\,\partial_t\varphi + \int_\Omega \bigl(g_2 - F''(\varphi^*)\,\varphi\bigr)\partial_t\varphi\,.\end{aligned}$$

At this point, we add these equalities to each other and notice several cancellations. The left-hand side then becomes

$$\int_\Omega |\nabla\mu|^2 + \frac12\,\frac{d}{dt}\int_\Omega |\nabla\varphi|^2\,, \tag{5.18}$$

and we need to estimate the terms on the corresponding right-hand side. However, we only treat the most delicate of them, since the others are simple to handle. By recalling, in particular, that $\nabla\varphi^*$ is bounded owing to the regularity in (2.28), we have that

$$\begin{aligned}&\int_\Omega \bigl(-\nabla\mu + \chi_\varphi\nabla n\bigr)\cdot\nabla\bigl(g_2 - F''(\varphi^*)\varphi\bigr)\\&\le \frac14\int_\Omega |\nabla\mu|^2 + c\int_\Omega |\nabla n|^2 + c\int_\Omega |\nabla g_2|^2 + c\int_\Omega |\varphi\,F'''(\varphi^*)\nabla\varphi^*|^2 + c\int_\Omega |F''(\varphi^*)\nabla\varphi|^2\\&\le \frac14\int_\Omega |\nabla\mu|^2 + c\int_\Omega |\nabla n|^2 + c\int_\Omega |\nabla g_2|^2 + c\int_\Omega |\varphi|^2 + c\int_\Omega |\nabla\varphi|^2\,,\end{aligned}$$

and we account for (5.15). By combining it with the already proved estimates, we conclude that

$$\|\mu\|_{L^2(0,T;V)} + \|\varphi\|_{L^\infty(0,T;V)} \le c \sum_{i=1}^{5} \|g_i\|_{L^2(Q)} + c \,\|\nabla g_2\|_{L^2(Q)} \,. \tag{5.19}$$

Conclusion. Now that μ is estimated in $L^2(0,T;V)$, a comparison argument in (5.7), using also (5.19), yields a similar estimate for $\partial_t\varphi$ in $L^2(0,T;V^*)$, and the existence part of the proof is complete.

Uniqueness. By linearity, we consider only the homogeneous problem. We want to come back to the proof of (5.15) and show that the procedure used there can be made rigorous. On account of the regularity (5.2)–(5.6), we notice that the equations (5.8), (5.10), and (5.11), can be written in strong form. For instance, (5.8) with $g_2 = 0$ becomes

$$-\Delta\varphi = -F''(\varphi^*)\varphi + \mu \quad \text{a.e. in } Q\,. \tag{5.20}$$

Hence, instead of testing (5.8) by $-\Delta\varphi$, we can multiply (5.20) by $-\Delta\varphi$ and integrate over Ω. Therefore, we can still arrive at (5.15), which yields $(\varphi, a, n, \sigma) = (0,0,0,0)$, from which, (5.20) implies that $\mu = 0$ as well. □

6 The control problem

In this section, we give the first result on the control problem presented in the Introduction. For the reader's convenience, we recall the definitions of the cost functional $\mathcal{J}$ and of the set $\mathcal{U}_{\rm ad}$ of the admissible controls:

$$\mathcal{J}(\varphi,u) := \frac{b_1}{2}\int_Q |\varphi-\varphi_Q|^2 + \frac{b_2}{2}\int_\Omega |\varphi(T)-\varphi_\Omega|^2 + \frac{b_3}{2}\int_Q |u|^2$$
$$\text{for } \varphi \in C^0([0,T];H) \text{ and } u \in L^2(Q), \tag{6.1}$$

$$\mathcal{U}_{\rm ad} := \bigl\{u \in \mathcal{U} : 0 \le u \le u_{\max} \text{ a.e. in } Q\bigr\}, \quad \text{where} \quad \mathcal{U} := L^\infty(Q). \tag{6.2}$$

We make the following assumptions:

$$b_i \in [0,+\infty) \text{ for } i = 1,2,3, \text{ with } b_3 > 0; \quad \varphi_Q \in L^2(Q),\ \varphi_\Omega \in V. \tag{6.3}$$

$$u_{\max} \in L^\infty(Q) \text{ is nonnegative.} \tag{6.4}$$

Then the control problem is given by:

$$\text{Minimize } \mathcal{J}(\varphi,u) \text{ subject to } u \in \mathcal{U}_{\rm ad} \text{ and to the constraint that}$$
$$(\varphi,\mu,a,n,\sigma) \text{ is the solution to the system (2.22)–(2.27).} \tag{6.5}$$

In the remainder of the paper, it is understood that the above assumptions are in force, as well as those on the structure and the data (with the same $u_{\max}$ as here, of course) that ensure well-posedness for the state system (see Theorem 2.2). We therefore do not recall them in any of the following statements. Besides, by virtue of Theorem 2.2, we can introduce the *control-to-state* operator $\mathcal{S}$ as

$$\mathcal{S} := (\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3,\mathcal{S}_4,\mathcal{S}_5) \text{ mapping } u \in \mathcal{U}_{\rm ad} \text{ into } (\varphi,\mu,a,n,\sigma) \in \mathcal{Y},$$

where $\mathcal{Y}$ is the regularity space defined in (2.29). The same operator is Lipschitz continuous from $\mathcal{U}_{\rm ad}$ into the space $\mathcal{X}$ specified by (5.13), in the sense of the continuous dependence estimate (2.32).

Theorem 6.1. *The control problem* (6.5) *has at least one solution, that is, there exists at least one* $u^* \in \mathcal{U}_{\rm ad}$ *that satisfies*

$$\mathcal{J}(\varphi^*, u^*) \le \mathcal{J}(\varphi, u) \quad \textit{for every } u \in \mathcal{U}_{\rm ad}, \tag{6.6}$$

where φ^* *and* φ *are the first components of the solutions to the state system* (2.22)–(2.27) *corresponding to* u^* *and* u*, respectively.*

Proof. We use the direct method of the calculus of variations. To begin with, we recall (6.1) and observe that $\mathcal{J}(\varphi, u)$ is bounded from below as it is nonnegative. Thus, we term Λ the infimum of the cost functional under the constraints of the control problem and fix a minimizing sequence $\{u_k\}$ and the sequence of the corresponding solutions $(\varphi_k, \mu_k, a_k, n_k, \sigma_k) \in \mathcal{Y}$ as given by Theorem 2.2. Namely, we have that

$$\lim_{k\to\infty} \mathcal{J}(\varphi_k, u_k) = \Lambda \, .$$

Now, $\mathcal{U}_{\rm ad}$ is bounded in $L^\infty(Q)$, and all of the solutions $(\varphi_k, \mu_k, a_k, n_k, \sigma_k)$ satisfy the stability estimate (2.30). Hence, we can use well-known compactness results to obtain that, possibly for a nonrelabeled subsequence, as $k \to \infty$,

$$u_k \to u^* \quad \text{weakly star in } L^\infty(Q), \tag{6.7}$$

$$(\varphi_k, \mu_k, a_k, n_k, \sigma_k) \to (\varphi^*, \mu^*, a^*, n^*, \sigma^*) \quad \text{weakly star in } \mathcal{Y}, \tag{6.8}$$

for some limiting functions u^* and $\varphi^*, \mu^*, a^*, n^*, \sigma^*$. At this point, strong convergence properties are needed, and we apply [26, Sect. 8, Cor. 4] several times. First, we notice that $\{\varphi_k\}$ weakly star converges in $\mathcal{Y}_1$ that is compactly embedded in $C^0(\overline{Q})$. We thus infer that, as $k \to \infty$,

$$\varphi_k \to \varphi \quad \text{strongly in } C^0(\overline{Q}), \quad \text{whence also}$$
$$F'(\varphi_k) \to F'(\varphi) \quad \text{and} \quad \mathbb{h}(\varphi_k) \to \mathbb{h}(\varphi^*) \quad \text{strongly in } C^0(\overline{Q}),$$

since the functions φ_k satisfy the separation property (2.31) and F' and $\mathbb{h}$ are Lipschitz continuous in the interval $[r_-, r_+]$. Moreover, (6.8) (cf. (2.19)) implies that a_k converges to a^* strongly in $C^0([0,T]; H) \cap L^2(0,T;V)$. In view of (2.37), we deduce that, as $k \to \infty$, $a_k \to a^*$ strongly in $L^4(Q)$ and consequently

$$(a_k)^2 \to (a^*)^2 \quad \text{strongly in } L^2(Q) \, .$$

Next, from (6.8) and (2.21) it follows that, as $k \to \infty$,

$$\sigma_k \to \sigma^* \quad \text{strongly in } C^0(\overline{Q}) \cap L^4(0,T;W^{1,4}(\Omega)),$$

whence

$$a_k\sigma_k \to a^*\sigma^* \quad \text{and} \quad a_k \nabla\sigma_k \to a^*\nabla\sigma^* \quad \text{strongly in } L^2(Q) \text{ and } L^2(Q)^2.$$

Collecting all this information, and passing to the limit in the variational equalities (2.22)–(2.26) written for $(\varphi_k, \mu_k, a_k, n_k, \sigma_k)$, we deduce that $(\varphi^*, \mu^*, a^*, n^*, \sigma^*)$ solves the state system corresponding to u^*. This shows that $(\varphi^*, \mu^*, a^*, n^*, \sigma^*)$ is actually $\mathcal{S}(u^*)$. Thus, we have that

$$\mathcal{J}(\varphi^*, u^*) \le \liminf_{k\to\infty} \mathcal{J}(\varphi_k, u_k) = \Lambda \, ,$$

so that $\mathcal{J}(\varphi^*, u^*) = \Lambda$ and u^* is an optimal control. □

7 Necessary conditions for optimality

To obtain significant necessary optimality conditions, we prove some differentiability property of the control-to-state operator $\mathcal{S}$. To this end, recall the definitions of the spaces $\mathcal{Y}_i$ and $\mathcal{X}_i$, $i = 1, ..., 4$, given in (2.17)–(2.21) and (5.2)–(5.6), respectively. Notice that $\mathcal{Y}_i \hookrightarrow \mathcal{X}_i$ for every $i = 1, ..., 4$. We then recall the definition (5.13) of $\mathcal{X}$ and consider the mapping $\mathcal{S} : \mathcal{U}_{\text{ad}} \to \mathcal{X}$ by observing that

$$\text{for } u \in \mathcal{U}_{\text{ad}},\ \mathcal{S}(u) = (\varphi, \mu, a, n, \sigma) \text{ is the solution to (2.22)–(2.27).} \tag{7.1}$$

Often, it is possible to extend $\mathcal{S}$ to an open subset of $\mathcal{U}$ and prove the differentiability of this extension. However, we cannot develop this idea in our scenario. Indeed, although the constraint $u \le u_{\max}$ could be replaced by $\|u\|_\infty < K$ (with any prescribed $K > \|u_{\max}\|_\infty$) without any significant change in our previous proofs, we cannot avoid the constraint $u \ge 0$. Therefore, we can only prove some kind of *sectorial* differentiability that is close to Fréchet differentiability but intrinsically involves a constraint for the increments. Also in the present case, a crucial role is played by the linearized system we introduce at once. To this end, we fix $u^* \in \mathcal{U}_{\text{ad}}$ and $(\varphi^*, \mu^*, a^*, n^*, \sigma^*) := \mathcal{S}(u^*)$. By accounting for the separation property (2.31) and for the smoothness of F on the interval $[r_-, r_+]$, we infer that

$$F^{(j)}(\varphi^*) \in L^\infty(Q) \quad \text{and} \quad \|F^{(j)}(\varphi^*)\|_\infty \le K_1' \quad \text{for } j \in \{0, ..., 4\}, \tag{7.2}$$

where K_1' is similar to the constant K_1 appearing in (2.30).

Let us come to the linearized problem associated with u^*. For a given $h \in L^2(Q)$, it consists in looking for a quintuple $(\psi, \eta, \alpha, \nu, \omega)$ with the regularity

$$(\psi, \eta, \alpha, \nu, \omega) \in \mathcal{X} \tag{7.3}$$

that solves the variational equations

$$\langle \partial_t \psi, v \rangle + \int_\Omega \nabla \eta \cdot \nabla v - \chi_\varphi \int_\Omega \nabla \nu \cdot \nabla v$$

$$= -m \int_\Omega \psi\, v + \int_\Omega \mathbb{h}'(\varphi^*)\, \psi v\,, \tag{7.4}$$

$$\int_\Omega \nabla \psi \cdot \nabla v + \int_\Omega F''(\varphi^*)\, \psi v = \int_\Omega \eta v\,, \tag{7.5}$$

$$\langle \partial_t \alpha, v \rangle + \int_\Omega \nabla \alpha \cdot \nabla v - \chi_a \int_\Omega (\alpha \nabla \sigma^* + a^* \nabla \omega) \cdot \nabla v$$

$$= \int_\Omega (\alpha - 2a^* \alpha) v + \int_\Omega h v\,, \tag{7.6}$$

$$\int_\Omega \partial_t \nu\, v + \int_\Omega \nabla \nu \cdot \nabla v - \chi_\varphi \int_\Omega \psi v = \int_\Omega (c_\varphi \psi + c_n \nu + c_\sigma \omega) v\,, \tag{7.7}$$

$$\int_\Omega \partial_t \omega\, v + \int_\Omega \nabla \omega \cdot \nabla v = \int_\Omega (-\omega + \chi_a \alpha - \alpha \sigma^* - a^* \omega) v\,, \tag{7.8}$$

for every $v \in V$ and a.e. in $(0, T)$, and satisfies the initial condition

$$(\psi, \alpha, \nu, \omega)(0) = (0, 0, 0, 0)\,. \tag{7.9}$$

We have the following result concerning well-posedness.

Theorem 7.1. *Let $u^* \in \mathcal{U}_{\rm ad}$ be given and $(\varphi^*, \mu^*, a^*, n^*, \sigma^*) := \mathcal{S}(u^*)$. Then, for every $h \in L^2(Q)$, problem* (7.4)–(7.9) *has a unique solution $(\psi, \eta, \alpha, \nu, \omega)$ satisfying* (7.3)*, and the estimate*

$$\|(\psi, \eta, \alpha, \nu, \omega)\|_{\mathcal{X}} \leq K_3 \, \|h\|_{L^2(Q)} \tag{7.10}$$

holds true with a constant $K_3 > 0$ that depends only on Ω, T, the structure and the data of the state system, and $u_{\max}$.

Proof. It suffices to apply Theorem 5.1. Indeed, problem (7.4)–(7.9) is the particular case of problem (5.7)–(5.12), where $g_4 = h$ and $g_1 = g_2 = g_3 = g_5 = 0$. □

Let us move to prove some differentiability property of the solution operator $\mathcal{S}$. As said before, we cannot speak of Fréchet differentiability. However, the above result implies that the linear mapping $h \mapsto (\psi, \eta, \alpha, \nu, \omega)$ is continuous from $L^2(Q)$ into $\mathcal{X}$, and we are now going to see that it plays a similar role as a Fréchet derivative. Indeed, we can prove the following result.

Theorem 7.2. *Let $u^* \in \mathcal{U}_{\rm ad}$ and $(\varphi^*, \mu^*, a^*, n^*, \sigma^*) := \mathcal{S}(u^*)$. For every $h \in L^2(Q)$, let $(\psi, \eta, \alpha, \nu, \omega)$ be the solution to the corresponding linearized system* (7.4)–(7.9)*. Then we have that*

$$\frac{\|\mathcal{S}(u^*+h) - \mathcal{S}(u^*) - (\psi, \eta, \alpha, \nu, \omega)\|_{\mathcal{X}}}{\|h\|_{L^2(Q)}} \quad \text{tends to zero}$$
$$\text{as } \|h\|_{L^2(Q)} \text{ tends to zero under the constraint that } u^* + h \in \mathcal{U}_{\rm ad}. \tag{7.11}$$

Proof. We assume that $u^* + h$ belongs to $\mathcal{U}_{\rm ad}$ and introduce $(\varphi^h, \mu^h, a^h, n^h, \sigma^h) := \mathcal{S}(u^* + h)$ and the quintuplet $(\phi, \rho, \gamma, \lambda, \xi) \in \mathcal{X}$ defined by

$$\begin{aligned} &\phi := \varphi^h - \varphi^* - \psi \,, \quad \rho := \mu^h - \mu^* - \eta \,, \quad \gamma := a^h - a^* - \alpha \,, \\ &\quad \lambda := n^h - n^* - \nu \quad \text{and} \quad \xi := \sigma^h - \sigma^* - \omega \,, \end{aligned} \tag{7.12}$$

so that

$$\mathcal{S}(u^* + h) - \mathcal{S}(u^*) - (\psi, \eta, \alpha, \nu, \omega) = (\phi, \rho, \gamma, \lambda, \xi) \,.$$

Then, $(\phi, \rho, \gamma, \lambda, \xi)$ satisfies the variational equations

$$\begin{aligned} &\langle \partial_t \phi, v \rangle + \int_\Omega \nabla \rho \cdot \nabla v - \chi_\varphi \int_\Omega \nabla \lambda \cdot \nabla v \\ &= -m \int_\Omega \phi v + \int_\Omega \big[\mathbb{h}(\varphi^h) - \mathbb{h}(\varphi^*) - \mathbb{h}'(\varphi^*) \psi \big] \, v \,, \end{aligned} \tag{7.13}$$

$$\int_\Omega \nabla \phi \cdot \nabla v = \int_\Omega \rho v - \int_\Omega \big[F'(\varphi^h) - F'(\varphi^*) - F''(\varphi^*) \psi \big] \, v \,, \tag{7.14}$$

$$\begin{aligned} &\langle \partial_t \gamma, v \rangle + \int_\Omega \nabla \gamma \cdot \nabla v - \chi_a \int_\Omega \big[a^h \nabla \sigma^h - a^* \nabla \sigma^* - \alpha \nabla \sigma^* - a^* \nabla \omega \big] \cdot \nabla v \\ &= \int_\Omega \gamma v - \int_\Omega \big[(a^h)^2 - (a^*)^2 - 2 a^* \alpha \big] \, v \,, \end{aligned} \tag{7.15}$$

$$\int_\Omega \partial_t \lambda \, v + \int_\Omega \nabla \lambda \cdot \nabla v - \chi_\varphi \int_\Omega \phi v = \int_\Omega (c_\varphi \phi + c_n \gamma + c_\sigma \xi) v \,, \tag{7.16}$$

$$\int_\Omega \partial_t \xi \, v + \int_\Omega \nabla \xi \cdot \nabla v = \int_\Omega \big(-\xi + \chi_a \gamma - \big[a^h \sigma^h - a^* \sigma^* - \alpha \, \sigma^* - a^* \omega \big] \big) v \,, \tag{7.17}$$

all for every $v \in V$ and a.e. in $(0,T)$, and the initial condition

$$(\phi, \gamma, \lambda, \xi)(0) = (0,0,0,0) \quad \text{a.e. in } \Omega\,. \tag{7.18}$$

We transform some terms on the right-hand sides, using Taylor expansions in three cases. We have that

$$\begin{aligned}
&\mathbb{h}(\varphi^h) - \mathbb{h}(\varphi^*) - \mathbb{h}'(\varphi^*)\psi = \mathbb{h}'(\varphi^*)\,\phi + R_1(\varphi^h - \varphi^*)^2 \\
&\quad \text{with} \quad R_1 = \int_0^1 (1-s)\mathbb{h}''(\varphi^* + s(\varphi^h - \varphi^*))\,ds\,, \\
&F'(\varphi^h) - F'(\varphi^*) - F''(\varphi^*)\psi = F''(\varphi^*)\,\phi + R_2(\varphi^h - \varphi^*)^2 \\
&\quad \text{with} \quad R_2 = \int_0^1 (1-s)F'''(\varphi^* + s(\varphi^h - \varphi^*))\,ds\,, \\
&a^h\nabla\sigma^h - a^*\nabla\sigma^* - \alpha\nabla\sigma^* - a^*\nabla\omega = (a^h - a^*)(\nabla\sigma^h - \nabla\sigma^*) + \gamma\nabla\sigma^* + a^*\nabla\xi\,, \\
&(a^h)^2 - (a^*)^2 - 2a^*\alpha = 2a^*\gamma + R_3(a^h - a^*)^2 \\
&\quad \text{with} \quad R_3 = \int_0^1 (1-s)\,2(\varphi^* + s(\varphi^h - \varphi^*))\,ds\,, \\
&a^h\sigma^h - a^*\sigma^* - \alpha\sigma^* - a^*\omega = (a^h - a^*)\,(\sigma^h - \sigma^*) + \gamma\sigma^* + a^*\xi\,.
\end{aligned} \tag{7.19}$$

We notice that both φ^* and φ^h satisfy the separation condition (2.31), so that the same holds for $\varphi^* + s(\varphi^h - \varphi^*)$ for every $s \in [0,1]$. Hence, the quantities under the above integrals over $(0,1)$ are bounded, and all the remainders $R_1, \ldots, R_3$ are uniformly bounded. At this point, we notice that problem (7.13)–(7.18) takes the form (5.7)–(5.12) with

$$\begin{aligned}
&g_1 = R_1\,(\varphi^h - \varphi^*)^2\,, \quad g_2 = -R_2\,(\varphi^h - \varphi^*)^2\,, \quad \boldsymbol{g}_3 = -\chi_a(a^h - a^*)\,\nabla(\sigma^h - \sigma^*)\,, \\
&g_4 = -R_3\,(a^h - a^*)^2 \quad \text{and} \quad g_5 = -(a^h - a^*)\,(\sigma^h - \sigma^*)\,.
\end{aligned}$$

Hence, we can apply Theorem 5.1, to infer that

$$\|(\phi, \rho, \gamma, \lambda, \xi)\|_{\mathcal{X}} \le K_3 \Big(\sum_{i=1}^5 \|g_i\|_{L^2(Q)} + \|\nabla g_2\|_{L^2(Q)} \Big)\,. \tag{7.20}$$

Thus, it remains to estimate the right-hand side of (7.20). In doing this, we also account for Theorem 2.3 and apply (2.32) to $(\varphi^h, \mu^h, a^h, n^h, \sigma^h)$ and $(\varphi^*, \mu^*, a^*, n^*, \sigma^*)$, possibly combined with the last inequality in (2.37). We have that

$$\begin{aligned}
&\|g_1\|_{L^2(Q)}^2 + \|g_2\|_{L^2(Q)}^2 \le c \int_Q |\varphi^h - \varphi^*|^4 \le c\,\|h\|_{L^2(Q)}^4\,, \\
&\|\boldsymbol{g}_3\|_{L^2(Q)}^2 \le \int_\Omega |a^h - a^*|^2\,|\nabla(\sigma^h - \sigma^*)|^2 \\
&\quad \le \|a^h - a^*\|_{L^4(Q)}^2\,\|\nabla(\sigma^h - \sigma^*)\|_{L^4(Q)}^2 \le c\,\|h\|_{L^2(Q)}^4\,, \\
&\|g_4\|_{L^2(Q)}^2 \le c \int_Q |a^h - a^*|^4 \le c\,\|h\|_{L^2(Q)}^4\,, \\
&\|g_5\|_{L^2(Q)}^2 \le \int_\Omega |a^h - a^*|^2\,|\sigma^h - \sigma^*|^2 \\
&\quad \le \|a^h - a^*\|_{L^4(Q)}^2\,\|\sigma^h - \sigma^*\|_{L^4(Q)}^2 \le c\,\|h\|_{L^2(Q)}^4\,.
\end{aligned}$$

Finally, since it turns out that $|\nabla R_2| \le c(|\nabla\varphi^*| + |\nabla\varphi^h|) \le c$, we also have that

$$\begin{aligned}&\|\nabla g_2\|^2_{L^2(Q)} = \|(\varphi^h - \varphi^*)^2\,\nabla R_2 + 2R_2(\varphi^h - \varphi^*)\nabla(\varphi^h - \varphi^*)\|^2_{L^2(Q)}\\&\le c\left(\|\varphi^h - \varphi^*\|^4_{L^4(Q)} + \|\varphi^h - \varphi^*\|^2_{L^4(Q)}\,\|\nabla(\varphi^h - \varphi^*)\|^2_{L^4(Q)} \le c\,\|h\|^4_{L^2(Q)}\,.\end{aligned}$$

Therefore, (7.20) implies that

$$\|(\phi, \rho, \gamma, \lambda, \xi)\|_{\mathcal{X}} \le c\,\|h\|^2_{L^2(Q)}\,,$$

and (7.11) readily follows. □

Thanks to the above result and the convexity of $\mathcal{U}_{\rm ad}$, a standard argument leads to the following necessary condition for an element $u^* \in \mathcal{U}_{\rm ad}$ to be an optimal control:

$$\begin{aligned}&b_1\int_Q (\varphi^* - \varphi_Q)\,\psi + b_2\int_\Omega \bigl(\varphi^*(T) - \varphi_\Omega\bigr)\,\psi(T) + b_3\int_Q u^*\,(u - u^*) \ge 0\\&\text{for every } u \in \mathcal{U}_{\rm ad}\,,\end{aligned} \tag{7.21}$$

where ψ is the first component of the solution to the linearized problem (7.4)–(7.9) corresponding to $h := u - u^*$. However, this condition is problematic, since it requires to solve the linearized problem infinitely many times because u is arbitrary in $\mathcal{U}_{\rm ad}$. As usual, this trouble is overcome by introducing a proper adjoint problem associated with a given $u^* \in \mathcal{U}_{\rm ad}$. In order to simplify its presentation, we use some abbreviations: for some pairs (i, j), with $i, j \in \{0, ..., 5\}$, we define $f_{i,j}$ as follows:

$$f_{1,0} = b_1(\varphi^* - \varphi_Q)\,,\ f_{1,1} = m - \mathbb{h}'(\varphi^*)\,,\ f_{1,2} = F''(\varphi^*)\,,\ f_{1,4} = -\chi_\varphi - c_\varphi\,, \tag{7.22}$$

$$f_{3,3} = -1\,,\ f_{3,5} = \sigma^* - \chi_a\,, \tag{7.23}$$

$$f_{5,4} = -c_\sigma\,,\ f_{5,5} = 1\,, \tag{7.24}$$

and put to zero all the other cases. We notice that $f_{1,0} \in L^2(Q)$ and that every other $f_{i,j}$ is a bounded function. At this point, we can write the adjoint problem associated with u^*. It consists in looking for a quintuple $(p_1, p_2, p_3, p_4, p_5)$ with the regularity properties

$$p_1 \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W \cap H^3(\Omega))\,, \tag{7.25}$$

$$p_2 \in L^2(0,T;V)\,, \tag{7.26}$$

$$p_3 \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\,, \tag{7.27}$$

$$p_4 \in H^1(0,T;V) \cap L^2(0,T;W \cap H^3(\Omega))\,, \tag{7.28}$$

$$p_5 \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V)\,, \tag{7.29}$$

that solves the variational equations

$$- \langle \partial_t p_1, v \rangle + \int_\Omega \nabla p_2 \cdot \nabla v + \sum_{j=1}^{5} \int_\Omega f_{1,j}\, p_j\, v = \int_\Omega f_{1,0}\, v \,, \tag{7.30}$$

$$\int_\Omega \nabla p_1 \cdot \nabla v = \int_\Omega p_2\, v \,, \tag{7.31}$$

$$\begin{aligned} &- \int_\Omega \partial_t p_3\, v + \int_\Omega \nabla p_3 \cdot \nabla v - \chi_a \int_\Omega (\nabla \sigma^* \cdot \nabla p_3) v \\ &\quad + \int_\Omega 2a^*\, p_3\, v + \sum_{j=1}^{5} \int_\Omega f_{3,j}\, p_j\, v = 0 \,, \end{aligned} \tag{7.32}$$

$$- \int_\Omega \partial_t p_4\, v + \int_\Omega \nabla p_4 \cdot \nabla v - \chi_\varphi \int_\Omega \nabla p_1 \cdot \nabla v - c_n \int_\Omega p_4\, v = 0 \,, \tag{7.33}$$

$$\begin{aligned} &- \langle \partial_t p_5, v \rangle + \int_\Omega \nabla p_5 \cdot \nabla v - \chi_a \int_\Omega a^*\, \nabla p_3 \cdot \nabla v \\ &\quad + \int_\Omega a^*\, p_5\, v + \sum_{j=1}^{5} \int_\Omega f_{5,j} p_j \; v = 0 \,, \end{aligned} \tag{7.34}$$

for every $v \in V$ and a.e. in $(0,T)$, and the final conditions

$$p_1(T) = b_2 \big(\varphi^*(T) - \varphi_\Omega \big) \quad \text{and} \quad (p_3, p_4, p_5)(T) = (0,0,0) \,. \tag{7.35}$$

Theorem 7.3. *Let $u^* \in \mathcal{U}_{\rm ad}$. Then, the adjoint problem* (7.30)–(7.35) *has a unique solution satisfying* (7.25)–(7.29).

Proof. As for the existence of a solution, also for this problem one can start from a Faedo–Galerkin scheme constructed by means of the eigenfunctions of the Laplace operator with homogeneous Neumann boundary conditions. However, for brevity, we just perform the relevant formal estimates.

First a priori estimate. We test the above equations by p_1, $p_1 - p_2$, p_3, p_4, and p_5, respectively. In addition, we test (7.32) by $-\Delta p_3$. We obtain the identities

$$\begin{aligned} &- \frac{1}{2} \frac{d}{dt} \int_\Omega |p_1|^2 + \int_\Omega \nabla p_2 \cdot \nabla p_1 + \sum_{j=1}^{5} \int_\Omega f_{1,j}\, p_j\, p_1 = \int_\Omega f_{1,0}\, p_1 \,, \\ &\int_\Omega |\nabla p_1|^2 - \int_\Omega \nabla p_1 \cdot \nabla p_2 + \int_\Omega |p_2|^2 = \int_\Omega p_2\, p_1 \,, \\ &- \frac{1}{2} \frac{d}{dt} \int_\Omega |p_3|^2 + \int_\Omega |\nabla p_3|^2 - \chi_a \int_\Omega (\nabla \sigma^* \cdot \nabla p_3) p_3 + \int_\Omega 2a^*\, |p_3|^2 + \sum_{j=1}^{5} \int_\Omega f_{3,j}\, p_j\, p_3 = 0 \,, \\ &- \frac{1}{2} \frac{d}{dt} \int_\Omega |p_4|^2 + \int_\Omega |\nabla p_4|^2 - \chi_\varphi \int_\Omega \nabla p_1 \cdot \nabla p_4 - c_n \int_\Omega |p_4|^2 = 0 \,, \\ &- \frac{1}{2} \frac{d}{dt} \int_\Omega |p_5|^2 + \int_\Omega |\nabla p_5|^2 - \chi_a \int_\Omega a^*\, \nabla p_3 \cdot \nabla p_5 + \int_\Omega a^*\, |p_5|^2 + \sum_{j=1}^{5} \int_\Omega f_{5,j}\, p_j\, p_5 = 0 \,, \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla p_3|^2 + \int_\Omega |\Delta p_3|^2 + \chi_a \int_\Omega (\nabla\sigma^* \cdot \nabla p_3)\Delta p_3 \\
&- \int_\Omega 2a^*\, p_3\, \Delta p_3 - \sum_{j=1}^5 \int_\Omega f_{3,j}\, p_j\, \Delta p_3 = 0\,.
\end{aligned}
$$

Now, we sum up, notice some cancellations, and rearrange a little. Then we obtain the left-hand side

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_\Omega \bigl(|p_1|^2 + |p_3|^2 + |\nabla p_3|^2 + |p_4|^2 + |p_5|^2\bigr) \\
&+ \int_\Omega \bigl(|\nabla p_1|^2 + |p_2|^2 + |\nabla p_3|^2 + |\Delta p_3|^2 + |\nabla p_4|^2 + a^*\, |p_5|^2 + |\nabla p_5|^2\bigr)\,, \qquad (7.36)
\end{aligned}
$$

where we recall that a^* is nonnegative. We now have to estimate the terms of the corresponding right-hand side, with the intention of applying the (backward-in-time) Gronwall lemma. However, since for many of them it suffices to apply Young's inequality, we just deal with the ones that need some other treatment. We consider only one of the easy integrals, using the assumption that $\chi_\varphi \in (0,1)$: we have

$$
\chi_\varphi \int_\Omega \nabla p_1 \cdot \nabla p_4 \le \frac{1}{2}\int_\Omega |\nabla p_1|^2 + \frac{1}{2}\int_\Omega |\nabla p_4|^2\,,
$$

and the last two integrals are dominated by the corresponding expressions occurring in (7.36). As for the nontrivial terms, we recall the regularity for $\nabla\sigma^*$ ensured by (2.21) and that $a^* \in L^4(Q)$, which implies that the function $t \mapsto \|a^*(t)\|_4$ belongs to $L^4(0,T)$. We also account for the second inequality in (2.41). Hence, we have that

$$
\chi_a \int_\Omega (\nabla\sigma^* \cdot \nabla p_3) p_3 \le \|\nabla\sigma^*\|_\infty\, \|\nabla p_3\|_2\, \|p_3\|_2 \le c\|\sigma^*\|_{H^3(\Omega)} \Bigl(\int_\Omega |\nabla p_3|^2 + \int_\Omega |p_3|^2\Bigr),
$$

$$
\begin{aligned}
&\chi_a \int_\Omega a^*\, \nabla p_3 \cdot \nabla p_5 \le \|a^*\|_4\, \|\nabla p_3\|_4\, \|\nabla p_5\|_2 \\
&\le \frac{1}{2}\int_\Omega |\nabla p_5|^2 + c\, \|a^*\|_4^2\, \|p_3\|_V \bigl(\|p_3\| + \|\Delta p_3\|\bigr) \\
&\le \frac{1}{2}\int_\Omega |\nabla p_5|^2 + c\, \|a^*\|_4^2\, \|p_3\|_V^2 + c\, \|a^*\|_4^2\, \|p_3\|_V\, \|\Delta p_3\| \\
&\le \frac{1}{2}\int_\Omega |\nabla p_5|^2 + \frac{1}{4}\int_\Omega |\Delta p_3|^2 + c\, \bigl(\|a^*\|_4^2 + \|a^*\|_4^4\bigr) \int_\Omega (|p_3|^2 + |\nabla p_3|^2),
\end{aligned}
$$

$$
\begin{aligned}
&-\chi_a \int_\Omega (\nabla\sigma^* \cdot \nabla p_3)\Delta p_3 \le \|\nabla\sigma^*\|_\infty\, \|\nabla p_3\|_2\, \|\Delta p_3\|_2 \\
&\le \frac{1}{4}\int_\Omega |\Delta p_3|^2 + c\|\sigma^*\|_{H^3(\Omega)}^2 \int_\Omega |\nabla p_3|^2\,.
\end{aligned}
$$

By combining the above estimates, observing that the function $t \mapsto \|\sigma^*(t)\|_{H^3(\Omega)}^2$ is bounded in $L^1(0,T)$, integrating over (t,T) with respect to time, and applying Gronwall's lemma, we conclude that

$$
\begin{aligned}
&\|p_1\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|p_2\|_{L^2(0,T;H)} + \|p_3\|_{L^\infty(0,T;V)\cap L^2(0,T;W)} \\
&+ \|p_4\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|p_5\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\,, \qquad (7.37)
\end{aligned}
$$

and a comparison in (7.31) and in (7.32), along with elliptic regularity, yields that also

$$
\|p_1\|_{L^2(0,T;W)} + \|p_3\|_{H^1(0,T;H)} \le c\,. \qquad (7.38)
$$

Second a priori estimate. We test (7.30) by p_2 and (7.31) by $-\partial_t p_1$ and sum up. Since a cancellation occurs, we obtain that

$$\int_\Omega |\nabla p_2|^2 - \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla p_1|^2 = -\sum_{j=1}^{5}\int_\Omega f_{1,j}\, p_j\, p_2 + \int_\Omega f_{1,0}\, p_2 \,.$$

By virtue of the last assumption in (6.3), we notice that $p_1(T) \in V$. Thus, integrating with respect to time, and applying the Gronwall lemma, we immediately infer that

$$\|\nabla p_1\|_{L^\infty(0,T;H)} + \|\nabla p_2\|_{L^2(0,T;H)} \le c \,.$$

Therefore, we can conclude from (7.37) that

$$\|p_1\|_{L^\infty(0,T;V)} + \|p_2\|_{L^2(0,T;V)} \le c \,. \tag{7.39}$$

Consequences. Next, we recall that $p_1(T) \in V$ and $p_4(T) = 0$. Hence, we can first compare in (7.31), and then apply the parabolic regularity in (7.32) and (7.33), to see that

$$\|p_1\|_{L^2(0,T;H^3(\Omega))} + \|p_3\|_{H^1(0,T;H)} + \|p_4\|_{H^1(0,T;V)\cap L^2(0,T;H^3(\Omega))} \le c \,. \tag{7.40}$$

Further a priori estimates. Finally, we test (7.30) and (7.34) by a generic $v \in L^2(0,T;V)$ and integrate over $(0,T)$. Owing to the previous estimates, we easily conclude that

$$\|\partial_t p_1\|_{L^2(0,T;V^*)} + \|\partial_t p_5\|_{L^2(0,T;V^*)} \le c \,. \tag{7.41}$$

We just comment on a term involved in the second test: we have that

$$\chi_a \int_Q a^* \nabla p_3 \cdot \nabla v \le \|a^*\|_{L^4(Q)}\, \|\nabla p_3\|_{L^4(Q)}\, \|\nabla v\|_{L^2(Q)} \le c\, \|v\|_{L^2(0,T;V)} \,.$$

This concludes the formal proof of the existence of a solution $(p_1, p_2, p_3, p_4, p_5)$ satisfying (7.25)–(7.29).

Uniqueness. By linearity, we just have to consider the homogeneous problem, i.e., we replace $f_{1,0}$ by zero and assume $p_1(T) = 0$ in place of the first terminal value condition in (7.35). Once more, we make one of the formal estimates rigorous. Namely, we come back to the derivation of (7.37), where we have tested (7.32) by $-\Delta p_3$. Instead of doing this, we account for the regularity of the solution, write (7.32) in its strong form, multiply it by $-\Delta p_3$, and integrate over Ω. Then, the same estimates can be performed. After applying the Gronwall lemma, we conclude that $(p_1, p_2, p_3, p_4, p_5) = (0, 0, 0, 0, 0)$. □

Our final result is the first-order necessary optimality condition, expressed in the following theorem.

Theorem 7.4. *Let $u^* \in \mathcal{U}_{\rm ad}$ be an optimal control, and let $(p_1, p_2, p_3, p_4, p_5)$ be the solution to the associated adjoint problem* (7.30)–(7.35). *Then, there holds the following variational inequality:*

$$\int_Q (p_3 + b_3 u^*)(u - u^*) \ge 0 \quad \textit{for every } u \in \mathcal{U}_{\rm ad} \,. \tag{7.42}$$

In particular, being $b_3 > 0$, the optimal control u^* is the L^2-orthogonal projection of $-p_3/b_3$ on $\mathcal{U}_{\rm ad}$.

Proof. We fix $u \in \mathcal{U}_{\rm ad}$, set $h := u - u^*$, and consider both the linearized problem (7.4)–(7.9) associated with u^* and h and the adjoint problem. We test the equations of the former by p_1, p_2, p_3, p_4 and p_5, respectively, and those of the latter by $-\psi$, $-\eta$, $-\alpha$, $-\nu$ and $-\omega$, respectively. By also recalling (7.22)–(7.24), we have that

$$
\begin{aligned}
&\langle \partial_t\psi, p_1\rangle + \int_\Omega \nabla\eta\cdot\nabla p_1 - \chi_\varphi \int_\Omega \nabla\nu\cdot\nabla p_1 = -m\int_\Omega \psi\, p_1 + \int_\Omega \mathbb{h}'(\varphi^*)\,\psi\, p_1\,,\\
&\int_\Omega \nabla\psi\cdot\nabla p_2 + \int_\Omega F''(\varphi^*)\,\psi\, p_2 = \int_\Omega \eta\, p_2\,,\\
&\langle \partial_t\alpha, p_3\rangle + \int_\Omega \nabla\alpha\cdot\nabla p_3 - \chi_a\int_\Omega (\alpha\nabla\sigma^* + a^*\nabla\omega)\cdot\nabla p_3 = \int_\Omega (\alpha - 2a^*\alpha + h)p_3\,,\\
&\int_\Omega \partial_t\nu\, p_4 + \int_\Omega \nabla\nu\cdot\nabla p_4 - \chi_\varphi\int_\Omega \psi\, p_4 = \int_\Omega (c_\varphi\psi + c_n\nu + c_\sigma\omega)p_4\,,\\
&\int_\Omega \partial_t\omega\, p_5 + \int_\Omega \nabla\omega\cdot\nabla p_5 = \int_\Omega (-\omega + \chi_a\alpha - \alpha\sigma^* - a^*\omega)p_5\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \partial_t p_1, \psi\rangle - \int_\Omega \nabla p_2\cdot\nabla\psi - \int_\Omega \bigl((m - \mathbb{h}'(\varphi^*))p_1 + F''(\varphi^*)\,p_2 - (\chi_\varphi + c_\varphi)p_4\bigr)\psi\\
&= -b_1\int_\Omega (\varphi^* - \varphi_Q)\,\psi\,,\\
&-\int_\Omega \nabla p_1\cdot\nabla\eta = -\int_\Omega p_2\,\eta\,,\\
&\int_\Omega \partial_t p_3\,\alpha - \int_\Omega \nabla p_3\cdot\nabla\alpha + \chi_a\int_\Omega (\nabla\sigma^*\cdot\nabla p_3)\alpha\\
&\quad - \int_\Omega 2a^*\,p_3\,\alpha - \int_\Omega \bigl(-p_3 + (\sigma^* - \chi_a)p_5\bigr)\alpha = 0\,,\\
&\int_\Omega \partial_t p_4\,\nu - \int_\Omega \nabla p_4\cdot\nabla\nu + \chi_\varphi\int_\Omega \nabla p_1\cdot\nabla\nu + c_n\int_\Omega p_4\,\nu = 0\,,\\
&\langle \partial_t p_5, \omega\rangle - \int_\Omega \nabla p_5\cdot\nabla\omega + \chi_a\int_\Omega a^*\,\nabla p_3\cdot\nabla\omega - \int_\Omega a^*\,p_5\,\omega - \int_\Omega \bigl(-c_\sigma\,p_4 + p_5\bigr)\,\omega = 0\,.
\end{aligned}
$$

At this point, we take the sum of all these identities. Just a few terms do not cancel out. Namely, we find that

$$
\begin{aligned}
&\langle \partial_t\psi, p_1\rangle + \langle \partial_t p_1, \psi\rangle + \langle \partial_t\alpha, p_3\rangle + \langle \partial_t p_3, \alpha\rangle + \langle \partial_t\nu, p_4\rangle + \langle \partial_t p_4, \nu\rangle\\
&\quad + \langle \partial_t\omega, p_5\rangle + \langle \partial_t p_5, \omega\rangle = \int_\Omega h\,p_3 - b_1\int_\Omega (\varphi^* - \varphi_Q)\,\psi\,.
\end{aligned}
$$

Now, we integrate over $(0,T)$ and apply the well-known integration-by-parts formula for functions belonging to $H^1(0,T;V^*)\cap L^2(0,T;V)$. On account of the initial and final conditions (7.9) and (7.35), and recalling the choice of h, we then conclude that

$$
b_2\int_\Omega \bigl(\varphi^*(T) - \varphi_\Omega\bigr)\,\psi(T) = \int_Q (u - u^*)\,p_3 - b_1\int_Q (\varphi^* - \varphi_Q)\,\psi\,.
$$

Combining this identity with (7.21), we obtain (7.42), and the proof is complete. □

Remark 7.5. Since the optimal control problem under study is nonconvex, it may have many local minima. We claim that the variational inequality (7.42) has to be valid also for all such locally optimal controls. In this connection, recall that a control $u^* \in \mathcal{U}_{\rm ad}$ is termed *locally optimal in the sense of* $L^p(Q)$ for $1 \le p \le \infty$ if and only if there exists some $\varepsilon > 0$ such that

$$\mathcal{J}(u^*, \mathcal{S}_1(u^*)) \le \mathcal{J}(u, \mathcal{S}_1(u)) \quad \text{for all } u \in \mathcal{U}_{\rm ad} \text{ with } \|u - u^*\|_{L^p(Q)} \le \varepsilon.$$

Note also that every locally optimal control in the sense of $L^p(Q)$ for some $1 \le p < \infty$ is also locally optimal in the sense of $L^\infty(Q)$. Now, it is easily seen that any locally optimal control in the sense of $L^\infty(Q)$ satisfies the variational inequality (7.21). Hence, by the same argument as in the preceding proof, it must satisfy also (7.42).

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