Survival and extinction for a contact process with a density-dependent birth rate

Jonas Köppl[1], Nicolas Lanchier[2], Max Mercer[2]

submitted: April 24, 2024

[1]   Weierstrass Institute
    Mohrenstr. 39
    10117 Berlin
    Germany
    E-Mail: jonas.koeppl@wias-berlin.de

[2]   School of Mathematical and Statistical Sciences
    Arizona State University
    Tempe, AZ 85287
    USA
    E-Mail: nlanchie@asu.edu
            mamerce1@asu.edu

WI
AS

Survival and extinction for a contact process with a density-dependent birth rate

Jonas Köppl, Nicolas Lanchier, Max Mercer

Abstract

To study later spatial evolutionary games based on the multitype contact process, we first focus in this paper on the conditions for survival/extinction in the presence of only one strategy, in which case our model consists of a variant of the contact process with a density-dependent birth rate. The players are located on the d-dimensional integer lattice, with natural birth rate λ and natural death rate one. The process also depends on a payoff $a_{11} = a$ modeling the effects of the players on each other: while players always die at rate one, the rate at which they give birth is given by λ times the exponential of a times the fraction of occupied sites in their neighborhood. In particular, the birth rate increases with the local density when $a > 0$, in which case the payoff a models mutual cooperation, whereas the birth rate decreases with the local density when $a < 0$, in which case the payoff a models intraspecific competition. Using standard coupling arguments to compare the process with the basic contact process (the particular case $a = 0$), we prove that, for all payoffs a, there is a phase transition from extinction to survival in the direction of λ. Using various block constructions, we also prove that, for all birth rates λ, there is a phase transition in the direction of a. This last result is in sharp contrast with the behavior of the nonspatial deterministic mean-field model in which the stability of the extinction state only depends on λ. This underlines the importance of space (local interactions) and stochasticity in our model.

1 Introduction

The field of evolutionary game theory was developed by Maynard Smith [22], and first appeared in his work with Price [23]. The basic idea in this field is to reinterpret the different strategies as species and their payoff as fitness to create realistic dynamical systems with density-dependent birth and/or death rates. The most popular (nonspatial deterministic) model in evolutionary game theory is the replicator equation. Having an $n \times n$ payoff matrix $A = (a_{ij})$ where a_{ij} represents the payoff a type i player receives from a type j player, and letting u_i denote the density of type i players in the population, the payoff of each type i player is given by

$$\phi_i = \phi_i(u_1, u_2, \ldots, u_n) = a_{i1}u_1 + a_{i2}u_2 + \cdots + a_{in}u_n.$$

Reinterpreting the payoff as fitness (a birth rate when the payoff is positive or minus a death rate when the payoff is negative) and assuming that each individual produced replaces a player chosen uniformly at random, while each individual removed is replaced by a player chosen uniformly at random, result in the following so-called replicator equation [16]:

$$u_i' = (\phi_i u_i)(\textstyle\sum_{j \neq i} u_j) - (\textstyle\sum_{j \neq i} \phi_j u_j)\, u_i = \textstyle\sum_{j \neq i}(\phi_i - \phi_j)\, u_i u_j \quad \text{for} \quad i = 1, 2, \ldots, n.$$

This system of coupled differential equations can be turned into a spatially explicit stochastic process following the modeling approach of [26, 27]. More precisely, to include a spatial structure in the form of

local interactions, we first assume that the players are located on the d-dimensional integer lattice $\mathbb{Z}^d$, so the state at time t is a configuration

$$\xi_t : \mathbb{Z}^d \longrightarrow \{1, 2, \ldots, n\} \quad \text{where} \quad \xi_t(x) = \text{strategy of the player at site } x.$$

Then, writing $x \sim y$ to indicate that the two lattice points x and y are nearest neighbors (distance one apart), the payoff of the player at site x is defined as

$$\phi(x, \xi_t) = \sum_{i,j} a_{ij} f_j(x, \xi_t) \mathbf{1}\{\xi_t(x) = i\} \quad \text{where} \quad f_j(x, \xi_t) = \sum_{y \sim x} \mathbf{1}\{\xi_t(y) = j\}/2d \tag{1.1}$$

denotes the fraction of nearest neighbors of site x following strategy j, i.e., the payoff only depends on the strategy of the neighbors. The fitness is then given by

$$\Phi(x, \xi_t) = (1 - w) \times 1 + w \times \phi(x, \xi_t) = (1 - w) \times 1 + w \times \text{payoff}, \tag{1.2}$$

where the parameter $w \in (0, 1]$ represents the strength of selection. Weak selection refers to the case where w is small, while strong selection means $w = 1$. The most popular models that fall under this framework are the birth-death updating process and the death-birth updating process introduced in [28]. In the birth-death updating process, the fitness is interpreted as a birth rate, and offspring replace a neighbor of the parent's site chosen uniformly at random, so the rate at which site x switches from strategy i to strategy j is given by

$$c_{i \to j}(x, \xi_t) = \sum_{y \sim x} \Phi(y, \xi_t) \mathbf{1}\{\xi_t(y) = j\}/2d \quad \text{for all} \quad i \neq j. \tag{1.3}$$

In contrast, in the death-birth updating process, players die at rate one and are instantaneously replaced by the offspring of a neighbor chosen at random with a probability proportional to its fitness, so the local transition rates are now given by

$$c_{i \to j}(x, \xi_t) = \sum_{y \sim x} \Phi(y, \xi_t) \mathbf{1}\{\xi_t(y) = j\}/\sum_{y \sim x} \Phi(y, \xi_t) \quad \text{for all} \quad i \neq j. \tag{1.4}$$

Taking $w = 0$, the transition rates in (1.3)–(1.4) simplify to $f_j(x, \xi_t)$, showing that the birth-death and the death-birth updating processes both reduce to the voter model [6, 17]. In the presence of weak selection, these two processes were studied in [4, 5, 21] in the context of two-strategy games, while [7, 10, 24] also considered games with more strategies. In the weak selection limit $w \to 0$, voter model perturbations techniques developed in [8] can be used to have a precise description of the phase structure of the processes. In the presence of strong selection $w = 1$, more qualitative aspects such as the existence of phase transitions were proved in [12, 18]. Other natural variants with discontinuous transition rates were also studied rigorously in the presence of strong selection: the best-response dynamics [11], and the death-birth of the fittest process [13].

In the previous models, each birth/death induces the instantaneous death/birth of a neighbor so each site is occupied by exactly one player at all times. In particular, these models, as well as all the models of interacting particle systems of interest in evolutionary game theory that have been studied in the probability literature, consist of variants of the voter model with density-dependent birth and/or death rates in which all the sites are occupied. As far as we know, the only exception is the variant of the multitype contact process [25] introduced in [19]. This process, however, was only designed to model the interactions among cooperators and defectors in the prisoner's dilemma rather than general games described by a payoff matrix. Our main objective is to initiate the study of spatial evolutionary games based more realistically on Neuhauser's multitype contact process instead of the voter model. In particular, the state at time t is now

$$\xi_t : \mathbb{Z}^d \longrightarrow \{0, 1, 2, \ldots, n\} \quad \text{where} \quad \xi_t(x) = \text{strategy of the player at site } x,$$
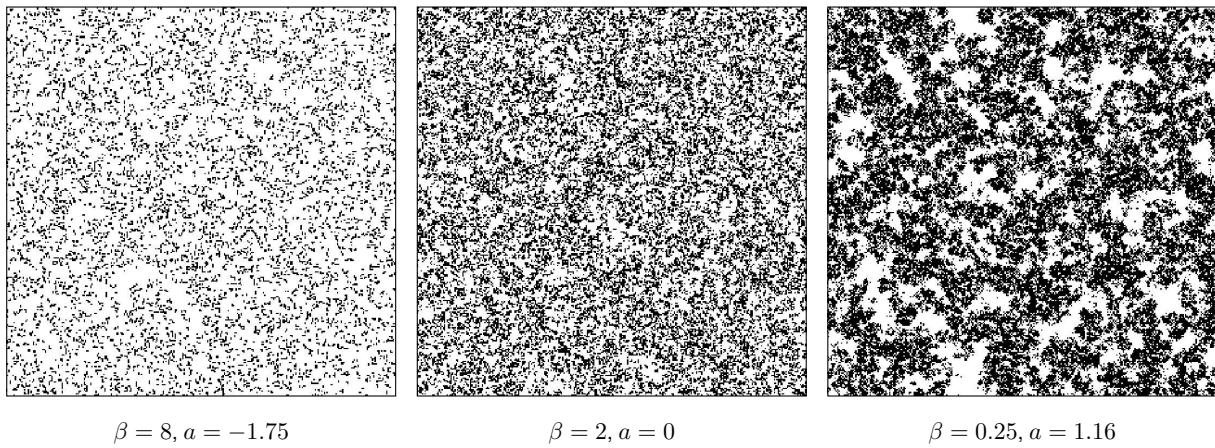
$$\beta = 8, a = -1.75 \qquad \beta = 2, a = 0 \qquad \beta = 0.25, a = 1.16$$

Figure 1: Snapshots at time 1000 of the two-dimensional single-type contact process with various birth parameters λ and payoffs a. Middle: When $a = 0$, the process reduces to the basic contact process. Left: When $a < 0$, small clusters have a reduced birth rate, which results in a more scattered configuration at equilibrium. Right: In contrast, when $a > 0$, small clusters have an increased birth rate, which results in stronger spatial correlations.

with the convention 0 = empty. The dynamics combines the multitype contact process and the birth-death updating process (1.3) dynamics. Like in the neutral multitype contact process, we assume that, regardless of their strategy, the players have the same natural birth rate λ and the same natural death rate one, and that births onto already occupied sites are suppressed. Having a payoff matrix $A = (a_{ij})$, each player now receives a payoff from its occupied neighbors while empty neighbors have no effects, so the payoff can be defined as in (1.1) assuming that $a_{i0} = 0$, i.e., empty sites give a zero payoff. Like in the birth-death updating process, the payoff of the players affects their birth rate. To have a well-defined positive birth rate even when the payoff is negative, instead of using the fitness function (1.2), we assume that the natural birth rate λ is multiplied by the exponential of the payoff. In particular, the transition rates are given by

$$c_{0 \to i}(x, \xi_t) = \sum_{y \sim x} \Phi(y, \xi_t)\, \mathbf{1}\{\xi_t(y) = i\}/2d \quad \text{and} \quad c_{i \to 0}(x, \xi_t) = 1,$$

where the fitness function is defined as

$$\Phi(x, \xi_t) = \lambda \exp(\phi(x, \xi_t)) = \lambda \exp(\textstyle\sum_{i,j \neq 0} a_{ij}\, f_j(x, \xi_t)\, \mathbf{1}\{\xi_t(x) = i\}).$$

In particular, players with no neighbors give birth at rate λ, while players with a positive payoff give birth at a higher rate and players with a negative payoff give birth at a lower rate.

Before looking at the competition among multiple strategies, the first step is to study the conditions for survival in the presence of only one strategy, say strategy 1, which is the objective of this paper. In this case, the model depends on two parameters: the natural birth rate λ, and the single payoff coefficient a_{11} that we simply denote by a. A player at x gives birth at rate

$$\Phi(x, \xi_t) = \lambda \exp(a f_1(x, \xi_t)), \tag{1.5}$$

and dies at rate one. See Figure 1 for simulation pictures. Note that this function is nondecreasing with respect to the natural birth rate λ and the payoff coefficient a. In particular, one expects the probability of survival, i.e., the probability that, starting with a single player, there is at least one player at all times, to be nondecreasing with respect to the parameters λ and a. Standard coupling arguments, however, fail to prove this result when $a < 0$. Indeed, increasing the birth rate (1.5) increases the local density
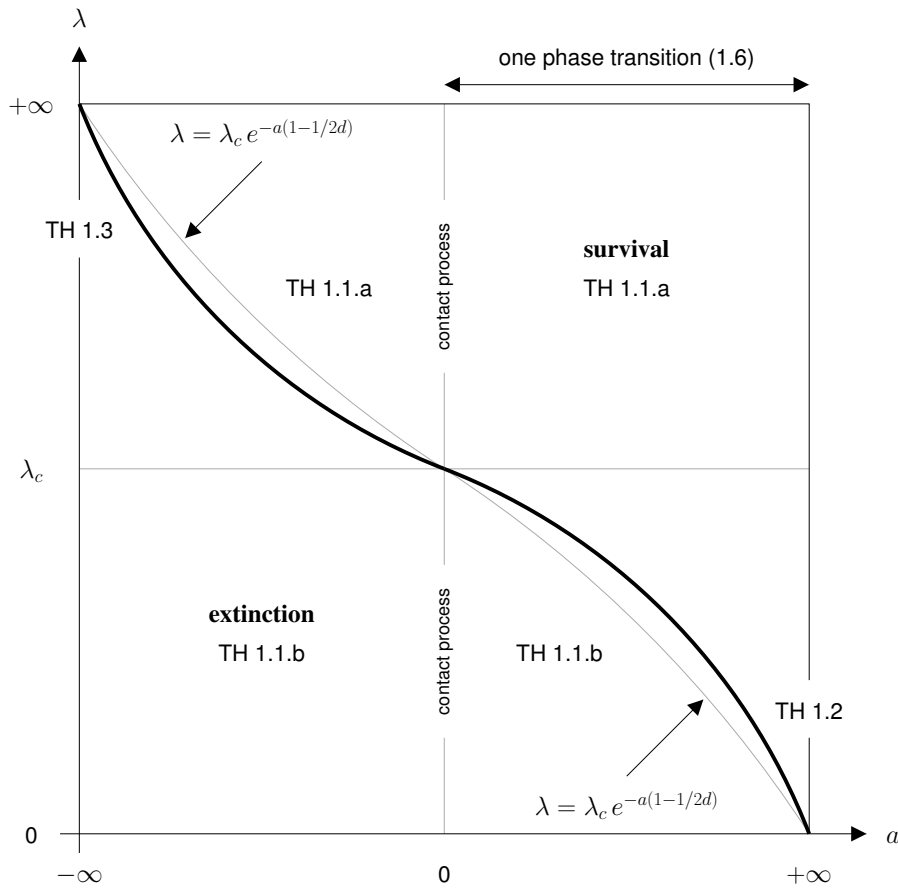
Figure 2: Phase structure of the contact process with a density-dependent birth rate.

of occupied sites, which decreases the birth rate of the surrounding players. Coupling processes with different parameters/initial configurations, we can prove the following result: letting $\mathbb{P}_{\lambda,a}^{\xi}$ be the law of the process with parameters λ and a, starting from $\xi_0 = \xi$,

$$\lambda_1 \leq \lambda_2, \;\; a_2 \geq a_1 \vee 0, \;\; \xi^1 \subset \xi^2 \;\; \Longrightarrow \;\; \mathbb{P}_{\lambda_1,a_1}^{\xi^1}[\xi_t \neq \varnothing \; \forall t] \leq \mathbb{P}_{\lambda_2,a_2}^{\xi^2}[\xi_t \neq \varnothing \; \forall t]. \tag{1.6}$$

Looking at particular cases with the same initial configuration with a single player and/or the same natural birth rate and/or the same payoff coefficient, the implication in (1.6) shows that the process is attractive and monotone with respect to λ and a in the parameter region where $a \geq 0$. In particular, there is at most one phase transition from extinction to survival in the direction of each of the two parameters. The implication also shows that the probability of survival for $a \geq 0$ is no less than the probability of survival for $a \leq 0$. In addition, in the special case $a = 0$, the birth rate (1.5) becomes λ regardless of the configuration, therefore the process reduces to the basic contact process [14]. In particular, letting $\lambda_c = \lambda_c(\mathbb{Z}^d)$ denote the critical value of the contact process on $\mathbb{Z}^d$, and using other coupling arguments, we can prove the following result.

Theorem 1.1. *For every fixed $a \in \mathbb{R}$, the process*

(a) *survives when $(a > 0$ and $\lambda > \lambda_c)$ or $(a < 0$ and $\lambda > \lambda_c\, e^{-a(1-1/2d)})$,*

(b) *dies out when $(a < 0$ and $\lambda \leq \lambda_c)$ or $(a > 0$ and $\lambda \leq \lambda_c\, e^{-a(1-1/2d)})$.*

The theorem shows that, for each fixed payoff $-\infty < a < \infty$, there exists at least one (and exactly one when $a \geq 0$) phase transition in the direction of the birth rate λ. Proving that, for each fixed birth

rate $0 < \lambda < \infty$, there exists a phase transition in the direction of the payoff a is more complicated. Even when $\lambda > 0$ is small, for $a < \infty$ large, adjacent players have a large birth rate due to cooperation. In particular, it can be proved that, with high probability, a small block of players quickly doubles in size. Using also attractiveness when the payoff is positive, and a block construction, implies survival when a is sufficiently large.

Theorem 1.2 (Survival). *For every birth rate $\lambda > 0$, there exists an $a_+ = a_+(\lambda, d) < \infty$ such that the process survives for all payoffs $a > a_+$.*

In contrast, even when $\lambda < \infty$ is large, for $a = -\infty$, adjacent players cannot give birth before one of them dies due to competition. In particular, it can be proved that the family generated by a single player decays exponentially in space and time. Using another block construction and a perturbation argument implies extinction when a is sufficiently small.

Theorem 1.3 (Extinction). *For every birth rate $\lambda < \infty$, there exists an $a_- = a_-(\lambda, d) > -\infty$ such that the process dies out for all payoffs $a < a_-$.*

We refer to the phase diagram in Figure 2 for a summary/visualization of our main results. Before going into the proofs, we point out that Theorem 1.3 cannot be deduced from general ergodicity results like the $(M - \epsilon)$-criterion [20, Theorem I.4.1] because we always have $\epsilon = 1$ while M is lower bounded by the corresponding value for the contact process. We also note that, although the technical details in the proofs of Theorems 1.2 and 1.3 differ significantly, the intuition behind both results is the same. Even if λ is very small, once a player gives birth, this player and its offspring form an adjacent pair with an arbitrarily large birth rate when a is large. Similarly, even if λ is very large, once a player gives birth, this player and its offspring form an adjacent pair with an arbitrarily small birth rate when a is small. In both cases, the conclusion (survival/extinction) is due to the presence of local interactions: the players place their offspring in their neighborhood, while their payoff is also determined by their neighbors. In particular, the two theorems are expected to fail in the absence of local interactions. Indeed, a simple analysis of the mean-field model shows that whether the trivial extinction fixed point 0 is stable or unstable, and so whether the population dies out or survives starting at low density, depends on λ but not on a.

The rest of the paper is devoted to the proofs. Section 2 gives a brief analysis of the mean-field model, focusing on the local stability of the trivial fixed point, but also on the size of its basin of attraction. Section 3 relies on various coupling arguments to prove monotonicity and attractiveness in the case where $a \geq 0$. Similar couplings are used to compare the process with the basic contact process, and deduce Theorem 1.1. Section 4 uses a block construction to prove Theorem 1.2. Finally, Section 5 establishes some exponential decay to deduce extinction of the process with $a = -\infty$ from a block construction. Theorem 1.3 is then deduced by using a perturbation argument.

2 Mean-field model

This section gives a brief analysis of the nonspatial deterministic mean-field model, which describes the process in the large population limit when the system is homogeneously mixing. Letting u be the density of occupied sites, the mean-field model reduces to the differential equation

$$u' = \phi(u) = \lambda e^{au} u(1 - u) - u.$$

Because of the exponential form of the birth rate, we cannot obtain the exact expression of the fixed points. However, the stability of the trivial fixed point 0, corresponding to the extinction state, as well as

the existence and stability of additional (interior) fixed points, can be studied. To begin with, observe that, for all fixed $-\infty < a < \infty$,

$$u \approx 0 \implies \phi(u) \approx \lambda u - u = (\lambda - 1)u. \tag{2.1}$$

This shows that the stability of the trivial fixed point, and so whether the population survives or dies out when starting at low density, depends on the natural birth rate λ but not on the payoff coefficient a, which is in sharp contrast with Theorems 1.2 and 1.3.

Extinction phase. It follows from (2.1) that, when $\lambda < 1$, the trivial fixed point is locally stable, so the population dies out ($u \to 0$ starting at low density) even when a is very large. This contrasts with Theorem 1.2, which states that, even starting with a finite number of players (density zero), the population survives with positive probability. However, for all $\lambda > 0$ and $\bar{u} < 1/2$,

$$a > \ln(2/\lambda)/\bar{u} \implies \phi(\bar{u}) > \lambda e^{\ln(2/\lambda)} \bar{u}(1 - \bar{u}) - \bar{u} = 2\bar{u}(1 - \bar{u}) - \bar{u} \geq 0.$$

This shows that $\phi(u)$ becomes positive at some unstable fixed point $u_* \in (0, \bar{u})$ therefore, starting at a density $> u_*$, the population converges to a limit $u^* > 0$. In conclusion, when $\lambda < 1$, there is extinction in the sense that the trivial fixed point is locally stable, but starting from a fixed positive density, the population survives for all a sufficiently large (depending on that density).

Survival phase. It follows from (2.1) that, when $\lambda > 1$, the trivial fixed point is unstable, so the population survives (starting from a positive density, u converges to a positive limit) even when a is very small. This contrasts with Theorem 1.3, which states that, regardless of the initial configuration, the density of occupied sites vanishes to zero. However, for all $\lambda < \infty$ and $\bar{u} < 1/2$,

$$a < \ln(1/\lambda)/\bar{u} \implies \phi(\bar{u}) < \lambda e^{\ln(1/\lambda)} \bar{u}(1 - \bar{u}) - \bar{u} = \bar{u}(1 - \bar{u}) - \bar{u} \leq 0.$$

This shows that $\phi(u)$ becomes negative at some stable fixed point $u^* \in (0, \bar{u})$ therefore, starting at low density, the population converges to a limit that cannot exceeds $\bar{u}$. In conclusion, when $\lambda > 1$, there is survival in the sense that the trivial fixed point is unstable, but starting at low density, the limiting density can be made arbitrarily small by taking a sufficiently small.

3 Proof of Theorem 1.1 (monotonicity and attractiveness)

The proofs of Theorem 1.1 and (1.6) are based on standard coupling arguments. Processes with different parameters and/or different initial configurations can be coupled by constructing them jointly on the same graphical representation [15]. However, the graphical representation of the contact process with a density-dependent birth rate is somewhat complicated. Instead, we use the classical comparison result [20, Theorem III.1.5], which we restate for the reader's convenience.

Theorem 3.1. *Let ξ_t^1 and ξ_t^2 be two interacting particle systems with state space $\Omega = \{0, 1\}^{\mathbb{Z}^d}$, and assume that, whenever $\xi^1 \subset \xi^2$, we have the inequalities*

$$c_{0 \to 1}(x, \xi^1) \leq c_{0 \to 1}(x, \xi^2) \quad and \quad c_{1 \to 0}(x, \xi^1) \geq c_{1 \to 0}(x, \xi^2).$$

Then, there is a coupling of the two processes such that

$$\xi^1 \subset \xi^2 \implies \mathbb{P}^{(\xi^1, \xi^2)}[\xi_t^1 \subset \xi_t^2 \,\forall t] = 1.$$

Using this result, we can prove (1.6) about the monotonicity and the attractiveness of the contact process with a density-dependent birth rate.

Proof of (1.6). For $i = 1, 2$, let ξ_t^i be the process with natural birth rate λ_i, payoff coefficient a_i, and initial configuration ξ^i. Because the death rate of the processes is always equal to one, the second inequality in Theorem 3.1, which is in fact an equality, is always satisfied. To prove the first inequality, recall that the birth rate of the process is of the form

$$c_{0\to1}(x,\xi) = \sum_{y\sim x} \Phi(y,\xi)\,\xi(y)/2d = \sum_{y\sim x} \lambda \exp(af_1(y,\xi))\,\xi(y)/2d = \psi(\lambda, a, \xi).$$

The function $\psi(\lambda, a, \xi)$ is nondecreasing with respect to λ, a, and ξ when $a \geq 0$. In particular, according to Theorem 3.1, the processes can be coupled in such a way that

$$\lambda_1 \leq \lambda_2, \ \ 0 \leq a_1 \leq a_2, \ \ \xi^1 \subset \xi^2 \ \implies \ \mathbb{P}^{(\xi^1,\xi^2)}[\xi_t^1 \subset \xi_t^2 \ \forall t] = 1. \tag{3.1}$$

Note also that, for all $\lambda_1 \leq \lambda_2$, $a_1 \leq 0 \leq a_2$, and $\xi^1 \subset \xi^2$,

$$\psi(\lambda_1, a_1, \xi_1) \leq \sum_{y\sim x} \lambda_1 \xi^1(y)/2d \leq \sum_{y\sim x} \lambda_2 \xi^2(y)/2d \leq \psi(\lambda_2, a_2, \xi_2),$$

from which we deduce that

$$\lambda_1 \leq \lambda_2, \ \ a_1 \leq 0 \leq a_2, \ \ \xi^1 \subset \xi^2 \ \implies \ \mathbb{P}^{(\xi^1,\xi^2)}[\xi_t^1 \subset \xi_t^2 \ \forall t] = 1. \tag{3.2}$$

The implication in (1.6) follows from the existence of the couplings (3.1) and (3.2). $\qquad\square$

Using again Theorem 3.1, we can also prove Theorem 1.1.

Proof of Theorem 1.1. Let η_t and ζ_t be the contact processes with parameter λ and $\lambda e^{-a(1-1/2d)}$, respectively, and let ξ_t be the density-dependent contact process. Observing that the density-dependent contact process with payoff $a = 0$ reduces to the basic contact process with the same natural birth rate, it follows from (3.2) that the process ξ_t dominates η_t when $a \geq 0$ but is dominated by η_t when $a \leq 0$. To compare the processes ξ_t and ζ_t, observe that, when site x is empty, its neighbors have at most $2d - 1$ occupied neighbors, therefore

$$a \leq 0, \ \xi \supset \zeta \ \implies \ c_{0\to1}(x,\xi) \geq \sum_{y\sim x} \lambda e^{a(1-1/2d)} \xi(y)/2d = \lambda e^{a(1-1/2d)} f_1(x,\xi)$$
$$\geq \lambda e^{a(1-1/2d)} f_1(x,\zeta) = c_{0\to1}(x,\zeta),$$
$$a \geq 0, \ \xi \subset \zeta \ \implies \ c_{0\to1}(x,\xi) \leq \sum_{y\sim x} \lambda e^{a(1-1/2d)} \xi(y)/2d = \lambda e^{a(1-1/2d)} f_1(x,\xi)$$
$$\leq \lambda e^{a(1-1/2d)} f_1(x,\zeta) = c_{0\to1}(x,\zeta).$$

In particular, by Theorem 3.1, the process ξ_t dominates ζ_t when $a \leq 0$ but is dominated by ζ_t when $a \geq 0$. In conclusion, there are couplings of the three processes such that

$$a \leq 0, \ \eta \supset \xi \supset \zeta \ \implies \ \mathbb{P}^{(\eta,\xi,\zeta)}[\eta_t \supset \xi_t \supset \zeta_t \ \forall t] = 1,$$
$$a \geq 0, \ \eta \subset \xi \subset \zeta \ \implies \ \mathbb{P}^{(\eta,\xi,\zeta)}[\eta_t \subset \xi_t \subset \zeta_t \ \forall t] = 1.$$

Since [2, Theorem 1] implies that the contact process η_t survives if and only if $\lambda > \lambda_c$, while the contact process ζ_t survives if and only if $\lambda > \lambda_c\, e^{-a(1-1/2d)}$, the theorem follows. $\qquad\square$

4   Proof of Theorem 1.2 (survival for $a < \infty$ large)

This section is devoted to the proof of Theorem 1.2, which states that, for all birth rates $\lambda > 0$ even small, the process survives provided the payoff coefficient a is sufficiently large. The basic idea is that, though it might be likely that isolated players die before they have a chance to give birth, players with at least one neighbor have a large birth rate when a is large, so a small fully occupied cube will double in size with probability close to one in a small deterministic time. This and a block construction imply survival. More precisely, letting

$$\Lambda_- = \{0, 1\}^d \quad \text{and} \quad \Lambda_+ = \{-1, 0, 1, 2\}^d,$$

the goal is to prove that, if the small cube Λ_- is initially fully occupied then, with probability arbitrarily close to one when a is large, the larger cube Λ_+ will be fully occupied at some fixed deterministic time τ. The density-dependent process with $a \geq 0$ is attractive according to (1.6), and the birth rate of players with at least one occupied neighbor is larger than $\lambda e^{a/2d}$. In particular, it suffices to prove the result for the process $\bar{\xi}_t$ with transition rates

$$c_{0 \to 1}(x, \bar{\xi}_t) = \sum_{y \sim x} \lambda e^{a/2d}\, \mathbf{1}\{f_1(y, \bar{\xi}_t) \neq 0\}\, \bar{\xi}_t(y)/2d \quad \text{and} \quad c_{1 \to 0}(x, \bar{\xi}_t) = 1,$$

modified so that births outside Λ_+ are suppressed. The transition rates indicate that isolated players cannot give birth, while players with at least one occupied neighbor give birth at rate $\lambda e^{a/2d}$, and that players die at rate one. This process can be constructed graphically as follows:

- **Births**. Equip each $\vec{xy}$, $x \sim y$, with an exponential clock with rate $\lambda e^{a/2d}/2d$. At the times t the clock rings, draw an arrow $(x, t) \to (y, t)$ to indicate that, if x is occupied and has at least one occupied neighbor, and y is empty, then y becomes occupied.

- **Deaths**. Equip each x with an exponential clock with rate one. At the times t the clock rings, put a cross $\times$ at (x, t) to indicate that, if x is occupied, then it becomes empty.

We denote by $\bar{\mathbb{P}}_{\lambda,a}$ the law of this process. To prove survival, we first show that, with high probability, there are no death marks $\times$ in Λ_+ by some small time τ.

Lemma 4.1. *For all $\epsilon > 0$, there exists $\tau > 0$ such that*

$$\bar{\mathbb{P}}_{\lambda,a}[\text{no death marks } \times \text{ in } \Lambda_+ \times [0, \tau]] \geq 1 - \epsilon/2.$$

Proof. The number D_τ of death marks in the space-time box $\Lambda_+ \times [0, \tau]$ is Poisson distributed with parameter $4^d \tau$. In particular, the probability of no death marks satisfies

$$\mathbb{P}[D_\tau = 0] = e^{-4^d \tau} = 1 - \epsilon/2 \iff \tau = -\ln(1 - \epsilon/2)/4^d > 0. \tag{4.1}$$

This completes the proof. $\square$

The next step is to prove invasion $\Lambda_- \to \Lambda_+$ by time τ. Because the players in the small cube Λ_- cannot immediately give birth onto the corners of the larger cube Λ_+ when $d > 1$, we divide the problem into d steps by considering the sequence of spatial regions

$$\Lambda_i = \{y \in \Lambda_+ : \min_{x \in \Lambda_-} \|x - y\|_1 = i\} \quad \text{for} \quad i = 0, 1, \ldots, d.$$

Figure 3 shows a picture in the $d = 2$ case. Note that $\Lambda_- = \Lambda_0$ and $\Lambda_+ = \cup_i \Lambda_i$. In addition, each site in one region has at least one neighbor in the previous region, so a fully occupied region can immediately invade the next region. We are now ready to prove invasion.
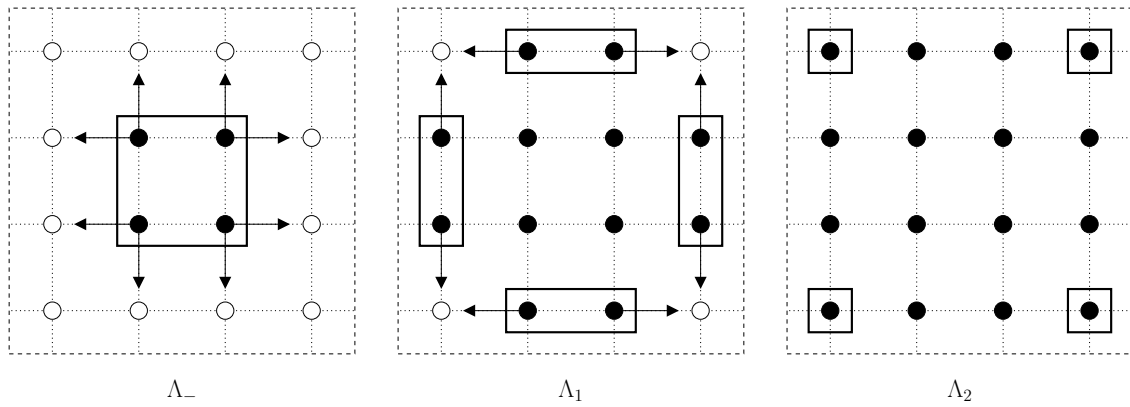
Figure 3: Illustration of the d-step process used in the proof of Theorem 1.2. The region delimited with bold lines represents Λ_- on the left, Λ_1 in the middle, and Λ_2 on the right. In all three pictures, the dashed box is Λ_+.

Lemma 4.2. *For all $\epsilon > 0$ and $\tau > 0$ like in* (4.1)*, there exists $a < \infty$ large such that*

$$\bar{\mathbb{P}}^{\Lambda_-}_{\lambda,a}[\bar{\xi}_\tau = \Lambda_+] \geq 1 - \epsilon.$$

Proof. As previously mentioned, we proceed in d steps, and prove that, conditional on no deaths, each invasion $\Lambda_i \to \Lambda_{i+1}$ occurs in less then τ/d units of time with high probability. Given that there are no deaths by time τ, and that Λ_i is fully occupied, the probability that Λ_{i+1} becomes fully occupied in less than τ/d units of time is larger than the probability that, for each site $y \in \Lambda_{i+1}$, there is at least one birth arrow $x \to y$ for some $x \in \Lambda_i$ in this time window. Because each region has less than 4^d sites, and birth arrows occur along each directed edge at rate $\lambda e^{a/2d}/2d$, this is larger than the probability that $X_1, X_2, \ldots, X_{4^d}$ = independent exponential random variables with that rate are all less than τ/d. In particular, for all $i = 0, 1, \ldots, d-1$,

$$\begin{aligned}
\bar{\mathbb{P}}_{\lambda,a}[\bar{\xi}_t \supset \Lambda_{i+1} \, \forall t \in [(i+1)\tau/d, \tau] \,|\, \bar{\xi}_{i\tau/d} = \Lambda_i, D_\tau = 0] \\
= \bar{\mathbb{P}}^{\Lambda_i}_{\lambda,a}[\bar{\xi}_{\tau/d} \supset \Lambda_{i+1} \,|\, D_\tau = 0] \geq \mathbb{P}[X_1, X_2, \ldots, X_{4^d} \leq \tau/d] \\
= (1 - \exp(-\lambda\tau e^{a/2d}/2d^2))^{4^d}.
\end{aligned} \tag{4.2}$$

Some basic algebra shows that this is $\geq 1 - \epsilon/2d$ for all a larger than

$$a_+ = 2d \ln(-2d^2 \ln(1 - (1 - \epsilon/2d)^{4^{-d}})/\lambda\tau) < \infty. \tag{4.3}$$

In particular, for all $a \geq a_+$, it follows from (4.2) and Lemma 4.1 that

$$\begin{aligned}
\bar{\mathbb{P}}^{\Lambda_-}_{\lambda,a}[\bar{\xi}_\tau \neq \Lambda_+] &\leq \bar{\mathbb{P}}^{\Lambda_-}_{\lambda,a}[\bar{\xi}_\tau \neq \Lambda_+ \,|\, D_\tau = 0] + \mathbb{P}[D_\tau \neq 0] \\
&\leq \sum_{i<d} \bar{\mathbb{P}}^{\Lambda_i}_{\lambda,a}[\bar{\xi}_{\tau/d} \not\supset \Lambda_{i+1} \,|\, D_\tau = 0] + \mathbb{P}[D_\tau \neq 0] \\
&\leq d(1 - (1 - \epsilon/2d)) + \epsilon/2 = \epsilon,
\end{aligned}$$

which proves the lemma. $\qquad\square$

The rest of the proof of Theorem 1.2 is standard and relies on a block construction. This technique first appeared in [3] and is explained in detail in [9]. The basic idea is to compare the process properly rescaled in space and time with supercritical oriented site percolation. Let

$$\mathscr{L} = \{(m, n) \in \mathbb{Z}^d \times \mathbb{N} : m_1 + \cdots + m_d + n \text{ is even}\},$$

and turn $\mathscr{L}$ into a directed graph $\vec{\mathscr{L}}$ by placing an edge

$$(m, n) \to (m', n') \quad \text{if and only if} \quad m'_i = m_i \pm 1 \text{ for all } i \text{ and } n' = n + 1.$$

Fix $\epsilon > 0$, let τ as in (4.1), and call $(m, n) \in \mathscr{L}$ a good site whenever

$$E_{m,n} = \{\text{the box } m + \Lambda_- \text{ is fully occupied at time } n\tau\} \quad \text{occurs.}$$

Lemma 4.3. *Let a_+ as in (4.3). Then, for all $a \geq a_+$, the set of good sites dominates stochastically the set of wet sites in an oriented site percolation process on $\vec{\mathscr{L}}$ with parameter $1 - \epsilon$.*

Proof. Because the process ξ_t dominates the process $\bar{\xi}_t$, and the evolution rules of the process ξ_t are translation invariant, it follows from Lemma 4.2 that, for all sites $(m', n') \leftarrow (m, n)$,

$$\mathbb{P}_{\lambda,a}[\xi_{(n+1)\tau} \supset m' + \Lambda_- \,|\, \xi_{n\tau} \supset m + \Lambda_-]$$
$$\geq \mathbb{P}_{\lambda,a}[\xi_{(n+1)\tau} \supset m + \Lambda_+ \,|\, \xi_{n\tau} \supset m + \Lambda_-] \geq 1 - \epsilon \quad \text{for all} \quad a \geq a_+.$$

This shows the existence of a collection of good events $G_{m,n}$ such that

(a) $\mathbb{P}_{\lambda,a}[G_{m,n}] \geq 1 - \epsilon$ and (b) $E_{m,n} \cap G_{m,n} \subset E_{m',n'}$ for all $(m', n') \leftarrow (m, n)$.

The lemma then follows from [9, Theorem A.4] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To deduce the theorem, we start the process ξ_t with a single player at the origin. Because $\lambda > 0$, there is a positive probability that Λ_- is fully occupied at time one, so we may assume that the process starts from $\xi_0 = \Lambda_-$ instead. In addition, because Lemma 4.2 applies to the modified process $\bar{\xi}_t$, the events $G_{m,n}$ in Lemma 4.3 can be made measurable with respect to the graphical representation in a bounded space-time box. This shows that the range of dependence of the percolation process is finite, so we can fix $\epsilon > 0$ small to make the percolation process supercritical. Since in addition the set of good sites dominates the set of wet sites for all $a \geq a_+$, the process survives.

5   Proof of Theorem 1.3 (extinction for $a > -\infty$ small)

This section is devoted to the proof of Theorem 1.3, which states that, for all $\lambda < \infty$ even large, the process dies out provided the payoff coefficient a is sufficiently small. Like in Theorem 1.2, the basic idea is that isolated players and players with at least one neighbor may have fitnesses that differ strongly, except that the effects are now reversed: though it might be likely that isolated players give birth quickly, once they give birth and have one neighbor (their offspring), they are no longer likely to give birth when a is small. The proof again relies on a block construction, but the technical details are somewhat more complicated. To begin with, we consider the process starting with a single individual in the limit $a = -\infty$. In this case, adjacent individuals cannot give birth and the process essentially behaves like a symmetric random walk that dies after a geometric number of jumps. This implies that the length of the invasion path decays exponentially in both space and time (radius and time to extinction). Because the birth rate decreases with the local density of individuals, the process starting from a general configuration is dominated stochastically by a system of such independent random walks starting with one particle per site. The exponential decay implies that, regardless of the configuration outside a large space-time box and with probability close to one, a large space-time region around the center of the box is not reached by any of the random walks/invasion paths (dead region). This is

used to prove percolation of the dead regions under a suitable space-time rescaling. Once the space and time scales are fixed, we can use a perturbation argument to prove that percolation still occurs for all $a > -\infty$ sufficiently small. Extinction of the process then follows from the fact that the percolation parameter can be chosen close enough to one to ensure the lack of percolation of the closed sites (corresponding to potentially occupied blocks), and the fact that individuals cannot appear spontaneously.

Fast extinction for $a = -\infty$**.** We first study the process in the limiting case $a = -\infty$ starting with a single player at the origin, whose law is denoted by $\mathbb{P}^0_{\lambda,-\infty}$. Let

$$T = \inf\{t : \xi_t = \varnothing\} = \text{time to extinction}.$$

Lemma 5.1. *For every* $\lambda \geq 0$*, there exists a constant* $\beta = \beta(\lambda, d) > 0$ *such that*

$$\mathbb{P}^0_{\lambda,-\infty}[T \geq t] \leq e^{-\beta t} \quad \text{for all} \quad t \geq 0.$$

Proof. Since $a = -\infty$, two adjacent players cannot give birth, so there can be at most two players alive at the same time. By the superposition property, the time until one of the two dies is distributed as an exponential random variable with parameter two. Now, if there is just a single player, the probability that it dies before it reproduces is equal to $1/(1 + \lambda)$. Therefore, the total number of generations N in which exactly two players are alive is geometrically distributed:

$$\mathbb{P}^0_{\lambda,-\infty}[N > n] \leq \left(1 - \frac{1}{1+\lambda}\right)^n = \left(\frac{\lambda}{1+\lambda}\right)^n = \left(1 + \frac{1}{\lambda}\right)^{-n}, \quad n \in \mathbb{N}. \tag{5.1}$$

In particular, T can be written as a sum of independent exponential random variables:

$$T = \sum_{i=1}^{N} (T_i + S_i) + U_{N+1}$$

where the $(T_i)_{i\in\mathbb{N}}$ are exponential random variables with parameter two, the $(S_i)_{i\in\mathbb{N}}$ are exponential random variables with parameter λ, and the $(U_i)_{i\in\mathbb{N}}$ are exponential random variables with parameter one. In particular, T is a hypoexponential random variable. Since we are interest in the case where λ is large (and so typically $\lambda > 1$), and we do not need very precise bounds, we can upperbound the time to extinction T by the sum $\tilde{T}_N$ of $2N + 1$ exponential random variables with parameter one. Then, $\tilde{T}_N$ is a Gamma$(2N + 1, 1)$ random variable. For a deterministic n, one can use the following Chernoff bound for the tails of the Gamma distribution:

$$\mathbb{P}^0_{\lambda,-\infty}[\tilde{T}_n \geq t] \leq \frac{e^{-\theta t}}{(1-\theta)^{2n+1}}. \tag{5.2}$$

Combining (5.1) and (5.2) implies that, for any $\theta < 1$ and $m \in \mathbb{N}$,

$$\mathbb{P}^0_{\lambda,-\infty}[T \geq t] \leq \sum_{n=1}^{m} \mathbb{P}[\tilde{T}_n \geq t] + \mathbb{P}[N > m] \leq m\left(\frac{e^{-\theta t}}{(1-\theta)^{2m+1}}\right) + e^{-m\ln(1+1/\lambda)}.$$

Taking for example $\theta = 1/2$ and $m = \lfloor tr \rfloor$ with $r < 1/4\ln(2)$ gives the result. □

Looking closely at the proof of the previous lemma, one easily deduces the following result, which also gives us an exponential decay of the radius of the invasion paths.

Lemma 5.2. *Let $\xi^0 \subset \mathbb{Z}^d$ be the (random) subset of vertices that are ever occupied. Then, for every birth rate $\lambda \geq 0$, there exists a constant $\delta = \delta(\lambda, d) > 0$ such that*

$$\mathbb{P}^0_{\lambda, -\infty}[|\xi^0| > n] \leq e^{-\delta n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. Recall from the proof of Lemma 5.1 that the number of generations N for which our process survives is a geometric random variable with parameter $1/(1 + \lambda)$. Since we can only visit at most one new site per generation, it follows from (5.1) that

$$\mathbb{P}^0_{\lambda, -\infty}[|\xi^0| > n] \leq \mathbb{P}[N > n] = (1 + 1/\lambda)^{-n} = e^{-n \ln(1 + 1/\lambda)},$$

which is exactly the exponential decay we were looking for. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Comparison to non-interacting copies. In order to use union bounds, it will be convenient to compare the process with arbitrary initial condition to a family of non-interacting copies of the process started with a single occupied site. For all $A \subset \mathbb{Z}^d$, we let ξ^A_t denote the process starting from the set A occupied and, as previously, simply write ξ^z_t when $A = \{z\}$. Having a collection $\{(\xi^x_t)_{t \geq 0} : x \in A\}$ of non-interacting copies of our process, we let

$$\Xi^A_t(x) = \sum_{z \in A} \xi^z_t(x) \quad \text{for all} \quad x \in \mathbb{Z}^d.$$

With this notation at hand, we are ready to state the following comparison result.

Lemma 5.3. *There exists a coupling such that $\mathbb{P}_{\lambda, -\infty}[\xi^A_t \leq \Xi^A_t \ \forall t] = 1$.*

In particular, the set of sites occupied by the original process at time t is always contained in the set of sites occupied by at least one of the non-interacting copies.

Proof. To compare the two processes, we differentiate the players assuming that, for all $z \in A$, site z is initially occupied by a type z player. Then, in the limiting case $a = -\infty$, we can construct all the processes from the following graphical representation:

- ■ **Births**. For each $z \in A$, equip each $\vec{xy}$, $x \sim y$, with a rate $\lambda/2d$ exponential clock. At the times t the clock rings, draw an arrow $(x, t) \xrightarrow{z} (y, t)$.

- ■ **Deaths**. For each $z \in A$, equip each x with a rate one exponential clock. At the times t the clock rings, put a cross $\times_z$ at the space-time point (x, t).

The crosses have the same effects on both processes: a cross $\times_z$ at site x kills a type z particle at that site. The arrows, however, have different effects. The process ξ^A_t is constructed by assuming that, if the tail x of a type z arrow is occupied by a type z player, and none of the neighbors of x is occupied, then the head y of the arrow becomes occupied by a type z player. In contrast, the system of non-interacting copies Ξ^A_t is constructed by assuming that, if the tail of a type z arrow is occupied by a type z player, and none of the neighbors of site x is occupied by a type z player, then the head y of the arrow becomes occupied by a type z player. Because the condition for giving birth in the first process is more restrictive (no players of any type in the neighborhood as opposed to no type z players), if there is a type z player at (x, t) in the first process then there is a type z player at (x, t) in the second process, which proves the lemma. Note that there is at most one type z player at each site in both processes, but sites can be occupied by multiple players with different types in the system of non-interacting copies. $\qquad\qquad\qquad\qquad\qquad\qquad\square$
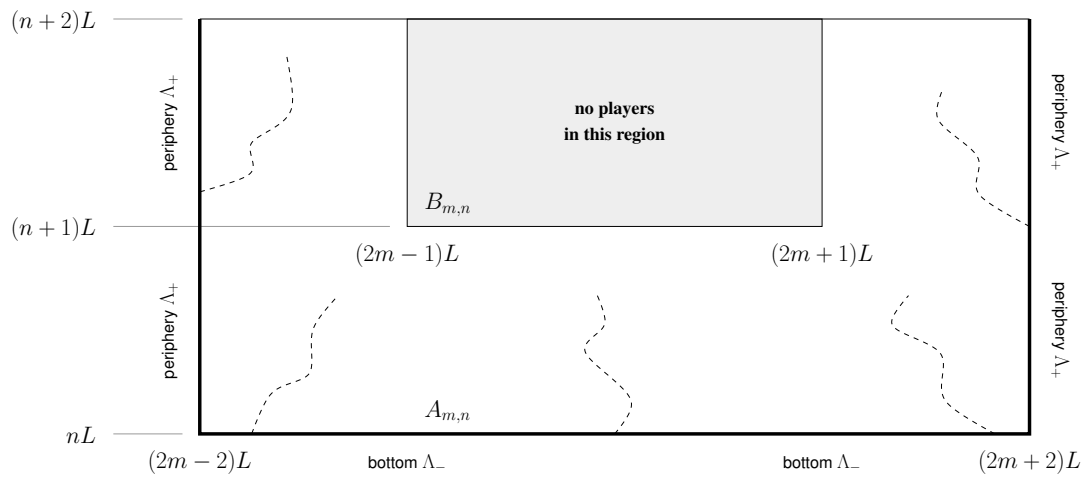
Figure 4: Illustration of the event $E_{m,n}$ when $d = 1$. With high probability, the invasion paths starting from the bottom or the periphery of the big space-time box $A_{m,n}$ do not reach the smaller space-time box $B_{m,n}$.

Block construction. Using the exponential decay (in space and time) of the invasion paths and the stochastic domination in the previous two sections, we can now use a block construction to prove extinction (and more importantly control the rate of extinction) of the process. Let $\mathscr{L}_d = \mathbb{Z}^d \times \mathbb{N}$, which we turn into a directed graph $\vec{\mathscr{L}}_d$ by placing an edge

$$(m, n) \to (m', n') \quad \text{if and only if} \quad |m_1 - m'_1| + \cdots + |m_d - m'_d| + |n - n'| = 1 \text{ and } n \leq n'.$$

In other words, starting from each site (m, n), there are $2d$ "horizontal" arrows that we can think of as potential invasions in space, and one "vertical" arrow that we can think of as a potential persistence in time. To rescale the interacting particle system in space and time, we let L be a large integer, and define the space-time blocks

$$A_{m,n} = (2mL, nL) + [-2L, 2L]^d \times [0, 2L],$$
$$B_{m,n} = (2mL, nL) + [-L, L]^d \times [L, 2L],$$

for all $(m, n) \in \mathscr{L}_d$. See Figure 4 for a picture. We call (m, n) a good site whenever

$$E_{m,n} = \{\text{the space-time block } B_{m,n} \text{ is empty}\} \quad \text{occurs.}$$

We now prove that, when $a = -\infty$ and regardless of the configuration outside the space-time box $A_{m,n}$, the event $E_{m,n}$ occurs with probability close to one when L is large.

Proposition 5.4. *Let $\epsilon > 0$. Then, $\mathbb{P}_{\lambda,-\infty}[E_{m,n}] \geq 1 - \epsilon/2$ for sufficiently large $L \in \mathbb{N}$.*

Proof. In view of the stochastic domination in Lemma 5.3, it suffices to prove the result for the process that evolves according to ξ_t outside the space-time block $A_{m,n}$ but according to Ξ_t inside the block. The basic idea is to use Lemma 5.1 to prove that the invasion paths starting from the bottom of the block cannot live too long, and Lemma 5.2 to prove that the invasion paths starting from the periphery cannot go too far. More precisely, let

$$\Lambda_- = \{(x, t) \in A_{m,n} : t = nL\} = \text{bottom of } A_{m,n},$$
$$\Lambda_+ = \{(x, t) \in A_{m,n} : \|x - 2mL\|_\infty = 2L\} = \text{periphery of } A_{m,n},$$

and write $\Lambda_\pm \to B_{m,n}$ to indicate an invasion path going from $\Lambda_\pm$ to the block $B_{m,n}$. Because players cannot appear spontaneously, we have

$$\mathbb{P}_{\lambda,-\infty}[E_{m,n}^c] \leq \mathbb{P}_{\lambda,-\infty}[\Lambda_- \to B_{m,n}] + \mathbb{P}_{\lambda,-\infty}[\Lambda_+ \to B_{m,n}]. \tag{5.3}$$

Now, by the domination in Lemma 5.3 and the exponential decay in Lemma 5.1,

$$\mathbb{P}_{\lambda,-\infty}[\Lambda_- \to B_{m,n}] \leq |\Lambda_-| \times \mathbb{P}_{\lambda,-\infty}[\xi_L^0 \neq \varnothing] \leq (4L+1)^d \, e^{-\beta L}. \tag{5.4}$$

Dealing with the invasion paths coming in the block $A_{m,n}$ through its periphery Λ_+ is more complicated because the number of such paths is random. Note that the number paths is dictated by the process ξ_t whereas their length is upperbounded by their counterpart for Ξ_t. To deal with the number of invasion paths coming in through the periphery, let

$$H_{m,n} = \{\text{there are less than } 2e\,4dL(4L+1)^{d-1}\lambda$$
$$\text{players outside } A_{m,n} \text{ giving birth onto } \Lambda_+\}.$$

Because the births in the process ξ_t occur at rate at most λ, the number of births onto Λ_+ is dominated by the Poisson random variable with parameter

$$\lambda|\Lambda_+| \leq 2d(4L+1)^{d-1} \times 2L\lambda.$$

Using the standard tail bound for the Poisson distribution, we get

$$\mathbb{P}_{\lambda,-\infty}[H_{m,n}^c] \leq 2\exp(-4Ld(4L+1)^{d-1}\lambda - 2e\,4Ld(4L+1)^{d-1}\lambda\log(2)).$$

This, together with Lemmas 5.2 and 5.3 and a union bound, implies that

$$\begin{aligned}
\mathbb{P}_{\lambda,-\infty}[\Lambda_- \to B_{m,n}] &\leq \mathbb{P}_{\lambda,-\infty}[\Lambda_- \to B_{m,n} \,|\, H_{m,n}] + \mathbb{P}_{\lambda,-\infty}[H_{m,n}^c] \\
&\leq 2e\,4dL(4L+1)^{d-1}\lambda\,\mathbb{P}_{\lambda,-\infty}^0[|\xi^0| > L] + \mathbb{P}_{\lambda,-\infty}[H_{m,n}^c] \\
&\leq 2e\,4dL(4L+1)^{d-1}\lambda\,e^{-\delta L} \\
&\quad + 2\exp(-4dL(4L+1)^{d-1}\lambda - 2e\,4dL(4L+1)^{d-1}\lambda\log(2)).
\end{aligned} \tag{5.5}$$

Finally, plugging (5.4) and (5.5) into (5.3), and noticing that both can be made arbitrarily small by choosing L sufficiently large yield the claim. $\qquad\square$

Perturbation argument. It follows from the block construction that the set of good sites = empty blocks $B_{m,n}$ dominates the set of wet sites (m, n) in the percolation process. Because ϵ can be chosen arbitrarily small, in which case the set of dry (not wet) sites does not percolate, and players cannot appear spontaneously, this shows extinction of the process. To complete the proof of Theorem 1.3, the last step is to transport this result from the limiting case $a = -\infty$ to the case where the payoff coefficient a is small but finite. In the latter case, an individual with a neighbor can give birth, but it is unlikely, so (the graphical representations of) the two processes in a space-time block should agree with high probability, as long as one chooses a sufficiently small, depending on the size of the block. Instead of working with the graphical representation in Lemma 5.3, which was designed to compare our process with the system of non-interacting copies, we construct the processes with $a > -\infty$ and $a = -\infty$ using the following graphical representation:

- **Births**. For each $i = 0, 1, \ldots, 2d$, equip each $\vec{xy}$, $x \sim y$, with a rate $\lambda e^{ia/2d}/2d$ exponential clock. At the times t the clock rings, draw an arrow $(x, t) \xrightarrow{i} (y, t)$.

■ **Deaths**. Equip each x with a rate one exponential clock. At the times t the clock rings, put a cross $\times$ at the space-time point (x, t).

The crosses have the same effects on both processes: a cross $\times$ at site x kills a particle/player at that site. The process with $a > -\infty$ is constructed by assuming that, if the tail x of a type i arrow is occupied, the head y is empty, and exactly i of the neighbors of x are occupied, then the head y of the arrow becomes occupied. The process with $a = -\infty$ is constructed similarly but using only the type 0 arrows, since the other arrows occur at rate zero in the limit. Using this graphical representation, we can now extend Proposition 5.4 to the process with $a > -\infty$ small.

Lemma 5.5. *Let $\epsilon > 0$. Then, $\mathbb{P}_{\lambda,a}[E_{m,n}] \geq 1 - \epsilon$ for all $L \in \mathbb{N}$ large and $a > -\infty$ small.*

Proof. In view of Proposition 5.4, it suffices to show that, with probability arbitrarily close to one, the process with $a = -\infty$ and the process with $a > -\infty$ small agree in the block $A_{m,n}$, i.e., there are no type i arrows for $i \neq 0$ in the block, which prevents players with at least one neighbor to give birth. The overall rate of all the type i arrows for $i \neq 0$ starting at a given site is

$$\lambda e^{a/2d} + \lambda e^{2a/2d} + \lambda e^{3a/2d} + \cdots + \lambda e^{2da/2d} \leq 2\lambda e^{a/2d}$$

for all $a > -\infty$ small. This implies that the number of such arrows that point at the space-time block $A_{m,n}$ is dominated by the Poisson distribution X with parameter

$$|A_{m,n}| \times 2\lambda e^{a/2d} = 2L(4L+1)^d \times 2\lambda e^{a/2d}.$$

Now, the scale parameter $L \in \mathbb{N}$ being fixed as in Proposition 5.4, we define

$$a_- = 2d \ln \left(\frac{-\ln(1 - \epsilon/2)}{4L(4L+1)^d \lambda} \right) > -\infty.$$

Finally, since the event $E_{m,n}$ occurs for the process with $a < a_-$ whenever it occurs for the process with $a = -\infty$ and the two processes agree in the block, it follows from Proposition 5.4 that

$$\begin{aligned} \mathbb{P}_{\lambda,a}[E_{m,n}] &\geq \mathbb{P}_{\lambda,-\infty}[E_{m,n}] \times \mathbb{P}[X = 0] \\ &\geq (1 - \epsilon/2) \times \exp(-2L(4L+1)^d \times 2\lambda e^{a_-/2d}) = (1 - \epsilon/2)^2 \geq 1 - \epsilon \end{aligned}$$

for all $a \leq a_-$. This completes the proof. $\qquad \square$

Using Lemma 5.5 and an idea of [1], we can now conclude the proof of the theorem.

Proof of Theorem 1.3. It follows from Lemma 5.5 that there is a collection of good events $G_{m,n}$ that only depend on the graphical representation in the slightly enlarged space-time blocks

$$A_{m,n}^+ = [-2L - 1, 2L + 1]^d \times [0, 2L], \quad (m, n) \in \mathscr{L}_d,$$

such that, for every $\epsilon > 0$, we can choose the scale parameter $L \in \mathbb{N}$ large, then the payoff coefficient $a > -\infty$ small, to guarantee

$$\mathbb{P}_{\lambda,a}[G_{m,n}] \geq 1 - \epsilon \quad \text{and} \quad G_{m,n} \subset E_{m,n}.$$

This implies that the set of good sites dominates stochastically the set of wet sites in the oriented site percolation process on $\vec{\mathscr{L}}_d$ with parameter $1 - \epsilon$. If $\epsilon > 0$ is small enough, not only the set of wet sites percolates, but also the probability of a path of closed sites with length at least n starting from $(0, 0)$ decays exponentially with n (see [1, Section 8] for a proof). Because the players cannot appear spontaneously, the presence of a player in a space-time block $B_{m,n}$ implies the existence of a path of closed sites to (m, n), which shows extinction of our process. $\qquad \square$

References

[1] J. van den Berg, G. R. Grimmett, and R. B. Schinazi. Dependent random graphs and spatial epidemics. *Ann. Appl. Probab.*, 8(2):317–336, 1998.

[2] C. Bezuidenhout and G. Grimmett. The critical contact process dies out. *Ann. Probab.*, 18(4):1462–1482, 1990.

[3] M. Bramson and R. Durrett. A simple proof of the stability criterion of Gray and Griffeath. *Probab. Theory Related Fields*, 80(2):293–298, 1988.

[4] Y.-T. Chen. Sharp benefit-to-cost rules for the evolution of cooperation on regular graphs. *Ann. Appl. Probab.*, 23(2):637–664, 2013.

[5] Y.-T. Chen. Wright-Fisher diffusions in stochastic spatial evolutionary games with death-birth updating. *Ann. Appl. Probab.*, 28(6):3418–3490, 2018.

[6] P. Clifford and A. Sudbury. A model for spatial conflict. *Biometrika*, 60:581–588, 1973.

[7] J. T. Cox and R. Durrett. Evolutionary games on the torus with weak selection. *Stochastic Process. Appl.*, 126(8):2388–2409, 2016.

[8] J. T. Cox, R. Durrett, and E. A. Perkins. Voter model perturbations and reaction diffusion equations. *Astérisque*, (349):vi+113, 2013.

[9] R. Durrett. Ten lectures on particle systems. In *Lectures on probability theory (Saint-Flour, 1993)*, volume 1608 of *Lecture Notes in Math.*, pages 97–201. Springer, Berlin, 1995.

[10] R. Durrett. Spatial evolutionary games with small selection coefficients. *Electron. J. Probab.*, 19:no. 121, 64, 2014.

[11] Evilsizor, S. and Lanchier, N. (2014). Evolutionary games on the lattice: best-response dynamics. *Electron J. Probab.* **19**, no. 75, 12 pp.

[12] S. Evilsizor and N. Lanchier. Evolutionary games on the lattice: death-birth updating process. *Electron. J. Probab.*, 21:Paper No. 17, 29, 2016.

[13] E. Foxall and N. Lanchier. Evolutionary games on the lattice: death and birth of the fittest. *ALEA Lat. Am. J. Probab. Math. Stat.*, 14(1):271–298, 2017.

[14] T. E. Harris. Contact interactions on a lattice. *Ann. Probability*, 2:969–988, 1974.

[15] T. E. Harris. Additive set-valued Markov processes and graphical methods. *Ann. Probability*, 6(3):355–378, 1978.

[16] J. Hofbauer and K. Sigmund. *Evolutionary games and population dynamics*. Cambridge University Press, Cambridge, 1998.

[17] R. A. Holley and T. M. Liggett. Ergodic theorems for weakly interacting infinite systems and the voter model. *Ann. Probability*, 3(4):643–663, 1975.

[18] N. Lanchier. Evolutionary games on the lattice: payoffs affecting birth and death rates. *Ann. Appl. Probab.* **25** 1108–1154, 2015.

[19] N. Lanchier. Local interactions promote cooperation in cooperation-defector systems. *ALEA Lat. Am. J. Probab. Math. Stat.*, 16(1):399–427, 2019.

[20] T. M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[21] R. Ma and R. Durrett. A simple evolutionary game arising from the study of the role of IGF-II in pancreatic cancer. *Ann. Appl. Probab.*, 28(5):2896–2921, 2018.

[22] J. Maynard Smith. *Evolution and the Theory of Games*. Cambridge university press, 1982.

[23] J. Maynard Smith and G. R. Price. The logic of animal conflict. *Nature*, 246(5427):15–18, 1973.

[24] M. Nanda and R. Durrett. Spatial evolutionary games with weak selection. *Proc. Natl. Acad. Sci. USA*, 114(23):6046–6051, 2017.

[25] C. Neuhauser. Ergodic theorems for the multitype contact process. *Probab. Theory Related Fields*, 91(3-4):467–506, 1992.

[26] M. A. Nowak. *Evolutionary dynamics*. The Belknap Press of Harvard University Press, Cambridge, MA, 2006. Exploring the equations of life.

[27] M. A. Nowak and R. M. May. Evolutionary games and spatial chaos. *Nature*, 359(6398):826–829, 1992.

[28] H. Ohtsuki, C. Hauert, E. Lieberman, and M. A Nowak. A simple rule for the evolution of cooperation on graphs and social networks. *Nature*, 441(7092):502–505, 2006.