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Cluster sizes in subcritical soft Boolean models

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Abstract

We consider the soft Boolean model, a model that interpolates between the Boolean model and long-range percolation, where vertices are given via a stationary Poisson point process. Each vertex carries an independent Pareto-distributed radius and each pair of vertices is assigned another independent Pareto weight with a potentially different tail exponent. Two vertices are now connected if they are within distance of the larger radius multiplied by the edge weight. We determine the tail behaviour of the Euclidean diameter and the number of points of a typical maximally connected component in a subcritical percolation phase. For this, we present a sharp criterion in terms of the tail exponents of the edge-weight and radius distributions that distinguish a regime where the tail behaviour is controlled only by the edge exponent from a regime in which both exponents are relevant. Our proofs rely on fine path-counting arguments identifying the precise order of decay of the probability that far-away vertices are connected.

1 Introduction

Sparse random-graph models that combine *heavy-tailed degree distributions* and *long-range effects* have attracted a lot of attention in recent years due to their ability to describe many real-world networks. For these random graphs vertices are embedded into the d-dimensional Euclidean space via a homogeneous *Poisson point process* with unit intensity and additionally each vertex is assigned an i.i.d. weight or *interaction radius*. Given the vertices and their radii, edges are then drawn independently such that edges to vertices with a large radius or to vertices nearby are more probable. The overall number of edges is controlled by an intensity parameter $\beta > 0$ that influences the connectivity in such a way that increasing β leads to more edges on average. Although short edges are more likely, the connection mechanism still allows edges between arbitrarily far-apart vertices to occur occasionally. An important question then is whether there exists a non-trivial critical intensity β_c such that the graph contains an infinite connected component with positive probability if $\beta > \beta_c$ but does not if $\beta < \beta_c$. The ergodicity entailed in the model ensures that the existence of an infinite connected component is a 0-1-event and any existing infinite connecting component is unique [5, 23]. We call $(0, \beta_c)$ the *subcritical phase* and (β_c, ∞) the *supercritical phase*. Once the two phases are identified one aims for more detailed results about the behaviour of the graph within each of those phases. This article is devoted to the analysis of the tail distribution of the Euclidean diameter as well as the number of points of the typical connected component in the *soft Boolean model* in the subcritical regime. This random graph was recently introduced in [16] as a model that interpolates between the *(Poisson) Boolean model* and the *long-range random connection model*, arguably the two most studied models in the continuum-percolation literature. For the diameter of the typical cluster in the soft Boolean model we identify the decay exponent for the whole parameter regime where a subcritical phase exists and find a sharp criterion when the long-range effects, or the radii, or the combination of both determine the decay. For the number of points in the typical cluster we derive complementary results, at least in parts of the parameter regime, that show a strong qualitative difference to the behaviour of the diameter.

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The idea of constructing graphs on random point clouds in the continuum and then connecting the vertices based on their distances goes back to [13], where a model is proposed in which any two Poisson points are connected by an edge whenever their distance is smaller than a threshold $\beta > 0$, which can be seen as a continuum equivalent to nearest-neighbour percolation on the lattice. This model is commonly referred to as the *Gilbert graph* or *random geometric graph* [40]. The connection rule can be also seen as assigning a ball of radius $\beta/2$ to each vertex and connecting any pair of vertices whose balls intersect. Later on, this idea was generalised to random radii in the physics literature [43, 29] and rigorously in [21]. Nowadays, this model is commonly known under the name *(Poisson) Boolean model*. It is a, by now, classical result that there always is a non-trivial phase transition in dimensions $d \geq 2$, and no supercritical phase in dimension $d = 1$, in both cases, whenever the radius distribution has finite d-th moment [21, 36]. This especially includes heavy-tailed radius distributions which are of particular interest to us as they lead to heavy-tailed (or scale-free) degree distributions. In 2008, it was shown that there exists another critical intensity $\widehat{\beta}_c \leq \beta_c$ such that the Euclidean diameter of the typical connected component has finite s-th moment if and only if the radius distribution has finite $(d+s)$-th moment whenever $\beta < \widehat{\beta}_c$, see [14]. The same holds for the number of points in the typical cluster [25]. For the planar Boolean model in $d = 2$, it was shown that $\beta_c = \widehat{\beta}_c$ for all radius distributions with finite d-th moment in [1]. Under additional moment conditions, this was generalised to all dimensions $d \geq 2$ via randomised algorithms and the OSSS inequality in [11]. Very recently, this result was extended to the class of Pareto distributed radii with some technical exceptions in [7]. We further discuss the potential equality of critical intensities below in Section 2.6.

Another generalisation of Gilbert's model is to replace the hard-threshold condition and instead connect any pair of vertices with a probability determined by a non-increasing function of the distance of the vertices. This was introduced in continuum percolation under the name *(Poisson) random connection model* in [41, 42] and further studied in [34, 35]. The generalised connection mechanism particularly includes connection functions that decay polynomially. The idea of connecting any pair of vertices with a probability given by a negative power of their spatial distance was first introduced in [44] as a lattice model that exhibits a non-trivial percolation phase transition even in dimension $d = 1$. More precisely, any pair of lattice points is connected independently with a probability decaying polynomially in the distance of the vertices with exponent $\delta > 1$, and there is a non-trivial phase transition $\beta_c \in (0, \infty)$ in dimension $d = 1$ if and only if $\delta \leq 2$, see [39, 10]. Moreover, for $\delta = 2$, the percolation function is discontinuous [2], which is rather atypical. This generalises to the continuum version. It was shown that there always exists a non-trivial phase transition in $d \geq 2$ as long as the connection function is integrable [37]. Further, in the whole subcritical regime the expected number of points in a typical connected component is finite [34], which is sometimes also referred to as a *sharp phase transition*. For a detailed overview of the random connection model, we refer the reader to [37]. Since we are particularly interested in the long-range effects coming from a connection function that decays polynomially in the distance, we will stick with the name of long-range percolation throughout the paper.

Besides the soft Boolean model, other models combining heavy-tailed degree distributions and long-range effects and their percolation behaviour have been studied recently in the literature. Examples include *scale-free percolation* [6] and its continuum version [9], also studied under the name *geometric inhomogeneous random graphs* [3, 4]. Here, the radii play the role of weights and two vertices are connected with a probability that decays polynomially in their distance divided by the product of both weights. This model also appears as a generalised weak local limit of *hyperbolic random graph models* [31], cf. [30]. Another model is the *age-dependent random connection model* [15] which appears as the weak local limit of a preferential-attachment type model, where the influence of a vertex is determined by its age. These models, together with the soft Boolean model, and the above mentioned classical continuum percolation models as well as the *ultra-small scale-free geometric network* [48] are contained in the class of *weight-dependent random connection models*, introduced in [17]. The existence of subcritical and supercritical percolation phases for these models are shown in [19, 18, 24]. In [16], the graph distances in supercritical phases are identified. In [38] it is shown that, for strong enough long-range effects, the percolation function on weight-dependent random connection models is continuous, a property that is believed to be generally true. Closely related to the weight-dependent random connection model, only using a different parametrisation, are the recently introduced *kernel-based spatial random graphs* [26]. In that article, the authors show, under some additional assumptions, that the tail of the cluster-size distribution of a finite component in a supercritical regime decays stretched exponentially.

Our article is organised as follows: In the next section, we give a description of the soft Boolean model and our main results. Our main result is Theorem 2.1 describing the asymptotics of the Euclidean diameter. Additionally, we derive results on the cardinality of the component of the origin, see Theorem 2.2, in comparison. In Section 2.4 we give a formal construction of the model. In Section 2.6 we further discuss our results, compare it with known results and elaborate on some open problems. We present the proofs of our results in Section 3. The main contribution here is the proof of the upper bound of the Euclidean diameter in Section 3.2. Due to the number of long edges in our model in addition to the inhomogeneity coming from the radii of the vertices, classical renormalisation arguments for the Boolean model or long-range percolation cannot be applied. Instead we rely on fine moment bounds applied to carefully chosen paths to derive our results.

2 Setting and main results

2.1 Description of the model

We now introduce the soft Boolean model [16] that combines both heavy-tailed degree distributions and long-range effects. The vertex set is given by a homogeneous Poisson point process X in $\mathbb{R}^d$ with intensity one. Note that fixing the intensity is no restriction for our results by rescaling. Next, we assign to each vertex $x \in X$ an independent radius R_x distributed according to a Pareto distribution with tail exponent $1/\gamma$ for some $\gamma \in (0,1)$, that is $\mathbb{P}(R_x > r) = 1 \wedge r^{-1/\gamma}$. Further, given X, we assign to each pair of distinct vertices $\{x, y\} \subset X$ an independent edge weight $W(x,y)$ that is also Pareto distributed with tail exponent δ for some $\delta > 1$. Then, given X and the collection of radii and edge weights, each pair of vertices $x, y \in X$ is connected by an edge if and only if

$$|x-y|^d \leq \beta W(x,y)\big(R_x \vee R_y\big), \tag{1}$$

where $\beta > 0$ is the edge intensity mentioned above. Here and throughout the article, $|\cdot|$ denotes the Euclidean norm. In words, two vertices are connected if the vertex with the smaller radius, say x, is contained in the ball of radius $\beta W(x,y) R_y$ around the vertex with larger radius y, see Figure 1. We denote the resulting graph by $\mathscr{G}^\beta$.

A variant of the model is given by replacing (1) by the connection rule $|x-y|^d \leq \beta W(x,y)\big(R_x + R_y\big)$, where now any two vertices are connected when their enlarged balls intersect. However, since $R_x \vee R_y \leq R_x + R_y \leq 2(R_x \vee R_y)$, there is no qualitative change of behaviour between the two model when it comes to the existence of global phase transitions. We therefore stick to the model (1). Also, our parameters are chosen in a way that allows us to introduce the soft Boolean model as a

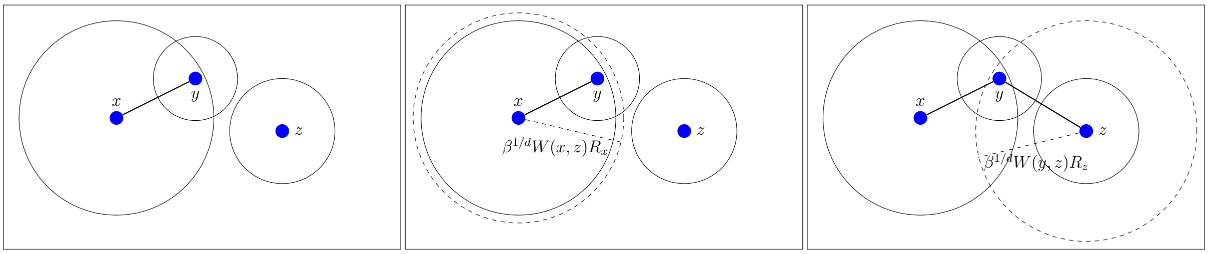

Figure 1: Example for the connection mechanism of the soft Boolean model in two dimensions. The solid lines represent the graph's edges. Left: the black circles represent the individual interaction balls governed by R_x, R_y, R_z which already gives rise to an edge (x, y). Middle: the enlarged dashed circle centred at x is due to the edge weight $W(x, y)$, which in this case does not lead to a new edge. Right: On the other hand, the enlarged dashed circle centred at z now includes the point y, leading to an edge (y, z).

special instance of the weight-dependent random connection model [17] in Section 2.4. It is easy to see that the soft Boolean model interpolates between long-range percolation and the heavy-tailed Boolean model: The parameter δ controls the strength of the geometric restrictions of the model such that the restrictions become stronger when δ increases. Put differently, the smaller δ, the less a vertex feels the geometry. The requirement $\delta > 1$ is needed to remain sparse in the sense that every vertex has finite expected degree. By sending $\delta \to \infty$, one obtains a variant of the classical Boolean model. Similarly, the parameter $\gamma \in (0, 1)$ measures the size of the vertex radii. The larger γ, the larger the radii and hence the more powerful the vertices are that one can find in the graph. Since the radii are heavy-tailed, $\mathscr{G}^\beta$ has a heavy-tailed degree distribution with power-law exponent $\tau = 1 + 1/\gamma$, see [33, 17]. Again the constraint $\gamma < 1$ is needed to maintain sparseness. By setting $\gamma = 0$, one obtains a continuum version of long-range percolation.

2.2 Results on the Euclidean diameter

We are interested in the size of the component of a typical vertex in the subcritical regime. Let $x \leftrightarrow y$ denote the event that two vertices are connected in $\mathscr{G}^\beta$ by a finite path of directly connected vertices. Let further (o, R_o) be a vertex placed at the origin, assigned with an independent radius and added to the graph according to the rule (1). Denote the underlying law after the vertex o has been added by $\mathbb{P}_o$. According to Palm theory, the vertex o plays the role of a typical vertex shifted to the origin [32] and we study its component in the following. Denote by

$$\mathscr{C}_\beta := \big\{x \in X : o \leftrightarrow x \text{ in } \mathscr{G}^\beta\big\}$$

the component of the origin. When there is no infinite component in the graph, the component of the origin must be finite. Vice versa, if there is an infinite component, the origin is part of it with a positive probability. We can therefore write the critical percolation intensity as

$$\beta_c := \beta_c(\gamma, 0, \delta) := \sup\big\{\beta > 0 \colon \mathbb{P}_o(\sharp\mathscr{C}_\beta = \infty) = 0\big\},$$

where the notation of $\beta_c(\gamma, 0, \delta)$ is made clear in Section 2.4 once we introduced the more general framework. It was shown in [19] that $\beta_c > 0$ if $\gamma < \delta/(\delta+1)$ but that $\beta_c = 0$ if $\gamma > \delta/(\delta+1)$. Hence, we focus on the first parameter regime in the following. We are interested in the Euclidean diameter of the component of the origin in a subcritical regime as it measures how far a typical component still spreads out. We define

$$\mathscr{M}_\beta := \sup\big\{|x|^d \colon x \in \mathscr{C}_\beta\big\} \tag{2}$$

and call it the Euclidean diameter of $\mathscr{C}_\beta$ with a slight abuse of notation. (Note that the actual diameter is, up to constants, given by $\mathscr{M}_\beta^{1/d}$ but we have chosen a parametrisation that allows our results to be independent of the dimension.) Let us state our main result about the tail probability of the Euclidean diameter of a typical cluster in the subcritical percolation regime.

Theorem 2.1 (Subcritical Euclidean diameter)**.** *Let $d \geq 1$, $\delta > 1$, and $0 < \gamma < \delta/(\delta+1)$. Then, there exists $\widetilde{\beta}_c := \widetilde{\beta}_c(\gamma, 0, \delta) > 0$ such that, for all $\beta < \widetilde{\beta}_c$, there exist constants $c_1, C_2, c_3, C_4, c_5, C_6 \in (0,\infty)$, depending on β, γ, and δ, such that for all $m > 1$,*

(i) if $\gamma < 1/(\delta+1)$, we have

$$c_1 m^{1-\delta} \leq \mathbb{P}_o(\mathscr{M}_\beta > m) \leq C_2 m^{1-\delta},$$

(ii) if $\gamma = 1/(\delta+1)$, we have

$$c_3 m^{1-\delta} \leq \mathbb{P}_o(\mathscr{M}_\beta > m) \leq C_4 \log(m) m^{1-\delta}, \text{ and}$$

(iii) if $1/(\delta+1) < \gamma < \delta/(\delta+1)$, we have

$$c_5 m^{1-\frac{\delta-1+\gamma}{\gamma\delta}} \leq \mathbb{P}_o(\mathscr{M}_\beta > m) \leq C_6 \log(m)^{1\vee d(\delta-1)} m^{1-\frac{\delta-1+\gamma}{\gamma\delta}}.$$

For the critical intensity $\widetilde{\beta}_c$, we deduce the following bounds from our proofs

$$\widetilde{\beta}_c \geq \begin{cases} \frac{1}{2^{d\delta+3}+1}\cdot\frac{\delta-1}{\omega_d\delta}\big(1-\gamma\frac{\delta+1}{\delta}\big), & \text{if } 1/2 \leq \gamma < \delta/(\delta+1),\\ \frac{\delta-1}{\omega_d\delta}\big(\frac{1}{1-2\gamma}+\frac{1}{1-\gamma}\big)^{-1} \vee \frac{1}{2^{d\delta+3}+1}\cdot\frac{\delta-1}{\omega_d\delta}\big(1-\gamma\frac{\delta+1}{\delta}\big), & \text{if } 0 < \gamma < 1/2.\end{cases}$$

Here, ω_d denotes the volume of the d-dimensional unit ball. We present the proof in Section 3 and discuss the result further in the context of various other critical values in Section 2.6 below. Theorem 2.1 shows how the soft Boolean model interpolates between the long-range and the power-law degree distribution effects. When $\gamma < 1/(\delta+1)$, the effect of the radii is too weak to qualitatively change the behaviour compared to long-range percolation. Notably, when sending $\delta \to \infty$, one observes the exponent $1 - 1/\gamma$ in the whole parameter regime $\gamma \in (0,1)$, matching the result in [14] translated to our setting. Theorem 2.1, Part (iii) then describes the situation when both effects matter. In fact the lower bounds remain true for all values of β and even for all $\delta > 1$ and $\gamma \in (0,1)$. This is due to the fact that the lower bounds are derived by considering a single long edge in Part *(i)* or by a path of length two in Part *(iii)*. If β is additionally sufficiently small, the matching upper bounds then show that there is no better strategy to connect the origin to a large distance. For the same reason the exponent in Part *(iii)* does not equal zero at the critical value $\gamma = \delta/(\delta+1)$ when the soft Boolean model loses its subcritical phase. This is in contrast with the two extremal models since these lose their subcritical phase at the same point they lose finite degrees (which happens at $\delta = 1$ for long-range percolation and $\gamma = 1$ for the Boolean model). Contrarily, the degrees in the soft Boolean model remain finite even in the $\gamma > \delta/(\delta+1)$ regime when a subcritical phase no longer exists. The exponent in Part *(iii)* indeed takes the value one for $\gamma = 1$ or $\delta = 1$ when also the considered soft model loses the finite-degree property. With reference to the established lower bounds, let us further mention that the phase transition at the critical point $\gamma = 1/(\delta+1)$ is of second order.

We can further replace the Pareto distributed radii by more general heavy-tailed radii. As long as the new radius distributions have the same tail exponent, our Results *(i)* and *(iii)* still remain true in the sense that the decay exponent of the Euclidean diameter does not change. However, this may come at the cost of additional slowly varying correction terms. Only in Result *(ii)*, dealing with the critical point of the phase transition, may the precise form of the radius distribution matter. The same also applies to our results on the number of points, presented in the following section.

2.3 Results on the number of points

Another important quantity to describe the subcritical typical component is the number of vertices it contains. Combined with the Euclidean diameter both quantities tell us how far the component spreads out and how many vertices can be reached. Let us denote the cardinality of the cluster of the origin by

$$\mathscr{N}_\beta := \sharp \mathscr{C}_\beta. \tag{3}$$

While the Euclidean diameter is driven by the longest edges, the cardinality is driven by the highest degrees in the cluster. Indeed, due to subcriticality, there are no 'long' paths (in the sense of the graph distance) in the component and hence, in order to reach many vertices with short paths only, there must be at least a few high-degree vertices in the cluster. There is however one direct connection between $\mathscr{N}_\beta$ and $\mathscr{M}_\beta$. Since the number of Poisson points in a ball of radius $m^{1/d}$ concentrate exponentially fast around its volume, $\mathscr{N}_\beta > m$, for large m, can only occur with a large probability if also $\mathscr{M}_\beta > m$ occurs. For the standard Boolean model it is shown that indeed both quantities are of the same order [25, 14]. This is due to the fact that the tail probability for both $\mathscr{M}_\beta$ and $\mathscr{N}_\beta$ are driven by the vertex connected to the origin with largest radius since the length of edges and degrees are both linked directly to the radii in that model. The situation is different in long-range percolation. Here, for β small enough, the Euclidean diameter is heavy tailed with tail exponent $\delta - 1$ whereas the tail of the number of vertices decays exponentially fast by a branching-process argument [34, 37]. This is due to the fact that the long-range effects connect vertices at large distances, yielding a heavy-tailed Euclidean diameter, but the overall degree distribution is not strong enough to guarantee a heavy-tailed number of vertices in the component of the origin.

In the soft Boolean model, now both effects mix. The required large degrees are however still mostly driven by large radii. Indeed, our parametrisation is such that the degree distribution in the soft Boolean model and in the classical Boolean model have the same power-law exponent $1 + 1/\gamma$, see [33, 17]. Therefore, the degree of 'large-ball' vertices is in both models of the same order of magnitude. The additional neighbours of such a vertex in the soft model may however be at a really large distance due to the long-range effects. These far away neighbours significantly contribute to the Euclidean diameter but they do not qualitatively contribute to the number of vertices contained in the cluster. Hence, we expect the tail distribution for $\mathscr{N}_\beta$ in the soft Boolean model to be of the same order as in the classical model. We are able to derive this in the finite-variance regime of the degree distribution as stated in the following theorem.

Theorem 2.2 (Subcritical cardinality). *Let $d \geq 1$, $\delta > 1$, and $0 < \gamma < 1$.*

(i) If $\gamma < 1/2$, then there exists $\widetilde{\beta}_c := \widetilde{\beta}_c(\gamma, 0, \delta) > 0$ such that, for all $0 < \beta < \widetilde{\beta}_c$, there exist constants $c, C \in (0, \infty)$ such that, for all $m > 1$,

$$cm^{1-1/\gamma} \leq \mathbb{P}_o(\mathscr{N}_\beta > m) \leq Cm^{1-1/\gamma}.$$

(ii) If $\gamma > 1/2$, then $\mathbb{E}\mathscr{N}_\beta = \infty$ for all $\beta > 0$.

We comment on the critical density and the conjectured behaviour in Section 2.6 and give the proof in Section 3.3. Theorem 2.2 highlights the qualitative difference between the Euclidean diameter and the cardinality of the cluster of the origin in the soft Boolean model. First of all, for $\gamma < 1/2$, the tail behaviour for the number of points depends on γ only, whereas the Euclidean diameter always depends on δ. Also, for $\gamma < 1/2$ and β small enough, we have $\mathbb{E}\mathscr{N}_\beta < \infty$ by Part *(i)* while for

$\delta \in (1,2)$ and all $\gamma < 1/2$ or for $\delta > 2$ and $(\delta-1)/(2\delta-1) < \gamma < 1/2$, we have $\mathbb{E}\mathscr{M}_\beta = \infty$ by Theorem 2.1.

Let us finally note that the lower bound of Part *(i)* in Theorem 2.2 is valid for all $\gamma \in (0,1)$ by a direct comparison of the soft version with the classical model. Additionally, in order to prove the upper bound of the theorem, we couple the soft Boolean model with a version of continuous scale-free percolation [9, 6] in which the minimum $R_x \vee R_y$ in (1) is replaced by the product R_xR_y. As a result, we immediately observe the same bounds for this model as well. Interestingly, this model only has a subcritical component for $\gamma \le 1/2$, see [9, 33]. Hence, in that case the result covers the whole parameter regime for which a subcritical phase exists.

Theorem 2.3 (Subcritical cardinality in scale-free percolation)**.** *Let $d \ge 1$, $\delta > 1$, and $\gamma < 1/2$. Consider the graph constructed via connection rule* (1) *but with $R_x \vee R_y$ replaced by R_xR_y and denote the cardinality of the component of the origin by $\mathscr{N}_{\beta,\gamma,\gamma,\delta}$. Then, there exists $\widetilde{\beta}_c := \widetilde{\beta}_c(\gamma,\gamma,\delta)$ such that, for all $0 < \beta < \widetilde{\beta}_c$, there exist constants $c, C \in (0,\infty)$ such that, for all $m > 1$,*

$$cm^{1-1/\gamma} \le \mathbb{P}_o(\mathscr{N}_{\beta,\gamma,\gamma,\delta} \ge m) \le Cm^{1-1/\gamma}.$$

2.4 Formal construction

We formally introduce the soft Boolean model as a special instance of the *weight-dependent random connection model* [17] as mentioned in Section 1. Recall that X denotes a unit-intensity Poisson point process on $\mathbb{R}^d$. We can write $X = (X_i : i \in \mathbb{N})$, c.f. [32] and call the elements of X the *vertex locations*. Let $\mathcal{U} = (U_i : i \in \mathbb{N})$ be a family of independent random variables distributed uniformly on $(0,1)$ that we call the *vertex marks*. Let us further write

$$\mathbf{X} := \big(\mathbf{X}_i = (X_i, U_i) \in X \times \mathcal{U} : i \in \mathbb{N}\big)$$

for the set of vertices and note that $\mathbf{X}$ is a unit-intensity Poisson point process on $\mathbb{R}^d \times (0,1)$. Finally, let $\mathcal{V} = (V_{i,j} : i < j \in \mathbb{N})$ be another independent sequence of uniformly-distributed random variables on $(0,1)$ that we call the *edge marks*. We define

$$\xi := \big((\{\mathbf{X}_i, \mathbf{X}_j\}, V_{i,j}) \in \mathbf{X}^{[2]} \times (0,1), i < j \in \mathbb{N}\big), \tag{4}$$

where $\mathbf{X}^{[2]}$ denotes the set of all subsets of $\mathbf{X}$ of size two. We call ξ an *independent edge marking* in accordance with the construction in [22]. Note that ξ is an ergodic point process on $(\mathbb{R}^d \times (0,1))^2 \times (0,1)$. Further note that the law of ξ does not depend on the ordering of the points and that $\mathbf{X}$ as well as X and $\mathcal{U}$ can be recovered from it.

Now, fix $\beta > 0$, $\gamma \in [0,1)$, $\alpha \in [0, 2-\gamma)$ and $\delta > 1$. We define the *interpolation kernel*

$$g_{\gamma,\alpha}(s,t) := (s \wedge t)^\gamma (s \vee t)^\alpha, \quad s,t \in (0,1), \tag{5}$$

as introduced in [18] and the *profile function*

$$\rho(x) := \rho_\delta(x) = 1 \wedge x^{-\delta}, \quad x \in (0,\infty).$$

The undirected graph $\mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi)$ is then defined through its vertex set $\mathbf{X}$ and edge set

$$E\big(\mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi)\big) = \Big\{\{\mathbf{X}_i, \mathbf{X}_j\} : V_{i,j} \le \rho\big(\beta^{-1} g_{\gamma,\alpha}(U_i,U_j)|X_i - X_j|^d\big), i < j \in \mathbb{N}\Big\}. \tag{6}$$

Note that for $\alpha = 0$, the soft Boolean model $\mathscr{G}^\beta$ defined above and the graph $\mathcal{G}^{\beta,\gamma,0,\delta}(\xi)$ have the same law as

$$V_{i,j} \le \big(\beta^{-1}(U_i \wedge U_j)^\gamma |X_i - X_j|^d\big)^{-\delta} \qquad \Longleftrightarrow \qquad |X_i - X_j|^d \le \beta V_{i,j}^{-1/\delta}\big(U_j^{-\gamma} \vee U_j^{-\gamma}\big).$$

We therefore work from now on explicitly on the probability space on which the edge marking ξ lives and denote the underlying probability measure by $\mathbb{P}$ and the corresponding expectation by $\mathbb{E}$. To shorten notation, we abbreviate $\mathcal{G}^\beta := \mathcal{G}^{\beta,\gamma,0,\delta}(\xi)$ whenever we work exclusively on the soft Boolean model and no confusion of parameters is possible.

The same parametrisation can also be used to consider models without long-range effects, particularly the random geometric graph or the standard Boolean model. To this end, only the function ρ in (6) has to be replaced by the indicator $\mathbb{1}_{[0,1]}$. We identify this choice of profile function with $\delta = \infty$ as it arrives as the $\delta \to \infty$ limit of our previous choice. In this notion, the random geometric graph with edge length β is given by $\mathcal{G}^{\beta,0,0,\infty}(\xi)$, and the version of the Boolean model where connections are formed when the ball of the stronger vertex contains the weaker vertex is given by $\mathcal{G}^{\beta,\gamma,0,\infty}(\xi)$.

Next, in order to formulate our main Theorem 2.1, we have to add the origin to the graph. To do so formally, let us denote by $\mathbf{x}_0 = (x_0, u_0)$ a vertex placed at x_0 and with a given vertex mark $u_0 \in (0,1)$. Let $\mathbf{X}_{\mathbf{x}_0} = \mathbf{X} \cup \{(x_0, u_0)\}$ denote the vertex set with the additional vertex added. Note that almost surely no vertex in X has been placed at $\mathbf{x}_0$ before. Let us further extend the edge marks by a sequence of independent uniform random variables $(V_{0,j} : j \in \mathbb{N})$ and denote the resulting sequence by $\mathcal{V}_{\mathbf{x}_0}$. Finally, we define the edge marking containing the extra vertex

$$\xi_{\mathbf{x}_0} := \xi \cup \big\{\big(\{\mathbf{x}_0, \mathbf{X}_i\}, V_{0,i}\big) : i \in \mathbb{N}\big\}.$$

The graph containing the extra vertex is then $\mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi_{\mathbf{x}_0})$. Again, if the graph we are considering is the one corresponding to the soft Boolean model, we abbreviate and refer to it as $\mathcal{G}^\beta_{\mathbf{x}_0}$. We denote the probability measure and expectation governing $\xi_{\mathbf{x}_0}$ by $\mathbb{P}_{\mathbf{x}_0}$ or $\mathbb{P}_{(x_0,u_0)}$ and $\mathbb{E}_{\mathbf{x}_0}$ respectively. If the vertex mark of $\mathbf{x}_0$ is not fixed but uniformly distributed independently from everything else, we also denote the vertex by $\mathbf{x}_0 = (x_0, U_0)$ and define $\mathbb{P}_{x_0} := \mathbb{P}_{(x_0,u)}\mathrm{d}u$. Then, for $(x_0, U_0) = (o, U_o)$ the origin, $\mathbb{P}_o$ represents the probability measure that describes the Palm version of the graph, which can be seen as the graph shifted such that a typical (i.e., uniformly chosen) vertex is located at the origin [32, Chapter 9]. Note that this is consistent with the notation in Section 2 whenever the considered graph coincides with the soft Boolean model. If this is the case, we may also index our objects as before by o.

In the same way, finitely-many given vertices $\mathbf{y}_1 = (y_1, t_1), \mathbf{y}_2 = (y_2, t_2), \dots$ can be added to the graph using negative indices and writing $y_i = x_{-i}$ and $t_i = u_{-i}$. We then write $\xi_{\mathbf{y}_1,\mathbf{y}_2,\dots}$, and $\mathbb{P}_{\mathbf{y}_1,\mathbf{y}_2,\dots}$, etc.

It is important to note that in the formal construction in the previous section the ordering of the Poisson points was important. However, the precise ordering does not change any distributional properties. Therefore, we drop this notation from here on and denote given vertices by $\mathbf{x} = (x, u_x)$ or $\mathbf{y} = (y, u_y)$ but stick with the notation $\mathbf{o} = (o, u_o)$ for the origin. We further write $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ for any sequence of given vertices without referring to the ordering above. For two given vertices, we denote by $\{\mathbf{x} \sim \mathbf{y} \text{ in } \mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi_{\mathbf{x},\mathbf{y}})\}$ the event that $\mathbf{x}$ and $\mathbf{y}$ are connected by an edge in the graph $\mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi_{\mathbf{x},\mathbf{y}})$. If the graph is clear from the context, we simply write $\{\mathbf{x} \sim \mathbf{y}\}$. Similarly, we denote by $\{\mathbf{x} \leftrightarrow \mathbf{y} \text{ in } \mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi_{\mathbf{x},\mathbf{y}})\}$ (resp. $\{\mathbf{x} \leftrightarrow \mathbf{y}\}$) the event that $\mathbf{x}$ and $\mathbf{y}$ are connected by a finite path in the graph. We denote by $\mathscr{C}_{\beta,\gamma,\alpha,\delta}$ the component of the origin in $\mathcal{G}^{\beta,\gamma,\alpha,\delta}_{\mathbf{o}}$, by $\mathscr{M}_{\beta,\gamma,\alpha,\delta}$ its Euclidean diameter (to the power d as above) and by $\mathscr{N}_{\beta,\gamma,\alpha,\delta}$ the cardinality of the component. In accordance with Section 2, we denote by $\mathscr{C}_\beta$, $\mathscr{M}_\beta$, and $\mathscr{N}_\beta$ the component of the origin, its Euclidean diameter, and the cardinality of the component of the origin in the soft Boolean model $\mathcal{G}^\beta_{\mathbf{o}}$.

2.5 Strategy of proof

We briefly explain the strategy of the proof in this section. We restrict ourselves to the proof of Theorem 2.1 and put special emphasis on the proof of Part *(iii)* as it is the main contribution of this paper. In order to prove the lower bounds one has to consider a potentially optimal connection strategy to derive the desired Euclidean diameter and calculate its tail behaviour. As in long-range percolation all edges are independent, the order of magnitude of the Euclidean diameter is driven by the longest edge incident to the origin. By our construction as a weight-dependent random connection model (6), there is a direct coupling between long-range percolation $\mathscr{G}^{\beta,0,0,\delta}(\xi)$ and the soft Boolean model $\mathscr{G}^{\beta,\gamma,0,\delta}(\xi)$. Hence, the long-range percolation lower bound immediately transfers to the soft Boolean model giving the lower bound in Part *(i)*. For the classical Boolean model results of Gouéré imply that, in order to obtain a Euclidean diameter of order m, one has to find a vertex at distance at most $m^{1/d}$ of the origin with radius of order $m^{1/d}$ or equivalently with a mark of order $m^{-1/\gamma}$. Put differently, the Euclidean diameter of the component of the origin is driven by the largest radius of the neighbours of the origin. To derive the lower bound in Part *(iii)* we also search for a powerful vertex in the vicinity of the origin. However, due to the additional long-range effect it suffices if this vertex has radius of order $m^{(\delta-1)/(d\delta)}$, respectively a mark of order $m^{-(\delta-1)/(\gamma\delta)}$. The probability of existence of such a vertex then gives the lower bound.

To derive the corresponding upper bound, one has to show that the combination of all possibilities to build paths connecting the origin to distance $m^{1/d}$ do not lead to a larger probability of achieving the desired Euclidean diameter. We carry this out in four steps which we exemplarily describe for Part *(iii)*.

Step (A) In order to get decent bounds on the probability that certain paths exist, we introduce the concept of the skeleton of a path. The skeleton vertices can be seen as the key vertices of the path and allow us to decompose each path in a set of skeleton vertices and a set of connectors building the subpaths between two skeleton vertices. We can then reduce the probability of existence of a path to the probability that its skeleton vertices already build a paths by themselves times an exponential term in the path length depending on β that vanishes for small enough β. This is carried out in Section 3.2.1.

Step (B) We use 2.5 to deduce that in a subcritical regime the probability that a vertex with radius of order $m^{(\delta-1)/\delta}$ is present is of the same order as the lower bound when this vertex was required to be a direct neighbour, cf. Lemma 3.7.

Step (C) On the complementary event of 2.5 we deduce that the probability that a vertex in the component of the origin that is located within distance $m^{1/d}$ has a neighbour at distance $2m^{1/d}$, is of the same order than the lower bound, cf. Lemma 3.8.

Step (D) On the complementary events of the 2.5 and 2.5 a path that leads to a diameter of order m cannot use too strong vertices and has to stop within distance $2m^{1/d}$ of the origin. We use a calculation comparable to 2.5 to obtain that there is no long edge contained in the component of the origin restricted to this area with a probability of the desired order. Hence, the paths cannot use long edges and have to visit many vertices and we show that the probability of the origin starting a path that visits many vertices is of the desired order, cf. Lemma 3.9 and Proposition 3.10.

2.6 Further discussion

In this section, we discuss some further details which particularly concern the logarithmic term in the upper bound in Theorem 2.1 and the critical intensities β_c and $\widetilde{\beta}_c$.

2.6.1 Critical intensities

Let us consider the standard definition of the critical percolation intensity for $\mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi_{\mathbf{o}})$. That is,

$$\begin{aligned}\beta_c(\gamma,\alpha,\delta) &:= \sup\big\{\beta>0\colon \mathbb{P}_o(\sharp\mathscr{C}_{\beta,\gamma,\alpha,\delta}=\infty)=0\big\}\\ &= \sup\big\{\beta>0\colon \lim_{n\to\infty}\mathbb{P}_o\big(o \text{ starts a path of length } n \text{ in } \mathcal{G}^{\beta,\gamma,\alpha,\delta}\big)=0\big\},\end{aligned} \tag{7}$$

where the function $\theta(\beta):=\mathbb{P}_o(\sharp\mathscr{C}_{\beta,\gamma,\alpha,\delta}=\infty)$ is also called the percolation function. To prove the existence of a subcritical percolation phase in the standard Boolean model $\mathcal{G}^{\beta,\gamma,0,\infty}$ and to obtain the tails of the Euclidean diameter in this phase, Gouéré introduced in [14] a new critical intensity (translated to our setting) as

$$\begin{aligned}&\widehat{\beta}_c(\gamma,\alpha,\delta)\\ &\quad:= \sup\big\{\beta>0\colon \liminf_{m\to\infty}\mathbb{P}_o\big(\exists \mathbf{x},\mathbf{y}\in\mathbf{X}: |x|^d<m, |y|^d>2^dm,\ \mathbf{x}\leftrightarrow\mathbf{y} \text{ in } \mathcal{G}^{\beta,\gamma,\alpha,\delta}\big)=0\big\}.\end{aligned}$$

Generally, $\widehat{\beta}_c(\gamma,\alpha,\delta)$ can be seen as the critical *annulus-crossing* intensity. For the Boolean model, $\widehat{\beta}_c(\gamma,0,\infty)$ can further be interpreted as the critical intensity needed for the probability to be bounded away from zero of the event that the diameter of the origin exceeds 2^dm when setting the radius of the origin to $m^{1/d}$. Since $\widehat{\beta}_c(\gamma,\alpha,\delta)\le\beta_c(\gamma,\alpha,\delta)$, the positivity of the first critical intensity implies the existence of a subcritical phase. One advantage of considering $\widehat{\beta}_c(\gamma,0,\infty)$ is the possibility to apply a multi-scale scheme to determine bounds for the Euclidean diameter's tail that roughly works as follows. Let us denote the defining event of $\widehat{\beta}_c(\gamma,0,\infty)$ by $G'(m^{1/d})$. Consider the event $G'(Cm^{1/d})$ for some suitable constant $C>1$. Then either one vertex located in the centred ball $\mathcal{B}(2Cm^{1/d})$ is incident to an edge longer than $m^{1/d}$, or the path connecting a vertex located in the ball $\mathcal{B}(Cm^{1/d})$ to some vertex located in $\mathcal{B}(2Cm^{1/d})^c$ can only use shorter edges. However, on the letter event, up to shifts, the event $G'(m^{1/d})$ occurs twice independently and hence using shift invariance

$$\begin{aligned}&\mathbb{P}\big(G'(Cm^{1/d})\big)\\ &\quad\le c\mathbb{P}\big(G'(m^{1/d})\big)^2+\mathbb{P}\big(\exists\mathbf{x}\colon |x|^d<(2C)^dm, \mathbf{x} \text{ is incident to an edge longer than } m^{1/d}\big),\end{aligned} \tag{8}$$

for some constant $c>1$. Hence, if the second probability on the right-hand side converges to zero, $\mathbb{P}(G'(m^{1/d}))$ converges to zero as well. This is due to the fact that on the initial scale $m=1$ the probability $\mathbb{P}(G'(C))$ can be made arbitrarily small by choosing β sufficiently small. The result then follows by iterating (8). To derive bounds for the tail of the Euclidean diameter, one observes that if $\widehat{\beta}_c(\gamma,\alpha,\delta)>0$, then (8) implies for each $\beta<\widehat{\beta}_c(\gamma,\alpha,\delta)$ that $\mathbb{P}(G'(m^{1/d}))$ converges to zero and that the rate of convergence is determined by the probability of existence of long edges. In the classical Boolean model, the probability of the occurrence of such long edges can be expressed in terms of the moments of the radius distribution R (which is Pareto with exponent d/γ in our case, but may also be considered to be more general) and one derives

$$\mathbb{P}\big(\exists\mathbf{x}\colon |x|^d<(2C)^dm, \mathbf{x} \text{ is incident to an edge } \ge m^{1/d} \text{ in } \mathcal{G}^{\beta,\gamma,0,\infty}\big)\le c\mathbb{E}[R^d\mathbb{1}\{R^d>m\}].$$

Calculating the right-hand side for our parametrisation then yields

$$\mathbb{P}_o(\mathscr{M}_{\beta,\gamma,0,\infty} > m) \le C m^{1-1/\gamma}.$$

Results for the Boolean model in [1, 11, 7] indicate further that $\widehat{\beta}_c(\gamma, 0, \infty) = \beta_c(\gamma, 0, \infty)$. In fact, it is shown in [1] that this equality holds for all radius distributions in $d = 2$ and in [7] that $\widehat{\beta}_c(\gamma, 0, \infty) = \beta_c(\gamma, 0, \infty)$ in all dimensions $d \ge 2$ for all but at most countably many γ. Hence, in these situations the above upper bound for the Euclidean diameter's tail distributions holds for all $\beta < \beta_c(\gamma, 0, \infty)$.

The above renormalisation argument (8) shows that we can effectively think of attaching a growing radius of length $m^{1/d}$ to the origin in the Boolean model at no extra cost. However, as we can see from our proofs below, this argument is no longer possible in the soft Boolean model. The reason is that there are simply too many long edges in the graph. Assigning a radius of length $m^{1/d}$ to the origin combined with the long-range effects dramatically increases (by order m) the number of long edges contained in the component of the origin. We are therefore restricted to work with the original radius of the origin and its original component at all times when dealing with similar events as outlined above in 2.5 and 2.5. More precisely, to quantify the number of long edges, which is essential for the multi-scale argument, the authors in [24] make use of the coefficient δ_{eff}, recently introduced in [18], to measure the effect of the weights of the vertices on the appearance of long edges and therefore quantify the number of long edges crossing annuli of radii $m^{1/d}$ and $2m^{1/d}$ in weight-dependent random-connection-type models. In [24], the authors show that in general $\widehat{\beta}_c(\gamma, \alpha, \delta) > 0$ if $\delta_{\text{eff}} > 2$, but $\widehat{\beta}_c(\gamma, \alpha, \delta) = 0$ if $\delta_{\text{eff}} < 2$. Unfortunately, we have $\delta_{\text{eff}} < 2$ whenever $\delta < 2$, or $\gamma > (\delta - 1)/\delta$. Also, in the $\delta_{\text{eff}} > 2$ regime, the observed upper bound for the exponent in [24] does not match the bound established in this article, in fact they differ by a summand of size $1/\delta$. Summarising, for the soft Boolean model the situation of critical intensities is less clear.

In order to go one step further, let us consider another critical intensity. To prove the existence of a subcritical percolation phase for the age-dependent random connection model $\mathcal{G}^{\beta,\gamma,1-\gamma,\delta}(\xi_{\mathbf{o}})$ of [15] and models dominated by it, the authors in [19] consider

$$\beta_1(\gamma, \alpha, \delta) := \sup\Big\{\beta > 0\colon \sum_{n \in \mathbb{N}} \mathbb{P}_o\Big(\substack{\mathbf{o} \text{ starts a shortcut-free path in } \mathcal{G}^{\beta,\gamma,\alpha,\delta} \text{ of length } n \\ \text{whose end vertex has the smallest mark in the path}}\Big) < \infty\Big\}.$$

Here, a path $P = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ is called *shortcut-free* if it contains no shorter subpath $Q \subset P$ also connecting $\mathbf{x}_0$ and $\mathbf{x}_n$. The restriction to shortcut-free paths is possible as the existence of an infinite shortcut-free path is equivalent to the existence of an infinite path since all degrees are finite. However note that in general the existence of a path of length n does not necessarily imply the existence of a shortcut-free paths of the same length. The idea behind the definition is to make use of the fact that the vertices with highest degree are those with smallest marks which can be seen as the skeleton vertices from 2.5. To build an infinite path, one may want to use vertices with smaller and smaller marks to find sufficiently many vertices that have not been visited yet to continue the path. However, such a path contains infinitely many finite subpath ending in the vertex with smallest vertex mark. For $\beta < \beta_1(\gamma, \alpha, \delta)$ only finitely many such subpaths exist by the Borel–Cantelli Lemma. Hence, each potentially infinite path must have marks that are bounded from zero. The latter ultimately implies

$$\inf\{u_x\colon \mathbf{x} \in \mathscr{C}_{\beta,\gamma,\alpha,\delta}\} > 0,$$

which however is equivalent to $\sharp\mathscr{C}_{\beta,\gamma,\alpha,\delta} < \infty$. For the age-dependent random connection model it is then shown in [19] that $\beta_1(\gamma, 1-\gamma, \delta) > 0$ if $\gamma < \delta/(\delta+1)$ and $\beta_1(\gamma, 1-\gamma, \delta) = 0$ if $\gamma > \delta/(\delta+1)$ and the same holds for the soft Boolean model. From our proofs in Lemma 3.8 and 3.12 below it is easy to see that

$$\beta_1(\gamma,0,\delta) \geq \begin{cases} \frac{(\delta-1)(\delta-\gamma(\delta+1))}{(2^{d\delta+3}+1)\omega_d\delta^2}, & \text{if } 1/2 \leq \gamma < \delta/(\delta+1), \\ \frac{\delta-1}{\omega_d\delta}\big(\frac{1}{1-2\gamma}+\frac{1}{1-\gamma}\big)^{-1} \vee \frac{(\delta-1)(\delta-\gamma(\delta+1))}{(2^{d\delta+3}+1)\omega_d\delta^2}, & \text{if } 1/(\delta+1) < \gamma < 1/2, \end{cases}$$

matching the results of [19] when their proof is specialised to the soft Boolean model.

On a technical level, the necessity of β being small enters our proof in the 2.5 to 2.5 when potentially arbitrarily long paths have to be considered. Based on the nature of our proof, one could suspect that the statement of our main theorem holds for all $\beta < \beta_1(\gamma,0,\delta)$. Particularly in 2.5 only paths ending in its most powerful vertex are considered. Unfortunately, especially bounding (21) requires slightly more than bounding the number of paths occurring in the definition of $\beta_1(\gamma,0,\delta)$. As (21) is the result of a moment bound, we believe that our results hold at least up to $\beta_1(\gamma,0,\delta)$.

Conjecture 2.4. *Let $\widetilde{\beta}_c$ be the critical intensity of Theorem 2.1, then $\widetilde{\beta}_c \geq \beta_1(\gamma,0,\delta)$.*

Although, we have already commented on the fact that the decay of the probability of existence of shortcut-free paths of length n generally gives no information about the decay of the probability of a general path, in order to deduce the probability of $\{\mathscr{M}_\beta > m\}$ one can always reduce to shortcut-free paths. Therefore, it seems to us that there is no particular reason indicating that increasing β within the subcritical regime would change the tail of the distribution of $\mathscr{M}_\beta$, so that $\widetilde{\beta}_c = \beta_c(\gamma,0,\delta)$. An immediate consequence would also be $\beta_1(\gamma,0,\delta) = \beta_c(\gamma,0,\delta)$. We therefore conjecture:

Conjecture 2.5. *Let $\widetilde{\beta}_c$ be the critical intensity of Theorem 2.1, then $\widetilde{\beta}_c = \beta_c(\gamma,0,\delta) = \beta_1(\gamma,0,\delta)$.*

2.6.2 The logarithmic term in the upper bound.

Heuristically, we see in our proof that, in probability, the most promising strategy to achieve a large Euclidean diameter should be to connect the origin to a powerful vertex. This strategy then yields our polynomial tail without the logarithmic correction. The logarithmic term in our upper bound in Theorem 2.1 is however a consequence of our last proof step in which we consider the cases that either certain long edges are present or the path connecting the origin to distance $m^{1/d}$ must contain many steps. This distinction is technically necessary to control the sheer number of connection possibilities when none of the extremely powerful vertices can be used. However, on the desired order $m^{1-(\delta-1+\gamma)/\gamma\delta}$ only edges longer than $m^{1/d}$ are absent and as a result the path length and the radius of the ball under consideration live on the same scale which is not sufficient to bound the existence of a long path appropriately with our method. We therefore work on the event that also slightly shorter edges are absent, resulting in the logarithmic term. We however believe that this error term is a result of our method and should not appear in the case of Pareto-distributed radii. In fact it seems reasonable to conjecture the following.

Conjecture 2.6. *Let $\widetilde{\beta}_c$ be the critical intensity of Theorem 2.1, then, for all $\beta < \widetilde{\beta}_c$, $\delta > 1$, and $1/(\delta+1) < \gamma < \delta/(\delta+1)$, there exists $C > 0$ such that, for all $m > 1$, we have that*

$$\mathbb{P}_o(\mathscr{M}_\beta > m) \leq C m^{1-\frac{\delta-1+\gamma}{\gamma\delta}}.$$

The other reason why we believe in Conjecture 2.6 is that we can show that the error term indeed does not occur if $\gamma < 1/2$. This is due to that fact that, in this case, the expected number of paths decays exponentially, cf. Lemma 3.13. Hence, we can use a truncated moment bound on the number of paths connecting $\mathbf{o}$ with some vertex in $\mathcal{B}(m^{1/d})^c$ in the same way we prove Proposition 3.1 Part *(ii)*, which is the upper bound of Theorem 2.1 Part *(i)*. The truncation here refers to the vertex marks, where we use an error event to ensure that no atypically strong vertex lies on the path. We formulate this finding in the following proposition which is an immediate consequence of Lemma 3.12 at the end of Section 3.2.

Proposition 2.7. *Let* $\delta > 1$, $0 < \gamma < 1/2$ *and* $0 < \beta < \frac{\delta-1}{\omega_d \delta}\big(\frac{1}{1-2\gamma} + \frac{1}{1-\gamma}\big)^{-1}$. *Then, there exists* $C > 0$ *such that, for all* $m > 1$,

$$\mathbb{P}_o(\mathscr{M}_\beta > m) \le Cm^{1-\delta \wedge \frac{\delta-1+\gamma}{\gamma\delta}}.$$

2.6.3 Cardinality of the cluster of the origin

In Section 3.3 we derive results for the cardinality of the component for the $\gamma < 1/2$ regime and yet another threshold for β. The new threshold is on the one hand a result of the coupling with scale-free percolation coinciding with $\alpha = \gamma$ in (5). On the other hand, we use a coupling of the component of the origin with a multi-type branching process that requires a finite second moment of the degree distribution and a small enough β to control the exponential growth of integration constants associated with the second moment. While these branching-process arguments work very well for non-spatial random graphs, they have the disadvantage of not seeing the spatial clustering. Hence, we cannot expect to get precise results in a spatial setting where the effect of clustering is highly relevant. The relevance of clustering in the soft Boolean model can be seen for example in the fact that a subcritical phase still exists for parameter regimes with degree distributions with infinite variance ($\gamma > 1/2$) while non-spatial models are always robust in the infinite-variance regime [47]. For scale-free percolation the effect of clustering seems to be less relevant as many changes of behaviour happen again at $\gamma = 1/2$ when the second moment of the degree distribution becomes infinite. Still we do not see any reason why the established tail in Theorem 2.3 should not be valid in the whole subcritical regime.

Conjecture 2.8. *Consider* $\beta_c(\gamma, \alpha, \delta)$ *as defined in* (7).

(i) *Consider the soft Boolean model* $\mathscr{G}^{\beta,\gamma,0,\delta}$ *for* $\delta > 1$, $\gamma < \delta/(\delta+1)$, *and* $\beta < \beta_c(\gamma, 0, \delta)$. *Then, there exists a constant* $C \in (0, \infty)$ *such that, for all* $m > 1$,

$$\mathbb{P}_o(\mathscr{N}_{\beta,\gamma,0,\delta} \ge m) \le Cm^{1-1/\gamma}.$$

(ii) *Consider scale-free percolation* $\mathscr{G}^{\beta,\gamma,\gamma,\delta}$ *for* $\delta > 1$, $\gamma < 1/2$, *and* $\beta < \beta_c(\gamma, \gamma, \delta)$. *Then, there exists a constant* $C \in (0, \infty)$ *such that, for all* $m > 1$,

$$\mathbb{P}_o(\mathscr{N}_{\beta,\gamma,\gamma,\delta} \ge m) \le Cm^{1-1/\gamma}.$$

Let us further comment on Part *(i)* of the conjecture. As already mentioned above, in order to achieve an Euclidean diameter of order m in the classical Boolean model, one has to find a vertex with radius $m^{1/d}$ in the neighbourhood of the origin. However, this vertex then also has of order m neighbours itself leading to the same tail behaviour for the Euclidean diameter and the cardinality of the component of the origin. In contrast, it suffices to find a vertex with radius $m^{(\delta-1)(d\delta)}$ in the neighbourhood of the origin in order to have an Euclidean diameter of order m in the soft model. This vertex however does not have order m neighbours so that it cannot guarantee a cardinality of order m alone. To achieve this with a single vertex again a vertex with radius of order $m^{1/d}$ is required. This leads to the same lower bound in the classical and in the soft Boolean model. To prove a matching upper bound, one can easily adapt 2.5 to vertices of that strength in the component of the origin. Further, our diameter result can be used to derive that the whole component is contained in a ball of volume $m^{\delta/(\delta-1)}$, with an error probability of the right order. Now, one can think of the component of the origin in the soft model as a collection of classical Boolean clusters which are connected via long-range edges. On the event that no powerful vertex is present in the component of the origin, one can use Gouéré's method to deduce that no Boolean cluster in the considered ball has cardinality no larger than $m/\log(m)$ implying that the the component of the origin decomposes in at least $\log m$ many clusters when the long-range edges are removed. Additionally, we can adapt the bound on long paths of 2.5 to observe with the right error probability that the component's depth is larger than $\log m$. Hence, it remains to prove that it is unlikely enough to connect at least $\log m$ small Boolean clusters with long-range edges without creating shortest paths longer than $\log m$. Since the tail of the cardinality in long-range percolation decays exponentially fast one would get the desired result if one thought of the Boolean clusters as nodes of a long-range percolation model. Unfortunately, this ignores correlations between the edges and we couldn't find a convincing way yet to control these. We do however believe that it should be possible.

Let us finally mention the recent results in [26, 28]. Here, the authors derive stretched exponential decay of the tail distribution of finite clusters in *supercritical* regimes. The fundamental difference between their and our work is the presence of an infinite connected component. Since we explicitly work in subcritical regimes all components must be finite. This leaves the distribution of the radii unchanged and a single exceptionally large radius alone can lead to a relatively large cluster. Therefore, the cluster sizes are still heavy-tailed in our regime. In a supercritical regime however, this is no longer true as a too large radius makes it way too likely to be part of the infinite cluster. Hence, a large but finite cluster can only use small radii vertices leading to the drastically smaller tail. An exception to this argument is long-range percolation where no radii are present, a special case the authors consider in [27].

2.6.4 The constants

Let us finally remark that from our proofs we also deduce bounds for the constants of the leading order term appearing in the theorems when m grows large. The constants c_1, c_3, and c_5 appearing in the lower bounds of Theorem 2.1 are given in Section 3.1. Further, for the constants of the upper bounds, we derive C_2 in Proposition 3.6, C_4 in Proposition 3.11, and C_6 in Proposition 3.10. Finally, the constants appearing in Theorem 2.2 are given in Section 3.3.

3 Proofs

In this section we present the proofs of our results. We will use the common notation $f \sim g$ for two positive functions satisfying $f(x)/g(x) \to 1$ as $x \to 0$, and $f = o(g)$, if $f(x)/g(x) \to 0$ as well as $g \asymp f$, if f/g is bounded from zero and infinity. We further denote by $\mathcal{B}(x,r)$ the ball of radius r centred in x. If $x = o$, we simply write $\mathcal{B}(r) = \mathcal{B}(o,r)$ for the ball around the origin and by ω_d the volume of $\mathcal{B}(1)$. For a Borel set $B \subset \mathbb{R}^d$, we also write $B^c = \mathbb{R}^d \setminus B$, and we use the same notation for the complement of an event as usual. Finally, we denote by $\sharp A$ the number of elements in an at most countable set A.

3.1 Proof of the lower bounds in Theorem 2.1

We present two strategies to connect the origin to a vertex at a large distance in this section. The first strategy is to use a long edge incident to the origin. This strategy corresponds to the long-range percolation case. The second strategy corresponds to the classical Boolean model case where a powerful intermediate vertex connected to both the origin and the distant vertex is used. Both strategies combined give the lower bounds of our theorem.

We start with the proof of the 'long-range percolation' case when only one long edge is used giving the lower bound in Part *(i)*. The reason here is that for $\gamma < 1/(\delta+1)$, the effect of the radii on far connections is too small to play a significant role. Recall that ω_d denotes the volume of a d-dimensional unit ball.

Proposition 3.1. *Let $\beta > 0$, $\delta > 1$, and $\gamma \in (0,1)$ and consider $\mathcal{G}^\beta_{\mathbf{o}} = \mathcal{G}^{\beta,\gamma,0,\delta}(\xi_{\mathbf{o}})$. Then, for all $m > \beta$,*

$$\mathbb{P}_o\big(\exists \mathbf{x} = (x,u_x)\colon |x|^d > m \text{ and } \mathbf{x} \sim \mathbf{o} \text{ in } \mathcal{G}^\beta_{\mathbf{o}}\big) \geq 1 - \exp\big(\tfrac{\beta^\delta \omega_d}{\delta-1} m^{1-\delta}\big) \sim \tfrac{\beta^\delta\omega_d}{\delta-1} m^{1-\delta}.$$

Proof. Since a version of the random connection model is given by choosing $\gamma = \alpha = 0$ in the interpolation kernel (5) and $g_{0,0}(s,t) = 1 \geq g_{\gamma,0}(s,t)$, each edge that is present in a realisation of $\mathcal{G}^{\beta,0,0,\delta}(\xi_{\mathbf{o}})$ is also present in $\mathcal{G}^\beta_{\mathbf{o}}$ by the construction rule (6). (Note that the profile function ρ is decreasing and therefore a smaller value of $g_{\gamma,\alpha}(U_i,U_j)$ increases the probability of having an edge.) Hence,

$$\begin{aligned}&\mathbb{P}_o\big(\exists \mathbf{x} = (x,u_x)\colon |x|^d > m \text{ and } \mathbf{x} \sim \mathbf{o} \text{ in } \mathcal{G}^\beta_{\mathbf{o}}\big)\\ &\quad\geq \mathbb{P}_o\big(\exists \mathbf{x} = (x,u_x) \in \mathbf{X}\colon |x|^d > m \text{ and } \mathbf{x} \sim \mathbf{o} \text{ in } \mathcal{G}^{\beta,0,0,\delta}_{\mathbf{o}}\big).\end{aligned}$$

Since the number of the neighbours of the origin in $\mathcal{G}^{\beta,0,0,\delta}(\xi_{\mathbf{o}})$ at distance at least $m^{1/d}$ is Poisson distributed with parameter

$$\int_{|x|^d>m} \mathrm{d}x\, \rho(\beta^{-1}|x|^d) = \beta^\delta \int_{|x|^d>m} \mathrm{d}x\, |x|^{-d\delta} = \tfrac{\beta^\delta\omega_d}{\delta-1} m^{1-\delta},$$

assuming $m > \beta$ in the first equality, we deduce

$$\mathbb{P}_o(\mathscr{M}_\beta > m) \geq 1 - \exp\big(\tfrac{\beta^\delta\omega_d}{\delta-1} m^{1-\delta}\big),$$

yielding the desired lower bound. □

While in long-range percolation the order of the longest edge determines the order of the Euclidean diameter, this order is determined by the largest radius of a vertex connected to the origin in the classical Boolean model [14]. In order to connect the origin to a vertex at distance $m^{1/d}$, one searches for a vertex of radius at least $m^{1/2}$ within distance $m^{1/d}$. Put differently in our parametrisation, such a vertex has mark smaller than $m^{1/\gamma}$. We will show in the following proposition how to apply this connection strategy in the soft Boolean model. Here, only less powerful vertices are required due to the additional long-range effects. To make this precise, we write

$$\zeta := (\delta-1)/(\gamma\delta) \quad \text{and} \quad s_m := m^{-\zeta}$$

in the following. Note that the exponent in Part *(iii)* of Theorem 2.1 can be written as $-(1-\gamma)\zeta$.

Proposition 3.2. *Let $\beta > 0$, $\delta > 1$, and $\gamma > 0$ and consider $\mathcal{G}_o^\beta = \mathcal{G}^{\beta,\gamma,0,\delta}(\xi_{\mathbf{o}})$. Then, there exists $M > 0$ such that, for all $m \geq M$, we have*

$$\mathbb{P}_o\big(\exists \mathbf{x}, \mathbf{y}\colon |x|^d < m, u_x \leq s_m, m < |y|^d < 2^d m, u_y > 1/2 \text{ and } \mathbf{o} \sim \mathbf{x} \sim \mathbf{y}\big) \geq 1 - e^{-c_5 m^{-(1-\gamma)\zeta}} \sim c_5 m^{-(1-\gamma)\zeta},$$

where c_5 is given below in (9).

Proof. Since the considered event is monotone under the law of $u \mapsto \mathbb{P}_{(o,u)}$ in the sense that the smaller the mark of the origin the more connections it forms and hence the likelier the occurrence of the considered event, and the fact that we aim for a lower bound, we work under $\mathbb{P}_{(o,1)}$ in this proof. That is, we set the mark of the origin to be 1. Let now $\mathbf{Y}' \subset \mathbf{X}$ be the set of all vertices (y, u_y) with $m < |y|^d < 2m$ and $u_y > 1/2$. Then, $\mathbb{E}[\sharp Y'] = m\omega_d/2$ and a standard Poisson tail bound gives the existence of a constant $\tilde{c} > 0$ such that

$$\mathbb{P}(|Y'| \geq m\omega_d/4) \geq 1 - e^{-\tilde{c}m\omega_d}.$$

Now, consider a vertex $\mathbf{x} = (x, u_x) \in \mathbf{X}$ with $|x|^d < m$ and mark $u_x < s_m$. Such an $\mathbf{x}$ with mark $u_x < (3m/\beta)^{-1/\gamma}$ is automatically connected to the origin and to each $\mathbf{y} \in \mathbf{Y}'$ as $|x - y|^d \leq 3m$. Since there are of order $m^{1-1/\gamma}$ such vertices by the Poisson process properties, reproducing the known tail bound for the standard Boolean model, we focus on vertices with mark $(3m/\beta)^{-1/\gamma} < u_x < s_m$ in the following. Then, for m sufficiently large, the probability that this vertex $\mathbf{x}$ is connected to a particular vertex $\mathbf{y} \in \mathbf{Y}'$ is, by the monotonicity of ρ, lower bounded by

$$\rho(\beta^{-1}u_x^\gamma|x-y|^d) \geq \rho(\beta^{-1}u_x^\gamma 3m) \geq (\tfrac{\beta}{3})^\delta u_x^{-\gamma\delta} m^{-\delta} \geq (\tfrac{\beta}{3})^\delta m^{-1},$$

using the definition of s_m. Thus, on the event $\{\sharp\mathbf{Y}' \geq m\omega_d/4\}$, the number of vertices $\mathbf{y} \in \mathbf{Y}'$ connected to $\mathbf{x}$ is bounded from below by a Binomial random variable with $m\omega_d/4$ trials and success probability $(\beta/3)^\delta m^{-1}$. Therefore, by Poisson approximation, we infer for sufficiently large m that the probability of existence of at least one such $\mathbf{y}$ is no smaller than

$$1 - \exp\big(-\beta^\delta\omega_d/3^{\delta+1}\big) =: c.$$

Since this lower bound is independent of the vertices in $\mathbf{Y}'$ and of $\mathbf{x}$ itself, the number of neighbours of $(o, 1)$ in $\mathcal{B}(m^{1/d}) \times ((3m/\beta)^{-1/\gamma}, s_m)$, which are also connected to some vertex in $\mathbf{Y}'$, is bounded from below by the number of points of a Poisson point process in $\mathcal{B}(m^{1/d}) \times ((3m/\beta)^{-1/\gamma}, s_m)$ of intensity $c\rho(\beta^{-1}u^\gamma|x|^d)\mathrm{d}u\mathrm{d}x$. The expected number of vertices contained in this Poisson point process is given by

$$c\int_{m^{-1/\gamma}}^{s_m}\mathrm{d}u\int_{|x|^d<m}\mathrm{d}x\,\rho(\beta u_x^\gamma|x|^d) = c\beta\Big(\int_{m^{-1/\gamma}}^{s_m}\mathrm{d}u\,u^{-\gamma}\Big)\Big(\int_{|x|^d<u^\gamma m}\mathrm{d}x\,\rho(|x|^d)\Big) \geq \tfrac{c\beta\omega_d\delta}{2(1-\gamma)(\delta-1)}s_m^{1-\gamma},$$

for large enough m using $\int_{|x^d|<u^\gamma m}\rho(|x|^d) \to \omega_d\delta/(\delta-1)$. Therefore, the probability of existence of such an $\mathbf{x}$ is given by $1 - \exp(-c_5 m^{(1-\gamma)\zeta})$, where

$$c_5 = \tfrac{c\beta\omega_d\delta}{2(1-\gamma)(\delta-1)}\big(1 - \exp(-\tfrac{\beta^\delta\omega_d}{3^{\delta+1}})\big), \tag{9}$$

concluding the proof. □

Proof of the lower bounds in Theorem 2.1. Since $1-\delta \le -(1-\gamma)\zeta$ if and only if $\gamma \ge 1/(\delta+1)$, the lower bound in Part *(i)* is a consequence of Proposition 3.1, the lower bound in Part *(iii)* is a consequence of Proposition 3.2, and the lower bound in Part *(ii)* is a consequence of the combination of both, where the appearing constant is given by

$$c_3 := c_1 \vee c_5. \tag{10}$$

This concludes the proof of the lower bounds. □

3.2 Proof of the upper bounds in Theorem 2.1

In this section, we prove the upper bounds of our main theorem. To this end, we show that no strategy of connecting the origin to some vertex at distance $m^{1/d}$ works better in probability than the respective strategy in the lower bound. In order to do so, we rely on fine bounds for the probability that carefully chosen paths exist which we do by using a first-moment method. Note that if $\gamma > 1/2$, the degree distribution has infinite second moment and already the expected number of paths of length two starting at the origin is infinite. Hence, we cannot apply direct moment bounds in the whole parameter regime. Instead, we decompose each considered path in its 'powerful' vertices, the *skeleton* of the path, and the remaining 'weak' vertices. This concept was first introduced in [19] and gives a powerful tool to control the combinatorics of our path counts. Then, we combine this with a BK inequality for independent edge markings of [22] to reduce the probability of a path existing to a moment bound on the skeleton vertices that form a path themselves. This forms 2.5 of our proof.

3.2.1 The skeleton of a path

The concept of a skeleton of a path for weight-dependent random connection models was introduced in [19] and is based on a decomposition of a *shortcut-free* path into its powerful vertices, the *skeleton*, and connectors which are weaker vertices that connect the skeleton vertices with each other. Recall that a path $P = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ is called *shortcut free* if it contains no shorter subpath $Q \subset P$ also connecting $\mathbf{x}_0$ and $\mathbf{x}_n$. From now on, each path is considered to be shortcut free.

The idea is now the following: To have a long path it is important to have significantly powerful vertices which are those with smallest marks since a small mark corresponds to a large radius. Let $P = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ be a path of length n. We call $\mathbf{x}_i = (x_i, u_i)$ for $i \notin \{0, n\}$ a *local maximum* if $u_i > u_{i-1}$ and $u_i > u_{i+1}$. We now construct a new path without any local maxima, see Figure 2. For this, we first take the local maximum in P with greatest vertex mark, remove it from P and connect its former neighbours by an edge. In the resulting path, we take the new local maximum of greatest vertex mark, remove it, and connect its former neighbours, repeating until there is no local maximum left in P. We call the remaining vertices the *skeleton* of the path. Note that start and end vertex of a path can never be a local maximum and are therefore always part of the skeleton. Further, the constructed paths of skeleton vertices is not necessarily an actual path of the graph. In particular, the skeleton vertices of a shortcut-free path cannot themselves form a path unless the vertices of the path already form a skeleton. We have decomposed the path in two sets of vertices: the skeleton vertices and the local maxima that connect two consecutive skeleton vertices. From now on, we refer to the latter as *connectors*. In the following, we derive bounds for the probability that two given vertices are connected by a shortcut-free path consisting of connectors only. Afterwards, we state the BK inequality of [22] that can be used to decompose the whole path into its subpaths connecting two consecutive skeleton vertices to obtain bounds for the existence of paths.

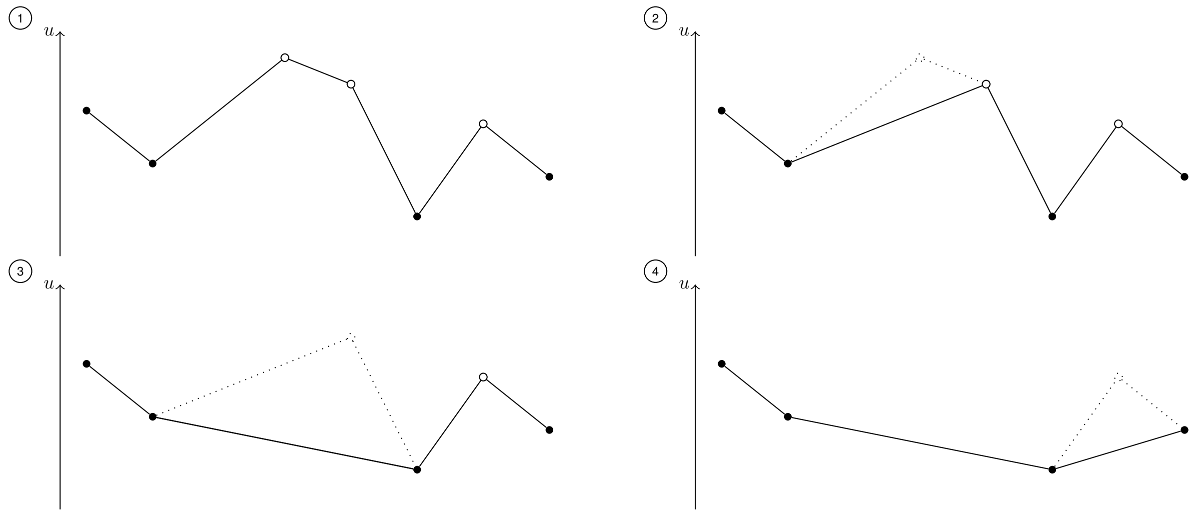

Figure 2: A path where the mark of a vertex is denoted on the u-axis and the spatial location of the vertices is not shown. The vertices of the skeleton are in black. We successively remove all local maxima, starting with the largest mark vertex, and replace them by direct edges until the path, only containing the skeleton vertices, is left.

Connecting two powerful vertices. In this paragraph, we consider two given vertices $\mathbf{x}$ and $\mathbf{y}$ and the probability that they are connected by a shortcut-free path in a weight-dependent random connection model $\mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi_{\mathbf{x},\mathbf{y}})$ of length n consisting of connectors only. That is, the paths skeleton is given by $\mathbf{x}$ and $\mathbf{y}$ only. Let us denote this event by $\{\mathbf{x} \underset{\mathbf{x},\mathbf{y}}{\overset{n}{\longleftrightarrow}} \mathbf{y}$ in $\mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi_{\mathbf{x},\mathbf{y}})\}$. Since by construction $\mathbf{x}$ and $\mathbf{y}$ are connected by an edge with probability

$$\rho\big(\beta^{-1}g_{\gamma,\alpha}(u_x,u_y)|x-y|^d\big) = \begin{cases} 1, & \text{if } |x-y|^d \leq \beta g_{\gamma,\alpha}(u_x,u_y)^{-1}, \\ \beta^\delta g_{\gamma,\alpha}(u_x,u_y)^{-\delta}|x-y|^{-d\delta}, & \text{if } |x-y|^d > \beta g_{\gamma,\alpha}(u_x,u_y)^{-1}, \end{cases}$$

we focus on given vertices $\mathbf{x}$ and $\mathbf{y}$ at distance $|x-y|^d > \beta g_{\gamma,\alpha}(u_x,u_y)^{-1}$ to fulfil the shortcut-free property. The next lemma is key in this section and it is a combination of [19, Lemma 2.2 and 2.3].

Lemma 3.3 (n-connection Lemma, [19])**.** *Let $\beta > 0$, $\gamma \in (0,1)$, $\alpha \in [0, 2-\gamma)$, and $\delta > 1$. Let further $\mathbf{x} = (x,u_x)$ and $\mathbf{y} = (y,u_y)$ be two vertices satisfying the distance condition $|x-y|^d > \beta g_{\gamma,\alpha}(u_x,u_y)^{-1}$. Assume that there is a constant $C > 0$ such that*

$$\begin{aligned} &\mathbb{E}_{\mathbf{x},\mathbf{y}}\big[\sharp\{\mathbf{z} = (z,u_z)\colon u_z > u_x \vee u_y, \text{ and } \mathbf{x} \sim \mathbf{z} \sim \mathbf{y} \text{ in } \mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi_{\mathbf{x},\mathbf{y}})\}\big] \\ &\quad \leq (\beta C)\rho\big(\beta^{-1}g_{\gamma,\alpha}(u_x,u_y)|x-y|^d\big). \end{aligned}$$

Then, for all $n \in \mathbb{N}$, we have

$$\mathbb{P}_{\mathbf{x},\mathbf{y}}\big(\mathbf{x} \underset{\mathbf{x},\mathbf{y}}{\overset{n}{\longleftrightarrow}} \mathbf{y} \text{ in } \mathcal{G}^{\beta,\gamma,\alpha,\delta}(\xi_{\mathbf{x},\mathbf{y}})\big) \leq (4\beta C)^{n-1}\rho\big(\beta^{-1}g_{\gamma,\alpha}(u_x,u_y)|x-y|^d\big).$$

The idea behind the lemma is the following. One can represent the inner path consisting of all vertices excluding $\mathbf{x}$ and $\mathbf{y}$ as a binary tree which is labelled by the Poisson points of the graph such that each child has a greater vertex mark than its parent. Here, a binary tree is a tree where each vertex has either no child, a left child, a right child, or a left and a right child. The underlying binary tree encodes important structural information of the path. Namely, it encodes the order of the local maxima where the precise vertex marks order is then given by the labelling. The proof works by induction. If

$n = 1$ there is nothing to show. For $n = 2$ the binary tree consists only of a root vertex which is the intermediate connector and the claim follows from the assumptions. Now, for any $n \geq 3$, fix a binary tree of $n-1$ vertices (recall that start and end vertex are not represented in the tree) and label it with Poisson points such that each child has mark larger than its parent and the distance condition for shortcut-free paths is fulfilled. This then represents a paths of length n. Pick the leaf with largest vertex mark in the tree. The vertex represented by this leaf is by necessity a connector of two vertices with smaller marks in the path. Therefore, we can apply the assumption and bound the expected number of possible labels for this leaf by βC times the probability that there is an edge between the vertices the leaf connects in the path. Since this yields a new path on $n-1$ vertices represented by the tree of $n-2$ vertices where the leaf was removed, the induction hypothesis applies and we infer that the expected number of possible labelings is bounded by $(\beta C)^{n-1}\rho\big(\beta^{-1}g_{\gamma,\alpha}(u_x,u_y)|x-y|^d\big)$. Now, the claim follows by applying a union bound over all unlabelled trees on $n-1$ vertices and the fact that there are at most 4^{n-1} such trees. For the details of the proof and particularly the tree representation, we refer the reader to [19].

To apply Lemma 3.3, the graph has to have the property that the expected number of weak vertices connecting two given stronger vertices at a large distance is bounded by a β depending constant times the probability of an edge existing. The soft Boolean model does not have this property. Indeed, the connection probability of two given vertices does only depend on the larger radius respectively the smaller mark. That is, the probability of an edge between the two vertices depends only on the stronger vertex. Whether the other vertex is rather strong itself has no effect. On the contrary, the expected number of weaker connectors is the intersection of both neighbourhoods restricted to radii smaller than both of the given ones. Hence, the stronger the weaker of the two given vertices, the larger this intersection. To tackle this issue and still make use of Lemma 3.3 and the skeleton strategy, we stochastically dominate the soft Boolean model by a graph that matches the assumption. To this end, consider the weight-dependent random connection model $\mathcal{G}^{\beta,\gamma,\gamma/\delta,\delta}(\xi)$ defined through its kernel

$$g_{\gamma,\gamma/\delta}(s,t) = (s\wedge t)^{\gamma}(s\vee t)^{\gamma/\delta},$$

which we also refer to as *two-connection kernel*. To lighten notation, we also write $\widehat{\mathcal{G}}^\beta = \mathcal{G}^{\beta,\gamma,\gamma/\delta,\delta}(\xi)$ in the following. The following lemma shows that $\widehat{\mathcal{G}}^\beta$ indeed dominates $\mathcal{G}^\beta$ and has the required n-connection property of Lemma 3.3.

Lemma 3.4. *Let $\beta > 0$, $0 < \gamma < \delta/(\delta+1)$, and $\delta > 1$ and consider the soft Boolean model $\mathcal{G}^\beta = \mathcal{G}^{\beta,\gamma,0,\delta}(\xi)$ and the graph $\widehat{\mathcal{G}}^\beta = \mathcal{G}^{\beta,\gamma,\gamma/\delta,\delta}(\xi)$. Then*

(i) the two edge sets satisfy $E(\mathcal{G}^\beta) \subset E(\widehat{\mathcal{G}}^\beta)$ almost surely and

(ii) for two given vertices $\mathbf{x} = (x,u_x)$ and $\mathbf{y} = (y,u_y)$ at distance $|x-y|^d > \beta g_{\gamma,\gamma/\delta}(u_x,u_y)^{-1}$ and all $n \in \mathbb{N}$, we further have

$$\mathbb{P}_{\mathbf{x},\mathbf{y}}\big(\mathbf{x} \underset{\mathbf{x},\mathbf{y}}{\overset{n}{\longleftrightarrow}} \mathbf{y} \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x},\mathbf{y}}\big) \leq (\beta\widehat{C})^{n-1}\rho\big(\beta^{-1}g_{\gamma,\gamma/\delta}(u_x,u_y)|x-y|^d\big),$$

where $\widehat{C} = \frac{2^{d\delta+3}\omega_d\delta^2}{(\delta-1)(\delta-\gamma(\delta+1))}$.

Proof. Since $\gamma/\delta > 0$ and therefore $g_{\gamma,0}(s,t) > g_{\gamma,\gamma/\delta}(s,t)$ and ρ is non increasing, each edge that is present in $\mathcal{G}^\beta$ is also present in $\widehat{\mathcal{G}}^\beta$ by construction (6), proving (i).

To prove (ii), we assume without loss of generality $u_x > u_y$, and calculate using Mecke's equation [32]

$$\begin{aligned}&\mathbb{E}_{\mathbf{x},\mathbf{y}}\big[\sharp\{\mathbf{z}=(z,u_z)\in\mathbf{X}: u_z>u_x\vee u_y,\text{ and }\mathbf{x}\sim\mathbf{z}\sim\mathbf{y}\text{ in }\widehat{\mathcal{G}}^{\beta}_{\mathbf{x},\mathbf{y}}\}\big]\\&\quad=\int_{\mathbb{R}^d}\mathrm{d}z\int_{u_x}^{1}\mathrm{d}u_z\,\rho\big(\beta^{-1}u_y^{\gamma}u_z^{\gamma/\delta}|y-z|^d\big)\rho\big(\beta^{-1}u_x^{\gamma}u_z^{\gamma/\delta}|x-z|^d\big).\end{aligned}$$

Now, either $|y-z|\geq\frac{1}{2}|x-y|$ or $|x-z|\geq\frac{1}{2}|x-y|$. Splitting $\mathbb{R}^d$ according to those cases yields

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathrm{d}z\int_{u_x}^{1}\mathrm{d}u_z\,\rho(\beta^{-1}u_y^{\gamma}u_z^{\gamma/\delta}|y-z|^d)\rho(\beta^{-1}u_x^{\gamma}u_z^{\gamma/\delta}|x-z|^d)\\&\qquad\leq\int_{u_x}^{1}\mathrm{d}u_z\,\rho\big((2^d\beta)^{-1}u_y^{\gamma}u_z^{\gamma/\delta}|y-x|^d\big)\int_{\mathbb{R}^d}dz\,\rho(\beta^{-1}u_x^{\gamma}u_z^{\gamma/\delta}|z|^d)\\&\qquad\quad+\int_{u_x}^{1}\mathrm{d}u_z\,\rho((2^d\beta)^{-1}u_x^{\gamma}u_z^{\gamma/\delta}|y-x|^d)\int_{\mathbb{R}^d}\mathrm{d}z\,\rho(\beta^{-1}u_y^{\gamma}u_z^{\gamma/\delta}|z|^d).\end{aligned}$$

For the first integral, we use the change of variables $w=(\beta^{-1}u_x^{\gamma}u_z^{\gamma/\delta})^{1/d}z$ and the distance condition to deduce

$$\begin{aligned}&\int_{u_x}^{1}\mathrm{d}u_z\,\rho\big((2^d\beta)^{-1}u_y^{\gamma}u_z^{\gamma/\delta}|y-x|^d\big)\int_{\mathbb{R}^d}dz\,\rho(\beta^{-1}u_x^{\gamma}u_z^{\gamma/\delta}|z|^d)\\&\qquad=2^{d\delta}\beta^{1+\delta}u_x^{-\gamma}|x-y|^{-d\delta}u_y^{-\gamma\delta}\Big(\int_{u_x}^{1}\mathrm{d}u_z\,u_z^{-\gamma-\gamma/\delta}\Big)\Big(\int_{\mathbb{R}^d}\mathrm{d}w\,\rho(|w|^d)\Big)\\&\qquad\leq\beta\tfrac{2^{d\delta}\omega_d\delta^2}{(\delta-1)(\delta-\gamma(\delta+1))}\rho\big(\beta^{-1}g_{\gamma,\gamma/\delta}(u_x,u_y)|x-y|^d\big),\end{aligned}$$

where we have used $\gamma<\delta/(\delta+1)$, and $\int\rho(|w|^d)\mathrm{d}w=\omega_d\delta/(\delta-1)$, together with the distance condition in the last step. A similar calculation for the second integral yields the same bound and summing both terms yields

$$\begin{aligned}&\mathbb{E}_{\mathbf{x},\mathbf{y}}\big[\sharp\{\mathbf{z}=(z,u_z)\in\mathbf{X}: u_z>u_x\vee u_y,\text{ and }\mathbf{x}\sim\mathbf{z}\sim\mathbf{y}\text{ in }\widehat{\mathcal{G}}^{\beta}_{\mathbf{x},\mathbf{y}}\}\big]\\&\quad\leq\tfrac{\beta\widehat{C}}{4}\rho\big(\beta^{-1}g_{\gamma,\gamma/\delta}(u_x,u_y)|x-y|^d\big).\end{aligned}$$

Hence, we can apply Lemma 3.3, finishing the proof. □

BK inequality. The last section was devoted to showing how a path whose skeleton consists of start and end vertex only can be reduced to a single edge in probability. Since we consider more general paths in the following, we need to decompose the whole path in its subpaths between two consecutive skeleton vertices to make use of the above results. To this end, we use a version of the BK inequality [45] for an independent edge marking of a Poisson point processes as in [22, Theorem 2.1] generalising the result in [20] for the classical Boolean model. The application to our setting is described in detail in [22, p. 14].

Let $\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k$ be given vertices with a vertex mark structure that form a skeleton. That is, their marks are decreasing until they reach the vertex with minimum mark and only increasing afterwards.

Let us denote by $\{\mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k]{n} \mathbf{x}_k \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k}\}$ the event that $\mathbf{x}_0$ and $\mathbf{x}_k$ are connected by a (shortcut-free) path of length n and skeleton $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k$ in $\widehat{\mathcal{G}}^\beta_{\mathbf{x}_o,\dots,\mathbf{x}_n}$, which is consistent with the previously used notation.

We start with the case $k = 2$ where the skeleton only consists of the three vertices $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$. We consider for $n_1+n_2 = n$ the two events $\{\mathbf{x}_{i-1} \xleftrightarrow[\mathbf{x}_{i-1},\mathbf{x}_i]{n_i} \mathbf{x}_i \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_{i-1},\mathbf{x}_i}\}$, for $i = 1, 2$. We are interested in the event that there exists a path from $\mathbf{x}_0$ to $\mathbf{x}_2$ with intermediate skeleton vertex $\mathbf{x}_1$ such that there are n_1 connectors used between $\mathbf{x}_0$ and $\mathbf{x}_1$ and n_2 connectors used between $\mathbf{x}_1$ and $\mathbf{x}_2$. Since all our paths are self avoiding and shortcut free, this event is the same as the *disjoint occurrence* of the two events $\mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0,\mathbf{x}_1]{n_1} \mathbf{x}_1$, and $\mathbf{x}_1 \xleftrightarrow[\mathbf{x}_1,\mathbf{x}_2]{n_2} \mathbf{x}_2$. Here, disjoint occurrence means that the two paths share no element of ξ. That is, no Poisson vertex is used twice. Note here that $\mathbf{x}_1$ is a given vertex and thus not a random element of $\mathbf{X}$. We denote the disjoint occurrence by $\circ$. Hence, the event of interest is given by

$$\Big\{\big(\mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0,\mathbf{x}_1]{n_1} \mathbf{x}_1 \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_0,\mathbf{x}_1}\big) \circ \big(\mathbf{x}_1 \xleftrightarrow[\mathbf{x}_1,\mathbf{x}_2]{n_2} \mathbf{x}_2 \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_1,\mathbf{x}_2}\big)\Big\}.$$

Further, the two events are *increasing* in the following sense: Given any realisation ω of, say, $\xi_{\mathbf{x}_0,\mathbf{x}_1}$ such that the event $\{\mathbf{x}_0 \xleftrightarrow[\mathbf{x}_1,\mathbf{x}_1]{n_1} \mathbf{x}_1 \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_1,\mathbf{x}_2}\}$ occurs on ω, it also occurs for on all realisations ω' with $\omega \subset \omega'$. That is, if there is such a path for some realisation, it will also be present if additional vertices are added to the graph. It is important to note that this notion of increasing refers to set inclusion only, i.e., we only may add additional vertices with their incident edges but we never add additional edges between already existing vertices as this may shorten the path of length n_1 due to the shortcut-free property. The application of the BK inequality [22, Theorem 2.1] then yields

$$\begin{aligned}\mathbb{P}_{\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2}&\Big(\big(\mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0,\mathbf{x}_1]{n_1} \mathbf{x}_1 \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_0,\mathbf{x}_1}\big) \circ \big(\mathbf{x}_1 \xleftrightarrow[\mathbf{x}_1,\mathbf{x}_2]{n_2} \mathbf{x}_2 \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_1,\mathbf{x}_2}\big)\Big)\\ &\le \mathbb{P}_{\mathbf{x}_0,\mathbf{x}_1}\big(\mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0,\mathbf{x}_1]{n_1} \mathbf{x}_1 \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_0,\mathbf{x}_1}\big)\mathbb{P}_{\mathbf{x}_1,\mathbf{x}_2}\big(\mathbf{x}_1 \xleftrightarrow[\mathbf{x}_1,\mathbf{x}_2]{n_2} \mathbf{x}_2 \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_1,\mathbf{x}_2}\big).\end{aligned}$$

Inductively, this applies to all $k \ge 2$ and $n_1 + \dots + n_k = n$ and therefore

$$\mathbb{P}_{\mathbf{x}_0,\dots,\mathbf{x}_k}\big(\mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0,\dots,\mathbf{x}_k]{n} \mathbf{x}_k \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_0,\dots,\mathbf{x}_k}\big) \le \sum_{\substack{n_1,\dots,n_k\in\mathbb{N}\\ n_1+\dots+n_k=n}} \prod_{i=1}^k \mathbb{P}_{\mathbf{x}_{i-1},\mathbf{x}_i}\big(\mathbf{x}_{i-1} \xleftrightarrow[\mathbf{x}_{i-1},\mathbf{x}_i]{n_i} \mathbf{x}_i \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_{i-1},\mathbf{x}_i}\big). \quad (11)$$

Observe that the above holds for all versions of the weight-dependent random connection model. However, we only apply it to $\widehat{\mathcal{G}}^\beta$ in the following and we have hence restricted ourselves to this version to lighten notation.

Bounds on the probability of paths in the soft Boolean model. In this paragraph, we combine the results of the two previous paragraphs to derive bounds on the probability that certain paths exist. Recall the notation of $\mathscr{C}_\beta = \mathscr{C}_{\beta,\gamma,0,\alpha}(\mathbf{o})$ for the component of the origin in the soft Boolean model and the notation $\mathscr{M}_\beta = \mathscr{M}_{\beta,\gamma,0,\delta}(\mathbf{o})$ for its Euclidean diameter. Now, the event $\mathscr{M}_\beta > m$ is equivalent to the existence of a path starting in $\mathbf{o}$ where all vertices are located in $\mathcal{B}(m^{1/d})$ except for the last vertex that is located outside the ball. We prepare bounds for that type of events in the following.

Let us denote by $\{\mathbf{x} \xleftrightarrow[D\times J]{} \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{x},\mathbf{y}}\}$ for two given vertices $\mathbf{x}$ and $\mathbf{y}$, a domain $D \subset \mathbb{R}^d$, and a measurable set $J \subset (0,1)$ the event that $\mathbf{x}$ and $\mathbf{y}$ are connected by a shortcut-free path in $\mathcal{G}^\beta$ where all skeleton vertices but $\mathbf{x}$ and $\mathbf{y}$ are elements of $D \times J$. If $J = (0,1)$, we simply write

$\{\mathbf{x} \underset{D}{\longleftrightarrow} \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{x},\mathbf{y}}\}$. We further denote by $\{\mathbf{x} \overset{n}{\underset{D\times J}{\longleftrightarrow}} \mathbf{y} \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x},\mathbf{y}}\}$ the event that $\mathbf{x}$ and $\mathbf{y}$ are connected by a shortcut-free path of *length* n in $\widehat{\mathcal{G}}^\beta$ where the skeleton has the same restriction as above. Note that now only the skeleton vertices are restricted to certain locations and marks. However, the results of the previous paragraphs tell us that this is what matters when it comes to bound the probability of a path. Note further that this notation is consistent with the one used above to describe paths with a *given* skeleton. Indeed, if the whole skeleton is given, the location domain D and the vertex marks J reduce to the given points. We remark that it will always be clear from the context whether we refer to a domain for the vertex locations or to a concrete skeleton. Making use of the domination derived in Lemma 3.4 Part *(i)*, we directly obtain

$$\mathbb{P}_{\mathbf{x},\mathbf{y}}\big(\mathbf{x} \underset{D\times J}{\longleftrightarrow} \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{x},\mathbf{y}}\big) \leq \sum_{n\in\mathbb{N}} \mathbb{P}_{\mathbf{x},\mathbf{y}}\big(\mathbf{x} \overset{n}{\underset{D\times J}{\longleftrightarrow}} \mathbf{y} \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x},\mathbf{y}}\big). \tag{12}$$

Using Mecke's equation [32], the BK-inequality (11), the n-connection property of Lemma 3.4 Part *(ii)*, and writing $\mathbf{x} = \mathbf{x}_0, \mathbf{y} = \mathbf{y}_k$ the probability on the right-hand side can further be bounded by

$$\begin{aligned}
&\mathbb{P}_{\mathbf{x},\mathbf{y}}\big(\mathbf{x} \overset{n}{\underset{D\times J}{\longleftrightarrow}} \mathbf{y} \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x},\mathbf{y}}\big) \\
&\leq \sum_{k=1}^n \mathbb{E}_{\mathbf{x}_0,\mathbf{x}_k}\Bigg[\sum^{\neq}_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_{k-1} \\ x_j\in D, u_j\in J, \forall j}} \mathbb{1}\{\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k \text{ form skeleton}\} \mathbb{P}_{\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k}\big(\mathbf{x} \overset{n}{\underset{\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k}{\longleftrightarrow}} \mathbf{y} \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_0,\dots,\mathbf{x}_k}\big)\Bigg] \\
&\leq \sum_{k=1}^n \sum_{\substack{n_1,\dots,n_k\in\mathbb{N} \\ n_1+\dots+n_k=n}} \int\limits_{\substack{(D\times J)^{k-1} \\ \text{skeleton structure}}} \bigotimes_{j=1}^{k-1} \mathrm{d}x_j \prod_{i=1}^k \mathbb{P}_{\mathbf{x}_{i-1},\mathbf{x}_i}\big(\mathbf{x}_{i-1} \overset{n_i}{\underset{\mathbf{x}_{i-1},\mathbf{x}_i}{\longleftrightarrow}} \mathbf{x}_i \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{x}_{i-1},\mathbf{x}_i}\big) \\
&\leq \sum_{k=1}^n \binom{n}{k} (\beta\widehat{C})^{n-k} \int\limits_{\substack{(D\times J)^{k-1} \\ \text{skeleton structure}}} \bigotimes_{j=1}^{k-1} \mathrm{d}x_j \prod_{i=1}^k \rho\big(\beta^{-1} g_{\gamma,\gamma/\delta}(u_{i-1},u_i)|x_{i-1}-x_i|^d\big),
\end{aligned} \tag{13}$$

where we have additionally used $\sharp\{n_1,\dots,n_k \in \mathbb{N}\colon n_1+\dots+n_k = n\} = \binom{n-1}{k-1} \leq \binom{n}{k}$ in the last step. Let us further recall that $\widehat{C} = \frac{2^{d\delta+3}\omega_d\delta^2}{(\delta-1)(\delta-\gamma(\delta+1))}$ is the constant given in Lemma 3.4. In conclusion, an upper bound for the left-hand side in (12) is given by summing the right-hand side of (13) over all $n \in \mathbb{N}$.

A bound on paths with decreasing vertex marks. We close this section with the following technical lemma that can be used to derive bounds for the expected number of paths whose skeleton is monotone in its vertex marks once the spatial influence has been integrated out. Besides Lemma 3.4, this is the second main ingredient that requires $\gamma < \delta/(\delta+1)$ as it shows that the number of such paths grows no faster than exponentially which is sufficient to derive the existence of a subcritical phase as pointed out above in Section 2.6 and proved in [19].

Lemma 3.5. *Let $\delta > 1$, and $\gamma < \delta/(\delta+1)$, then for all $n \in \mathbb{N}$,*

$$\int_0^1 \mathrm{d}u_1 \cdots \int_0^1 \mathrm{d}u_n \prod_{j=1}^n u_j^{-\gamma-\gamma/\delta} \leq \big(\tfrac{\delta}{\delta-\gamma(\delta+1)}\big)^n.$$

Proof. Since $\gamma < \delta/(\delta+1)$, we have $1-\gamma-\gamma/\delta > 0$ and therefore

$$\int_0^1 \mathrm{d}u_1 \cdots \int_0^1 \mathrm{d}u_n \prod_{j=1}^n u_j^{-\gamma-\gamma/\delta} \leq \tfrac{\delta}{\delta-\gamma(\delta+1)} \int_0^1 \mathrm{d}u_1 \cdots \int_0^1 \mathrm{d}u_n \prod_{j=1}^{n-1} u_j^{-\gamma-\gamma/\delta}$$

and the proof is completed by iteration. □

3.2.2 Proof of the upper bound in Part *(i)*

We prove the upper bound of Theorem 2.1 Part *(i)* in this section. That is, we assume $\gamma < 1/(\delta+1)$. Since the influence of the radii is rather weak compared to the long-range effects in this setting, we can apply the previously derived bounds directly to obtain the desired result which is summarised in the following proposition. To quantify the bound on β for which our results hold, we define, as already discussed in Section 2.6,

$$\beta_0 := \tfrac{1}{2^{d\delta+3}+1}\cdot\tfrac{\delta-1}{\omega_d\delta}\big(1-\gamma\tfrac{\delta+1}{\delta}\big) = \tfrac{(\delta-1)(\delta-\gamma(\delta+1))}{(2^{d\delta+3}+1)\omega_d\delta^2}.$$

As we consider various paths starting in the origin (o,u_o) in the following, we will also often refer to the origin as $\mathbf{x}_0=(0,u_0)$ in order to have a clean numeration of the vertices on the path.

Proposition 3.6. *Let $\beta<\beta_0$, $\delta>1$ and $\gamma<1/(\delta+1)$. Consider the soft Boolean model $\mathscr{G}^\beta_{\mathbf{o}} = \mathscr{G}^{\beta,\gamma,0,\delta}_{\mathbf{o}}$ and its Euclidean diameter $\mathscr{M}_\beta=\mathscr{M}_{\beta,\gamma,0,\delta}$. Then, for all $m>1$, we have*

$$\mathbb{P}_o\big(\mathscr{M}_\beta>m\big)\le C_2 m^{1-\delta},$$

where C_2 is given below in (16).

Proof. Using the notation from the previous section, writing $\mathbf{o}=\mathbf{x}_0$ we have

$$\begin{aligned}\{\mathscr{M}_\beta>m\} &\subset \bigcup_{n\in\mathbb{N}}\big\{\exists \mathbf{x}\colon |x|^d>m \text{ and } \mathbf{o}\xleftrightarrow[\mathcal{B}(m^{1/d})]{n}\mathbf{x} \text{ in } \widehat{\mathscr{G}}^\beta_{\mathbf{o}}\big\}\\ &=\bigcup_{n\in\mathbb{N}}\bigcup_{k=1}^{n}\big\{\exists \mathbf{x}_1,\dots,\mathbf{x}_k\colon |x_1|^d,\dots,|x_{k-1}|^d<m, |x_k|^d>m \text{ and } \mathbf{x}_0\xleftrightarrow[\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k]{n}\mathbf{x}_k \text{ in } \widehat{\mathscr{G}}^\beta_{\mathbf{o}}\big\}.\end{aligned}$$

From now on we assume that each path considered occurs in $\widehat{\mathscr{G}}^\beta_{\mathbf{o}}$. We deduce from Mecke's equation [32]

$$\begin{aligned}&\mathbb{P}_o(\mathscr{M}_\beta>m)\\ &\le\sum_{\substack{n\in\mathbb{N}\\k\le n}}\int_0^1\mathrm{d}u_0\int_{\substack{|x_k|^d>m\\u_k\in(0,1)}}\mathrm{d}\mathbf{x}_k\,\mathbb{P}_{\mathbf{x}_0,\mathbf{x}_k}\big(\exists\mathbf{x}_1,\dots,\mathbf{x}_{k-1}: |x_1|^d,\dots,|x_{k-1}|^d<m \text{ and } \mathbf{x}_0\xleftrightarrow[\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k]{n}\mathbf{x}_k\big).\end{aligned}$$

Let us focus on a fixed path of length n and skeleton length k. Then, we must have $|x_\ell-x_{\ell-1}|>m^{1/d}/k$ for some $\ell\in\{1,\dots,k-1\}$ or $|\mathbf{x}_k-\mathbf{x}_{k-1}|>|\mathbf{x}_k|/k$. Applying the probability bounds on paths derived in (13) for some fixed $\ell\in\{1,\dots,k-1\}$ therefore yields

$$\begin{aligned}&\int_0^1\mathrm{d}u_0\int_{\substack{|x_k|^d>m\\u_k\in(0,1)}}\mathrm{d}\mathbf{x}_k\,\mathbb{P}_{\mathbf{x}_0,\mathbf{x}_k}\big(\exists\mathbf{x}_1,\dots,\mathbf{x}_{k-1}\colon |x_1|^d,\dots,|x_{k-1}|^d<m, |x_\ell-x_{\ell-1}|^d>\tfrac{m}{k^d}, \mathbf{x}_0\xleftrightarrow[\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k]{n}\mathbf{x}_k\big)\\ &\le\binom{n}{k}(\beta\widehat{C})^{n-k}\beta^\delta k^{d\delta}m^{-\delta}\int_{\substack{(0,1)^{k+1}\\\text{skeleton structure}}}\bigotimes_{j=0}^{k}\mathrm{d}u_j\,g_{\gamma,\gamma/\delta}(u_\ell,u_{\ell-1})^{-\delta}\\ &\qquad\times\int_{\substack{\mathcal{B}(m^{1/d})^{k-1}\\|x_k|^d>m}}\bigotimes_{j=1}^{k}\mathrm{d}x_j\prod_{\substack{i=1\\i\ne\ell}}^{k}\rho\big(\beta^{-1}g_{\gamma,\gamma/\delta}(u_{i-1},u_i)|x_{i-1}-x_i|^d\big),\end{aligned}$$

where we used the independence of vertex locations and vertex marks, i.e., $\mathrm{d}\mathbf{x}_j = \mathrm{d}x_j \mathrm{d}u_j$, and the bound $\rho(\beta^{-1} g_{\gamma,\gamma/\delta}(u_{\ell-1}, u_\ell)|x_\ell - x_{\ell-1}|^d) \le \beta^\delta g_{\gamma,\gamma/\delta}(u_{\ell-1}, u_\ell)^{-\delta} m^\delta$. Focusing on the integral only, we perform the change of variables $z_i = \beta^{-1/d} g_{\gamma,\gamma/\delta}(u_i, u_{i-1})^{1/d}$, starting from $i = n$ and successively going to $i = \ell + 1$, and continuing from $i = \ell - 1$ and successively going to $i = 1$, yielding the upper bound

$$m^{-\delta}\beta^{k-1} \int_{\substack{(0,1)^{k+1}\\ \text{skeleton structure}}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \, g_{\gamma,\gamma/\delta}(u_\ell, u_{\ell-1})^{-\delta} \prod_{\substack{i=1\\ i\neq \ell}}^{k} g_{\gamma,\gamma/\delta}(u_i, u_{i-1})^{-1} \Big(\int_{B(m^{1/d})} \mathrm{d}x_\ell \Big) \Big(\int_{\mathbb{R}^d} \mathrm{d}z \, \rho(|z|^d) \Big)^{k-1}$$

$$\le \omega_d m^{1-\delta} \beta^{k-1} (\tfrac{\omega_d \delta}{\delta-1})^{k-1} \int_{\substack{(0,1)^{k+1}\\ \text{skeleton structure}}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \, g_{\gamma,\gamma/\delta}(u_\ell, u_{\ell-1})^{-\delta} \prod_{\substack{i=1\\ i\neq \ell}}^{k} g_{\gamma,\gamma/\delta}(u_i, u_{i-1})^{-1}.$$

Hence,

$$\int_0^1 \mathrm{d}u_0 \int_{\substack{|x_k|^d > m\\ u_k \in (0,1)}} \mathrm{d}\mathbf{x}_k \, \mathbb{P}_{\mathbf{x}_0, \mathbf{x}_k}\big(\exists \mathbf{x}_1, \dots, \mathbf{x}_{k-1} \colon |x_1|^d, \dots, |x_{k-1}|^d < m, |x_\ell - x_{\ell-1}|^d > \tfrac{m}{k^d}, \mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k]{n} \mathbf{x}_k\big)$$

$$\le m^{1-\delta} \binom{n}{k} \beta^{n+(\delta-1)} k^{d\delta} \widehat{C}^{n-k} (\tfrac{\omega_d \delta}{\delta-1})^k \int_{\substack{(0,1)^{k+1}\\ \text{skeleton structure}}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \, g_{\gamma,\gamma/\delta}(u_\ell, u_{\ell-1})^{-\delta} \prod_{\substack{i=1\\ i\neq \ell}}^{k} g_{\gamma,\gamma/\delta}(u_i, u_{i-1})^{-1}.$$

In the case that $|x_k - x_{k-1}|^d > |x_k|/k$, we obtain similarly

$$\int_0^1 \mathrm{d}u_0 \int_{\substack{|x_k|^d > m\\ u_k \in (0,1)}} \mathrm{d}\mathbf{x}_k \, \mathbb{P}_{\mathbf{x}_0, \mathbf{x}_k}\big(\exists \mathbf{x}_1, \dots, \mathbf{x}_{k-1} \colon |x_1|^d, \dots, |x_{k-1}|^d < m, |x_k - x_{k-1}|^d > \tfrac{|x_k|^d}{k^d}, \mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k]{n} \mathbf{x}_k\big)$$

$$\le \binom{n}{k} (\beta \widehat{C})^{n-k} \beta^\delta k^{d\delta} \int_{\substack{(0,1)^{k+1}\\ \text{skeleton structure}}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \, g_{\gamma,\gamma/\delta}(u_\ell, u_{\ell-1})^{-\delta} \Big(\int_{|x_k|^d > m} |x_k|^{-d\delta} \Big)$$

$$\times \int_{\substack{\mathcal{B}(m^{1/d})^{k-1}\\ |x_k|^d > m}} \bigotimes_{j=1}^{k} \mathrm{d}x_j \prod_{\substack{i=1\\ i\neq \ell}}^{k} \rho\big(\beta^{-1} g_{\gamma,\gamma/\delta}(u_{i-1}, u_i) |x_{i-1} - x_i|^d\big)$$

$$\le m^{1-\delta} \binom{n}{k} \beta^{n+(\delta-1)} k^{d\delta} \widehat{C}^{n-k} (\tfrac{\omega_d \delta}{\delta-1})^k \int_{\substack{(0,1)^{k+1}\\ \text{skeleton structure}}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \, g_{\gamma,\gamma/\delta}(u_\ell, u_{\ell-1})^{-\delta} \prod_{\substack{i=1\\ i\neq \ell}}^{k} g_{\gamma,\gamma/\delta}(u_i, u_{i-1})^{-1}.$$

Since both cases yield the same upper bound, we infer

$$\mathbb{P}_o(\mathscr{M}_\beta > m)$$

$$\le m^{1-\delta} \sum_{\substack{n\in\mathbb{N}\\ k \le n}} \beta^{n+(\delta-1)} k^{d\delta} \widehat{C}^{n-k} (\tfrac{\omega_d \delta}{\delta-1})^k \sum_{\ell=1}^{k} \int_{\substack{(0,1)^{k+1}\\ \text{skeleton}\\ \text{structure}}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \, g_{\gamma,\gamma/\delta}(u_\ell, u_{\ell-1})^{-\delta} \prod_{\substack{i=1\\ i\neq \ell}}^{k} g_{\gamma,\gamma/\delta}(u_i, u_{i-1})^{-1}. \tag{14}$$

It remains to calculate the integral with respect to the vertex marks. Denoting by h the index of the smallest vertex marks among the skeleton, the integral under consideration reads

$$\sum_{h=0}^{k} \int\limits_{\substack{1>u_0>u_1>\dots>u_h\\ u_h<u_{h+1}<\dots u_k}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \, g_{\gamma,\gamma/\delta}(u_\ell,u_{\ell-1})^{-\delta} \prod_{\substack{i=1\\ i\neq\ell}}^{k} g_{\gamma,\gamma/\delta}(u_i,u_{i-1})^{-1}.$$

We have to distinguish various cases for ℓ and h. Let us start with the easy case $h=k$. That is, we assume the skeleton's vertex marks are strictly decreasing. The above integral is then bounded from above for any ℓ, using the definition $g_{\gamma,\gamma/\delta}(u,v)=(u\vee v)^{\gamma/\delta}(u\wedge v)^{\gamma}$, cf. (5), and Lemma 3.5

$$\begin{aligned}\int\limits_{u_0>\dots>u_{\ell-2}} \bigotimes_{j=0}^{\ell-2} \mathrm{d}u_j \prod_{i=0}^{\ell-2} u_i^{-\gamma-\gamma/\delta} \int\limits_0^{u_{\ell-2}} \mathrm{d}u_{\ell-1}\, u_{\ell-1}^{-2\gamma} \int\limits_0^{u_{\ell-1}} \mathrm{d}u_\ell\, u_\ell^{-\gamma/\delta-\gamma\delta} \int\limits_{u_{\ell+1}>\dots>u_k} \bigotimes_{j=\ell+1}^{k} \mathrm{d}u_j \prod_{i=\ell+1}^{k} u_i^{-\gamma-\gamma/\delta}\\ \leq \frac{1}{(1-2\gamma)(1-\gamma(\delta+1/\delta))}\Big(\frac{\delta}{\delta-\gamma(\delta+1)}\Big)^k,\end{aligned}$$

since $\gamma<1/(\delta+1)<1/2<\delta/(\delta+1)$ implying particularly $1-\gamma(\delta+1/\delta)>0$ and $1-2\gamma>0$. It is easy to see that the case $h=0$ and any ℓ yields the same bound. We consider next, the case $h\notin\{0,k\}$. Here, our calculations depend on the relation between h and ℓ. We start with the case $\ell\in\{h+2,\dots,k\}$ and obtain the upper bound

$$\begin{aligned}\int\limits_{u_0>\dots>u_{h-1}} \bigotimes_{j=0}^{h-1} \mathrm{d}u_j \prod_{i=0}^{h-1} u_i^{-\gamma-\gamma/\delta} \int\limits_0^{u_{h-1}} \mathrm{d}u_h\, u_h^{-2\gamma} \int\limits_{u_{h+1}<\dots<u_{\ell-2}} \bigotimes_{j=h+1}^{\ell-2} \mathrm{d}u_j \prod_{i=h+1}^{\ell-2} u_i^{-\gamma-\gamma/\delta}\\ \times \int\limits_{u_{\ell-2}}^{1} \mathrm{d}u_{\ell-1}\, u_{\ell-1}^{-\gamma(\delta+1/\delta)} \int\limits_{u_{\ell-1}}^{1} \mathrm{d}u_\ell\, u_\ell^{-2\gamma} \int\limits_{u_{\ell+1}<\dots<u_k} \bigotimes_{j=\ell+1}^{k} \mathrm{d}u_j \prod_{i=\ell+1}^{k} u_i^{-\gamma-\gamma/\delta}\\ \leq \frac{1}{(1-2\gamma)^2(1-\gamma(\delta+1/\delta)}\Big(\frac{\delta}{\delta-\gamma(\delta+1)}\Big)^k.\end{aligned}$$

Again, the case $\ell\in\{1,\dots,h-1\}$ yields the same bound and only the cases $\ell\in\{h,h+1\}$ remain. We start with the case $\ell=h$. Then, the integral under consideration is bounded by

$$\begin{aligned}\int\limits_{u_0>\dots>u_{h-2}} \bigotimes_{j=0}^{h-2} \mathrm{d}u_j \prod_{i=0}^{h-2} u_i^{-\gamma-\gamma/\delta} \int\limits_0^{u_{h-2}} \mathrm{d}u_{h-1}\, u_{h-1}^{-2\gamma} \int\limits_0^{u_{h-1}} \mathrm{d}u_h\, u_h^{-\gamma-\gamma\delta} \int\limits_{u_{h+1}<\dots<u_k} \bigotimes_{j=h+1}^{k} \mathrm{d}u_j \prod_{i=h+1}^{k} u_i^{-\gamma-\gamma/\delta}\\ \leq \frac{1}{(1-2\gamma)(1-\gamma(\delta+1))}\Big(\frac{\delta}{\delta-\gamma(\delta+1)}\Big)^k,\end{aligned} \tag{15}$$

as $\gamma<1/(\delta+1)$. Finally, for $\ell=h+1$, we obtain the same bound with the same calculations. Hence, summing over h, we obtain

$$\sum_{h=0}^{k} \int\limits_{\substack{1>u_0>u_1>\dots>u_h\\ u_h<u_{h+1}<\dots u_k}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \, g_{\gamma,\gamma/\delta}(u_\ell,u_{\ell-1})^{-\delta} \prod_{\substack{i=1\\ i\neq\ell}}^{k} g_{\gamma,\gamma/\delta}(u_i,u_{i-1})^{-1} \leq \tfrac{k+1}{(1-2\gamma)^2(1-\gamma(\delta+1))}\Big(\tfrac{\delta}{\delta-\gamma(\delta+1)}\Big)^k.$$

Plugging this into (14) yields

$$\mathbb{P}_o(\mathscr{M}_\beta>m)\leq m^{1-\delta}\sum_{n\in\mathbb{N}} \tfrac{\beta^{\delta-1}(n+1)n^{d\delta}}{(1-2\gamma)^2(1-\gamma(\delta+1))}\beta^n \sum_{k=1}^{n}\binom{n}{k}\widehat{C}^{n-k}\Big(\tfrac{\omega_d\delta^2}{(\delta-1)(\delta-\gamma(\delta+1))}\Big)^k \leq C_2 m^{1-\delta},$$

where

$$C_2 = \frac{\beta^{\delta-1}}{(1-2\gamma)^2(1-\gamma(\delta+1))} \sum_{n\in\mathbb{N}} (n+1) n^{d\delta} (\tfrac{\beta}{\beta_0})^n \tag{16}$$

is a finite constant as $\beta < \beta_0$. □

3.2.3 Proof of the upper bound in Part *(iii)*

In this section, we prove the upper bound of Part *(iii)* of Theorem 2.1 following the strategy outlined in Section 2.5. We fix the following notation throughout this section. As above, we set

$$\beta_0 := \tfrac{1}{2^{d\delta+3}+1} \cdot \tfrac{\delta-1}{\omega_d \delta}\big(1-\gamma\tfrac{\delta+1}{\delta}\big) = \tfrac{(\delta-1)(\delta-\gamma(\delta+1))}{(2^{d\delta+3}+1)\omega_d\delta^2}.$$

Recall the notation of $\mathscr{C}_\beta = \mathscr{C}_{\beta,\gamma,0,\alpha}(\mathbf{o})$ for the component of the origin in the soft Boolean model and the notation $\mathscr{M}_\beta = \mathscr{M}_{\beta,\gamma,0,\delta}$ for its Euclidean diameter. Recall further

$$\zeta = (\delta-1)/(\gamma\delta) \quad \text{and} \quad s_m = m^{-\zeta}.$$

We start by bounding the probability of the presence of powerful vertices inside the component of the origin. Here, powerful refers to a vertex mark no larger than s_m as demanded in the lower bound. We show that for $\beta < \beta_0$, the presence of such a vertex in the whole component is as unlikely as its presence in the direct neighbourhood of the origin and hence the probability of this event matches the probability of the lower bound. Afterwards, we deduce bounds for all other paths connecting $\mathbf{o}$ to distance $m^{1/d}$ when no powerful vertex is present depending on the occurrence of long edges.

Powerful vertices contained in the component of the origin. Let us now define the event that there exists a powerful vertex in the component of the origin as

$$F(m) := \big\{\exists \mathbf{x} \in \mathscr{C}_\beta : u_x < s_m\big\} \cap \{U_o > s_m\} = \big\{\exists \mathbf{x} : u_x < s_m \text{ and } \mathbf{o} \xleftrightarrow[\mathbb{R}^d\times(s_m,1)]{} \mathbf{x}\big\} \cap \{U_o > s_m\}. \tag{17}$$

Here, we work on the event that, the origin is not itself a powerful vertex since this only happens with probability s_m which is of lower order than the desired one. For the second equality observe that if such a powerful vertex $\mathbf{x}$ belongs to the component of the origin, there must be a path connecting $\mathbf{o}$ and this $\mathbf{x}$ using only vertices with marks larger than s_m. If one of the previous vertices had mark smaller than s_m itself, we would have found such a powerful vertex in the component already and would not have had to consider the remaining part of the path. The following lemma coincides with 2.5.

Lemma 3.7. *Consider the soft Boolean model $\mathcal{G}^\beta_\mathbf{o} = \mathcal{G}^{\beta,\gamma,0,\delta}(\xi_\mathbf{o})$ with $\delta > 1$, $\gamma < \delta/(\delta+1)$, and $\beta < \beta_0$. Then, there exists $M > 1$ such that for all $m > M$, we have*

$$\mathbb{P}_o\big(F(m)\big) \le C_6^{(1)} m^{-(1-\gamma)\zeta},$$

where $C_6^{(1)}$ is given below in (18).

Proof. Observe that the skeleton of each path considered in $F(m)$ has necessarily a skeleton with decreasing vertex marks. This is due to the fact that the target vertex $\mathbf{x}$ is the most powerful vertex on the path. Using Mecke's equation [32], (12), and (13), we hence immediately infer

$$\mathbb{P}_o(F(m)) \le \int_{s_m}^{1} du_o \int_{\mathbb{R}^d\times(0,s_m)} \mathrm{d}\mathbf{x}\,\mathbb{P}_{\mathbf{o},\mathbf{x}}\big(\mathbf{o} \xleftrightarrow[B(m^{1/d})\times(s_m,1)]{} \mathbf{x} \text{ in } \mathcal{G}^{\beta}_{\mathbf{o},\mathbf{x}}\big)\Big]$$

$$\le \sum_{\substack{n\in\mathbb{N}\\ k\in\{1,\dots,n\}}} \binom{n}{k}(\beta\widehat{C})^{n-k}\int_{s_m}^{1}\mathrm{d}u_0 \int_{\substack{(\mathbb{R}^d)^k\\ u_0>u_1>\dots>u_{k-1}>s_m>u_k}} \bigotimes_{j=1}^{k}\mathrm{d}\mathbf{x}_j \prod_{i=1}^{k}\rho\big(\tfrac{1}{\beta}u_j^{\gamma}u_{j-1}^{\gamma/\delta}|x_j-x_{j-1}|^d\big),$$

where we again identified $\mathbf{o}=\mathbf{x}_0$. We again make use of the independence of vertex locations and marks and use that $\mathrm{d}\mathbf{x}_k=\mathrm{d}x_k\mathrm{d}u_k$ to perform the change of variables $z_i=\beta^{-1/d}u_i^{\gamma/d}u_{j-1}^{\gamma/d\delta}(x_i-x_{i-1})$ together with $\int\rho(|x|^d)\mathrm{d}x=(\omega_d\delta)/(\delta-1)$, and Lemma 3.5, to obtain

$$\begin{aligned}
\mathbb{P}_o(F(m)) &\le \sum_{\substack{n\in\mathbb{N}\\ k\in\{1,\dots,n\}}} \beta^n\binom{n}{k}\widehat{C}^{n-k}\big(\tfrac{\omega_d\delta}{\delta-1}\big)^k \int_{s_m}^{1}\mathrm{d}u_0\,u_0^{-\gamma/\delta}\int_{s_m}^{u_0}\mathrm{d}u_1\cdots\int_{s_m}^{u_{k-2}}\mathrm{d}u_{k-1}\prod_{j=1}^{k-1}u_j^{-\gamma-\gamma/\delta}\int_0^{s_m}\mathrm{d}u_k\,u_k^{-\gamma}\\
&\le \tfrac{\delta-\gamma(\delta+1)}{(\delta-\gamma))(1-\gamma)}s_m^{1-\gamma}\sum_{n\in\mathbb{N}}\beta^n\sum_{k=0}^{n}\binom{n}{k}\widehat{C}^{n-k}\big(\tfrac{\omega_d\delta^2}{(\delta-1)(\delta-\gamma(\delta+1))}\big)^k\\
&\le \tfrac{\delta-\gamma(\delta+1)}{(\delta-\gamma)(1-\gamma)}s_m^{1-\gamma}\sum_{n=0}^{\infty}\big(\tfrac{\beta}{\beta_0}\big)^n=\big(\tfrac{\delta-\gamma(\delta+1)}{(\delta-\gamma)(1-\gamma)}\cdot\tfrac{\beta_0}{\beta_0-\beta}\big)s_m^{1-\gamma},
\end{aligned}$$

where we used the definition of $\widehat{C}$ from Lemma 3.4 and $\beta<\beta_0$. Hence, choosing

$$C_6^{(1)}=\tfrac{(\delta-1)(\delta-\gamma(\delta+1))^2}{(\delta-\gamma)(1-\gamma)(2^{d\delta+3}+1)\omega_d\delta^2}\cdot\tfrac{1}{\beta_0-\beta} \tag{18}$$

concludes the proof. □

Let us shortly remark that further restricting the target vertex $\mathbf{x}$ in the event $F(m)$ to be located in $\mathcal{B}(m^{1/d})$ corresponds to a version of the lower-bound strategy. More precisely, we define

$$E(m^{1/d}):=\big\{\exists\mathbf{x}\in\mathbf{X}:|x|^d<m,u_x<s_m,\text{ and } \mathbf{o}\xleftrightarrow[B(m^{1/d})\times(s_m,1)]{}\mathbf{x}\text{ in }\mathcal{G}^{\beta}_{\mathbf{o}}\big\}\cap\{U_o>s_m\}.$$

Clearly, $E(m^{1/d})\subset F(m)$ and each vertex $\mathbf{x}$ located in $B(m^{1/d})$ fulfilling the event $E(m^{1/d})$ is connected to $\mathbf{o}$ but also to some vertex at distance $m^{1/d}$ of $\mathbf{o}$ with a constant probability by our calculations in the proof of Proposition 3.2.

The occurrence of long edges in the component of the origin. In this section, we deal with the case that the origin is connected to a vertex at distance $m^{1/d}$, where we can assume that no powerful vertex, as defined in the previous section, is used. More precisely, we now work on the event that $F(m)$ does not occur. In order to do so, we have to bound the occurrence of long edges in the component of the origin. Those edges play a crucial rule as they contribute significantly to the Euclidean diameter of the component. Since we have no access to very powerful vertices for the strategy considered in this section, it is not a priori clear, where these long edges occur in a path. As outlined in Section 2.5, we distinguish the cases whether the path connecting $\mathbf{o}$ to distance $m^{1/d}$ ends within distance $2m^{1/d}$ or further away. Additionally, we consider the event that there are no edges longer than $d(m)$ present in $\mathscr{C}^\beta \cap (\mathcal{B}(2m^{1/d}) \times (s_m, 1))$, where $d(m)$ is some real number in $(\beta m^{\gamma\zeta}, m]$ to be specified later. With this at hand, we define the three events

$$\begin{aligned}
G(m^{1/d}) &:= \big\{\exists \mathbf{x}\colon m < |x|^d \le 2^d m, u_x > s_m \text{ and } \mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{} \mathbf{x} \text{ in } \mathcal{G}^\beta_{\mathbf{o}}\big\} \cap \{U_o > s_m\},\\
H(m^{1/d}) &:= \big\{\exists \mathbf{x},\mathbf{y}\colon |x|^d > 2^d m, |y|^d < m; u_y, u_x > s_m \text{ with}\\
&\qquad \mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{} \mathbf{y},\ \text{and } \mathbf{y} \sim \mathbf{x} \text{ in } \mathcal{G}^\beta_{\mathbf{o}}\big\} \cap \{U_o > s_m\},\\
I(d(m)^{1/d}) &:= \big\{\exists \mathbf{x},\mathbf{y}\colon |x|^d, |y|^d \le 2^d m, u_x, u_y > s_m, |y-x|^d > d(m),\\
&\qquad \mathbf{o} \xleftrightarrow[\mathcal{B}(2m^{1/d})\times(s_m,1)]{} \mathbf{x} \sim \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{o}}\big\} \cap \{U_o > s_m\}.
\end{aligned} \tag{19}$$

Let us recall one last time that $\beta_0 := \frac{(\delta-1)(\delta-\gamma(\delta+1))}{(2^{d\delta+3}+1)\omega_d\delta^2}$, $\zeta = (\delta-1)/(\gamma\delta)$ and $s_m = m^{-\zeta}$. We start by bounding the probability of the event $H(m^{1/d})$ which is 2.5 in the outlined strategy.

Lemma 3.8. *Consider the soft Boolean model $\mathcal{G}^\beta_{\mathbf{o}} = \mathcal{G}^{\beta,\gamma,0,\delta}(\xi_{\mathbf{o}})$ with $\delta > 1$, $1/(\delta+1) < \gamma < \delta/(\delta+1)$, and $\beta < \beta_0$. Then, there exists $M > 1$ such that, for all $m > M$, we have*

$$\mathbb{P}_o\big(H(m^{1/d})\big) \le C_6^{(2)} m^{-(1-\gamma)\zeta},$$

where $C_6^{(2)}$ is given below in (31).

Proof. Since the path considered in the event $H(m^{1/d})$ lies in the ball $\mathcal{B}(m^{1/d})$ until the final step that then connects to some vertex at distance $2m^{1/d}$, we can write using the conditional independence of edges and Mecke's equation

$$\begin{aligned}
&\mathbb{P}_o\big(H(m^{1/d})\big)\\
&\le \int_{s_m}^1 \mathrm{d}u_o \int_{|x|^d>2^d m} \mathrm{d}x \int_{s_m}^1 \mathrm{d}u_x\, \mathbb{P}_{\mathbf{o},\mathbf{x}}\big(\exists \mathbf{y} : |y|^d < m, \mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{} \mathbf{y} \sim \mathbf{x} \text{ in } \mathcal{G}^\beta_{\mathbf{o},\mathbf{x}}\big)\\
&\le \int_{s_m}^1 \mathrm{d}u_o \int_{|y|^d<m} \mathrm{d}y \int_{s_m}^1 \mathrm{d}u_y \int_{|x|^d>2^d m} \mathrm{d}x \int_{s_m}^1 \mathrm{d}u_x\, \mathbb{P}_{\mathbf{o},\mathbf{y}}\big(\mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{} \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{o},\mathbf{y}}\big)\big(\tfrac{1}{\beta}(u_x \wedge u_y)^\gamma |x-y|^d\big)^{-\delta}\\
&\le \beta^\delta \int_{s_m}^1 \mathrm{d}u_o \int_{|y|^d<m} \mathrm{d}y \int_{s_m}^1 \mathrm{d}u_y \mathbb{P}_{\mathbf{o},\mathbf{y}}\big(\mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{} \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{o},\mathbf{y}}\big)\\
&\qquad\qquad \times \int_{s_m}^1 \mathrm{d}u_x (u_x \wedge u_y)^{-\gamma\delta} \int_{|x-y|^d>m} \mathrm{d}x\, |x-y|^{-d\delta}.
\end{aligned}$$

The integral with respect to $\mathbf{x} = (x, u_x)$ reads

$$\begin{aligned}
\int_{s_m}^1 \mathrm{d}u_x (u_x \wedge u_y)^{-\gamma\delta} \int_{|x-y|^d>m} \mathrm{d}x\, |x-y|^{-d\delta} &= \tfrac{\omega_d}{\delta-1} m^{1-\delta} \Bigg[\int_{s_m}^{u_y} \mathrm{d}u_x\, u_x^{-\gamma\delta} + \int_{u_y}^1 \mathrm{d}u_x\, u_y^{-\gamma\delta}\Bigg]\\
&\le \tfrac{\omega_d}{\delta-1} m^{1-\delta} \Big[\tfrac{u_y^{1-\gamma\delta} \vee s_m^{1-\gamma\delta}}{|1-\gamma\delta|} + u_y^{-\gamma\delta}\Big]\\
&\le \big(1 + \tfrac{1}{|1-\gamma\delta|}\big) \tfrac{\omega_d}{\delta-1} u_y^{-\gamma\delta} m^{1-\delta} + \tfrac{\omega_d}{|1-\gamma\delta|(\delta-1)} m^{1-\delta-\zeta(1-\gamma\delta)},
\end{aligned}$$

using $s_m = m^{-\zeta}$ in the last step. To finish the proof, we therefore have to bound the two terms

$$\frac{\beta^\delta \omega_d}{|1-\gamma\delta|(\delta-1)} m^{1-\delta-\zeta(1-\gamma\delta)} \int_{m^{-\zeta}}^{1} \mathrm{d}u_o \int_{|y|^d<m} \mathrm{d}y \int_{m^{-\zeta}}^{1} \mathrm{d}u_y \, \mathbb{P}_{\mathbf{o},\mathbf{y}}\big(\mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{} \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{o},\mathbf{y}}\big) \tag{20}$$

and

$$\big(1+\tfrac{1}{|1-\gamma\delta|}\big)\frac{\beta^\delta \omega_d}{\delta-1} m^{1-\delta} \int_{m^{-\zeta}}^{1} \mathrm{d}u_o \int_{|y|^d<m} \mathrm{d}y \int_{m^{-\zeta}}^{1} \mathrm{d}u_y \, u_y^{-\gamma\delta}\mathbb{P}_{\mathbf{o},\mathbf{y}}\big(\mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{} \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{o},\mathbf{y}}\big). \tag{21}$$

Step 1: Bounding the integral in (20). We use the skeleton strategy and deduce by (12) and (13) together with the same change of variables as above

$$\begin{aligned}
&\int_{s_m}^{1} \mathrm{d}u_o \int_{|y|^d<m} \mathrm{d}y \int_{s_m}^{1} \mathrm{d}u_y \, \mathbb{P}_{\mathbf{o},\mathbf{y}}\big(\mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{} \mathbf{y} \text{ in } \widehat{\mathcal{G}}^\beta_{\mathbf{o},\mathbf{y}}\big) \\
&\leq \sum_{\substack{n\in\mathbb{N}\\k\leq n}} \binom{n}{k} (\beta\widehat{C})^{n-k} \int_{s_m}^{1} \mathrm{d}u_0 \int_{\substack{(\mathcal{B}(m^{1/d})\times(s_m,1))^k\\ \mathbf{x}_0,\dots,\mathbf{x}_k \text{ form skeleton}}} \bigotimes_{j=1}^{k} \mathrm{d}\mathbf{x}_j \prod_{j=1}^{k} \rho\big(\beta^{-1} g_{\gamma,\gamma/\delta}(u_{j-1},u_j)|x_j-x_{j-1}|^d\big) \\
&\leq \sum_{\substack{n\in\mathbb{N}\\k\leq n}} \binom{n}{k} \beta^n \widehat{C}^{n-k} \big(\tfrac{\omega_d\delta}{\delta-1}\big)^k \int_{\substack{(s_m,1)^{k+1}\\ u_0,\dots,u_k \text{ have skeleton structure}}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \prod_{j=1}^{k} g_{\gamma,\gamma/\delta}(u_{j-1},u_j)^{-1}.
\end{aligned} \tag{22}$$

As above in the proof of Proposition 3.6, we decompose the integral by considering skeletons where the h-th vertex has the minimum mark and sum over all possible skeletons to obtain

$$\begin{aligned}
&\int_{\substack{(s_m,1)^{k+1}\\ u_0,\dots,u_k \text{ have skeleton structure}}} \bigotimes_{j=0}^{k} \mathrm{d}u_j \prod_{j=1}^{k} g_{\gamma,\gamma/\delta}(u_{j-1},u_j)^{-1} \\
&\leq \sum_{h=0}^{k} \int_{s_m}^{1} \mathrm{d}u_0 \int_{s_m}^{u_0} \mathrm{d}u_1 \cdots \int_{s_m}^{u_{h-1}} \mathrm{d}u_h \int_{u_h}^{1} \mathrm{d}u_{h+1} \cdots \int_{u_{k-1}}^{1} \mathrm{d}u_k \prod_{j=1}^{k} g_{\gamma,\gamma/\delta}(u_{j-1},u_j)^{-1}.
\end{aligned}$$

We have to distinguish the three cases, whether the minimum mark vertex is the first vertex, the last vertex, or some vertex in between. For $h=0$ the integral reads

$$\int_{s_m}^{1} \mathrm{d}u_0 \int_{u_0}^{1} \mathrm{d}u_1 \cdots \int_{u_{k-1}}^{1} \mathrm{d}u_k u_0^{-\gamma} \Big(\prod_{j=1}^{k-1} u_j^{-\gamma/\delta-\gamma}\Big) u_k^{-\gamma/\delta} \leq \tfrac{\delta}{\delta-\gamma}\big(\tfrac{\delta}{\delta-\gamma(\delta+1)}\big)^{k-1}\tfrac{1}{1-\gamma}$$

using Lemma 3.5. For $h=k$ similarly

$$\int_{s_m}^{1} \mathrm{d}u_0 \int_{s_m}^{u_0} \mathrm{d}u_1 \cdots \int_{s_m}^{u_{k-1}} \mathrm{d}u_k u_k^{-\gamma} \Big(\prod_{j=1}^{k-1} u_j^{-\gamma/\delta-\gamma}\Big) u_0^{-\gamma/\delta} \leq \tfrac{\delta}{\delta-\gamma}\big(\tfrac{\delta}{\delta-\gamma(\delta+1)}\big)^{k-1}\tfrac{1}{1-\gamma}.$$

Finally, for all other values of h, we infer

$$\int_{s_m}^{1}\mathrm{d}u_0\int_{s_m}^{u_0}\mathrm{d}u_1\cdots\int_{s_m}^{u_{h-1}}\mathrm{d}u_h\int_{u_h}^{1}\mathrm{d}u_{h+1}\cdots\int_{u_{k-1}}^{1}\mathrm{d}u_k u_0^{-\gamma}\Big(\prod_{j=1}^{h-1}u_j^{-\gamma/\delta-\gamma}\Big)u_h^{-2\gamma}\Big(\prod_{j=h+1}^{k-1}u_j^{-\gamma-\delta/\gamma}\Big)u_k^{-\gamma/\delta}$$
$$\leq \tfrac{\delta}{\delta-\gamma}\Big(\tfrac{\delta}{\delta-\gamma(\delta+1)}\Big)^{k-2}\tfrac{1}{1-\gamma}\tfrac{1}{|1-2\gamma|}\big(1\vee m^{-\zeta(1-2\gamma)}\big), \tag{23}$$

where we assumed $\gamma\neq 1/2$, and we comment on the excluded case below. If $\gamma<1/2$, the above does not depend on m and since $1-2\gamma<1-\gamma(1+1/\delta)$, the above calculations together with (22) yield the following bound on (20)

$$\tfrac{\beta^\delta\omega_d}{|1-\gamma\delta|(\delta-1)}m^{1-\delta-\zeta(1-\gamma\delta)}\int\limits_{s_m}^{1}\mathrm{d}u_o\int\limits_{|y|^d<m}\mathrm{d}y\int\limits_{s_m}^{1}\mathrm{d}u_y\,\mathbb{P}_{\mathbf{o},\mathbf{y}}\big(\mathbf{o}\xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{}\mathbf{y}\text{ in }\mathcal{G}^\beta_{\mathbf{o},\mathbf{y}}\big)$$
$$\leq m^{1-\delta-\zeta(1-\gamma\delta)}\tfrac{\beta^\delta\omega_d(\delta-\gamma(\delta+1))^2}{\delta|1-\gamma\delta|(\delta-1)(\delta-\gamma)(1-\gamma)|1-2\gamma|}\sum_{n\in\mathbb{N}}\beta^n\sum_{k=0}^{n}(k+1)\binom{n}{k}\widehat{C}^{n-k}\Big(\tfrac{\omega_d\delta^2}{(\delta-\gamma(\delta+1))(\delta-1)}\Big)^k,$$
$$\leq m^{1-\delta-\zeta(1-\gamma\delta)}\tfrac{\beta^\delta\omega_d(\delta-\gamma(\delta+1))^2}{\delta|1-\gamma\delta|(\delta-1)(\delta-\gamma)(1-\gamma)|1-2\gamma|}\sum_{n\in\mathbb{N}}(n+1)\Big(\tfrac{\beta}{\beta_0}\Big)^n, \tag{24}$$

using the form of $\widehat{C}$. Further, the sum is finite as $\beta<\beta_0$. Using $\zeta=(\delta-1)/(\gamma\delta)$, we find for the order in m

$$1-\delta-\zeta(1-\gamma\delta)=-\zeta<-(1-\gamma)\zeta.$$

For $1/2<\gamma<\delta/(\delta+1)$, we infer with the same calculations

$$\tfrac{\beta^\delta\omega_d}{|1-\gamma\delta|(\delta-1)}m^{1-\delta-\zeta(1-\gamma\delta)}\int\limits_{s_m}^{1}\mathrm{d}u_o\int\limits_{|y|^d<m}\mathrm{d}y\int\limits_{s_m}^{1}\mathrm{d}u_y\,\mathbb{P}_{\mathbf{o},\mathbf{y}}\big(\mathbf{o}\xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{}\mathbf{y}\text{ in }\mathcal{G}^\beta_{\mathbf{o},\mathbf{y}}\big)$$
$$\leq m^{1-\delta-\zeta(1-\gamma\delta)-\zeta(1-2\gamma)}\tfrac{\beta^\delta\omega_d(\delta-\gamma(\delta+1))^2}{\delta|1-\gamma\delta|(\delta-1)(\delta-\gamma)(1-\gamma)|1-2\gamma|}\sum_{n\in\mathbb{N}}(n+1)\Big(\tfrac{\beta}{\beta_0}\Big)^n \tag{25}$$

and it remains to check the order in m for which we find

$$1-\delta-\zeta(1-\gamma\delta)-\zeta(1-2\gamma)=-\zeta-\zeta(1-2\gamma)=-2(1-\gamma)\zeta<-(1-\gamma)\zeta.$$

If $\gamma=1/2$ in (24) only the $|1-2\gamma|^{-1}$ term in the constant has to be replaced by $\zeta\log(m)$. Hence, for $\beta<\beta_0$, there are constants $c,C>0$ and some $\zeta'>\zeta$ such that we find for (20)

$$cm^{1-\delta-\zeta(1-\gamma\delta)}\int\limits_{s_m}^{1}\mathrm{d}u_o\int\limits_{|y|^d<m}\mathrm{d}y\int\limits_{s_m}^{1}\mathrm{d}u_y\,\mathbb{P}_{\mathbf{o},\mathbf{y}}\big(\mathbf{o}\xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{}\mathbf{y}\text{ in }\mathcal{G}^\beta_{\mathbf{o},\mathbf{y}}\big)\leq Cm^{-(1-\gamma)\zeta'}. \tag{26}$$

Step 2: Bounding the integral in (21). We infer with the same arguments as above

$$\begin{aligned}
&\big(1+\tfrac{1}{|1-\gamma\delta|}\big)\tfrac{\beta^\delta\omega_d}{\delta-1}m^{1-\delta}\int\limits_{s_m}^{1}\mathrm{d}u_o\int\limits_{|y|^d<m}\mathrm{d}y\int\limits_{s_m}^{1}\mathrm{d}u_y\,u_y^{-\gamma\delta}\mathbb{P}_{\mathbf{o},\mathbf{y}}\big(\mathbf{o}\xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{}\mathbf{y}\text{ in }\mathcal{G}^\beta_{\mathbf{o},\mathbf{y}}\big)\\
&\quad\le\big(1+\tfrac{1}{|1-\gamma\delta|}\big)\tfrac{\beta^\delta\omega_d}{\delta-1}m^{1-\delta}\sum_{n\in\mathbb{N}}\sum_{k=1}^{n}\beta^n\binom{n}{k}\widehat{C}^{n-k}\big(\tfrac{\omega_d\delta}{\delta-1}\big)^k\\
&\qquad\qquad\times\sum_{h=0}^{k}\int\limits_{\substack{u_0\ge\dots\ge u_h\ge s_m\\ u_h\le u_{h+1}\le\dots\le u_k}}\bigotimes_{j=0}^{k}\mathrm{d}u_j\,u_k^{-\gamma\delta}\prod_{j=1}^{k}g_{\gamma,\gamma/\delta}(u_{j-1},u_j)^{-1},
\end{aligned}\tag{27}$$

and we focus on the integral on the right-hand side. Again, we have to consider the three different cases depending on where the most powerful vertex is located within the path. We start with $h=0$ and infer, again using Lemma 3.5,

$$\int_{s_m}^1\mathrm{d}u_0\int_{u_0}^1\mathrm{d}u_1\cdots\int_{u_{k-1}}^1\mathrm{d}u_k\,u_0^{-\gamma}\Big(\prod_{j=1}^{k-1}u_j^{-\gamma/\delta-\gamma}\Big)u_k^{-\gamma\delta-\gamma/\delta}\le\tfrac{1\vee m^{-\zeta(1-\gamma(\delta+1/\delta))}}{|1-\gamma(\delta+1/\delta)|}\big(\tfrac{\delta}{\delta-\gamma(\delta+1)}\big)^{k-1}\tfrac{1}{1-\gamma}.$$

Hence, with the same arguments as above, combining this with (27), we infer for some constants $c,C>0$ that

$$\begin{aligned}
&cm^{1-\delta}\sum_{n\in\mathbb{N}}\beta^n\sum_{k=1}^{n}\binom{n}{k}\widehat{C}^{n-k}\big(\tfrac{\omega_d\delta}{\delta-1}\big)^k\int_{s_m}^1\mathrm{d}u_0\int_{u_0}^1\mathrm{d}u_1\cdots\int_{u_{k-1}}^1\mathrm{d}u_k\,u_0^{-\gamma}\Big(\prod_{j=1}^{k-1}u_j^{-\gamma/\delta-\gamma}\Big)u_k^{-\gamma\delta-\gamma/\delta}\\
&\quad\le C\big(m^{1-\delta}\vee m^{1-\delta-\zeta(1-\gamma(\delta+1/\delta))}\big)\tfrac{\beta_0}{\beta_0-\beta},
\end{aligned}\tag{28}$$

since $\beta<\beta_0$. To deduce the order in m, we only have to check the case when $m^{1-\delta-\zeta(1-\gamma(\delta+1/\delta))}$ is the dominant term. In the other case we immediately have $m^{1-\delta}<m^{-(1-\gamma)\zeta}$ since $\gamma>1/(\delta+1)$. As

$$1-\delta-\zeta(1-\gamma(\delta+\tfrac{1}{\delta}))=-\zeta-\tfrac{\delta-1}{\delta^2}=-\zeta(1+\tfrac{\gamma}{\delta})<-\zeta(1-\gamma),$$

we find that (28) is bounded by $Cm^{-(1-\gamma)\zeta'}$ for some $\zeta'>\zeta$.

The next case we consider is $1\le h\le k-1$. Then, the integral on the right-hand side of (27) reads

$$\int_{s_m}^1\mathrm{d}u_0\int_{s_m}^{u_0}\mathrm{d}u_1\cdots\int_{s_m}^{u_{h-1}}\mathrm{d}u_h\int_{u_h}^1\mathrm{d}u_{h+1}\cdots\int_{u_{k-1}}^1\mathrm{d}u_ku_0^{-\gamma/\delta}\Big(\prod_{\substack{j=1\\ j\ne h}}^{k-1}u_j^{-\gamma-\gamma/\delta}\Big)u_h^{-2\gamma}u_k^{-\gamma\delta-\gamma/\delta}.$$

We only consider the case when $1-\gamma\delta-\gamma/\delta<0$, since the other case coincides with (23) with a slightly changed constant. Now integrating out u_k and then using that $u_{k-1}\ge u_h$ we infer

$$\begin{aligned}
&\int_{s_m}^1\mathrm{d}u_0\int_{s_m}^{u_0}\mathrm{d}u_1\cdots\int_{s_m}^{u_{h-1}}\mathrm{d}u_h\int_{u_h}^1\mathrm{d}u_{h+1}\cdots\int_{u_{k-1}}^1\mathrm{d}u_k\,u_0^{-\gamma/\delta}\Big(\prod_{\substack{j=1\\ j\ne h}}^{k-1}u_j^{-\gamma-\gamma/\delta}\Big)u_h^{-2\gamma}u_k^{-\gamma\delta-\gamma/\delta}\\
&\le\tfrac{\delta}{\gamma(\delta^2+1)-\delta}\big(\tfrac{\delta}{\delta-\gamma(\delta+1)}\big)^{k-h-1}\int_{s_m}^1\mathrm{d}u_0\int_{s_m}^{u_0}\mathrm{d}u_1\dots\int_{s_m}^{u_{h-1}}\mathrm{d}u_h\,u_0^{-\gamma/\delta}\Big(\prod_{j=1}^{h-1}u_j^{-\gamma-\gamma/\delta}\Big)u_h^{1-2\gamma-\gamma\delta-\gamma/\delta}\\
&\le\tfrac{\delta}{\gamma(\delta^2+1)-\delta}\big(\tfrac{\delta}{\delta-\gamma(\delta+1)}\big)^{k-2}\tfrac{\delta}{\delta-\gamma}m^{-\zeta(2-2\gamma-\gamma\delta-\gamma/\delta)},
\end{aligned}$$

where we have again restricted ourselves to the case $2-2\gamma-\gamma\delta-\gamma/\delta<0$ since otherwise the $\gamma<1/2$ case of (23) applies again. As above, combining with (27) and since $\beta<\beta_0$, we find constants $c, C>0$ such that

$$\begin{aligned} cm^{1-\delta}\sum_{n\in\mathbb{N}}\beta^n\sum_{k=1}^{n}\binom{n}{k}\widehat{C}^{n-k}\big(\tfrac{\omega_d\delta}{\delta-1}\big)^k\sum_{h=1}^{k-1}\int\limits_{\substack{u_0\geq\cdots\leq u_h\geq s_m\\ u_h\leq u_{h+1}\leq\cdots\leq u_k}}\bigotimes_{j=0}^{k}\mathrm{d}u_j\; u_k^{-\gamma\delta}\prod_{j=1}^{k}g_{\gamma,\gamma/\delta}(u_{j-1},u_j)^{-1} \\ \leq Cm^{1-\delta-\zeta(2-2\gamma-\gamma\delta-\gamma/\delta)}<Cm^{-(1-\gamma)\zeta'} \end{aligned} \tag{29}$$

for some $\zeta'>\zeta$ since

$$1-\delta-\zeta(2-2\gamma-\gamma\delta-\gamma/\delta)=-2\zeta(1-\gamma)+\tfrac{\gamma}{\delta}\zeta<-(1-\gamma)\zeta,$$

as $\gamma<\delta/(\delta+1)$.

It remains to bound the case when the most powerful vertex of the path is the end vertex. This case can be seen as a pendant strategy to the one used in the lower bound respectively Lemma 3.7: Namely, instead of connecting the origin to a vertex with mark slightly more powerful than s_m, which then is connected to distant vertices, we now connect to vertices slightly less powerful. We shall see however that this strategy dominates in probability all former ones and is of the same order as the one from the lower bound. For $h=k$ the integral in (27) under consideration reads

$$\int_{s_m}^{1}\!\mathrm{d}u_0\int_{s_m}^{u_0}\!\mathrm{d}u_1\cdots\int_{s_m}^{u_{k-1}}\!\mathrm{d}u_k\;u_0^{-\gamma/\delta}\Big(\prod_{j=1}^{k-1}u_j^{-\gamma-\gamma/\delta}\Big)u_k^{-\gamma\delta-\gamma}\leq\tfrac{1}{\gamma(\delta+1)-1}\big(\tfrac{\delta}{\delta-\gamma(\delta+1)}\big)^{k-1}\tfrac{\delta}{\delta-\gamma}m^{-\zeta(1-\gamma(\delta+1))},$$

where we have used that $\gamma>1/(\delta+1)$. Plugging this into the right-hand side of (27) yields, by a similar calculation as before,

$$\begin{aligned} &\big(1+\tfrac{1}{|1-\gamma\delta|}\big)\tfrac{\beta^\delta\omega_d}{\delta-1}m^{1-\delta}\sum_{n\in\mathbb{N}}\beta^n\sum_{k=1}^{n}\binom{n}{k}\widehat{C}^{n-k}\big(\tfrac{\omega_d\delta}{\delta-1}\big)^k \\ &\qquad\times\int_{s_m}^{1}\mathrm{d}u_0\int_{s_m}^{u_0}\mathrm{d}u_1\cdots\int_{s_m}^{u_{k-1}}\mathrm{d}u_k\bigotimes_{j=0}^{k}\mathrm{d}u_j\;u_0^{-\gamma/\delta}\Big(\prod_{j=1}^{k-1}u_j^{-\gamma-\gamma/\delta}\Big)u_k^{-\gamma\delta-\gamma} \\ &\leq m^{-(1-\gamma)\zeta}\big(1+\tfrac{1}{|1-\gamma\delta|}\big)\tfrac{\beta^\delta\omega_d(\delta-\gamma(\delta+1))}{(\delta-1)(\gamma(\delta+1)-1)(\delta-\gamma)}\cdot\tfrac{\beta_0}{\beta_0-\beta}, \end{aligned} \tag{30}$$

using again $\beta<\beta_0$ and $1-\delta-\zeta(1-\gamma(\delta+1))=(1-\gamma)\zeta$. Hence, combining (28), (29) and (30), we find that the term (21) is bounded by

$$Cm^{-(1-\gamma)\zeta'}+m^{-(1-\gamma)\zeta}\big(1+\tfrac{1}{|1-\gamma\delta|}\big)\tfrac{\beta^\delta(\delta-\gamma(\delta+1))^2}{(\gamma(\delta+1)-1)(\delta-\gamma)(1+2^{d\delta+3})\delta^2}\cdot\tfrac{1}{\beta_0-\beta},$$

where $\zeta'>\zeta$ and hence $m^{-(1-\gamma)\zeta'}$ is of strictly smaller order than $m^{-(1-\gamma)\zeta}$. We set

$$C_6^{(2)}:=2\big(1+\tfrac{1}{|1-\gamma\delta|}\big)\tfrac{\beta^\delta(\delta-\gamma(\delta+1))^2}{(\gamma(\delta+1)-1)(\delta-\gamma)\delta^2}\cdot\tfrac{1}{\beta_0-\beta} \tag{31}$$

and infer for $\beta<\beta_0$ and sufficiently large m

$$\mathbb{P}_o\big(H(m^{1/d})\big)\leq C_6^{(2)}m^{-(1-\gamma)\zeta},$$

as claimed. □

Our next result is to bound the probability of $I(d(m)^{1/d})$, the event that the component of the origin restricted to vertices in $\mathcal{B}(2m^{1/d})$ contains no edges longer than $d(m)^{1/d}$, where $d(m) \in (\beta m^{\gamma\zeta}, m]$, preparing 2.5.

Lemma 3.9. *Consider the soft Boolean model $\mathscr{G}^\beta_{\mathbf{o}} = \mathscr{G}^{\beta,\gamma,0,\delta}(\xi_{\mathbf{o}})$ with $\delta > 1$, $1/(\delta+1) < \gamma < \delta/(\delta+1)$, and $\beta < \beta_0$. Then, there exists $M > 1$ such that, for all $m > M$, we have*

$$\mathbb{P}_o\big(I(d(m)^{1/d})\big) \leq C_6^{(2)}\big(m/d(m)\big)^{\delta-1} m^{-(1-\gamma)\zeta},$$

where $C_6^{(2)}$ is the constant given above in (31)*.*

Proof. We calculate similarly to above

$$\begin{aligned}
&\mathbb{P}_o\big(I(d(m)^{1/d})\big)\\
&\leq \int_{s_m}^{1} \mathrm{d}u_o \int_{|x|^d<2^d m} \mathrm{d}x \int_{s_m}^{1} \mathrm{d}u_x \int_{\substack{|y|^d<2^d m\\ |x-y|^d>d(m)}} \mathrm{d}y \int_{s_m} \mathrm{d}u_y\, \mathbb{P}_{\mathbf{o},\mathbf{x}}\big(\mathbf{o} \xleftrightarrow[\mathcal{B}(2m^{1/d})\times(s_m,1)]{} \mathbf{x} \text{ in } \mathscr{G}^\beta_{\mathbf{o},\mathbf{x}}\big) \tfrac{\beta^\delta}{(u_x\wedge u_y)^{\gamma\delta}|x-y|^{d\delta}}\\
&\leq \beta^\delta \tfrac{\omega_d}{\delta-1} d(m)^{1-\delta} \int_{s_m}^{1} \mathrm{d}u_o \int_{|x|^d<2^d m} \mathrm{d}x \int_{s_m}^{1} \mathrm{d}u_x\, \mathbb{P}_{\mathbf{o},\mathbf{x}}\big(\mathbf{o} \xleftrightarrow[\mathcal{B}(2m^{1/d})\times(s_m,1)]{} \mathbf{x} \text{ in } \mathscr{G}^\beta_{\mathbf{o},\mathbf{x}}\big) \int_{s_m}^{1} \mathrm{d}u_y (u_x\wedge u_y)^{-\gamma\delta}\\
&\leq \big(\tfrac{d(m)}{m}\big)^{1-\delta} \tfrac{\beta^\delta\omega_d}{\delta-1}\Big(\big(1+\tfrac{1}{|1-\gamma\delta|}\big)m^{1-\delta}u_x^{-\gamma\delta} + \tfrac{m^{1-\delta-\zeta(1-\gamma\delta)}}{|1-\gamma\delta|}\Big)\\
&\qquad\times \int_{s_m}^{1} \mathrm{d}u_o \int_{|x|^d<2^d m} \mathrm{d}x \int_{s_m}^{1} \mathrm{d}u_x\, \mathbb{P}_{\mathbf{o},\mathbf{x}}\big(\mathbf{o} \xleftrightarrow[\mathcal{B}(2m^{1/d})\times(s_m,1)]{} \mathbf{x} \text{ in } \mathscr{G}^\beta_{\mathbf{o},\mathbf{x}}\big).
\end{aligned}$$

Therefore, we can perform for the integral the exact same computations as in the previous proof and we infer for $\beta < \beta_0$ and sufficiently large m,

$$\mathbb{P}_o\big(I(d(m)^{1/d})\big) \leq C_6^{(2)}\big(m/d(m)\big)^{\delta-1} m^{-(1-\gamma)\zeta},$$

where $C_6^{(2)}$ is the constant given above in (31). This concludes the proof. □

Finalising the proof of the upper bound in Part *(iii)*. With the results of the previous sections, we have all tools at hand needed to finish the proof of Theorem 2.1. Recall the events $F(m)$ introduced in (17) as well as $G(m^{1/d})$, $H(m^{1/d})$, and $I(m^{1/d})$ introduced in (19). The upper bound in Theorem 2.1 Part *(iii)* is an immediate consequence of the following proposition.

Proposition 3.10. *Consider the soft Boolean model $\mathscr{G}^\beta_{\mathbf{o}} = \mathscr{G}^{\beta,\gamma,0,\delta}(\xi_{\mathbf{o}})$ with $\delta > 1$, $1/(\delta+1) < \gamma < \delta/(\delta+1)$, and $\beta < \beta_0$. Then, there exists $M > 1$ such that, for all $m > M$, we have*

$$\mathbb{P}_o(\mathscr{M}_\beta > m) \leq C_6 m^{-(1-\gamma)\zeta},$$

where C_6 is given below in (32)*.*

Proof. For large m by Lemma 3.7

$$\begin{aligned}
\mathbb{P}_o(\mathscr{M}_\beta > m) &\leq \mathbb{P}_o(U_o \leq s_m) + \mathbb{P}_o(\{\mathscr{M}_\beta > m\} \cap \{U_o > s_m\})\\
&\leq s_m + \mathbb{P}_o(F(m)) + \mathbb{P}_o\big(\{\mathscr{M}_\beta > m\} \cap F(m)^c \cap \{U_o > s_m\}\big)\\
&\leq \tfrac{3}{2} C_6^{(1)} m^{-(1-\gamma)\zeta} + \mathbb{P}_o\big(\{\mathscr{M}_\beta > m\} \cap F(m)^c \cap \{U_o > s_m\}\big).
\end{aligned}$$

To bound the remaining probability, we set $\kappa = \frac{(\gamma\vee(1-\gamma))\zeta}{\log\beta_0-\log\beta}$ and choose $d(m) = \frac{m}{(\kappa\log m)^d}$ in $I(d(m)^{1/d})$. We deduce, using Lemmas 3.8 and 3.9

$$\begin{aligned}&\mathbb{P}_o\big(\{\mathscr{M}_\beta > m\}\cap F(m)^c\cap\{U_o > s_m\}\big)\\ &\quad\le \mathbb{P}_o\big(G(m^{1/d})\big)+\mathbb{P}_o\big(H(m^{1/d})\big)\\ &\quad\le \mathbb{P}_o\big(G(m^{1/d})\cap I(d(m)^{1/d})^c\big)+\tfrac{3}{2}C_6^{(2)}\big(\kappa\log(m)\big)^{d(\delta-1)}m^{-(1-\gamma)\zeta}.\end{aligned}$$

for sufficiently large m. It hence remains to bound the probability of $G(m^{1/d})\cap I(m^{1/d})^c$ which then finishes 2.5. On this event, there exists some path of length at least $\kappa\log(m)$ where no vertex of the path has mark smaller than s_m. Hence, by (12)

$$\mathbb{P}_o\big(G(m^{1/d})\cap I(d(m)^{1/d})^c\big)\le\sum_{n\ge\kappa\log(m)}\sum_{k=1}^{n}\mathbb{E}_o\Big[\sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_k\in\mathbf{X}\\ u_i>s_m\,\forall i}}^{\neq}\mathbb{P}_{\mathbf{x}_0,\dots,\mathbf{x}_k}\big(\mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0,\dots,\mathbf{x}_k]{n} \mathbf{x}_k \text{ in } \widehat{\mathscr{G}}^\beta_{\mathbf{x}_0,\dots,\mathbf{x}_k}\big)\Big].$$

Using again Mecke's equation and the skeleton strategy (13) once more, we infer

$$\begin{aligned}&\sum_{k=1}^{n}\mathbb{E}_o\Big[\sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_k\in\mathbf{X}\\ u_i>s_m\,\forall i}}\mathbb{P}_{\mathbf{x}_0,\dots,\mathbf{x}_k}\big(\mathbf{x}_0 \xleftrightarrow[\mathbf{x}_0,\dots,\mathbf{x}_k]{n} \mathbf{x}_k \text{ in } \widehat{\mathscr{G}}^\beta_{\mathbf{x}_0,\dots,\mathbf{x}_k}\big)\Big]\\ &\quad\le\beta^n\sum_{k=1}^{n}\sum_{h=0}^{k}\binom{n}{k}\widehat{C}^{n-k}\big(\tfrac{\omega_d\delta}{\delta-1}\big)^k\int\limits_{\substack{u_0>u_1>\dots>u_h>s_m\\ u_h<u_{h+1}<\dots<u_k}}\bigotimes_{j=0}^{k}\mathrm{d}u_j\prod_{j=1}^{k}g_{\gamma,\gamma/\delta}(u_{j-1},u_j)^{-1}\\ &\quad\le\big(\tfrac{\delta}{\delta-\gamma}\big)^2\big(\tfrac{\delta-\gamma(\delta+1)}{\delta}\big)^2(1\vee m^{-\zeta(1-2\gamma)})(n+1)\big(\tfrac{\beta}{\beta_0}\big)^n,\end{aligned}$$

where we performed the same calculations as above in (23) to (24) to bound the integral. Since $\beta<\beta_0$, we have by our choice of κ

$$\sum_{n\ge\kappa\log(m)}(n+1)(\tfrac{\beta}{\beta_0})^n\le\int\limits_{\kappa\log(m)}^{\infty}(x+1)e^{-x\log(\beta_0/\beta)}\mathrm{d}x\le\tfrac{3}{2}\tfrac{\gamma\vee(1-\gamma)}{(\log\beta_0-\log\beta)^2}\log(m)m^{-(\gamma\vee(1-\gamma))\zeta}.$$

Therefore,

$$\begin{aligned}\mathbb{P}_o\big(G(m^{1/d})\cap I(d(m)^{1/d})^c\big)&\le\big(\tfrac{\delta}{\delta-\gamma}\big)^2\big(\tfrac{\delta-\gamma(\delta+1)}{\delta}\big)^2(1\vee m^{-\zeta(1-2\gamma)})\sum_{n\ge\kappa\log(m)}(n+1)\big(\tfrac{\beta}{\beta_0}\big)^n\\ &\le\tfrac{3}{2}\big(\tfrac{\delta}{\delta-\gamma}\big)^2\big(\tfrac{\delta-\gamma(\delta+1)}{\delta}\big)^2\tfrac{\gamma\vee(1-\gamma)}{(\log\beta_0-\log\beta)^2}\log(m)m^{-(1-\gamma)\zeta}.\end{aligned}$$

Hence, we set

$$C_6=\begin{cases}2\big(\frac{(\gamma\vee(1-\gamma))\zeta}{\log\beta_0-\log\beta}\big)^{d(\delta-1)}C_6^{(2)}, & \text{if } d(\delta-1)>1,\\ \frac{3}{2}\big(\frac{(\gamma\vee(1-\gamma))\zeta}{\log\beta_0-\log\beta}C_6^{(2)}+\big(\frac{\delta}{\delta-\gamma}\big)^2\big(\frac{\delta-\gamma(\delta+1)}{\delta}\big)^2\frac{\gamma\vee1-\gamma}{(\log\beta_0-\log\beta)^2}\big), & \text{if } d(\delta-1)=1, \text{ and}\\ 2\big(\frac{\delta}{\delta-\gamma}\big)^2\big(\frac{\delta-\gamma(\delta+1)}{\delta}\big)^2\frac{\gamma\vee1-\gamma}{(\log\beta_0-\log\beta)^2}, & \text{if } d(\delta-1)<1.\end{cases}\tag{32}$$

to infer

$$\mathbb{P}_o(\mathscr{M}_\beta>m)\le C_6\log(m)^{1\vee d(\delta-1)}m^{1-\zeta}$$

for large enough m, concluding the proof. □

3.2.4 The upper bound of Part *(ii)*

We shortly explain where we have to adapt our proofs to obtain the upper bound in the boundary case.

Proposition 3.11. *Consider the soft Boolean model $\mathcal{G}_{\mathbf{o}}^{\beta} = \mathcal{G}^{\beta,\gamma,0,\delta}(\xi_{\mathbf{o}})$ with $\delta > 1$, $1/(\delta+1) = \gamma$, and $\beta < \beta_0$. Then, there exists $M > 1$ such that, for all $m > M$, we have*

$$\mathbb{P}_{\beta}(\mathscr{M}_{\beta} > m) \le C_6 \log(m) m^{1-\delta},$$

where C_4 is given below in (33)*.*

Proof. First recall that for $\gamma = 1/(\delta+1)$ the exponents $1-\delta$ and $-(1-\gamma)\zeta$ coincide. Hence, we can work on the event $F(m)$ by Lemma 3.7. We then perform the same proof as for the case $\gamma < 1/(\delta+1)$ in Proposition 3.6 where only now the vertex marks of the skeleton are restricted to the interval $(s_m, 1)$. Observe that for $\gamma = 1/(\delta+1)$ we still have $\gamma < 1/2$ and $1-\gamma(\delta+1/\delta) > 0$. Hence, we only rely in (15) on the condition $\gamma < 1/(\delta+1)$. Performing the same calculations but with the lower integral bound replaced by s_m and using $\gamma = 1/(\delta+1)$, we obtain instead

$$\int\limits_{\substack{u_0 > \cdots > u_{h-2} \\ u_{h-2} > s_m}} \bigotimes_{j=0}^{h-2} \mathrm{d}u_j \prod_{i=s_m}^{h-2} u_i^{-\gamma-\gamma/\delta} \int\limits_{s_m}^{u_{h-2}} \mathrm{d}u_{h-1}\, u_{h-1}^{-2\gamma} \int\limits_{0}^{u_{h-1}} \mathrm{d}u_h\, u_h^{-\gamma-\gamma\delta} \int\limits_{\substack{s_m < u_{h+1} \\ u_{h+1} < \cdots < u_k}} \bigotimes_{j=h+1}^{k} \mathrm{d}u_j \prod_{i=h+1}^{k} u_i^{-\gamma-\gamma/\delta}$$
$$\le -\log(s_m) \frac{1}{1-2\gamma} \Big(\frac{\delta}{\delta - \gamma(\delta+1)}\Big)^k.$$

Due to the additional $-\log(s_m) = \zeta \log(m)$ term, this summand is now clearly the dominant one in the whole sum. Hence, finalising the proof from here as done in Proposition 3.6, we obtain

$$\mathbb{P}_o(\mathscr{M}_{\beta} > m) \le C_4 \log(m) m^{1-\delta},$$

where

$$C_4 := \tfrac{(\delta^2-1)(\delta+1)\beta^{\delta-1}}{\delta(\delta-1)} \sum_{n\in\mathbb{N}} (n+1) n^{d\delta} (\tfrac{\beta}{\beta_0})^n, \tag{33}$$

as $\zeta = (\delta^2-1)/\delta$ and $(1-2\gamma)^{-1} = (\delta+1)/(\delta-1))$, concluding the proof. □

3.2.5 The special case $\gamma < 1/2$

The final result of this section is the following lemma, showing that the logarithmic term in the upper bound of Theorem 2.1 Part *(iii)* can be omitted if $\gamma < 1/2$ at least for a changed critical intensity threshold, as discussed in Section 2.6. The proof of Theorem 2.7 follows immediately by summing the following lemma.

Lemma 3.12. *Define $K = \frac{\omega_d \delta}{\delta-1}\big(\frac{1}{1-2\gamma} + \frac{1}{1-\gamma}\big)$. Let $\beta < K^{-1}$, $\gamma < 1/2$ and $\delta > 1$. Then, there exist constants $K' > 0$ such that, for all $n \in \mathbb{N}$, we have*

(i) for $0 < \gamma < 1/(\delta+1)$

$$\mathbb{P}_o\big(\exists \mathbf{y} : |y|^d > m, u_y > s_m \text{ and } \mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})]{n} \mathbf{y} \text{ in } \mathcal{G}_{\mathbf{o}}^{\beta}\big) \le K' \beta^{\delta-1} n^{d\delta+1} (\beta K)^n \times m^{1-\delta},$$

(ii) for $1/(\delta+1) < \gamma < 1/2$

$$\mathbb{P}_o\big(\exists \mathbf{y} : |y|^d > m, u_y > s_m \text{ and } \mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})]{n} \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{o}}\big) \le K'\beta^{\delta-1}n^{d\delta+1}(\beta K)^n \times m^{-(1-\gamma)\zeta}.$$

Proving the lemma requires bounds on the expected number of paths similar to those in Lemma 3.5 but now for all paths rather than only those with skeleton structure as we no longer make use of the latter. Put differently, we require the following lemma.

Lemma 3.13. *Let* $\gamma < 1/2$ *and* $u_0 \in (0,1)$.

(a) For all $k \in \mathbb{N}$*, we have*

$$\int_0^1 \mathrm{d}u_1 \cdots \int_0^1 \mathrm{d}u_k \prod_{j=1}^k (u_{j-1}\wedge u_j)^{-\gamma} \le \tfrac{2-\gamma}{1-\gamma}\big(\tfrac{1}{1-2\gamma} + \tfrac{1}{1-\gamma}\big)^{k-1} u_0^{-\gamma}.$$

(b) Further, for all $k \in \mathbb{N}$*, we have*

$$\int_0^1 \mathrm{d}u_1 \cdots \int_0^1 \mathrm{d}u_k \Big(\prod_{j=1}^k (u_{j-1}\wedge u_j)^{-\gamma}\Big) u_k^{-\gamma} \le \big(\tfrac{1}{1-2\gamma} + \tfrac{1}{1-\gamma}\big)^{k-1} u_0^{-\gamma}.$$

Proof. We prove the Statement (a) by induction. For $k=1$, we have

$$\int_0^1 \mathrm{d}u_1\, (u_0\wedge u_1)^{-\gamma} \le \int_0^{u_0} u_1^{-\gamma}\mathrm{d}u_1 + u_0^{-\gamma} \le \tfrac{2-\gamma}{1-\gamma}u_0^{-\gamma}.$$

For $k \ge 2$ we have by using the induction hypothesis

$$\begin{aligned}\int_0^1 \mathrm{d}u_1 \cdots \int_0^1 \mathrm{d}u_k \prod_{j=1}^k (u_{j-1}\wedge u_j)^{-\gamma} &\le \tfrac{2-\gamma}{1-\gamma}\big(\tfrac{1}{1-2\gamma} + \tfrac{1}{1-\gamma}\big)^{k-2} \int_0^1 u_1^{-\gamma}(u_0\wedge u_1)^{-\gamma}\mathrm{d}u_1\\ &= \tfrac{2-\gamma}{1-\gamma}\big(\tfrac{1}{1-2\gamma} + \tfrac{1}{1-\gamma}\big)^{k-2}\Big(\int_0^{u_0} u_1^{-2\gamma}\mathrm{d}u_1 + \int_{u_0}^1 u_0^{-\gamma}u_1^{-\gamma}\mathrm{d}u_1\Big)\\ &\le \tfrac{2-\gamma}{1-\gamma}\big(\tfrac{1}{1-2\gamma} + \tfrac{1}{1-\gamma}\big)^{k-2}\big(\tfrac{u_0^{1-2\gamma}}{1-2\gamma} + \tfrac{u_0^{-\gamma}}{1-\gamma}\big)\\ &\le \tfrac{2-\gamma}{1-\gamma}\big(\tfrac{1}{1-2\gamma} + \tfrac{1}{1-\gamma}\big)^{k-1} u_0^{-\gamma}.\end{aligned}$$

For statement (b), we only show the induction start $k=1$ as the induction step works analogously as above. We have

$$\int_0^1 \mathrm{d}u_1 (u_0\wedge u_1)^{-\gamma}u_1^{-\gamma} = \int_0^{u_0} \mathrm{d}u_1\, u_1^{-2\gamma} + \int_{u_0}^1 \mathrm{d}u_1\, u_0^{-\gamma}u_1^{-\gamma} \le \big(\tfrac{1}{1-2\gamma} + \tfrac{1}{1-\gamma}\big)u_0^{-\gamma},$$

as desired. □

Proof of Lemma 3.12. We restrict ourselves to the proof of Part *(ii)*. The adaptation to the easier case of Part *(i)* is straightforward. With Lemma 3.13 it is also straight forward to adapt the proof of Lemma 3.7 to the case $\gamma < 1/2$ and $\beta < K^{-1}$ to derive

$$\sum_{k\to\infty} \mathbb{P}_o\big(\exists \mathbf{x} : u_x < s_m \text{ and } \mathbf{o} \xleftrightarrow[\mathbb{R}^d\times(s_m,1)]{k} \big) \le m^{-(1-\gamma)\zeta}K'\tfrac{1}{1-\beta K}.$$

Hence, from now on, we work on the assumption that all considered vertices have mark no smaller than s_m. On the path of length n, either one of the intermediate edges has length at least $m^{1/d}/n$, or the last edge is longer than $|y|/n$. Hence, following the arguments of the proof of Proposition 3.12, we infer

$$\begin{aligned}
&\mathbb{P}_o\big(\exists \mathbf{y} : |y|^d > m, u_y > s_m \text{ and } \mathbf{o} \xleftrightarrow[\mathcal{B}(m^{1/d})\times(s_m,1)]{n} \mathbf{y} \text{ in } \mathcal{G}^\beta_{\mathbf{o}}\big)\\
&\quad\le \sum_{\ell=1}^{n-1} \beta^{\delta-1} n^{d\delta}\big(\beta\omega_d\tfrac{\delta}{\delta-1}\big)^n m^{1-\delta} \int\limits_{(s_m,1)^{n+1}} \bigotimes_{j=0}^{n} \mathrm{d}u_j\,(u_{\ell-1}\wedge u_\ell)^{-\gamma\delta} \prod_{\substack{j=1\\ j\neq \ell}}^{n}(u_{j-1}\wedge u_j)^{-\gamma}\\
&\qquad + \beta^{\delta-1} n^{d\delta}\big(\beta\omega_d\tfrac{\delta}{\delta-1}\big)^{n-1} \int\limits_{|x_n|^d>m} \mathrm{d}x_n\,|x_n|^{-d\delta} \int\limits_{(s_m,1)^{n+1}} \bigotimes_{j=0}^{n} \mathrm{d}u_j\,(u_{n-1}\wedge u_n)^{-\gamma\delta} \prod_{j=1}^{n-1}(u_{j-1}\wedge u_j)^{-\gamma}\\
&\quad= \sum_{\ell=1}^{n} \beta^{\delta-1} n^{d\delta}\big(\beta\omega_d\tfrac{\delta}{\delta-1}\big)^n m^{1-\delta} \int\limits_{(s_m,1)^{n+1}} \bigotimes_{j=0}^{n} \mathrm{d}u_j\,(u_{\ell-1}\wedge u_\ell)^{-\gamma\delta} \prod_{\substack{j=1\\ j\neq \ell}}^{n}(u_{j-1}\wedge u_j)^{-\gamma}.
\end{aligned}$$

We have using Lemma 3.13

$$\begin{aligned}
&\int\limits_{(s_m,1)^{n+1}} \bigotimes_{j=0}^{n} \mathrm{d}u_j\,(u_{\ell-1}\wedge u_\ell)^{-\gamma\delta} \prod_{\substack{j=1\\ j\neq \ell}}^{n}(u_{j-1}\wedge u_j)^{-\gamma}\\
&\qquad\le \tfrac{2-\gamma}{1-\gamma}\big(\tfrac{1}{1-2\gamma}+\tfrac{1}{1-\gamma}\big)^{n-\ell-1} \int\limits_{(s_m,1)^{\ell}} \bigotimes_{j=0}^{\ell+1} \mathrm{d}u_j\, u_\ell^{-\gamma}(u_{\ell-1}\wedge u_\ell)^{-\gamma\delta} \prod_{j=1}^{\ell-1}(u_{j-1}\wedge u_j)^{-\gamma}.
\end{aligned}$$

Since $\gamma > 1/(\delta+1)$, we infer

$$\begin{aligned}
&\int_{s_m}^1 \mathrm{d}u_{\ell-1}(u_{\ell-1}\wedge u_{\ell-2})^{-\gamma}\Big[\int_{s_m}^{u_{\ell-1}} \mathrm{d}u_\ell\, u_\ell^{-\gamma(\delta+1)} + \int_{u_{\ell-1}}^1 \mathrm{d}u_\ell\, u_\ell^{-\gamma\delta} u_{\ell-1}^{-\gamma}\Big]\\
&\qquad\le \int_{s_m}^1 \mathrm{d}u_{\ell-1}(u_{\ell-1}\wedge u_{\ell-2})^{-\gamma}\Big[\tfrac{m^{-\zeta(1-\gamma(\delta+1))}}{\gamma(\delta+1)-1} + \tfrac{u_{\ell-1}^{1-\gamma(\delta+1)}\vee u_{\ell-1}^{-\gamma}}{|1-\gamma\delta|}\Big]\\
&\qquad\le \big(\tfrac{1}{\gamma(\delta+1)-1}+\tfrac{1}{|1-\gamma\delta|}\big) m^{-\zeta(1-\gamma(\delta+1))} \int_{s_m}^1 \mathrm{d}u_{\ell-1}(u_{\ell-1}\wedge u_{\ell-2})^{-\gamma} u_{\ell-1}^{-\gamma}.
\end{aligned} \tag{34}$$

Using Lemma 3.13 once more, we hence get

$$\int\limits_{(s_m,1)^{n+1}} \bigotimes_{j=0}^{n-1} \mathrm{d}u_j\,(u_{\ell-1}\wedge u_\ell)^{-\gamma\delta} \prod_{\substack{j=1\\ j\neq \ell}}^{n}(u_{j-1}\wedge u_j)^{-\gamma} \le K'\big(\tfrac{1}{1-2\gamma}+\tfrac{1}{1-\gamma}\big)^n m^{-\zeta(1-\gamma(\delta+1))}$$

for

$$K' = \tfrac{2-\gamma}{1-\gamma}\big(\tfrac{1}{\gamma(\delta+1)-1}+\tfrac{1}{|1-\gamma\delta|}\big)\big(\tfrac{1}{1-2\gamma}+\tfrac{1}{1-\gamma}\big)^{-2}.$$

Therefore,

$$\begin{aligned}
&\sum_{\ell=1}^{n} \beta^{\delta-1} n^{d\delta}\big(\beta\omega_d\tfrac{\delta}{\delta-1}\big)^n m^{1-\delta} \int\limits_{(s_m,1)^{n+1}} \bigotimes_{j=0}^{n} \mathrm{d}u_j\,(u_{\ell-1}\wedge u_\ell)^{-\gamma\delta} \prod_{\substack{j=1\\ j\neq \ell}}^{n}(u_{j-1}\wedge u_j)^{-\gamma}\\
&\qquad\le nK'\beta^{\delta-1} n^{d\delta}\big(\beta K\big)^n m^{-(1-\gamma)\zeta},
\end{aligned}$$

where K is given in the formulation of the lemma, concluding the proof of Part *(ii)*. The proof of Part *(i)* works analogously. However, the restrictions to vertices with not too small vertex marks is no longer necessary and we repeat the same computations without this restriction for the case $\gamma < 1/(\delta+1))$ to obtain the desired result. □

3.3 Number of points contained in the component of the origin

In this final section we prove our results about the number of points in a subcritical component. Part *(ii)* of Theorem 2.2, stating that the expected cardinality of the component of the origin is infinite whenever the degree distribution has no second moment, is only stated for completeness and directly follows from the corresponding result for the classical Boolean model [37, Theorem 3.2]. Hence, it suffices to prove Part *(i)*. Again, the lower bound $\mathbb{P}_o(\mathscr{N}_\beta \geq m) \geq cm^{1-1/\gamma}$ is an immediate consequence of the result for the classical Boolean model in [14, 25]. We however give the short proof for completeness.

Proof of the lower bound in Theorem 2.2, Part (i). Let us denote by $N(\mathbf{x})$ the number of neighbours of the vertex $\mathbf{x} = (x, u_x)$ in $\mathscr{G}^\beta_{\mathbf{x}}$ and let us write $N^>(\mathbf{x})$ for the number of neighbours of $\mathbf{x}$ with mark no smaller than u_x. It is clear that $N^>(\mathbf{x}) \leq N(\mathbf{x})$. Further, for given $\mathbf{x} = (x, u_x)$, $N^>(\mathbf{x})$ is Poisson distributed with parameter $\frac{\beta\omega_d\delta}{\delta-1}(u_x^{-\gamma} - 1)$ independently from its location by [33, Proposition 2.1]. Therefore, by the standard Poisson tail bound, each vertex with mark smaller than $c(2m)^{-1/\gamma}$ for $c = (\delta-1)/(\beta\omega_d\delta)$ has at least m neighbours with exponentially small error probability. Additionally, each vertex located in $\mathcal{B}(m^{1/d})$ with mark smaller than $c(2m)^{-1/\gamma}$ is connected to $\mathbf{o}$. Therefore,

$$\begin{aligned}
&\mathbb{P}_o\big(\mathscr{N}_\beta > m\big)\\
&\quad\geq \mathbb{P}\big(\exists \mathbf{x} \in \mathbf{X} \cap \big(\mathcal{B}(m^{1/d}) \times (0, c(2m)^{-1/\gamma})\big)\big)\\
&\qquad - \mathbb{P}\big(\exists \mathbf{x} \in \mathbf{X} \cap \big(\mathcal{B}(m^{1/d}) \times (0, c(2m)^{-1/\gamma})\big) : N^>(\mathbf{x}) < m\big)\\
&\quad\geq \tfrac{c\omega_d}{2^{1+1/\gamma}} m^{1-1/\gamma},
\end{aligned}$$

for all sufficiently large $m > c^{1/\gamma}$, since

$$\begin{aligned}
&\mathbb{P}\big(\exists \mathbf{x} \in \mathbf{X} \cap \big(\mathcal{B}(m^{1/d}) \times (0, c(2m)^{-1/\gamma})\big) : N^>(\mathbf{x}) < m\big)\\
&\quad\leq \int\limits_{|x|^d < m} \mathrm{d}x \int\limits_0^{c(2m)^{-1/\gamma}} \mathrm{d}u \, \mathbb{P}\big(N^<(\mathbf{x}) < m\big)\\
&\quad\leq \tfrac{c\omega_d}{2^{1/\gamma}} m^{1-1/\gamma} \, \mathrm{Pois}_{2m-1}(m) \leq \tfrac{c\omega_d}{2^{1+1/\gamma}} m^{1-1/\gamma},
\end{aligned}$$

where we have written Pois_{2m-1} for the distribution function of a Poisson random variable with parameter $2m-1$. This proves the lower bound. □

Consider now the scale-free percolation model corresponding to the choice of $\alpha = \gamma$ in the interpolation kernel (5). That is, $g_{\gamma,\gamma}(s,t) = s^\gamma t^\gamma$. Since $g_{\gamma,0} \geq g_{\gamma,\gamma}$, we have $E(\mathscr{G}^{\beta,\gamma,0,\delta}(\xi)) \subset E(\mathscr{G}^{\beta,\gamma,\gamma,\delta}(\xi))$ by (6). Put differently, the scale-free percolation model in this parametrisation contains all edges of the soft Boolean model and more. As a direct result, the lower bound for the tail probability of the cardinality of the component of the origin in the soft Boolean model is also valid for scale-free percolation. Contrarily, each upper bound for scale-free percolation is then also an upper bound in the soft Boolean model. Hence, it suffices to prove the upper bound for scale-free percolation in order to finish the proofs of the Theorems 2.2 and 2.3.

Proof of the upper bound in Theorem 2.3. We couple the component of the origin to a multi-type branching process starting at the origin which we now describe. The individuals of the branching process are nodes $\mathbf{x} = (x, u_x)$ that have a location in $\mathbb{R}^d$ and a type $u_x \in (0, 1)$. Let $\mathcal{Y}^{(1)}$ be a unit-intensity Poisson point process on $\mathbb{R}^d \times (0, 1) \times (0, 1)$, independent of everything else, the nodes of which we denote by (x, u_x, v_x). Given the origin and its type $\mathbf{o} = (o, u_o)$, the first generation consists of the points

$$\mathcal{Z}^{(1)} = \big\{(x, u) : \exists v \text{ such that } (x, u, v) \in \mathcal{Y}^{(1)} \text{ and } v \leq 1 \wedge (\beta^{-1} u_o^\gamma u^\gamma |x|^d)^{-\delta}\big\}.$$

Let us denote by $Z^{(1)}(A)$ the number of points in $\mathcal{Z}^{(1)}$ that have their type in A. Then, given $\mathbf{o} = (o, u_o)$, $Z^{(1)}(A)$ is Poisson distributed with mean

$$\mathbb{E}_{(o,u_o)} Z^{(1)}(A) = \int_A \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}x \; 1 \wedge (\beta^{-1} u^\gamma u_o^\gamma |x|^d)^{-\delta} = \beta \tfrac{\omega_d \delta}{\delta - 1} u_o^{-\gamma} \int_A \mathrm{d}u \; u^{-\gamma}. \qquad (35)$$

For simplicity we write $Z^{(1)} = Z^{(1)}((0, 1))$ in which case we have

$$\mu^{(1)}((o, u_o)) := \mathbb{E}_{(o,u_o)} Z^{(1)} = \beta \tfrac{\omega_d \delta}{(\delta-1)(1-\gamma)} u_o^{-\gamma} \leq \beta \tfrac{\omega_d \delta}{(\delta-1)(1-2\gamma)} u_o^{-\gamma}, \qquad (36)$$

using $\gamma < 1/2$ in the last step. It is easy to see that the number of individuals in the first generation of this process coincides with the number of neighbours the origin has in $\mathcal{G}_{\mathbf{o}}^{\beta,\gamma,\gamma,\delta}$. To construct the second generation, let the first generation $\{\mathbf{x}_1^{(1)}, \dots, \mathbf{x}_k^{(1)}\} = \mathcal{Z}^{(1)}$ be given. For $j \in \{1, \dots, k\}$ let $\mathcal{Y}_j^{(2)}$ be a unit-intensity Poisson point process on $\mathbb{R}^d \times (0, 1) \times (0, 1)$ independent of everything else. The children of $\mathbf{x}_j^{(1)} = (x_j^{(1)}, u_j^{(1)})$ are then given by

$$\mathcal{Z}_j^{(2)} = \big\{(x, u) : \exists v \text{ such that } (x, u, v) \in \mathcal{Y}_1^{(1)} \text{ and } v \leq 1 \wedge \big(\beta^{-1} u^\gamma (u_j^{(1)})^\gamma |x - x_j^{(1)}|^d\big)^{-\delta}\big\}.$$

Let as above $Z_j^{(2)}$ (resp. $Z_j^{(2)}(A)$) denote the number of children of the individual $\mathbf{x}_j$ (of type in A) and note that again the number of which is Poisson distributed with mean $\mu^{(1)}(\mathbf{x}_j)$, see (36). Given the whole first generation $\mathcal{Z}^{(1)}$, the expected size of the second generation is hence given by $\sum_{\mathbf{x} \in \mathcal{Z}^{(1)}} \mu^{(1)}(\mathbf{x})$. Using the Markov property of this process together with Mecke's equation, we infer that the expected size of the second generation, given the origin $\mathbf{o} = (o, u_o)$, is bounded by

$$\begin{aligned} \mu^{(2)}((o, u_o)) = \mathbb{E}_{(o,u_o)} \Big[\sum_{\mathbf{x} \in \mathcal{Z}^{(1)}} \mu^{(1)}(\mathbf{x}) \Big] &= \int_0^1 \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}x \; \big(1 \wedge (\beta^{-1} u^\gamma u_o^\gamma |x|^d)^{-\delta}\big) \mu^{(1)}((x, u)) \\ &\leq \big(\beta \tfrac{\omega_d \delta}{(\delta-1)(1-2\gamma)}\big)^2 u_o^{-\gamma}, \end{aligned}$$

where we used (36) and $\gamma < 1/2$. Again, it is easy to see that the branching process can be coupled with $\mathcal{G}_{\mathbf{o}}^{\beta,\gamma,\gamma,\delta}$ such that the second generation contains at least as many individuals as there are vertices at graph distance two away from $\mathbf{o}$ in $\mathcal{G}_{\mathbf{o}}^{\beta,\gamma,\gamma,\delta}$. We continue constructing the subsequent generations in the obvious way. Then, for each n, the n-th generation $\mathcal{Z}^{(n)}$ contains at least as many individuals as there are vertices at graph distance n away from $\mathbf{o}$ in $\mathcal{G}_{\mathbf{o}}^{\beta}$. Moreover, we obtain inductively by the Markov property of the process and Mecke's equation that

$$\mu^{(n)}((o, u_o)) := \mathbb{E}_{(o,u_o)} Z^{(n)} \leq \big(\beta \tfrac{\omega_d \delta}{(\delta-1)(1-2\gamma)}\big)^n u_o^{-\gamma}.$$

Choosing $\beta < \frac{(\delta-1)(1-2\gamma)}{\omega_d \delta}$ the sequence $\mu^{(n)}((o, u_o))$ is summable and hence

$$\mathbb{E} \mathcal{N}_{\beta,\gamma,\gamma,\delta} \leq \int_0^1 \mathrm{d}u \sum_n \mu^{(n)}(o, u_o) < \infty.$$

In particular, the constructed branching process is subcritical. Moreover, we can see in our calculations that the spatial embedding plays no particular role and only the polynomial types of the form $u^{-\gamma}$ influence the offspring distribution. Let us think in the following of the individual types as of the form $W_x = u_x^{-\gamma} \in (1,\infty)$. By the Poisson nature of the branching process, (35) and (36) together yield that an offspring of a type-w individual has type in $(1,z)$ with probability

$$\frac{\beta\frac{\omega_d\delta}{\delta-1}w\int_{z^{-1/\gamma}}^1 \mathrm{d}u\, u^{-\gamma}}{\beta\frac{\omega_d\delta}{(\delta-1)(1-\gamma)}w} = 1 - z^{1-1/\gamma},$$

independently from the ancestor's type. Hence, the constructed branching process has the same law as a single-type branching process with mixed-Poisson offspring distribution with mean $\beta C W$ where W is $\mathrm{Pareto}(1/\gamma)$ distributed and $C = \frac{\omega_d\delta}{(\delta-1)(1-\gamma)}$. This reduction to a single-type branching process is a crucial feature of the underlying product structure of the kernel $g_{\gamma,\gamma}$, see [47, Chapter 3.4.3]. In particular, the mixing parameter $W' = \beta C W$ is heavy tailed with tail distribution

$$\mathbb{P}(W' \geq w) = cw^{1-1/\gamma},$$

for an appropriate $c > 0$. But this also implies that a random variable Z that is distributed as mixed Poisson with mean W' is heavy tailed with tail distribution

$$cz^{1-1/\gamma} \leq \mathbb{P}(Z \geq z) \leq Cz^{1-1/\gamma},$$

for some constants $c, C > 0$, see [46, Chapter 6]. This in particular implies $\mathbb{P}(Z = z) \asymp z^{-1/\gamma}$ for all large $z \in \mathbb{N}$. Summarising, the cardinality of the component of the origin is stochastically dominated by the total progeny $\mathcal{C}$ of a branching process with offspring distribution identically to Z. Together with Dwass' Theorem [12] we obtain

$$\mathbb{P}_o(\mathscr{N}_{\beta,\gamma,\gamma,\delta} \geq m) \leq \mathbb{P}(\mathcal{C} \geq m) = \sum_{k\geq m}\mathbb{P}(\mathcal{C} = k) = \sum_{k\geq m}\frac{1}{k}\mathbb{P}(Z_1 + \cdots + Z_k = k-1), \tag{37}$$

where $Z_1, \dots, Z_k$ are i.i.d. copies of Z. Finally, recall that we have already shown the subcriticality of the process which particularly implies $\mathbb{E}Z_1 < 1$ (the integrability is due to $\gamma < 1/2$ and the boundedness is due to β being small enough). Hence, $n - 1 > n\mathbb{E}Z_1$ for large enough n and as $Z_1, Z_2, \dots$ are independent and heavy tailed with finite expectation, the *single big jump paradigm* [8, Theorem 9.1] yields

$$\mathbb{P}(Z_1 + \cdots + Z_n \geq n-1) \sim n\mathbb{P}(Z_1 \geq n-1) \asymp n^{2-1/\gamma},$$

as $\gamma < 1/2$. Again, this implies $\mathbb{P}(Z_1 + \cdots + Z_n = n-1) \asymp n^{1-1/\gamma}$ for sufficiently large n. Plugging this back into (37) we obtain, for large enough m, that

$$\mathbb{P}_o(\mathscr{N}_{\beta,\gamma,\gamma,\delta} \geq m) \leq \sum_{k\geq m}\frac{1}{k}\mathbb{P}(Z_1 + \cdots + Z_k = k-1) \asymp \sum_{k\geq m} k^{-1/\gamma} \asymp m^{1-1/\gamma}.$$

This concludes the proof. □

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